Statistical Signal Processing

Homework 1

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GitHub repository: `https://github.com/sararst/SSP`

The problem considered in this first homework is related to the roundtrip delay in a computer network between, in my case, my pc and "isl.stanford.edu". The delay is different each time the message is sent so that it can be modeled with a random variable y. Since there was not much other information about y at that moment, several distributions were considered, in particular:

- Gaussian distribution: $f_G\left(y \mid \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$
- Rayleigh distribution: $f_R\left(y \mid \sigma^2\right) = \begin{cases} 0 & , y < 0 \\ \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} & , y \geq 0 \end{cases}$
- Erlang distribution with m={0, 1, 2}: $f_{Em}(y \mid \lambda) = \begin{cases} 0 & , y < 0 \\ \frac{\lambda^{m+1}}{m!} y^m e^{-\lambda y} & , y \geq 0 \end{cases}$
- Shifted exponential density: $f_{\exp}(y \mid \lambda, \alpha) = \begin{cases} 0 & , y < \alpha \\ \lambda e^{-\lambda(y-\alpha)} & , y \geq \alpha \end{cases}$

The goal was to find the distribution that best fit the data.

Point i: Maximum Likelihood Estimation

Assuming that there are n iid measurements of y, all in the vector Y, I determined the Maximum Likelihood Estimate $\hat{\theta}_{ML}$, for each distribution, of the parameter (sometimes more than one, depending on the distribution considered) θ involved.
MLE is a method of estimating the parameters of a probability distribution, given some observed data. This is achieved by maximizing the log-likelihood function so that the observed data is most probable.

$$\hat{\theta}_{ML}(y) = \underset{\theta \in \Theta}{\arg\max} f(y \mid \theta), \quad y = [y_1, y_2, \ldots, y_m]^\top$$

0.1 Gaussian distribution

$$f_G(y \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$f(Y \mid \mu, \sigma^2) = \prod_{i=1}^{n} f(y_i \mid \mu, \sigma^2) = (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right]$$

$$L(\theta; y) = \ln l(\theta; y) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2$$

$$\begin{cases} \frac{\partial}{\partial \mu} L(\theta; Y) = 0 \\ \frac{\partial}{\partial \sigma^2} L(\theta; Y) = 0 \end{cases} \rightarrow \begin{cases} \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu) = 0 \\ -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2 = 0 \end{cases}$$

$$\begin{cases} \hat{\mu}_{ML} = \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y} \quad \text{(sample mean)} \\ \hat{\sigma}^2_{ML} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu)^2 = \frac{1}{m} \sum_{i=1}^{n} (y_i - \bar{y})^2 \quad \text{(sample variance)} \end{cases}$$

0.2 Rayleigh distribution

$$
\begin{aligned}
f_R(y \mid \sigma^2) &= \begin{cases} 0 & y < 0 \\ \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} & y \geq 0 \end{cases} \\
f(Y \mid \sigma^2) &= \prod_{i=1}^{n} \frac{y_i}{\sigma^2} e^{-\frac{y_i^2}{2\sigma^2}} \\
L(\sigma^2; Y) = \ln(\sigma^2; Y) &= \sum_{i=1}^{n} \ln\left(\frac{y_i}{\sigma^2} e^{-\frac{y_i^2}{2\sigma^2}}\right) = \sum_{i=1}^{n} \left[\ln(\frac{y_i}{\sigma^2}) - \frac{y_i^2}{2\sigma^2}\right] \\
\frac{\partial}{\partial\sigma^2} L(\sigma^2; Y) &= \sum_{i=1}^{n} \left[-\frac{1}{\sigma^2} + \frac{y_i^2}{2\sigma^4}\right] = -\frac{n}{\sigma^2} + \frac{\sum_{i=1}^{n} y_i^2}{(2\sigma^2)^2} = 0 \\
\sigma^2 &= \frac{\sum_{i=1}^{n} y_i^2}{2n}, \quad \hat{\sigma}_{ML} = \sqrt{\frac{\sum_{i=1}^{n} y_i^2}{2n}}
\end{aligned}
$$

0.3 Erlang distribution

0.3.1 m=0

$$
\begin{aligned}
f_{E0}(y \mid \lambda) &= \begin{cases} 0 & y < 0 \\ \lambda e^{-\lambda y} & y \geq 0 \end{cases} \\
f_{E0}(Y \mid \lambda) &= \prod_{i=1}^{n} \lambda e^{-\lambda y_i} \\
L(\lambda; Y) = \ln l(\lambda; Y) &= \sum_{i=1}^{n} \ln(\lambda e^{-\lambda y_i}) = \sum_{i=1}^{n} (\ln\lambda - \lambda y_i) \\
\frac{\partial L(\lambda; Y)}{\partial\lambda} &= \sum_{i=1}^{n} \left(\frac{1}{\lambda} - y_i\right) = 0 \\
\hat{\lambda}_{ML} &= \frac{n}{\sum_{i=1}^{n} y_i}
\end{aligned}
$$

0.3.2 m=1

$$
\begin{aligned}
f_{E1}(y \mid \lambda) &= \begin{cases} 0 & y < 0 \\ \lambda^2 y e^{-\lambda y} & y \geq 0 \end{cases} \\
f_{E1}(Y \mid \lambda) &= \prod_{i=1}^{n} \lambda^2 y_i e^{-\lambda y_i} \\
L(\lambda; Y) = \ln l(\lambda; Y) &= \sum_{i=1}^{n} [\ln\lambda^2 + \ln y_i - \lambda y_i] \\
\frac{\partial L(\lambda; Y)}{\partial\lambda} &= \sum_{i=1}^{n} \left(\frac{2}{\lambda} - y_i\right) = 0 \\
\hat{\lambda}_{ML} &= \frac{2n}{\sum_{i=1}^{n} y_i}
\end{aligned}
$$

0.3.3 m=2

$$
\begin{aligned}
f_{E2}(y \mid \lambda) &= \begin{cases} 0 & y < 0 \\ \frac{\lambda^3}{2} y^2 e^{-\lambda y} & y \geq 0 \end{cases} \\
f_{E2}(Y \mid \lambda) &= \prod_{i=1}^{n} \frac{\lambda^3}{2} y_i^2 e^{-\lambda y_i} \\
L(\lambda; Y) &= \ln l(\lambda; Y) = \sum_{i=1}^{n} [\ln \frac{\lambda^3}{2} + \ln y_i^2 - \lambda y_i] \\
\frac{\partial L(\lambda; Y)}{\partial \lambda} &= \sum_{i=1}^{n} \left(\frac{3}{\lambda} - y_i \right) = 0 \\
\hat{\lambda}_{ML} &= \frac{3n}{\sum_{i=1}^{n} y_i}
\end{aligned}
$$

0.3.4 General form

$$
f_{E_m}(Y \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{m+1}}{m!} y_i^m e^{-\lambda y_i}
$$

$$
\begin{aligned}
L(\lambda; Y) = \ln l(\lambda; Y) &= \sum_{i=1}^{n} ((m+1)\ln(\lambda) - \ln(m!) + m \ln(y_i) - \lambda y_i) \\
\frac{\partial L(\lambda; Y)}{\partial \lambda} &= \sum_{i=1}^{n} \left(\frac{m+1}{\lambda} - y_i \right) = 0 \\
\hat{\lambda}_{ML} &= \frac{n(m+1)}{\sum_{i=1}^{n} y_i}
\end{aligned}
$$

0.4 Shifted exponential

$$
f_{\exp}(y \mid \lambda, \alpha) = \begin{cases} 0 & y < \alpha \\ \lambda e^{-\lambda(y-\alpha)} & y \geq \alpha \end{cases}
$$

$$
f(Y \mid \lambda, \alpha) = \prod_{i=1}^{n} \lambda e^{-\lambda(y_i - \alpha)} \quad y \geq \alpha
$$

$$
L(\theta; Y) = \ln l(\theta; Y) = \sum_{i=1}^{n} [\ln \lambda - \lambda(y_i - \alpha)]
$$

$$
\begin{cases} \frac{\partial L(\theta;Y)}{\partial \lambda} = \sum_{i=1}^{n} \left[\frac{1}{\lambda} - y_i + \alpha \right] = 0 \\ \frac{\partial L(\theta;Y)}{\partial \alpha} = \sum_{i=1}^{n} \lambda = 0 \end{cases}
$$

$$
\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} (y_i - \alpha)}
$$

We can't estimate α from the second equation but, since we know that the distribution exists for $y \geq \alpha$, we can suppose that $\alpha\hat{_{ML}} = \min y_i$.

Point ii: shifted Rayleigh distribution

$$f_{SR}\left(y \mid \alpha, \sigma^2\right) = \left\{ \begin{array}{ll} 0 & , y < \alpha \\ \frac{y-\alpha}{\sigma^2} e^{-\frac{(y-\alpha)^2}{2\sigma^2}} & , \alpha \leq y \end{array} \right\} = \frac{y-\alpha}{\sigma^2} e^{-\frac{(y-\alpha)^2}{2\sigma^2}} \mathbf{1}_{[\alpha,\infty)}(y)$$

The values of the distribution depended on the parameters α and σ^2.

0.5 Point a

Here I calculated the mean of y as a function of the parameters α and σ^2.

$$m_y\left(\alpha, \sigma^2\right) = \int_\alpha^\infty y f_{SR}\left(\alpha, \sigma^2\right) dy = \int_\alpha^\infty y \cdot \frac{y-\alpha}{\sigma^2} e^{-\frac{(y-\alpha)^2}{2\sigma^2}} dy = -\left[y e^{-\frac{(y-\alpha)^2}{2\sigma^2}} \Big|_\alpha^\infty - \int e^{-\frac{(y-\alpha)^2}{2\sigma^2}} dy \right]$$

The second integral can be seen as the error function:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$\begin{aligned}
&\left(\tfrac{(y-\alpha)}{\sqrt{2}\sigma}\right)^2 = t^2 \rightarrow t = \tfrac{y-\alpha}{\sqrt{2}\sigma} \rightarrow y = \sqrt{2}\sigma t + \alpha \\
&\tfrac{dt}{dy} = \tfrac{1}{\sqrt{2}\sigma} \rightarrow dt = \tfrac{1}{\sqrt{2}\sigma} dy, dy = \sqrt{2}\sigma dt
\end{aligned}$$

$$\begin{aligned}
y = \alpha \rightarrow & t = 0; \quad y = \infty \rightarrow t = \infty \\
&= -\left[y e^{-\frac{(y-\alpha)^2}{2\sigma^2}} \Big|_\alpha^\infty - \int_{t=0}^{t=\infty} \sqrt{2}\sigma e^{-t^2} dt \right] \\
&= -\left[y e^{-\frac{(y-\alpha)^2}{2\sigma^2}} \Big|_\alpha^\infty - \frac{\sqrt{\pi}}{2} \cdot \sqrt{2}\sigma \cdot \frac{2}{\sqrt{\pi}} \int_{t=0}^{t=\infty} e^{-t^2} dt \right] \\
&= -\left[0 - \alpha - \sqrt{\frac{\pi}{2}}\sigma \right] = \sqrt{\frac{\pi}{2}}\sigma + \alpha
\end{aligned}$$

The obtained result is in line with the fact that the distribution is shifted.

0.6 Point b

$$\begin{aligned}
\sigma_y^2\left(\alpha, \sigma^2\right) &= E(Y^2) - (E(Y))^2 \\
E(Y^2) &= \int_\alpha^\infty y^2 fsR\left(\alpha, \sigma^2\right) dy = \int_\alpha^\infty y^2 \cdot \frac{y-\alpha}{\sigma^2} e^{-\frac{(y-\alpha)^2}{2\sigma^2}} dy \\
&= -\left[y^2 e^{-\frac{(y-\alpha)^2}{2\sigma^2}} \Big|_\alpha^\infty - \int_\alpha^\infty 2y e^{-\frac{(y-\alpha)^2}{2\sigma^2}} dy \right] \\
&= -[0 - \alpha^2 - 2 \int_\alpha^\infty y e^{-\frac{(y-\alpha)^2}{2\sigma^2}} dy \Big]
\end{aligned}$$

Now we can do the same substitutions explained in the point above

$$\begin{aligned}
&= -\left[0 - \alpha^2 - 2 \int_0^\infty (\sqrt{2}\sigma t + \alpha) e^{-t^2} \sqrt{2}\sigma dt \right] \\
&= \alpha^2 + 2 \int_0^\infty 2\sigma^2 t e^{-t^2} dt + \sqrt{\pi}\alpha\sqrt{2}\sigma \underbrace{\int_0^\infty \frac{2}{\sqrt{\pi}} e^{-t^2} dt}_{erf(\infty)} \\
&= \alpha^2 - 2\sigma^2 \int_0^\infty -2t e^{-t^2} dt + \sqrt{2\pi}\alpha\sigma \\
&= \alpha^2 - 2\sigma^2 \left[e^{-t^2} \Big|_0^\infty \right. + \sqrt{2\pi}\alpha\sigma \\
&= \alpha^2 + 2\sigma^2 + \sqrt{2\pi}\alpha\sigma
\end{aligned}$$

Taking $E(Y)$ from point b, we obtain:

$$\sigma_y^2\left(\alpha, \sigma^2\right) = \alpha^2 + 2\sigma^2 + \sqrt{2\pi}\alpha\sigma - (\sqrt{\frac{\pi}{2}}\sigma + \alpha)^2 = \sigma^2(2 - \frac{\pi}{2})$$

0.7 Point c

We now have n iid measurements y_i in the vector Y: $\quad Y = [y_1, y_2, \ldots, y_m]^\top$
I calculated the log-likelihood function, depending on the parameters α and σ^2.

$$f_{SR}\left(Y \mid \alpha, \sigma^2\right) = \prod_{i=1}^{n}\left[\frac{y_i-\alpha}{\sigma^2} e^{-\frac{(y_i-\alpha)^2}{2\sigma^2}} \mathbf{1}_{[\alpha,\infty)}\left(y_i\right)\right]$$

$$L\left(\alpha, \sigma^2; Y\right) = \ln l\left(\alpha, \sigma^2; Y\right) = \sum_{i=1}^{n}\left[\ln\left(\frac{y_i-\alpha}{\sigma^2}\right) - \frac{\left(y_i-\alpha\right)^2}{2\sigma^2} \cdot 1[\alpha,\infty)\left(y_i\right)\right]$$

$$\rightarrow \text{ Log-likelihood: } L\left(\alpha, \sigma^2; Y\right) = \sum_{i=1}^{n}\left[\ln\left(y_i-\alpha\right) - \ln\left(\sigma^2\right) - \frac{\left(y_i-\alpha\right)^2}{2\sigma^2}\mathbf{1}(\alpha,\infty)\left(y_i\right)\right]$$

0.8 Point d

To reduce the range of possible values for $\alpha \geq 0$ we need to consider the log-likelihood above. To be defined, the argument of the log must be greater than 0 $\sum_{i=1}^{n} \ln(y_i - \alpha)$ defined for $y_i > \alpha$ This implies that the range of α for which the log-likelihood takes on finite values is α ¡ $\min(y_i) \rightarrow \alpha \in [0, min(y_i)[$

0.9 Point e

$$L\left(\alpha, \sigma^2; Y\right) = \sum_{i=1}^{n}\left[\ln\left(y_i-\alpha\right) - \ln\left(\sigma^2\right) - \frac{\left(y_i-\alpha\right)^2}{2\sigma^2}\right]$$

$$\frac{\partial L\left(\alpha, \sigma^2; Y\right)}{\partial \sigma^2} = \sum_{i=1}^{n}\left[-\frac{1}{\sigma^2} + \frac{\left(y_i-\alpha\right)^2}{2} \cdot \frac{1}{\sigma^4}\right] = 0$$

$$\frac{n}{\sigma^2} = \sum_{i=1}^{n} \frac{\left(y_i-\alpha\right)^2}{2} \cdot \frac{1}{\sigma^4}$$

$$\hat{\sigma}^2(Y,\alpha) = \frac{1}{2n}\sum_{i=1}^{n}\left(y_i-\alpha\right)^2$$

0.10 Point f

$$L\left(\alpha, \hat{\sigma}^2(Y,\alpha); Y\right) = \sum_{i=1}^{n} \ln\left(y_i-\alpha\right) - n\ln\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_i-\alpha\right)^2\right] - \frac{\sum_{i=1}^{n}\left(y_i-\alpha\right)^2}{2} \cdot \frac{2n}{\sum_{i=1}^{n}\left(y_i-\alpha\right)^2}$$

$$= \sum_{i=1}^{n} \ln\left(y_i-\alpha\right) - n\ln\left[\frac{1}{2n}\sum_{i=1}^{n}\left(y_{i-\alpha}\right)^2\right] - n$$

$$= \sum_{i=1}^{n} \ln\left(y_i-\alpha\right) - n\ln\left[\frac{1}{2n}\right] - n\ln\sum_{i=1}^{n}\left(y_i-\alpha\right)^2 - n$$

$$\frac{\partial L}{\partial \alpha} = -\sum_{i=1}^{n}\left(\frac{1}{y_i-\alpha}\right) + \frac{2n}{\sum_{i=1}^{n}\left(y_i-\alpha\right)^2} = 0$$

$$\frac{2n\sum_{i=1}^{n}\left(y_i-\alpha\right)}{\sum_{i=1}^{n}\left(y_i-\alpha\right)^2} \stackrel{!}{=} \sum_{i=1}^{n}\left(\frac{1}{y_i-\alpha}\right)$$

The first requirement cannot be satisfied analytically since the problem is NP-hard. I tried to solve it using numerical methods, in particular with fmincon [1] , constraining the value of α in the range $[0, min(y_i)[$ found in point d, starting with an estimation of it equal to half of the minimum value of y_i.
The obtained value is $\alpha\hat{_{ML}} = 133.7798$.
The corresponding $\hat{\sigma^2}_{ML}(Y)$ is then given from point e above, substituting $\alpha\hat{_{ML}} \rightarrow \hat{\sigma^2}_{ML}(Y) = 236.8982$

[1] https://it.mathworks.com/help/optim/ug/fmincon.html

```
% Define the negative log-likelihood function
negative_log_likelihoodSR = @(alphaSR) -(-sum(1./(Y - alphaSR)) + (2 * n *
   sum(Y - alphaSR)) / sum((Y - alphaSR).^2));
% Initial guess for alpha
initial_alphaSR = min(Y) / 2;
lb = 0;
ub = min(Y);
% Optimize using fminsearch
[optimal_alphaSR, max_likelihoodSR] = fmincon(negative_log_likelihoodSR,
   initial_alphaSR, [], [], [], [], lb, ub, []);

alphaSR = optimal_alphaSR;
estimatedSR_sigma = 1/(2*n) * sum(Y-alphaSR).^2;
```

1 Point iii: roundtrip delay

After generating n=100 measurements of the roundtrip delay between my PC and 'isl.stanford.edu', I inserted them into the Y vector.
I substituted the previously derived expressions in point i and calculated the numerical values for each distribution parameter.

```
n = 100;

Y = pingstats('isl.stanford.com', n, 'v');

% Gaussian distribution
estimatedG_m = mean(Y);
estimatedG_sigma = std(Y);

% Rayleigh distribution
estimatedR_sigma = sqrt((sum(Y.^2))/(2*n));

% Erlang distribution
estimatedEm_lambda_val = zeros(1, 3);
for m = 0:2
    estimatedEm_lambda = n*(m+1)/sum(Y);
    estimatedEm_lambda_val(m+1) = estimatedEm_lambda;
end

% Shifted exponential distribution
estimatedExp_alpha = min(Y);
estimatedExp_lambda = n/(sum(Y-estimatedExp_alpha));

% Shifted Rayleigh, using alphaSR estimated before
estimatedSR_sigma = 1/(2*n) * sum(Y-alphaSR).^2;
```

I obtained the following values:

- Maximum Likelihood Estimator (MLE) for Gaussian distribution
  - $\mu_G = 154.18$
  - $\sigma_G = 7.6295$
- Maximum Likelihood Estimator (MLE) for Rayleigh distribution: $\sigma_R = 109.1538$
- Maximum Likelihood Estimator (MLE) for Erlang distribution
  - m = 0: $\lambda_{E0} = 0.0064859$
  - m = 1: $\lambda_{E1} = 0.012972$
  - m = 2: $\lambda_{E2} = 0.019458$
- Maximum Likelihood Estimator (MLE) for shifted exponential distribution:
  - $\alpha_{exp} = 146$
  - $\lambda_{exp} = 0.12225$
- Maximum Likelihood Estimator MLE for Shifted Rayleigh distribution: $\sigma_{SR} = 15.3915$

2 Point iv: plot

Figure 1 shows a plot of the histogram of the measurements $\{y_1, ..., y_n\}$ in a linspace between $y_{min} - 20$ and $y_{max} + 20$. That was superimposed to the plot for each distribution's marginal densities $f_i(y|\hat{\theta}_{ML,i}(Y)$, $i \in \{G, R, E0, E1, E2, exp, SR\}$.

```
% calculate the range for y
y_min = min(Y);
y_max = max(Y);
y_range = linspace(y_min-20, y_max+20, 1000);

% Gaussian distribution
pdf_gaussian = normpdf(y_range, estimatedG_m, estimatedG_sigma);

% Rayleigh distribution
pdf_rayleigh = raylpdf(y_range, estimatedR_sigma);

% Erlang distribution
pdf_erlang0 = exppdf(y_range, 1/estimatedEm_lambda_val(1));
pdf_erlang1 = estimatedEm_lambda_val(2).^(2)*y_range.*exp(-y_range*
   estimatedEm_lambda_val(2));
pdf_erlang2 = (estimatedEm_lambda_val(3).^(3)/2)*(y_range.^2).*exp(-y_range*
   estimatedEm_lambda_val(3));

% Shifted exponential distribution
pdf_exponential_shifted = estimatedExp_lambda*exp(-estimatedExp_lambda*(
   y_range-estimatedExp_alpha));

% Shidted Rayleigh distribution
pdfSR = (y_range - alphaSR) / estimatedSR_sigma^2 .* exp(-(y_range - alphaSR
   ).^2 / (2*estimatedSR_sigma^2));

% plot histogram
figure;
histogram(Y, 'Normalization', 'probability', 'EdgeColor', 'w');
hold on;

% superimpose graphs for marginal densities
plot(y_range, pdf_gaussian, 'LineWidth', 2.5, 'DisplayName', 'Gaussian');
plot(y_range, pdf_rayleigh, 'LineWidth', 2.5, 'DisplayName', 'Rayleigh');
plot(y_range, pdf_erlang0, 'LineWidth', 2.5, 'DisplayName', 'Erlang␣(m=0)');
plot(y_range, pdf_erlang1, 'LineWidth', 2.5, 'DisplayName', 'Erlang␣(m=1)');
plot(y_range, pdf_erlang2, 'LineWidth', 2.5, 'DisplayName', 'Erlang␣(m=2)');
plot(y_range, pdf_exponential_shifted, 'LineWidth', 2.5, 'DisplayName', '
   Shifted␣Exponential');
plot(y_range, pdfSR , 'LineWidth', 2.5, 'DisplayName', 'Shifted␣Rayleigh');

% add labels and legend
xlabel('Roundtrip␣delay');
ylabel('Probability␣density')
title('Histogram␣and␣superimposed␣densities')
legend('Histogram', 'Gaussian', 'Rayleigh', 'Erlang␣m=0', 'Erlang␣m=1', '
   Erlang␣m=2', 'Shifted␣Exponential', 'Shifted␣Rayleigh')
grid on;

hold off;
```

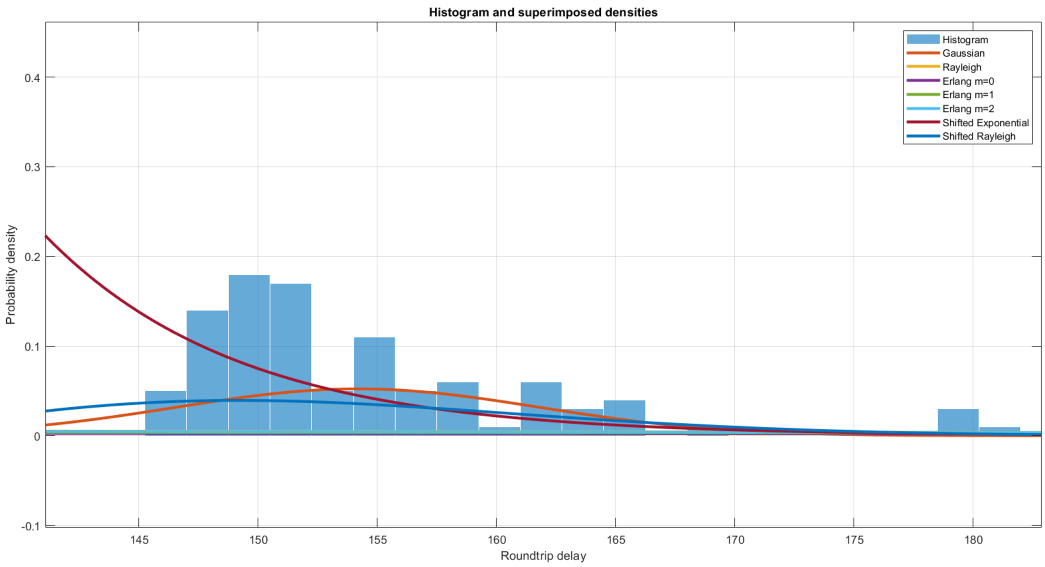

Figure 1: Hinstogram and superimposed densities

3 Point v: best distribution

To determine the best choice of distribution, based on the Maximum Likelihood Criterion, I calculated the values of all the log-likelihood functions of each distribution using as parameters the ones estimated above.

```
log_likelihood_gaussian = -n*log(2*pi)/2 - n*log(estimatedG_sigma.^2)/2 -(
    sum(Y-estimatedG_m).^2)/(2*estimatedG_sigma.^2);
log_likelihood_rayleigh = sum(log(Y/estimatedR_sigma^2) - Y.^2/(2*
    estimatedR_sigma^2));
log_likelihood_erlang0 = sum(log(estimatedEm_lambda_val(1)) -
    estimatedEm_lambda_val(1)*Y);
log_likelihood_erlang1 = sum(2*log(estimatedEm_lambda_val(2))+log(Y)-
    estimatedEm_lambda_val(2)*Y);
log_likelihood_erlang2 = sum(3*log(estimatedEm_lambda_val(3)) - log(2) + 2*
    log(Y) -estimatedEm_lambda_val(3)*Y);
log_likelihood_shiftedExp = sum(log(estimatedExp_lambda)-estimatedExp_lambda
    *(Y-estimatedExp_alpha));
log_likelihood_SR = sum(log(Y - alphaSR) -log(estimatedSR_sigma^2) - (Y-
    alphaSR).^2/(2*estimatedSR_sigma.^2));

% List of distribution names
distribution_names = {'Gaussian', 'Rayleigh', 'Erlang (m=0)', 'Erlang (m=1)'
    , 'Erlang (m=2)', 'Shifted Exponential', 'Shifted Rayleigh'};

log_likelihoods = [log_likelihood_gaussian, log_likelihood_rayleigh,
    log_likelihood_erlang0, log_likelihood_erlang1, log_likelihood_erlang2,
    log_likelihood_shiftedExp, log_likelihood_SR];

% Determine the best distribution
[best_log_likelihood, best_distribution_index] = max(log_likelihoods);

best_distribution_name = distribution_names{best_distribution_index};
```

Obtaining the following:

- Log-Likelihood for Gaussian distribution: -295.0965
- Log-Likelihood for Rayleigh distribution: -534.8547
- Log-Likelihood for Erlang distribution (m=0): -603.8121

- Log-Likelihood for Erlang distribution (m=1): -565.2978
- Log-Likelihood for Erlang distribution (m=2): -543.7735
- Log-Likelihood for shifted Exponential distribution: -310.1692
- Log-Likelihood for shifted Rayleigh distribution: -350.811

Here, the best distribution is Gaussian. This result is not completely in line with the plot of Fig. 1, in which it seems that maybe the Shifted Exponential distribution could fit well the data.
After that, I changed the value of n to $n = 500$ (Fig. 2) and $n = 1000$ (Fig. 3) and, after running the experiment, I obtained that best distribution is the Shifted Exponential.
In any case, in the plots the distributions don't fit the data completely well.

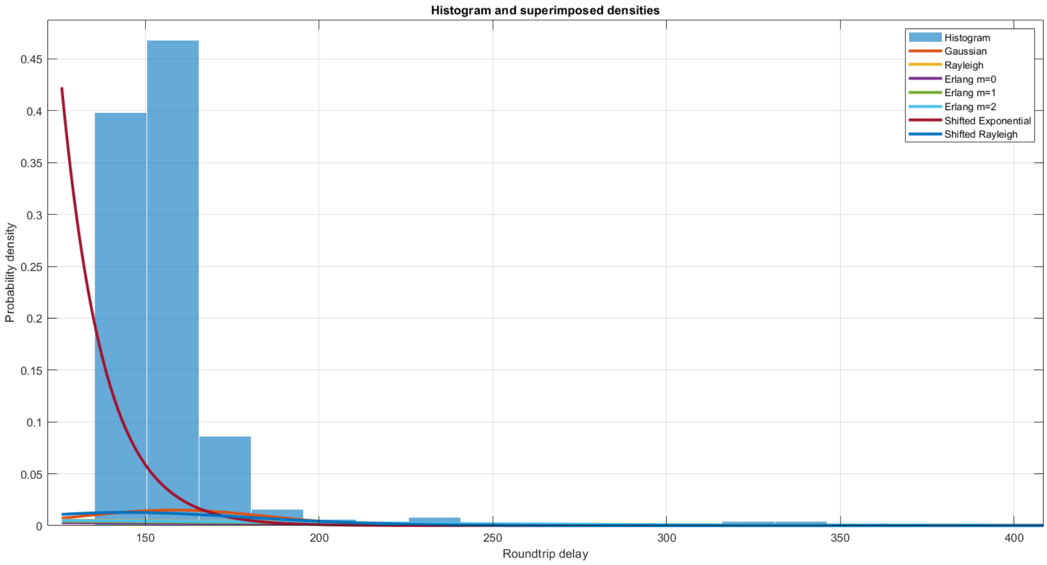

Figure 2: Hinstogram with superimposed densities, n=500

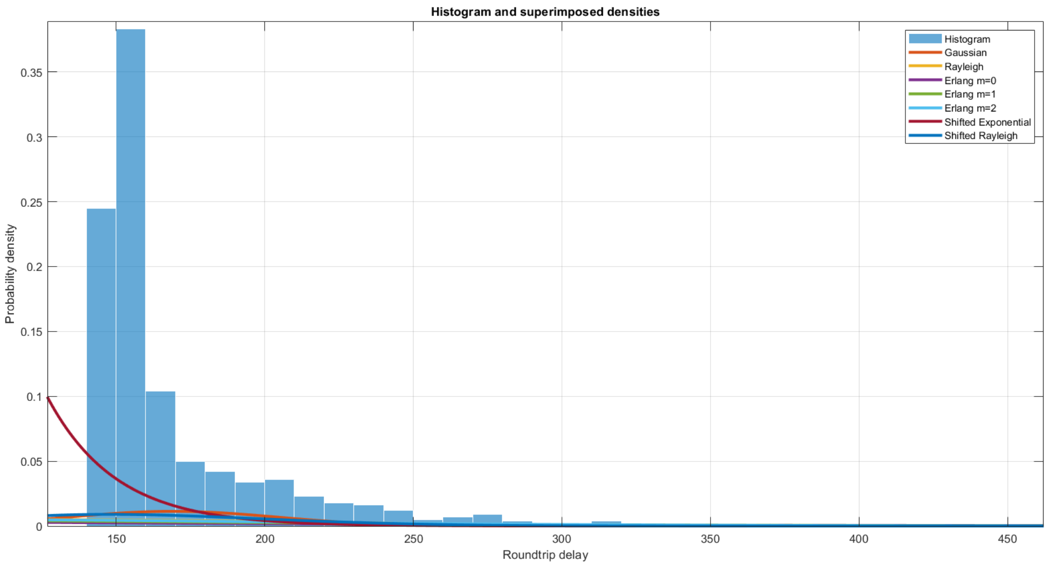

Figure 3: Hinstogram with superimposed densities, n=1000