University of Michigan
EECS 504: Foundations of Computer Vision
Winter 2020. Instructor: Andrew Owens.

Problem Set 1 Solution: Image filtering

Problem 1 Solution.1 *Properties of convolution*

Recall that 1D convolution between two signals $f, g \in \mathbb{R}^N$ is defined:

$$(f * g)[n] = \sum_{k=0}^{N-1} f[n-k]g[k]. \tag{1}$$

(a) Show that convolution (i) is commutative, i.e. $f * g = g * f$ and (ii) associative, i.e. $(f * g) * h = f * (g * h)$ (1 point).

Answer:

$$\begin{aligned}
(f * g)[n] &= \sum_{k=0}^{N-1} f[n-k]g[k] \\
&= \sum_{t=n-N+1}^{n} f[t]g[n-t] && (\text{let } t = n-k) \\
&= \sum_{t=0}^{n} f[t]g[n-t] && (g[t] = 0 \text{ when t} < 0) \\
&= \sum_{t=0}^{N-1} g[n-t]f[t] && (f[n-t] = 0 \text{ when t} > \text{n}) \\
&= (g * f)[n]
\end{aligned}$$

$$
\begin{aligned}
(f*g)*h &= (g*f)*h \\
&= \sum_{t=0}^{N-1} h[t] \sum_{k=0}^{N-1} g[n-t-k]f[k] \\
&= \sum_{t=0}^{N-1} \sum_{k=0}^{N-1} g[n-t-k]f[k]h[t] \\
&= \sum_{k=0}^{N-1} \sum_{t=0}^{N-1} g[n-t-k]f[k]h[t] \\
&= \sum_{k=0}^{N-1} f[k] \sum_{t=0}^{N-1} g[n-t-k]h[t] \\
&= (g*h)*f \\
&= f*(g*h)
\end{aligned}
$$

(b) Construct a matrix multiplication that produces the same result as convolution with a given filter. In other words, given a filter f describe a matrix H such that $f*g = Hg$ for any input g (1 point).

Answer:
The matrix $\mathbf{H} \in \mathbb{R}^{2N-1\times N}$ can be written as

$$
\mathbf{H} = \begin{bmatrix}
f[0] & 0 & 0 & \cdots & 0 \\
f[1] & f[0] & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f[N-1] & f[N-2] & f[N-3] & \cdots & f[0] \\
0 & f[N-1] & f[N-2] & \cdots & f[1] \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
0 & 0 & 0 & \cdots & f[N-1]
\end{bmatrix}
$$

(c) In class, we showed how convolution with a 2D Gaussian filter can be performed efficiently as a sequence of convolutions with 1D Gaussian filters. This idea also works with other kinds of filters. We say that a 2D filter $f \in \mathbb{R}^{N\times N}$ is separable if $f = uv^\top$ for some $u, v \in \mathbb{R}^N$. Show that if f is separable, then the 2D convolution $g*f$ can be computed as a sequence of two one-dimensional convolutions (1 point).

Answer:

$$
\begin{aligned}
(g * f)_{[n,n]} &= \sum_i \sum_j g[i,j] f[m-j, n-j] \\
&= \sum_i \sum_j g[i,j] u[m-i] \cdot v[n-j] \\
&= \sum_i u[m-i] \cdot \underline{\sum_j g[i,j] \cdot v[n-j]} && \text{(1-D convolution along the ith row)} \\
&= \sum_i u[m-i] \cdot (g[i,:] * v)[n] && \text{(denote } (g[i,:] * v)[n] \text{ as } k[n][i]) \\
&= \sum_i u[m-i] k[n][i] \\
&= (k[n] * u)[m]
\end{aligned}
$$

(d) (Optional) Show that cross-correlation,

$$
h[n] = \sum_{k=0}^{N-1} f[n+k] g[k], \tag{2}
$$

is *not* commutative (0 points). *Note: we will not grade this problem. It really is totally optional! Only do this problem if it interests you.*

Answer:

$$
\begin{aligned}
h[n] &= \sum_{k=0}^{N-1} f[n+k] g[k] = \sum_{k=0}^{N-n-1} f[n+k] g[k] && (f[n+k] = 0, \forall k > N-n-1) \\
h'[n] &= \sum_{k=0}^{N-1} g[n+k] f[k] = \sum_{k=0}^{N-n-1} g[n+k] f[k] && (g[n+k] = 0, \forall k > N-n-1)
\end{aligned}
$$

It's clear to see that when $n \geq 1$, $h[n] \neq h'[n]$.

Problem 1 Solution.2 *Pet edge detection*

Answer:
Please see the provided Colab notebook.