Numerical Analysis of the Reimann Problem for a Cosmological 2x2 Balance System

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Introduction

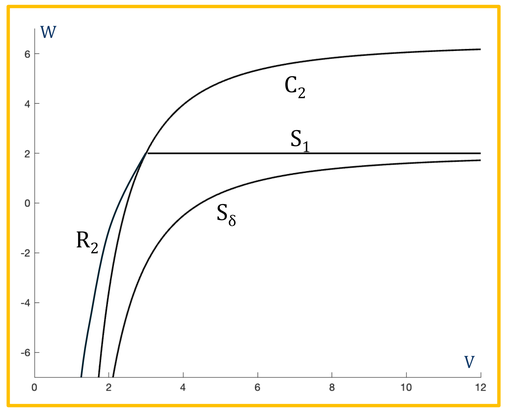
- Chaplygin gas is a cosmological model to connect dark matter and dark energy
- The given equation of state is:

$$p = A\rho^{\gamma}e^{\eta t}$$

- We combine this with a 2 x 2 Keyfitz -Kranzer type balance system in the form:

$$\rho_t + (\rho(u - p(\rho))_x = k\rho$$
$$(\rho u)_t + (\rho u(u - p(\rho))_x = \eta\rho u + \beta\rho$$

- With the initial left and right state known



- Transform ρ, u variables into V, W variables given by

$$\rho = Ve^{kt}, \quad u = (W + \frac{\beta}{\eta - k})e^{(\eta-k)t} - \frac{\beta}{\eta-k} \quad \{\eta \neq k\}$$

- Shock is a jump in states, Rarefaction is a smooth transition for one state to another
- Delta-shock characterized by "blowup" in a variable
- Cases defined by sign of $k(\gamma + 1)$, as well as the magnitude of γ
- Case 2 shown here, $-1 < \gamma < 0$, $k(\gamma + 1) < 0$
- Consists of 4 curves S_1 C_2 R_2 S_δ
- R_2 lies extremely close to C_2

Methods

- Numerical scheme used is local Lax-Friedrich's (LLF) method

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) + \frac{CFL}{2\lambda}(F_{j+1}^n - F_{j-1}^n)$$

- Wave-speed calculated locally using eigenvalues
- Utilized change of variables for numerical stability

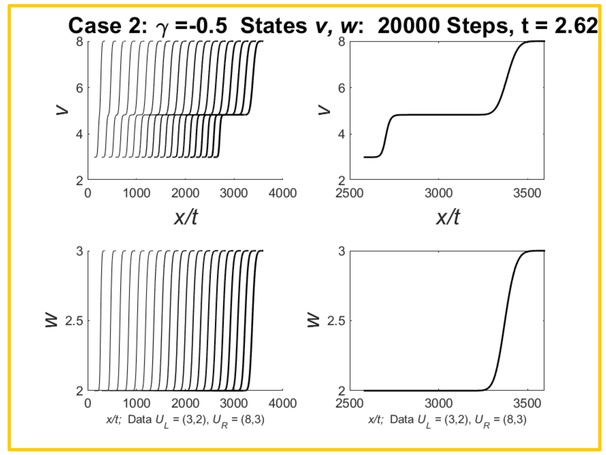
$$Y = WV + V\frac{\beta}{\eta - k}$$

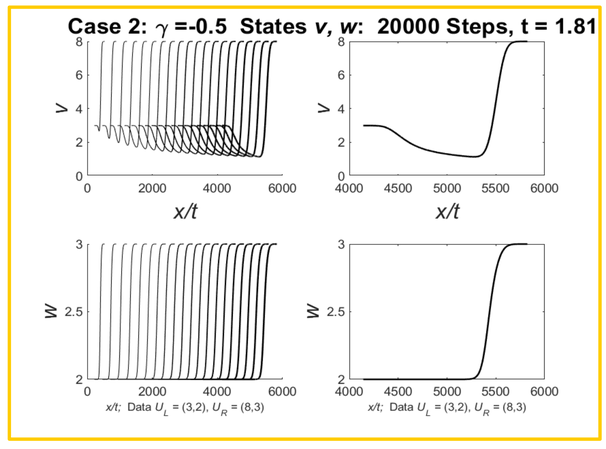
- CFL condition imposed to ensure convergence to physical weak solution

Results

Small Timescale

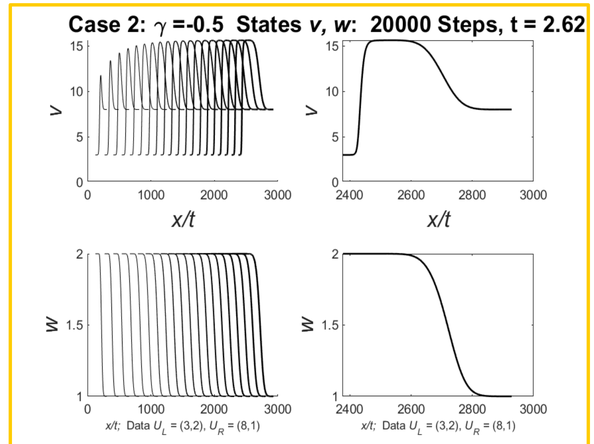
- Both points are initially S_1 C_2
- C_2 suffers from diffusion
- Small time simulated with k = -0.01

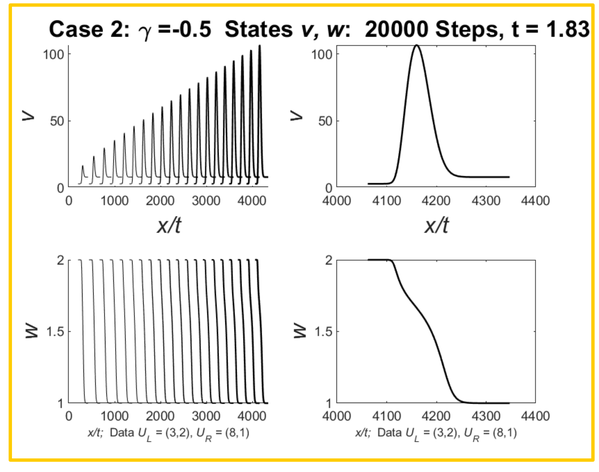




Large Timescale

- Point above S_1 shifts to R_2 C_2
- Movement in W is very small during R_2
- R_2 always appears to take priority over C_2
- Bottom-right point transforms into a delta-shock in large timescale
- Delta behavior appears only in V for all cases and all points in delta-shock regions





Discussion and Future Work

- Time-dependent numerical methods can be used for time-dependent fluxes given parameter restrictions that slow convergence
- Future work involves less diffusive, higher order schemes

Conclusion

- Successfully approximated physical weak solutions to the expanded varying Chaplygin gas system in time
- Validated analytical solutions derived by other members of our REU
- Studying the aforementioned system offers insights into the mechanics of the universe that we reside in

References and Acknowledgements

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Li, S. (2023). Delta-shocks for a 2 × 2 balance system of Keyfitz–Kranzer type with varying Chaplygin gas. Physics of Fluids, 35(7). https://doi.org/10.1063/5.0156662

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