COMPUTATIONAL METHODS IN PLASMA PHYSICS

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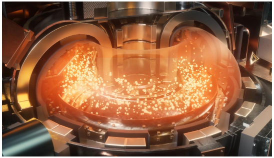
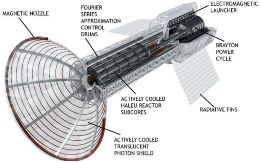
Introduction

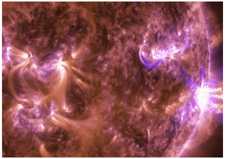
Plasma is a state of matter that results from a gas having undergone ionization. Accurate modeling of plasma is essential for the advancement of a broad range of technologies:

1. Advancing tokamak (a toroidal machine that utilizes magnetic fields to confine plasmas) technology could significantly boost the efficiency of fusion energy systems.
2. Current spacecraft designs are unable to sustain such complex plasma dynamics, but a deeper computational understanding could provide a blueprint for expansion.
3. Models of astronomical phenomena will have higher accuracy leading to a clearer understanding of our universe.

SOME AWESOME FIGURES







Experimental Advanced Superconducting Tokamak (EAST), Pulsed Plasma Rocket (PPR), The Sun's Plasma

Non-dimensionalize MHD equations using

- Alfvén velocity $V_A = \frac{B}{\sqrt{\mu_0 n M_i}}$
- Alfvén time $\tau_A = \frac{a}{V_A}$
- Time over which resistive effects are most important $\tau_R = \frac{\mu_0 a^2}{\eta}$

Terms relating to resistivity thus are multiplied by the inverse magnetic Lundquist number $S^{-1} \equiv \frac{\tau_R}{\tau_A}$. If we only care about fast time scales, we can neglect these terms. Which leads us to ...

A very rich set of equations!

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{2}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) + \nabla p = \mathbf{J} \times \mathbf{B} \tag{3}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0 \tag{4}$$

with current density $\mathbf{J} \equiv \mu_o^{-1} \nabla \times \mathbf{B}$ and ratio of specific heats $\gamma = \frac{5}{3}$. Non-dissipative solutions for large spatial scales are fundamentally unchanged.

Eigenvalue Problem

Idea: Suppose we have a function in variational formulation, i.e., find the function $U(x)$ which minimizes a given functional $\mathcal{F}(U)$. The most basic approach are the Galerkin methods. Essentially, they follow this game plan:

- Select a class of test functions in selected subspace, say $\varphi \in X$.
- Multiply φ through the equation.
- Integrate!!!

Utilize multivariate integration by parts to push derivatives from the partial differential equation to the test functions.

One use of finite elements is to tackle problems of the form

$$\mathcal{L}(U) = \lambda U$$

with $\mathcal{L}$ the self-adjoint operator, λ the scalar eigenvalue, and U the unknown. Observe that this is precisely the form of the linear ideal MHD equation of motion!

Idea: We can use the Galerkin method!

- Multiply $\mathcal{L}(U) = \lambda U$ by a test function φ then integrate by parts.
- Use the expansion $U = \sum_{j=1}^{N} a_j \phi_j$, with $\phi_1(x), \ldots, \phi_N(x)$ a basis that spans X, and require the weak form holds $\forall \varphi \in X$
- We get a matrix eigenvalue problem: $K_{jk} a_k = \lambda^k M_{jk} a_k$

Example Application

WHAT IS A PLASMA COLUMN?

A plasma column is a large aspect ratio toroidal plasma surrounded by a vacuum region which is surrounded by a conducting wall.

Think: fusion

The equations for the displacement and magnetic vector potential of the plasma in a spectral pollution minimizing torus is

$$\xi = R^2 \nabla U \times \nabla \phi + \omega R^2 \nabla \phi + R^{-2} \nabla_\perp \chi$$
$$\mathbf{A} = R^2 \nabla \phi \times \nabla f + \psi \nabla \phi - F_0 \ln R \hat{Z}$$

with (R, ϕ, Z) the cylindrical coordinates.

Plasma Column Exhibit at the Technorama Swiss Science Center

The associated eigenvalue problem is

$$\omega^2 \mathbf{A} \cdot \mathbf{x} = \mathbf{B} \cdot \mathbf{x}$$

where

$$A_{ij} = \rho_0 \int_0^a r dr \phi_i' \phi_j' + \rho_0 m^2 \int_0^a dr \frac{1}{r} \phi_i \phi_j$$

$$B_{ij} = \int_0^a r dr (F\phi_i)'(F\phi_j)' + m^2 \int_0^a dr \frac{1}{r} F^2 \phi_i \phi_j + mF^2 \left[\frac{1 + (a/b)^{2m}}{1 - (a/b)^{2m}} \right] \delta_{NN}$$

$$\mathbf{x} = [a_0, a_1, \ldots, a_N]$$

If equilibrium current density is constant, $J_0 = \Delta\psi_0$, $F(r) = F_0$ is constant. Thus, we have an analytic solution, i.e., there is just one unstable mode which is given by

$$\rho_0\gamma^2 = J_0 F - \frac{2F^2}{1-(a/b)^{2m}}$$

Research Directions

MHD-Kinetic Plasma Models

- Tokamaks are often modeled using coupled MHD-kinetic equations to capture both macroscopic and microscopic behavior.
- Because of the dependence on spatial scaling, it is imperative to accurately simulate the spatial scale-switch in modeling the MHD-kinetic equations.
- If this is not handled appropriately, there will be abrupt, nonphysical jumps in the solution of the system.

MY PROPOSED WORK

- Divergence-free (div-free) methods are more stable.
- I aim to investigate identifying a scale switching indicator between the MHD and kinetic portions.
- While existing hybrid methods rely on heuristic switchers, I predict that a theoretical error bound derived from the div-free hybridized discontinuous Galerkin formulation can be utilized.

Please reach out to me for more information, I would be happy to discuss my full proposal!

THANK YOU, PLEASE FEEL FREE TO ASK QUESTIONS!

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