Equality constrained quadratic programming problem

General problem

$$\min f(x) = \tfrac{1}{2}\, x^T Q x + c^T x$$

$$\text{s.t. } Ax = b$$

- Q symmetric positive semidefinite matrix of dimension $n \times n$
- c, x vectors of dimension $n \times 1$
- A full rank matrix of dimension $m \times n$
- b vector of dimension $m \times 1$

KKT conditions at a minimum x*:

$$
\begin{cases}
Qx^* + A^T\lambda^* = -c \\
Ax^* = b
\end{cases}
$$

We can introduce the matrix K and the vectors w*, d

$$
K = \begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix} \qquad w^* = \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} \qquad d = \begin{pmatrix} -c \\ b \end{pmatrix}
$$

and rewrite KKT conditions to QP obtaining the linear system $Kw^* = d$

Constrained problem

Considered problem and its constraints

Problem 2

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \textstyle\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} + \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & \text{the sum } x_1 + x_{1+K} + x_{1+2K} + \ldots \text{ should be } 1 \\ & \text{the sum } x_2 + x_{2+K} + x_{2+2K} + \ldots \text{ should be } 1 \\ & \vdots \\ & \text{the sum } x_K + x_{2K} + x_{3K} + \ldots \text{ should be } 1 \end{aligned}$$

Q = diag(2) + upper_diag(-1) + lower_diag(-1),   everything else 0
C = vector of ones
A = [diag(1) every K columns]

Methods used

Presentation of the methods used to solve the problem

FULL SYSTEM FACTORIZATION

- $K = LDL^T/LU$
- fill-in problem

GMRES

- Iterative solver
- $Kw^* = d$

SCHUR-COMPLEMENT APPROACH

- $\mathbf{Q} = AQ^{-1}A^T$
- Generally more efficient

NULL-SPACE METHOD

- Null-space matrix Z
- Computationally expensive

Evaluation metrics

- Elapsed time
- How close we are to the solution:
  - KKT_gradL_norm = $||Qx^* + A^T\lambda + c||$
- How well we respected the constraints:
  - KKT_eq_norm = $||Ax^* - b||$
- Value of the function in x*

Disclaimer

If $n=10^5$ the matrix is sparse, thus results obtained are not completely comparable with $n=10^4$.

$$\begin{bmatrix} X & X & X & \cdot & \cdot & \cdot & \cdot \\ X & X & \cdot & X & X & \cdot & \cdot \\ X & \cdot & X & \cdot & X & \cdot & \cdot \\ \cdot & X & \cdot & X & \cdot & X & \cdot \\ \cdot & X & X & \cdot & X & X & X \\ \cdot & \cdot & \cdot & X & X & X & \cdot \\ \cdot & \cdot & \cdot & \cdot & X & \cdot & X \end{bmatrix}$$

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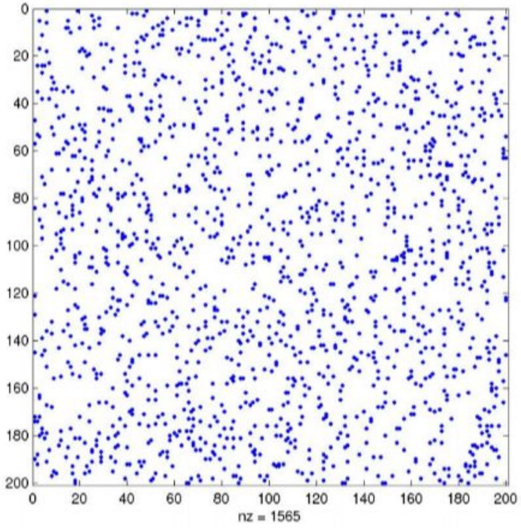
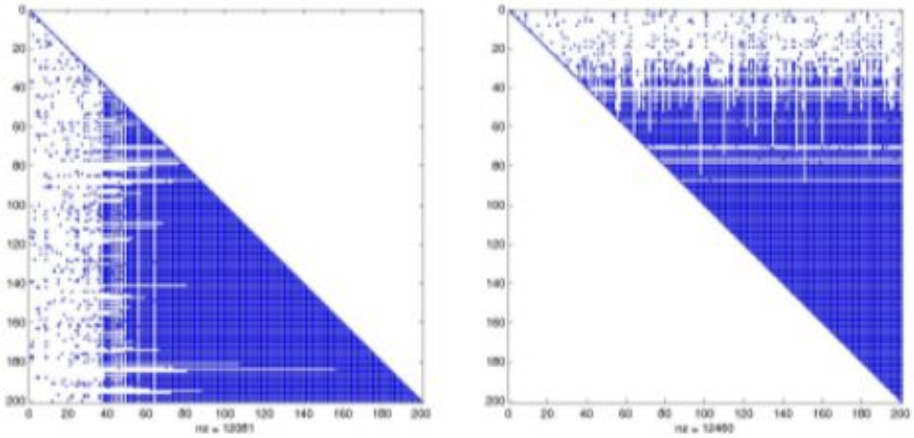
FULL SYSTEM FACTORIZATION

- Symmetric factorization $K=LDL^T/LU$
- Fill-in problem if n+m is very large and K sparse

Fill-in phenomenon



Matrix A



Matrices L, U

GMRES

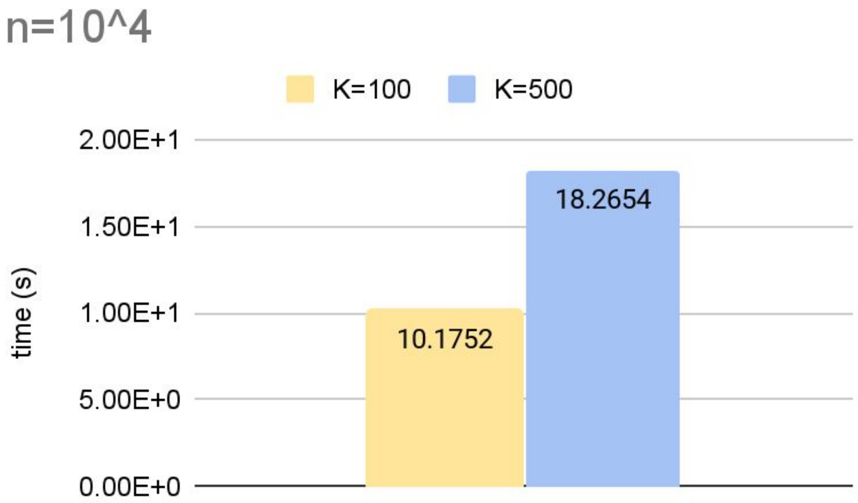
- Iterative solver
- Linearly increasing cost per iterations
- Solve the system Kw*=d

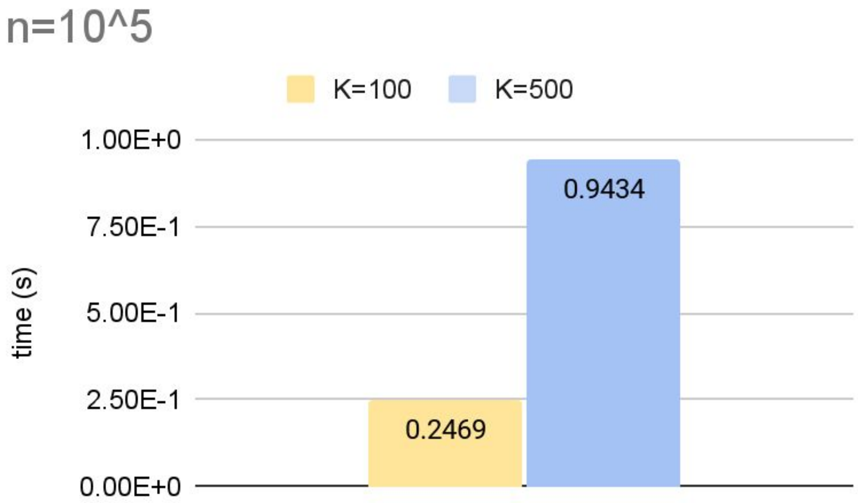
- Parameters used:

tol = 10^{-6}

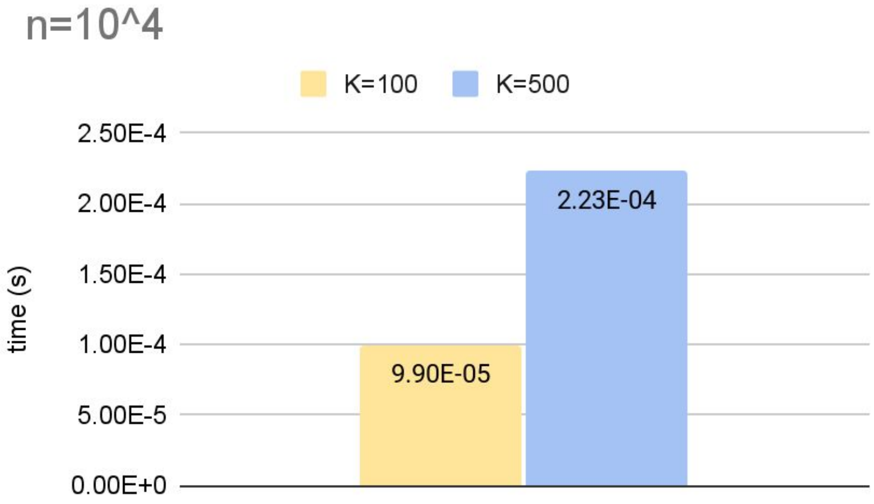
maxit = 200

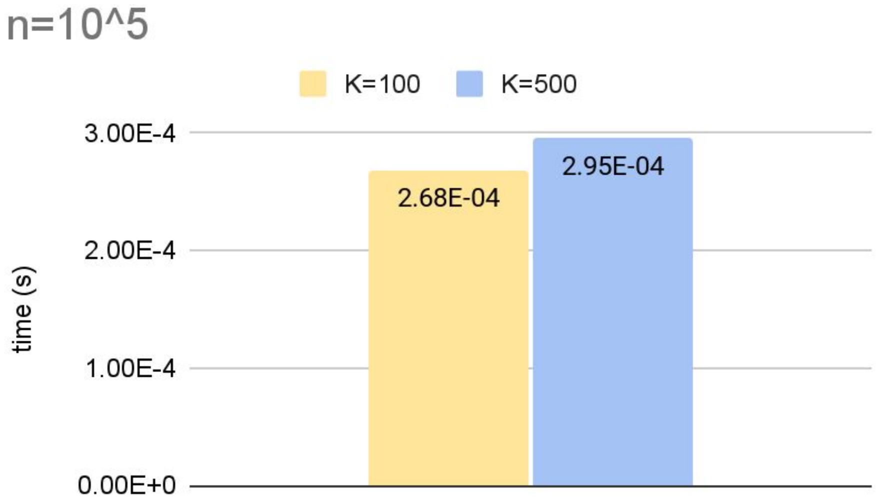
GMRES results: elapsed time



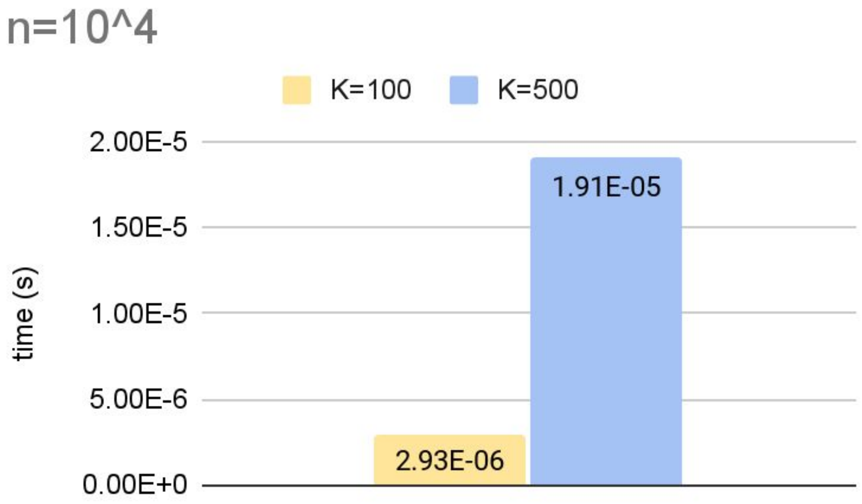


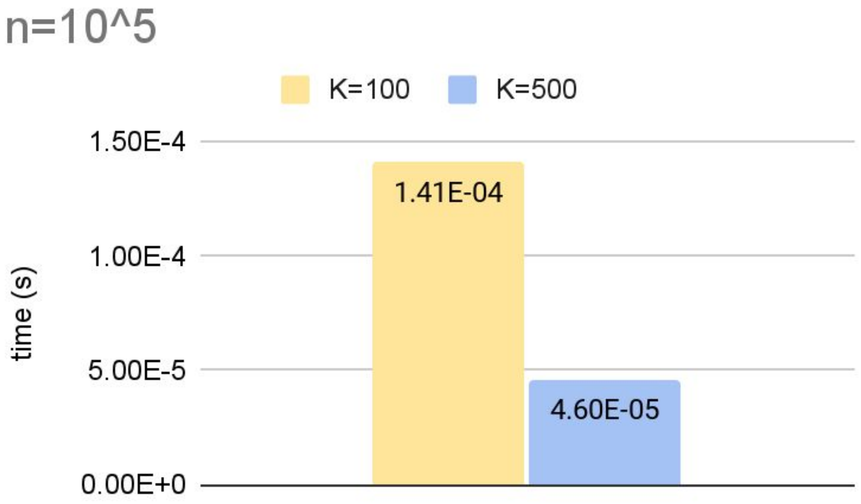
GMRES results: KKT_GradL_norm



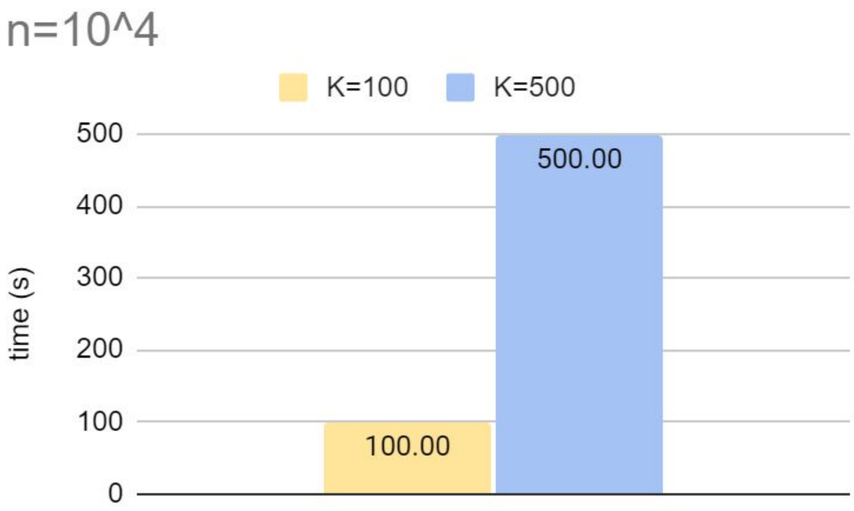


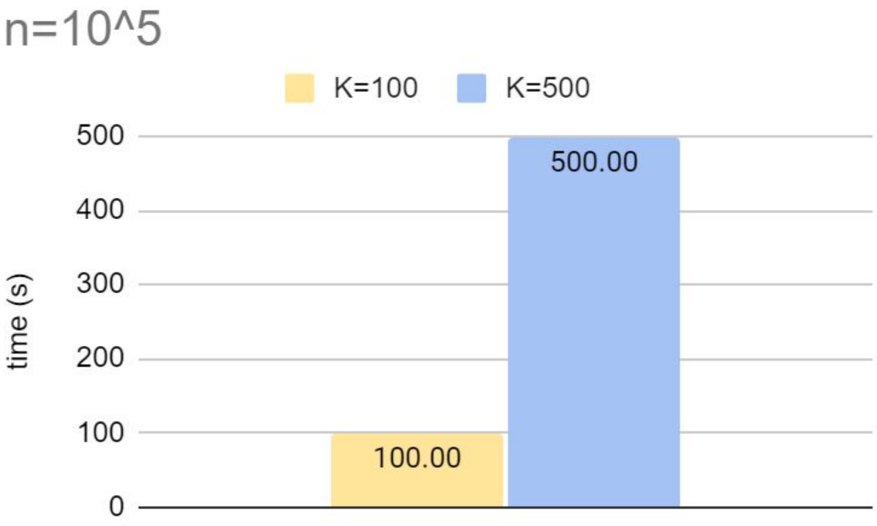
GMRES results: KKT_EQ_norm





GMRES results: f(x*)





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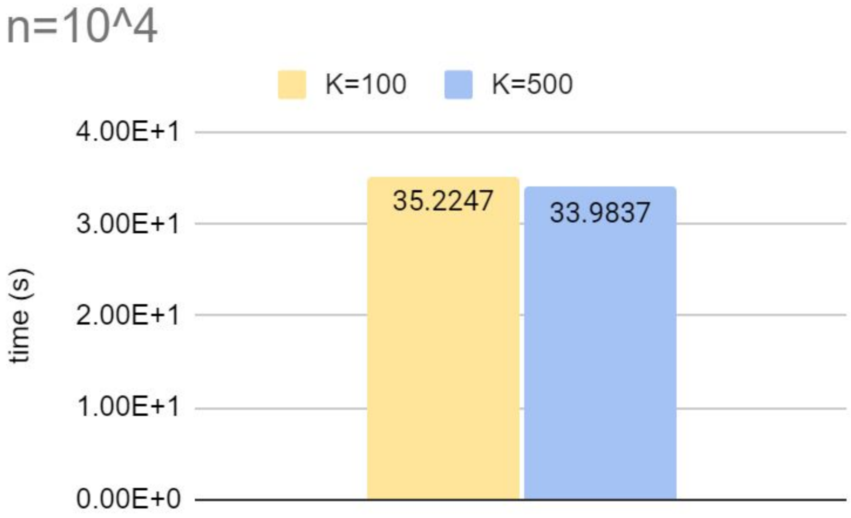
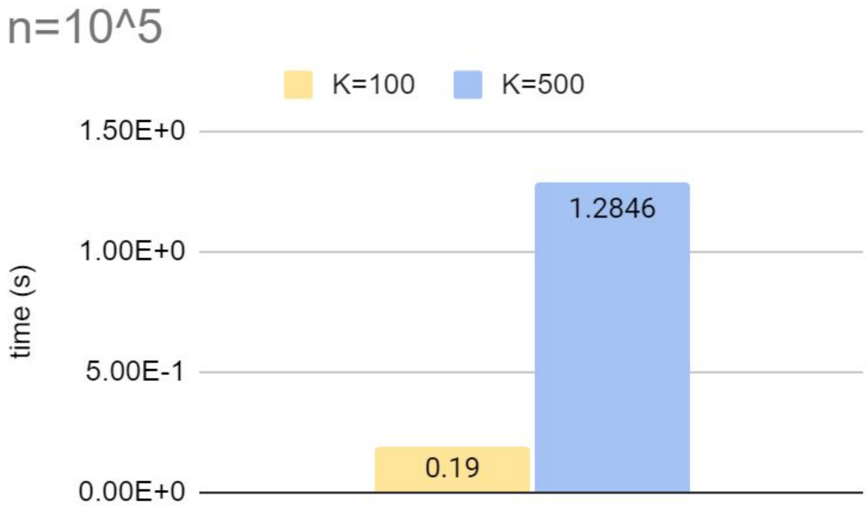
SCHUR-COMPLEMENT APPROACH

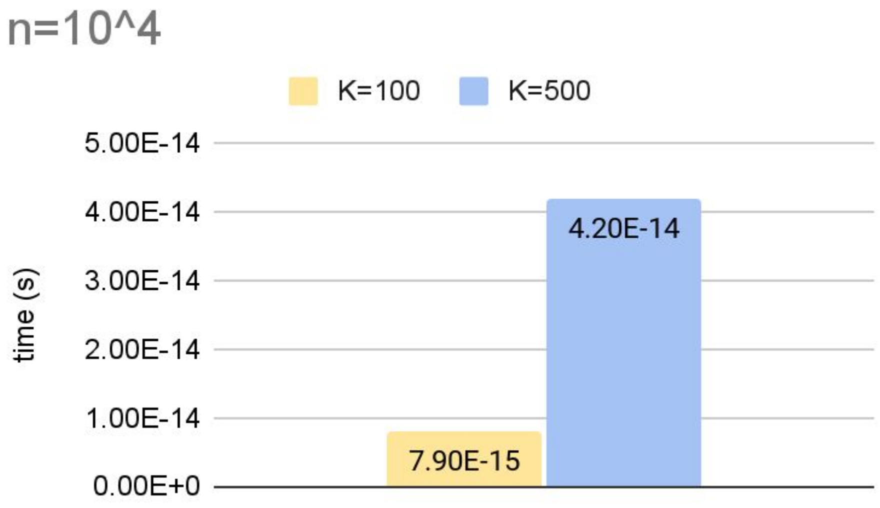
- Q symmetric positive semi-definite
- A full rank
- Efficient method
- $\boldsymbol{Q_{Schur}} = AQ^{-1}A^{T}$

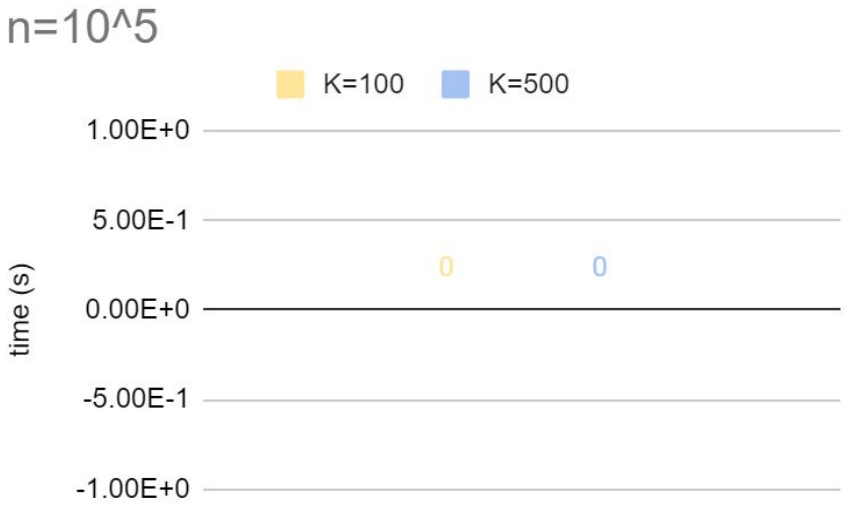
SCHUR results: elapsed time



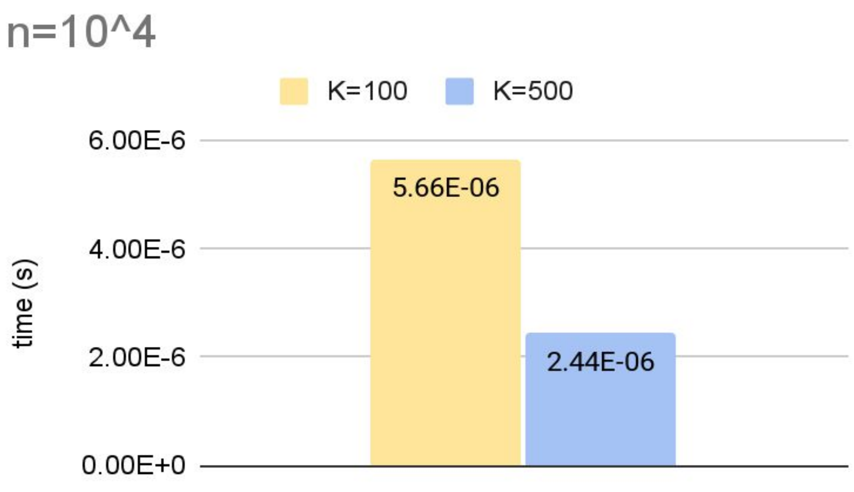


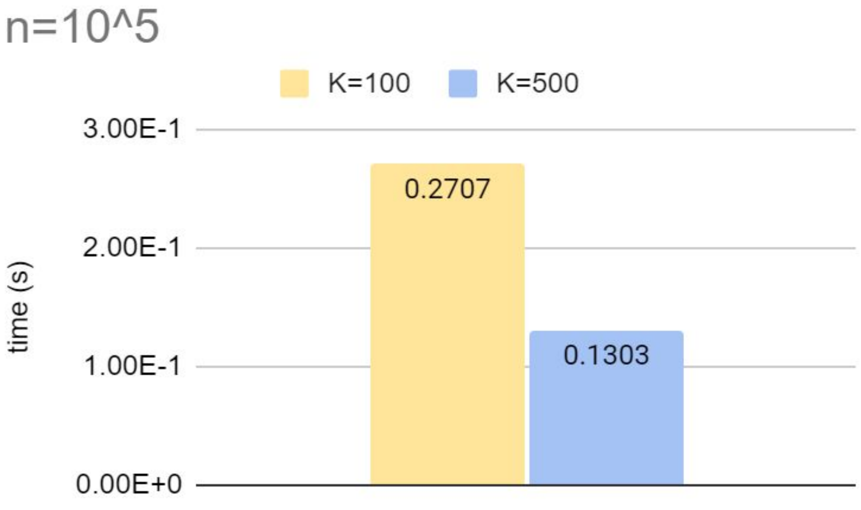
SCHUR results: KKT_GradL_norm



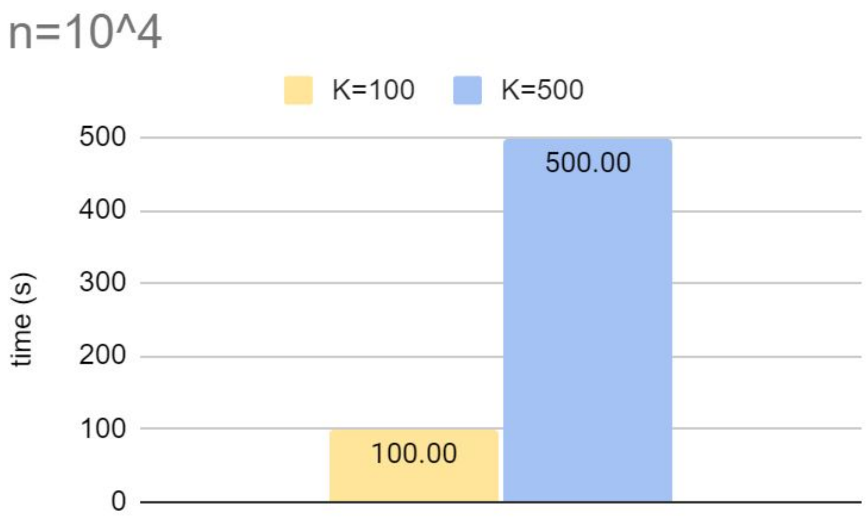


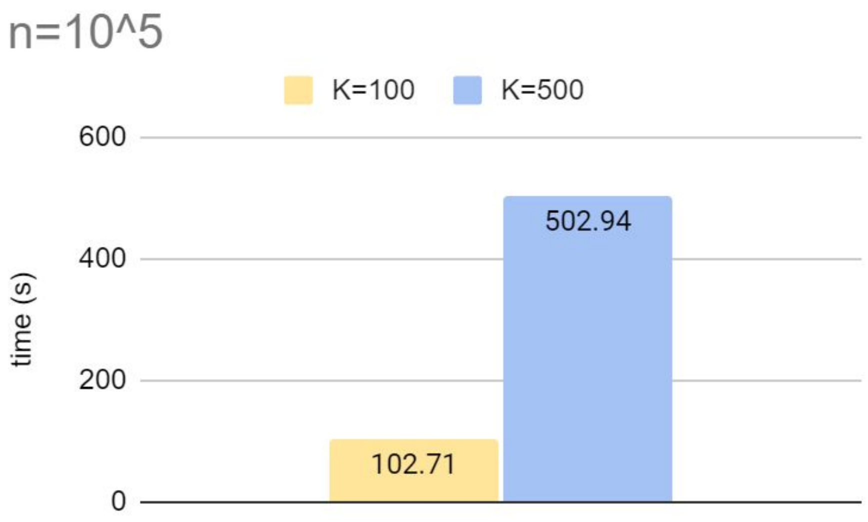
SCHUR results: KKT_EQ_norm





SCHUR results: f(x*)





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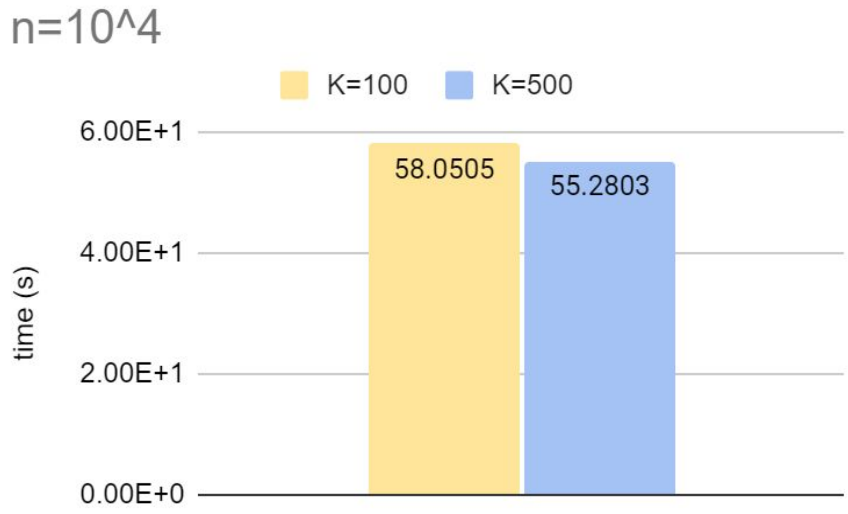
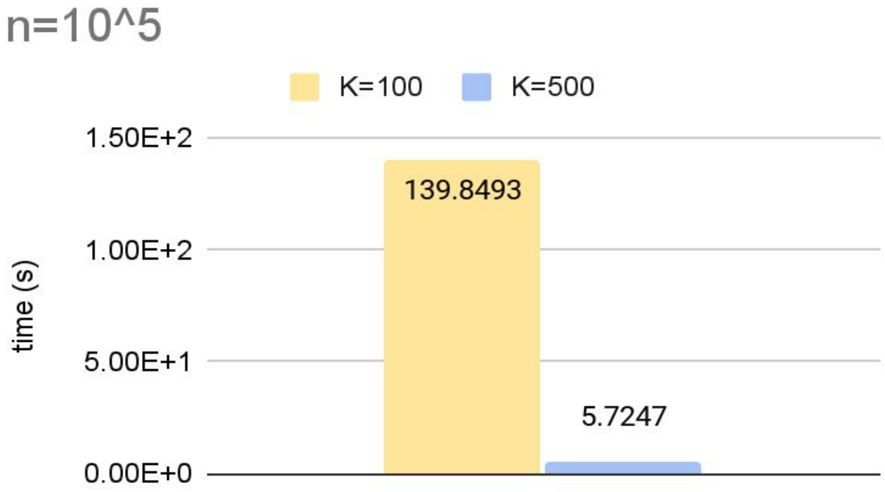
NULL-SPACE METHOD

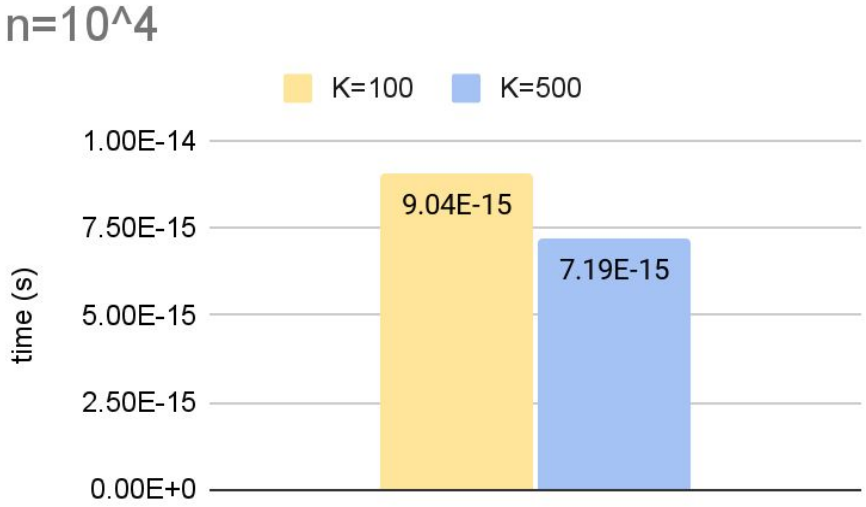
- Does not require non-singularity of Q
- A is a full-rank matrix
- Z has dimension nx(n-m)
- Z^TQZ is positive definite

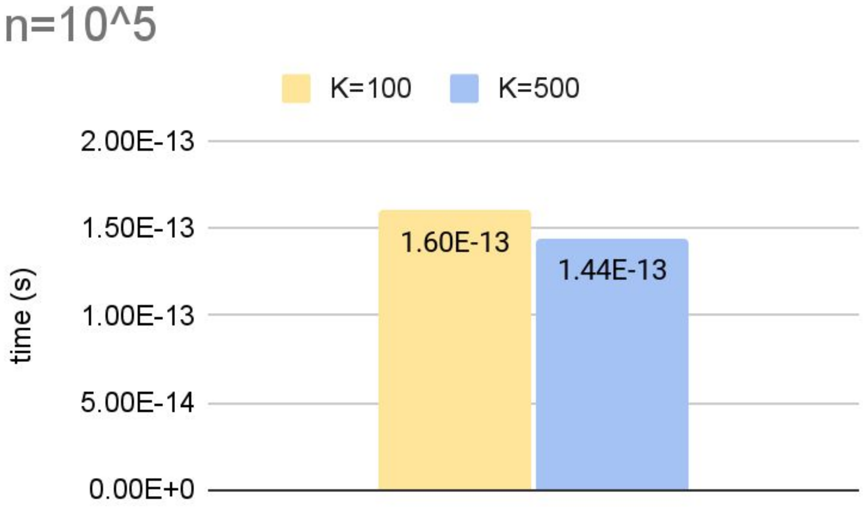
NULL-SPACE results: elapsed time



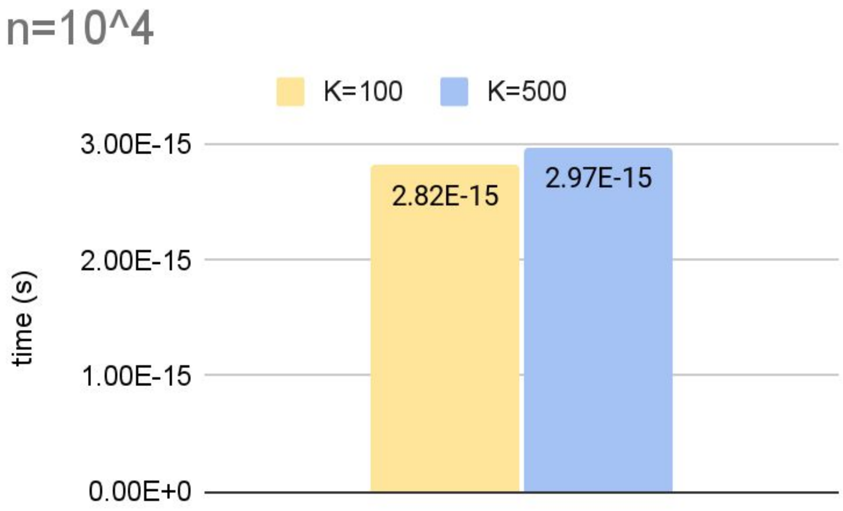


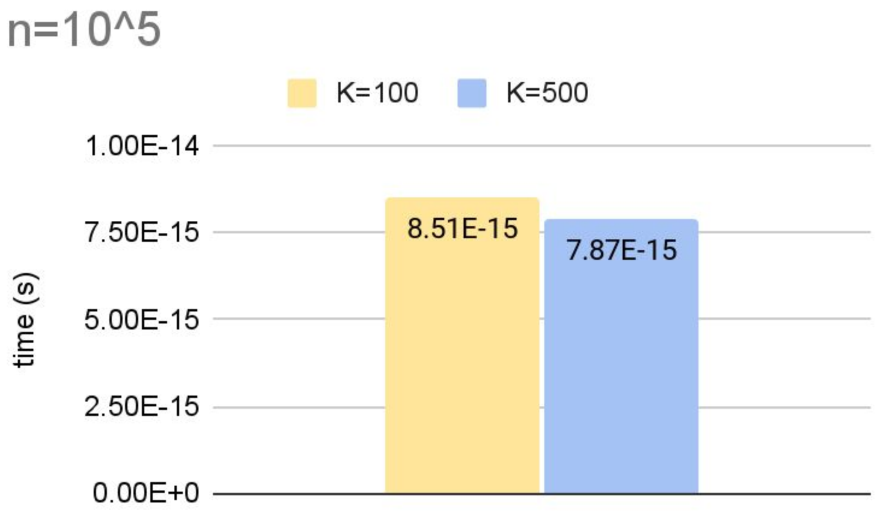
NULL-SPACE results: KKT_gradL_norm



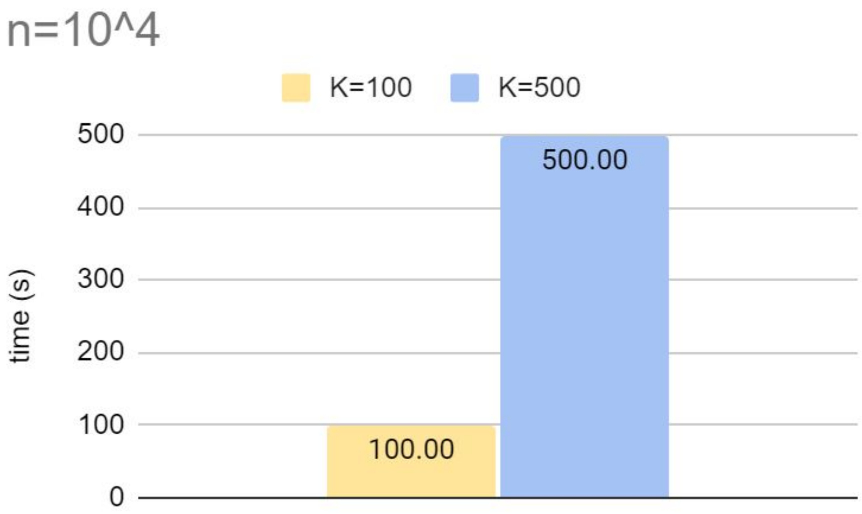


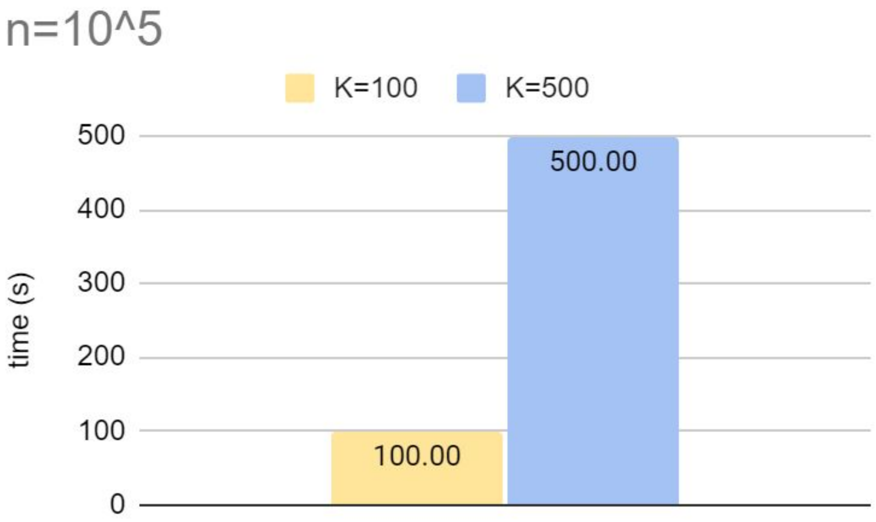
NULL-SPACE results: KKT_EQ_norm





NULL-SPACE results: f(x*)

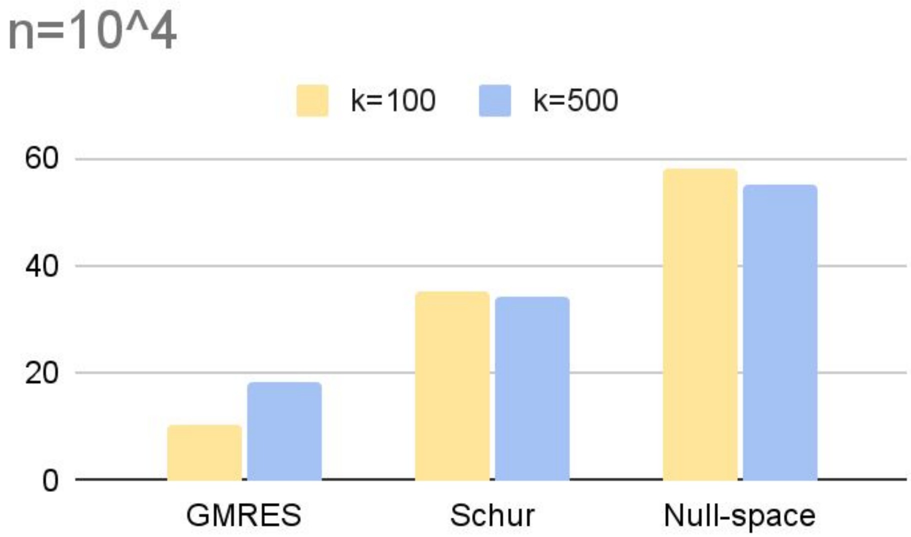


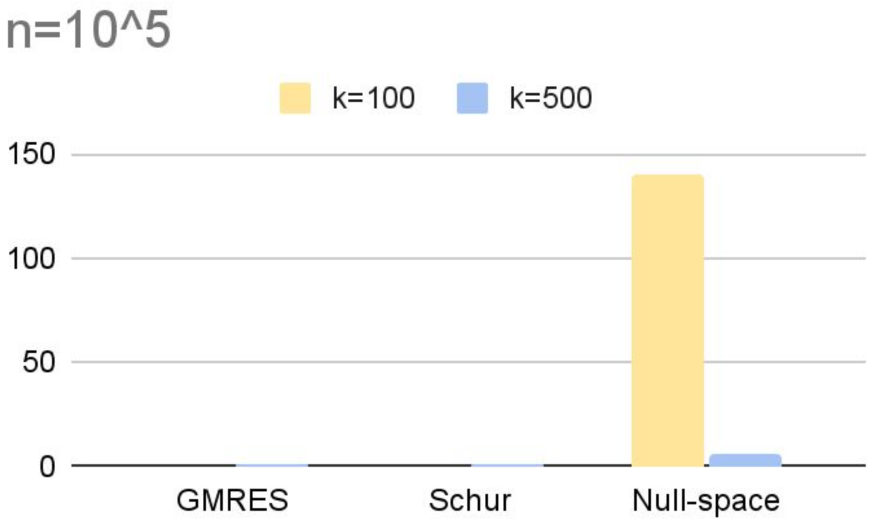


Conclusions

Final considerations and evaluations of the methods in this particular problem

Elapsed time comparison





Conclusions

GMRES

- Lots of iterations
- Lots of computational time to reach high precision

SCHUR

- Computationally efficient
- Fast
- Not completely precise when n is large

NULL-SPACE

- Generally efficient
- Precise
- Can be expensive for some matrix Z

There’s not a method always better, it highly depends on the problem faced and the constraints it has to satisfy.

Team Presentation

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Thanks for the attention!