Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER Journal of Economic Dynamics & Control 31 (2007) 3503–3544
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/jedc

Asset allocation under multivariate regime switching

Massimo Guidolin[a,b,*], Allan Timmermann[c]

[a]*Manchester Business School, University of Manchester, Crawford House, Booth Street East, Manchester M13 9PL, UK*
[b]*Federal Reserve Bank of St. Louis, USA*
[c]*University of California, San Diego, USA*

Received 17 November 2005; received in revised form 25 October 2006; accepted 4 December 2006
Available online 12 February 2007

Abstract

This paper studies asset allocation decisions in the presence of regime switching in asset returns. We find evidence that four separate regimes – characterized as crash, slow growth, bull and recovery states – are required to capture the joint distribution of stock and bond returns. Optimal asset allocations vary considerably across these states and change over time as investors revise their estimates of the state probabilities. In the crash state, buy-and-hold investors allocate more of their portfolio to stocks the longer their investment horizon, while the optimal allocation to stocks declines as a function of the investment horizon in bull markets. The joint effects of learning about state probabilities and predictability of asset returns from the dividend yield give rise to a non-monotonic relationship between the investment horizon and the demand for stocks. Out-of-sample forecasting experiments confirm the economic importance of accounting for the presence of regimes in asset returns.

JEL classification: G11; C22; C53

Keywords: Regime switching; Portfolio choice; Predictability

*Corresponding author. Manchester Business School, University of Manchester, Crawford House, Booth Street East, Manchester M13 9PL, UK. Tel.: +44 161 306 6406; fax: +44 161 275 4023.
E-mail address: Massimo.Guidolin@mbs.ac.uk (M. Guidolin).

0165-1889/$ - see front matter
doi:10.1016/j.jedc.2006.12.004

1. Introduction

For most investors the asset allocation decision – how much to invest in major asset classes such as cash, stocks and bonds – is a key determinant of their portfolio performance. Asset allocation decisions can only be made in the context of a model for the joint distribution of asset returns. Most studies assume that asset returns are generated by a linear process with stable coefficients so the predictive power of state variables such as dividend yields, default and term spreads does not vary over time. However, there is mounting empirical evidence that asset returns follow a more complicated process with multiple 'regimes', each of which is associated with a very different distribution of asset returns. Ang and Bekaert (2002a,b), Ang and Chen (2002), Garcia and Perron (1996), Gray (1996), Guidolin and Timmermann (2005a,b, 2006a–c), Perez-Quiros and Timmermann (2000), Turner et al. (1989) and Whitelaw (2001) all report evidence of regimes in stock or bond returns.

This paper characterizes investors' asset allocation decisions under a regime switching model for asset returns with four states that are characterized as crash, slow growth, bull and recovery states. Extending earlier work in the literature, we allow the states to be unobservable to investors who filter state probabilities from return observations and thus never know current or future states with certainty. The underlying states offer very different investment opportunities so investors' asset allocations vary significantly over time as they revise their beliefs about the underlying state probabilities.

The economic intuition for our main findings is most easily explained with reference to Fig. 1. For each month over the period January 1980–December 1999, panel (a) shows which of the four regimes was most likely at that point in time.[1] The first regime is a low return, highly volatile crash/bear state, regimes 2 and 3 are low-volatility, bullish states, while regime 4 is a high-volatility, recovery state which tends to follow crash regimes (more details are provided in Section 3). Regimes are seen to change frequently although the states are quite persistent.

Panels (b)–(d) show the evolution in the asset allocation for a buy-and-hold investor who updates the parameters of the four-state model and determines the portfolio weights recursively in time.[2] The portfolio weights of the short-term (12-month) investor in panel (b) vary considerably over time: bear states (e.g., 1983–1984) are associated with very moderate investments in stocks but large allocations to bonds and T-bills; in contrast, bull markets (e.g., 1993–1996) see substantial bets on equities – especially small stocks – and reduced allocations to bonds. Because regimes are persistent, short-horizon investors clearly attempt to time the market by reducing (increasing) the allocation to the riskiest assets when investment opportunities are poor (good).[3]

[1]This plot shows the smoothed state probabilities. The dashed lines surrounding the bullets provide a measure of the degree of uncertainty about the state.

[2]Details of the underlying modeling experiment are provided in Section 7.3.

[3]These are the optimal weights for an investor with power utility and constant relative risk aversion of 5.

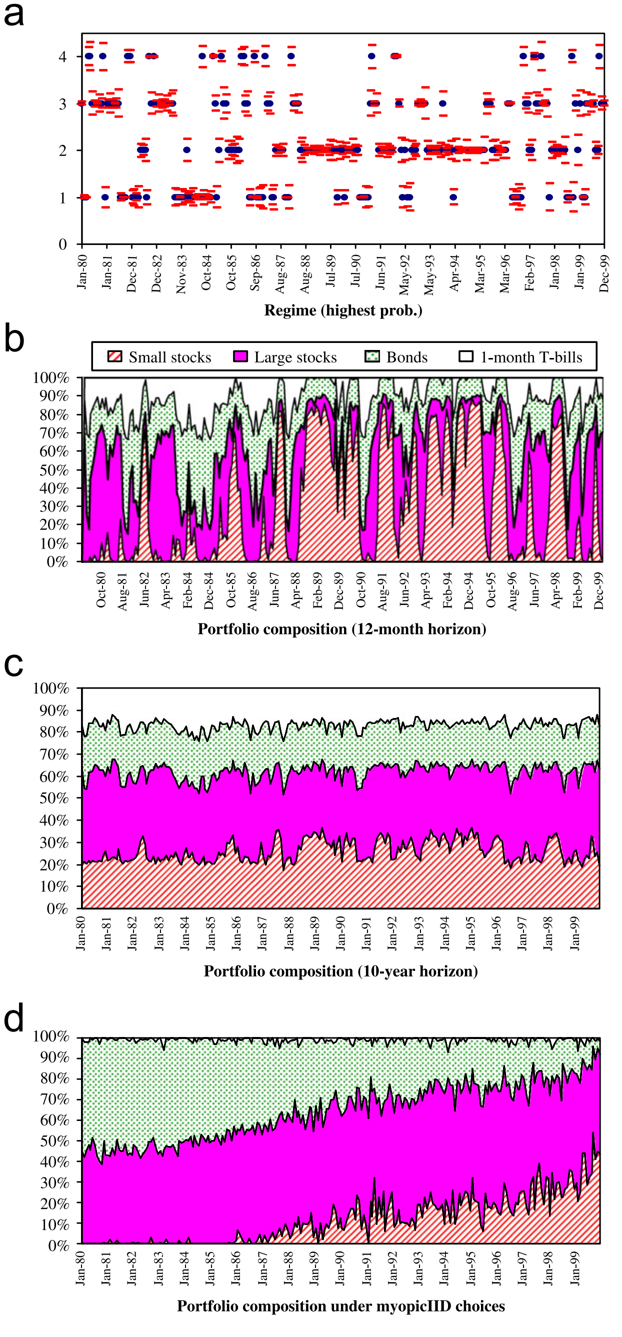

Fig. 1. Evolution in buy-and-hold investor' asset allocation weights.

The portfolio weights for a long-horizon (10-year) investor (panel (c)) are far more stable than those of the short-horizon investor. What matters for the long-run investor is not so much the perceived identity of the current state as the ability to correctly model the long-run return distribution. Interestingly, the long-run investor's asset allocation still differs substantially from those of an investor who ignores regimes and instead assumes that asset returns are drawn from a time-invariant distribution (panel (d)), albeit with parameters that get updated recursively over time and thus may induce a drift in the portfolio weights.

As indicated by Fig. 1, the relation between a buy-and-hold investor's investment horizon and the optimal portfolio allocation also varies significantly across states. Since stocks are not very attractive in the crash state, investors with a short horizon hold very little in stocks in this state. At longer investment horizons, there is a high chance that the economy will switch to a better state and so investors allocate more towards stocks. In the crash state the allocation to stocks is therefore an increasing function of the investment horizon. In the more persistent slow growth and bull states, investors with a short horizon hold large positions in stocks. At longer horizons investment opportunities will almost certainly worsen so investors hold less in stocks, thereby creating a downward sloping relation between stock holdings and the investment horizon.

Predictability of stock and bond returns has been documented by a large number of empirical studies, so we next extend the regime switching model to include predictability from state variables such as the dividend yield. Compared to a benchmark with constant expected returns, predictability from the dividend yield in a linear vector autoregression (VAR) reduces risk at longer horizon and leads to an increased demand for stocks, the longer the investment horizon. In contrast, regime switching leads to a positive correlation between return innovations and shocks to future expected returns, thereby increasing risk and lowering the long-run demand for stocks.

These findings can be understood in terms of two effects: investors' learning about the underlying state and mean reversion in the return generating process. To see this, consider the following simple two-period example that decomposes returns on a risky asset, r_{t+1}, into an expected component, $\mathrm{E}_t[r_{t+1}]$, and an innovation, u_{t+1}:

$$r_{t+1} = \mathrm{E}_t[r_{t+1}] + u_{t+1},$$

where $Var_t(r_{t+1}) = \sigma_u^2$. At the two-period horizon, cumulated returns become

$$r_{t+1} + r_{t+2} = \mathrm{E}_t[r_{t+1}] + u_{t+1} + \mathrm{E}_{t+1}[r_{t+2}] + u_{t+2},$$

and so the variance of two-period returns is

$$Var(r_{t+1} + r_{t+2}) = 2\sigma_u^2 + Var(\mathrm{E}_{t+1}[r_{t+2}]) + 2\,Cov(u_{t+1}, \mathrm{E}_{t+1}[r_{t+2}]).$$

Comparing single-period and two-period return variances, we have

$$\frac{Var(r_{t+1} + r_{t+2})}{2Var(r_{t+1})} = 1 + \frac{1}{2}\left(\frac{R^2}{1-R^2}\right)\beta,$$

where

$$R^2 = \frac{Var(\mathrm{E}_t[r_{t+1}])}{Var(\mathrm{E}_t[r_{t+1}]) + \sigma_u^2}, \quad \beta = \frac{Cov(u_{t+1}, \mathrm{E}_{t+1}[r_{t+2}])}{\sigma_u^2}.$$

Models of learning where investors revise their expectations of future returns upwards following positive return shocks imply that $\beta > 0$. For instance, in a pure regime switching model future positive shocks will induce belief revisions in favor of states with high expected returns. This implies that the variance of two-period returns exceeds twice the variance of the single-period return, suggesting that risk grows faster than the rate implied by the constant expected return model ($\beta = 0$). Since the investment demand is independent of the horizon under the constant expected return model, such learning effects tend to lead to a demand for the risky asset that declines in the investment horizon.

Conversely, models of predictable, mean-reverting returns imply $\beta < 0$. For example, a negative shock to returns in a VAR(1) model implies a higher value of the dividend yield and higher expected future returns. Hence the risk of stock returns grows at a slower rate than if expected returns were constant. This tends to lead to an increased demand for the risky asset, the longer the horizon.

These facts explain why we find many different shapes of the investment schedules relating portfolio weights to the investment horizon, depending on assumptions about the initial state probabilities and the form of the return generating process. As we go beyond models with constant expected returns with $\beta = 0$ to either models of predictable risk premia (a VAR(1) with $\beta < 0$) or models with learning (regime switching with $\beta > 0$), we obtain different and increasingly realistic asset allocation implications. When both learning and predictability are accounted for, non-monotonic relations between a buy-and-hold investor's allocation to stocks and the investment horizon appear. At short horizons the effect of regimes tends to dominate while at longer horizons the mean reverting component in returns tracked by the yield dominates and leads to an increasing demand for stocks.

Once rebalancing opportunities are introduced, the results are quite different. The allocation to different asset classes continues to differ across states, even for long investment horizons. However, as the rebalancing frequency increases, asset holdings respond far less to the investment horizon as a reflection of the possibility of changing asset allocations, in case investment opportunities change significantly before the end of the investment horizon.

The plan of the paper is as follows. Section 2 motivates the presence of regimes in the return distribution and reviews the existing literature. Section 3 introduces the multi-state model used to capture predictability and regime switching in asset returns and reports empirical findings. Section 4 sets up the investor's asset allocation problem while Section 5 presents asset allocation results. Section 6 extends the model to allow for predictability from the dividend yield and Section 7 presents utility cost calculations, investigates the effect of parameter uncertainty and examines the out-of-sample performance of alternative asset allocation schemes based on different models for the return distribution. This section also shows that a four-state regime switching model is not only supported by statistical evidence that two states are insufficient in our application, but also is a key determinant of the portfolio weights and (out-of-sample) return performance. Section 8 concludes.

2. Motivation for setup

2.1. Regimes in return distributions

There are good economic reasons why the equilibrium distribution of stock and bond returns should contain regimes. Whitelaw (2001) constructs an equilibrium model where consumption growth follows a two-state process so investors' intertemporal marginal rate of substitution also follows a regime process. Suppose that investors have constant relative risk aversion and that asset returns are determined from the standard no-arbitrage, equilibrium relation

$$\mathrm{E}_t[M_{t+1}(1+r_t)] = 1,$$

where M_{t+1} is the pricing kernel which is commonly restricted to be $M_{t+1} \equiv \beta(C_{t+1}/C_t)^{-\gamma}$, with β a discount factor and C_{t+1}/C_t real per-capita consumption growth. The risk premium (over and above the conditionally risk-free rate, r_t^{f}) is then given by

$$\mathrm{E}_t[r_{t+1} - r_t^{\mathrm{f}}] = -\frac{Cov_t[M_{t+1}, (r_{t+1} - r_t^{\mathrm{f}})]}{\mathrm{E}_t[M_{t+1}]}.$$

Suppose consumption growth follows a simple regime switching process, $g_{t+1} \sim \mathrm{N}(\mu_{s_{t+1}}, \sigma^2_{s_{t+1}})$ $(s_{t+1} = 1, \ldots, k)$, i.e. both the mean and the variance of consumption growth depend on the underlying state of the economy (e.g., expansions and recessions as found in Hamilton, 1989). This implies that the pricing kernel also follows a k-state process and so

$$\mathrm{E}_t[r_{t+1} - r_t^{\mathrm{f}}] = -\frac{\sum_{s_{t+1|t}=1}^{k} \pi_{s_{t+1|t}}\, Cov[M_{t+1}, (r_{t+1} - r_t^{\mathrm{f}})|s_{t+1}]}{\sum_{s_{t+1|t}=1}^{k} \pi_{s_{t+1|t}} \mathrm{E}[M_{t+1}|s_{t+1}]},$$

where $\pi_{s_{t+1|t}}$ is the predicted state probability at time $t+1$ given information at time t. This simple model thus implies that returns on risky assets in excess of the risk-free rate follow a regime switching process driven by states that reflect time-varying expected consumption growth and time-varying conditional covariances between asset returns and consumption growth.

There are also good reasons for incorporating time-variation in the relation between asset returns and state variables such as interest rates or dividend yields. Interest rates serve both as determinants of the discount rate used in calculating present values and are also likely to reflect expectations of future cash flows through the Federal Reserve's decisions on monetary policy – for example, higher interest rates may reflect beliefs of strong future growth. Similarly, as shown by Campbell and Shiller (1988), the (log) dividend–price ratio reflects expectations of future returns minus expected future dividend growth. If either risk premia or cash flows are subject to regimes (e.g., recessions and expansions), this should also show up in predictive return models that include the yield as a state variable. Integrating asset allocation decisions within a fully specified equilibrium framework is beyond the scope of our paper, but portfolio decisions are likely to be closely related to the evolving uncertainty about the underlying state of the economy, here captured through a regime switching process. Such regimes could be linked to business cycle variations in economic growth (cash flows) associated with the economic cycle, breaks in macroeconomic volatility (e.g., Lettau et al., 2005) large macroeconomic shocks (e.g., oil prices) or institutional changes.[4]

Our paper takes the presence of regimes in stock and bond returns as a starting point and proceeds to characterize asset allocation implications. Regime switching models can capture many properties of the return distribution. These models typically identify regimes with very different mean, variance and correlations across assets. As the underlying state probabilities change over time this leads to time-varying expected returns, volatility persistence and changing correlations and predictability in higher-order moments. This is consistent with Aït-Sahalia and Brandt (2001) who argue that higher-order moments of stock and bond returns are time-varying although different moments are typically predicted by different combinations of economic variables. The degree of predictability of mean returns can also vary significantly over time in regime switching models – a feature that seems present in stock return data (Bossaerts and Hillion, 1999). Finally, regime switching models are capable of capturing even complicated forms of heteroskedasticity, fat tails and skews in the underlying distribution of returns (Timmermann, 2000).

2.2. Existing results on asset allocation

Our paper is part of a growing literature that explores the asset allocation and utility cost implications of return predictability from the perspective of a small, expected utility maximizing investor with a multi-period horizon. In an analysis involving a single risky stock portfolio, Kandel and Stambaugh (1996) find that predictability can be statistically small yet still have a large effect on the optimal asset allocation. Barberis (2000) extends this result to long horizons. Campbell and Viceira (1999) derive closed-form expressions using log-linear approximations for a discrete-time consumption and portfolio choice problem with rebalancing and an infinite horizon. Brennan et al. (1997), Campbell and Viceira (2001, 2002) and Campbell et al. (2003) study strategic asset allocation and document large effects of predictability on asset holdings and welfare costs. Bielecki et al. (2000) show that

[4]Lettau et al. (2005) link the high stock prices experienced during the 1990s to a shift towards lower macroeconomic volatility levels using a two-state regime switching model fitted to the volatility and mean of consumption growth. They report evidence of a break in the volatility of consumption growth around 1992 and a shift in the mean around 1995 and calibrate a stochastic discount factor model to capture the implications for stock prices of these breaks. Calvet and Fisher (2005) propose an equilibrium model with regime shifts in the mean and volatility of consumption and dividend growth rates. Regime shifts are shown to affect asset prices that converge to a multifractal jump-diffusion process.

under an infinite horizon objective that depends only on the long-run growth rate of wealth and on variance, the optimal portfolio becomes a function of the factors predicting expected returns.

The papers whose modeling approach is most closely related to ours are Ang and Bekaert (2002a), Honda (2003), Detemple et al. (2003), Calvet and Fisher (2005) and Lettau et al. (2005). Ang and Bekaert (2002a) use a two-state model to evaluate the claim that the home bias observed in holdings of international assets can be explained by return correlations that increase in bear markets. Assuming observable states, they find that optimal portfolio weights depend both on the current regime and on the investment horizon and that the cost of ignoring regime switching is of the same order of magnitude as the cost of ignoring foreign equities in the optimal portfolio. While our paper shares a similar regime switching setup, we address a very different question, namely a US investor's asset allocation between bonds, stocks and cash. We find that a four-state model is required to capture the rich dynamics of the joint distribution of stock and bond returns. Furthermore, we model regimes as unobservable, calculate asset allocations under optimal filtering and therefore explicitly address the effects on hedging demands arising from investors' recursive updating in their beliefs about the underlying state probabilities.

Honda (2003) solves a continuous-time portfolio and consumption problem in which the expected return (drift) of a single risky asset depends on an unobservable regime governed by a continuous Markov chain with two states.[5] Optimal policies are computed using Monte Carlo methods. He finds that the shape of the function relating optimal portfolio weights to the investment horizon may depend on the perception of the current regime. Our paper shares a similar set up but extends Honda's results in several directions by investigating an allocation problem for a relatively rich asset menu including long-term bonds, by entertaining discrete-time regime switching models with more than two states, by accommodating heteroskedasticity in a regime-dependent fashion, and by jointly modeling regimes in excess asset returns and in factors (such as the dividend yield) that predict future asset returns. We also evaluate the real-time, out-of-sample performance of investment strategies that consider the impact of regimes.

Detemple et al. (2003) approach a wide class of portfolio choice problems in continuous time, including strategic asset allocation. Building on the widespread evidence that both interest rates and the market price of risk(s) follow non-linear processes, they investigate the asset allocation implications of non-linear predictability using simulation methods. They show that findings in the standard VAR framework – e.g., that the equity allocation should be higher the longer the investment horizon – may be overturned in the presence of non-linearities. For reasons similar to these authors, we resort to Monte Carlo methods to solve for the optimal asset allocation. However, we explore the asset allocation under a class of non-linear processes (multivariate regime switching) that is not nested in their framework.

[5] David (1997) also investigates optimal consumption and portfolio rules in a two-state continuous-time model.

3. Asset returns under regime switching

3.1. Data

Our analysis considers a US investor's choice among three major asset classes, namely stocks, bonds and T-bills. We further divide the stock portfolio into large and small stocks in light of the empirical evidence suggesting that these stocks have very different risk and return characteristics that vary across different regimes, see Ang and Chen (2002) and Perez-Quiros and Timmermann (2000).

Our analysis uses monthly returns on all common stocks listed on the NYSE, AMEX and NASDAQ. The first and second size-sorted CRSP decile portfolios are used to form a portfolio of small firm stocks, while deciles 9 and 10 are used to form a portfolio of large firm stocks. We also consider the return on the CRSP portfolio of 10-year T-bonds. Returns are continuously compounded and inclusive of any cash distributions. To obtain excess returns we subtract the 30-day T-bill rate from these returns. Dividend yields are also used in the analysis and are computed as dividends on a value-weighted portfolio of stocks over the previous 12 month period divided by the current stock price. Our sample is January 1954 – December 1999, a total of 552 observations. Consistent with the literature we only use data after the 1951 Treasury Accord. Data from 2000 to 2003 are not used for model selection or parameter estimation in order to keep a genuine post-sample period. All data are obtained from the Center for Research in Security Prices.

3.2. Model

To capture the possibility of regimes in the joint distribution of asset returns and predictor variables, consider an $(n+m) \times 1$ vector of asset returns in excess of the T-bill rate, $\mathbf{r}_t = (r_{1t}, r_{2t}, \ldots, r_{nt})'$ extended by a set of m predictor variables, $\mathbf{z}_t = (z_{1t}, \ldots, z_{mt})'$. Suppose that the mean, covariance and serial correlations in returns are driven by a common state variable, S_t, that takes integer values between 1 and k:

$$\begin{pmatrix} \mathbf{r}_t \\ \mathbf{z}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{s_t} \\ \boldsymbol{\mu}_{zs_t} \end{pmatrix} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t} \begin{pmatrix} \mathbf{r}_{t-j} \\ \mathbf{z}_{t-j} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{zt} \end{pmatrix}. \tag{1}$$

Here $\boldsymbol{\mu}_{s_t}$ and $\boldsymbol{\mu}_{zs_t}$ are intercept vectors for $\mathbf{r}_t$ and $\mathbf{z}_t$ in state s_t, $\{\mathbf{A}_{j,s_t}\}_{j=1}^{p}$ are $(n+m) \times (n+m)$ matrices of autoregressive coefficients in state s_t, and $(\boldsymbol{\varepsilon}_t' \; \boldsymbol{\varepsilon}_{zt}')' \sim \mathrm{N}(0, \boldsymbol{\Omega}_{s_t})$, where $\boldsymbol{\Omega}_{s_t}$ is an $(n+m) \times (n+m)$ covariance matrix. When $k=1$, Eq. (1) simplifies to a standard VAR. Our model thus nests as a special case the standard linear (single-state) model used in much of the asset allocation literature. This model gets selected if the data only supports a single regime.

Regime switches in the state variable, S_t, are assumed to be governed by the transition probability matrix, $\mathbf{P}$, with elements

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i,j = 1, \ldots, k. \tag{2}$$

Each regime is thus the realization of a first-order Markov chain with constant transition probabilities. S_t is not observable but a filtered estimate can be computed from time series data on $\mathbf{r}_t$ and $\mathbf{z}_t$.

While simple, this model is quite general and allows means, variances and correlations of asset returns to vary across states. Hence the risk-return trade-off can vary across states in a way that may have strong asset allocation implications. For example, knowing that the current state is a persistent bull state will make most risky assets more attractive than in a bear state. Estimation proceeds by optimizing the likelihood function associated with (1)–(2). Since the underlying state variable, S_t, is unobserved we treat it as a latent variable and use the EM algorithm to update our parameter estimates, cf. Hamilton (1989).

3.3. Choice of model specification

Guidolin and Timmermann (2005a) provide a detailed specification analysis to determine the statistical evidence in support of regimes in the univariate and joint distribution of stock and bond returns. Considering a range of values for the number of states, $k = 1, 2, 3, 4, 5, 6$ and the lag-order $p = 0, 1, 2, 3$, they use information criteria and specification tests to select a four-state model.

Using similar methods, we considered a range of model specifications with up to six states.[6] Although none of the models passes all tests, the most parsimonious model that captures the distribution of both large stock returns and bond returns is a four-state model with regime-dependent mean and covariance matrix. Some aspects of small firms' return distribution are not captured by this model, but most of the test statistics tend to be quite small (albeit statistically significant). Models with fewer states or constant volatility across states are clearly mis-specified, while models with more states have far more parameters so we select a specification with four states. Interestingly, no VAR terms are required. This is consistent with the common finding that asset returns are only weakly serially correlated.

3.4. Model estimates

Fig. 2 plots the state probabilities for the four-state model while Table 1 shows the parameter estimates for this model. Initially, we focus on the simplest case where $m = 0$ so no predictor variable is included to model the dynamics in asset returns.[7]

It is easy to interpret the four regimes. Regime 1 is a 'crash' state characterized by large, negative mean excess returns and high volatility. It includes the two oil price shocks in the 1970s, the October 1987 crash, the early 1990s, and the 'Asian flu'.

[6] We use the predictive density specification tests proposed by Berkowitz (2001) and based on the probability integral transform or z-score. If the model is correctly specified, the z-scores should be independently and identically distributed (IID) and uniform on the interval $[0, 1]$. Guidolin and Timmermann (2005a) provide detailed results for the data at hand.

[7] Attempts to simplify the number of parameters by imposing the restriction that mean returns are the same across the four states or that the covariance matrices are identical in the high volatility states (states 1 and 4) were clearly rejected at critical levels below 1%, cf. Guidolin and Timmermann (2005a).

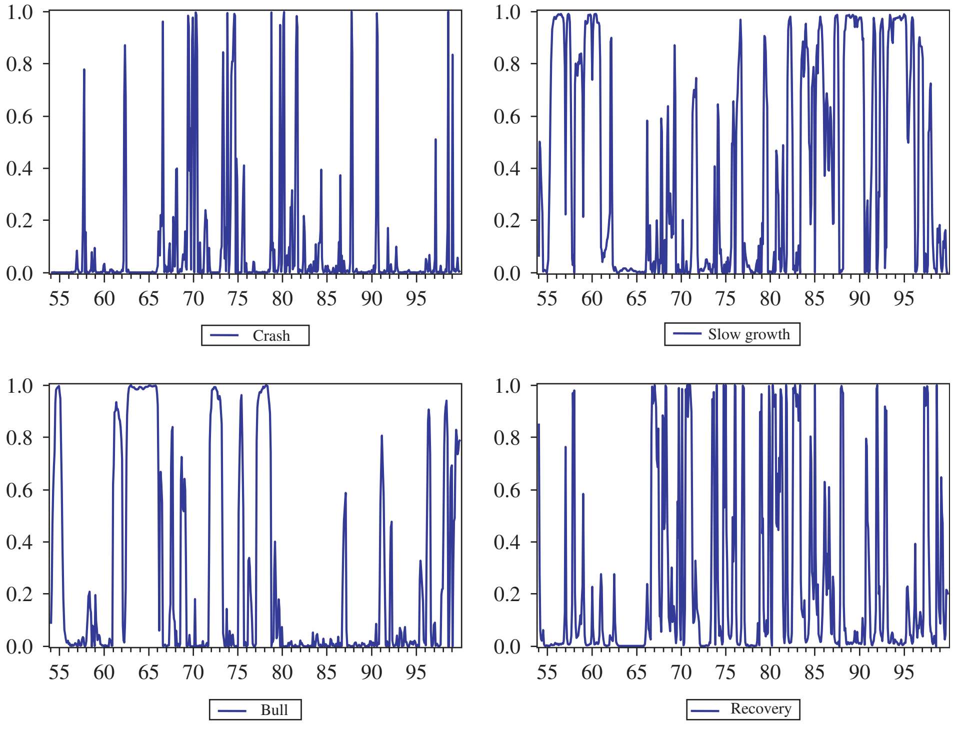

Fig. 2. Smoothed state probabilities: four-state model for stock and bond returns. The graphs plot the smoothed probabilities of regimes 1–4 for the multivariate regime switching model comprising returns on large and small firms and 10-year bonds all in excess of the return on 30-day T-bills.

Regime 2 is a low growth regime characterized by low volatility and small positive mean excess returns on all assets. Regime 3 is a sustained bull state where stock prices – especially those of small firms – grow rapidly on average. Mean excess returns on long-term bonds are negative in this state. States 2 and 3 identify a size effect in stock returns. In state 2 the mean return of large stocks exceeds that of small stocks by about 7% per annum, while this gets reversed in state 3. Regime 4 is a recovery state with strong market rallies and high volatility for small stocks and bonds.

Correlations between returns also appear to vary substantially across regimes. The estimated correlation between large and small firms' returns varies from a high of 0.82 in the crash state to a low of 0.50 in the recovery state. The correlation between returns on large stocks and bonds even changes signs across different regimes and varies from 0.37 in the recovery state to -0.40 in the crash state. Finally, the correlation between small stock and bond returns goes from -0.26 in the crash state to 0.12 in the slow growth state. This is consistent with the evidence of time-varying (regime-specific) correlations found in monthly equity returns by Ang and Chen (2002). The ability of our model to identify the negative correlation

Table 1
Estimates of regime switching model for stock and bond returns

	Large caps	Small caps	Long-term bonds
Panel A – single state model			
1. Mean excess return	0.0066 (0.0018)	0.0082 (0.0026)	0.0008 (0.0009)
2. Correlations/Volatilities			
Large caps	0.1428***		
Small caps	0.7215**	0.2129***	
Long-term bonds	0.2516	0.1196	0.0748***
Panel B – four state model			
1. Mean excess return			
Regime 1 (crash)	−0.0510 (0.0146)	−0.0410 (0.0219)	−0.0131 (0.0047)
Regime 2 (slow growth)	0.0069 (0.0027)	0.0008 (0.0033)	0.0009 (0.0016)
Regime 3 (bull)	0.0116 (0.0032)	0.0187 (0.0048)	−0.0023 (0.0007)
Regime 4 (recovery)	0.0519 (0.0055)	0.0478 (0.0098)	0.0136 (0.0033)
2. Correlations/Volatilities			
Regime 1 (*crash*):			
Large caps	0.1625***		
Small caps	0.8233***	0.2479***	
Long-term bonds	−0.4060*	−0.2590	0.0902***
Regime 2 (*slow growth*):			
Large caps	0.1118***		
Small caps	0.7655***	0.1099***	
Long-term bonds	0.2043***	0.1223	0.0688***
Regime 3 (*bull*):			
Largecaps	0.1133***		
Small caps	0.6707***	0.1730***	
Long-term bonds	0.1521	−0.0976	0.0261***
Regime 4 (*recovery*):			
Large caps	0.1479***		
Small caps	0.5013***	0.2429***	
Long-term bonds	0.3692***	−0.0011	0.1000***

3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (crash)	0.4940 (0.1078)	0.0001 (0.0001)	0.02409 (0.0417)	0.4818
Regime 2 (slow growth)	0.0483 (0.0233)	0.8529 (0.0403)	0.0307 (0.0110)	0.0682
Regime 3 (bull)	0.0439 (0.0252)	0.0701 (0.0296)	0.8822 (0.0403)	0.0038
Regime 4 (recovery)	0.0616 (0.0501)	0.1722 (0.0718)	0.0827 (0.0498)	0.6836

This table reports the estimation output for the regime switching model:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}_{s_t})$ is the vector of return innovations. The unobserved state variable S_t is governed by a first-order Markov chain that can assume $k = 4$ values. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The sample is 1954:01–1999:12. Panel A refers to the case ($k = 1$) and represents a single-state benchmark. The data reported on the diagonals of the correlation matrices are annualized volatilities. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

*Denotes significance at the 10% level.

**Denotes significance at the 5% level.

***Denotes significance at the 1% level.

between stock and bond returns in the crash state – which the linear model is unable to do, see panel (a) of Fig. 1 – is a sign of the potential value of adopting a multi-state model.[8]

Mean returns and volatilities are larger in absolute terms in the crash and recovery regimes, so it is perhaps unsurprising that the persistence of the states also varies considerably. The crash state has low persistence and on average only 2 months are spent in this regime. Interestingly, the transition probability matrix has a very particular form. Exits from the crash state are almost always to the recovery state and occur with close to 50% chance suggesting that, during volatile markets, months with large, negative mean returns cluster with months that have high positive returns. The slow growth state is far more persistent with an average duration of 7 months while the bull state is the most persistent state with an average duration of 8 months. Finally, the recovery state is again not very persistent and the market is expected to stay just over 3 months in this state. The steady state probabilities are 9% (state 1), 40% (state 2), 28% (state 3) and 23% (state 4). Hence, although the crash state is clearly not visited as often as the other states, it by no means only picks up extremely rare events.

It is interesting to relate these states to the underlying business cycle. Correlations between smoothed state probabilities and NBER recession dates are 0.32 (state 1), -0.13 (state 2), -0.21 (state 3), and 0.18 (state 4). This suggests that indeed, the high-volatility states – states 1 and 4 – occur around official recession periods.

4. The investor's asset allocation problem

We next study the asset allocation implications of regime dynamics in the joint distribution of stock and bond returns. First consider the 'pure' asset allocation problem for an investor with power utility defined over terminal wealth, W_{t+T}, coefficient of relative risk aversion $\gamma > 1$, and an investment horizon T:

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma}. \tag{3}$$

The investor is assumed to maximize expected utility by choosing at time t a portfolio allocation to large stocks, small stocks and bonds, $\boldsymbol{\omega}_t^T \equiv (\omega_t^l(T)\,\omega_t^s(T)\,\omega_t^b(T))'$, while $1-(\boldsymbol{\omega}_t^T)'\boldsymbol{\iota}_3$ is invested in riskless T-bills.[9] For simplicity we assume the investor has unit initial wealth. Portfolio weights are adjusted every $\varphi = T/B$ months at B equally spaced points t, $t+T/B$, $t+2T/B, \ldots, t+(B-1)T/B$. When $B=1$, $\varphi = T$ and the investor simply implements a buy-and-hold strategy.

[8] Recent work by Andersen et al. (2004) reaches the same conclusion: stock and bond returns move together insofar as the correlation is sizeable and important, but the correlation switches sign across different regimes and may appear spuriously small when averaged across states.

[9] Following standard practice we consider a partial equilibrium framework which takes the asset return process as exogeneous (see e.g., Ang and Bekaert, 2002a) and assume that the risk-free rate is constant and equal to the average 1-month T-bill yield over the sample period (5.3% per year).

Let $\boldsymbol{\omega}_b$ $(b = 0, 1, \ldots, B-1)$ be the portfolio weights on the risky assets at these rebalancing times. Then $1 - \boldsymbol{\omega}_b' \boldsymbol{\iota}_3$ is the weight on T-bills at time $t + bT/B$ and

$$u(W_B) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma} = \frac{W_B^{1-\gamma}}{1-\gamma}.$$

With regular rebalancing the investor's optimization problem is

$$\begin{aligned} &\max_{\{\boldsymbol{\omega}_j\}_{j=0}^{B-1}} \quad \mathrm{E}_t\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right] \\ &\quad \text{s.t.} \quad W_{b+1} = W_b\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp(\varphi r^{\mathrm{f}}) + \boldsymbol{\omega}_b' \exp(\mathbf{R}_{b+1} + \varphi r^{\mathrm{f}} \boldsymbol{\iota}_3)\}, \end{aligned} \tag{4}$$

$$\mathbf{R}_{b+1} \equiv \mathbf{r}_{t_b+1} + \mathbf{r}_{t_b+2} + \cdots + \mathbf{r}_{t_{b+1}}, \quad b = 0, 1, \ldots, B-1.$$

The wealth equation is exact when asset returns are continuously compounded and excess returns are computed as the difference between asset returns and the risk-free rate. Incorporating investors' use of predictor variables $\mathbf{z}_b$, at the decision times $b = 0, 1, \ldots, B-1$, we get the following derived utility of wealth:

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j\}_{j=b}^{B-1}} \mathrm{E}_{t_b}\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right]. \tag{5}$$

Here $\boldsymbol{\theta}_b = (\{\boldsymbol{\mu}_i, \boldsymbol{\mu}_{zi}, \boldsymbol{\Omega}_{i,b}^*, \{\mathbf{A}_{j,i,b}^*\}_{j=1}^p\}_{i=1}^k, \mathbf{P}_b)$ collects the parameters of the regime switching model and $\boldsymbol{\pi}_b$ is the (column) vector of probabilities for each of the k possible states conditional on information at time t_b. Consistent with common practice, we rule out short-selling. Let $\mathbf{e}_j$ be a 3×1 vector of zeros with a 1 in the jth place and $\boldsymbol{\iota}_3$ be a 3×1 vector of ones. No short sales then means that $\mathbf{e}_j'\boldsymbol{\omega}_b \in [0, 1]$ $(j = 1-3)$ and $\boldsymbol{\omega}_b'\boldsymbol{\iota}_3 \leqslant 1$.[10] We also ignore capital gains taxes and other frictions.

Under power utility the Bellman equation conveniently simplifies to

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) = \frac{W_b^{1-\gamma}}{1-\gamma} Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \quad (\gamma \neq 1). \tag{6}$$

Since the states are unobservable, investors' learning is incorporated in this setup by letting them optimally revise their beliefs about the underlying state at each point in

[10] Short-selling constraints only have a marginal impact on our results as they are not binding except at the very short investment horizons. This finding is similar to results in Detemple et al. (2003). The intuition is that nonlinear processes may imply long-run (ergodic) densities of the data that are far less 'extreme' (in terms of portfolio weights) than those obtained by iterating on linear VAR models of predictable expected returns over long horizons. As pointed out by Kandel and Stambaugh (1996), the portfolio can go bankrupt if it is fully invested in an asset with a return of −100%. With zero wealth, the investor's objective function becomes unbounded, preventing an interior solution from existing. We use a simple rejection algorithm to ensure that wealth remains positive at all horizons along all simulation paths. This is equivalent to truncating the joint density from which asset returns are drawn. In practice we never found that rejections occurred on the simulated paths.

time using the updating equation

$$\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) = \frac{(\boldsymbol{\pi}'_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}'_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k}, \tag{7}$$

where a 'hat' on top of a parameter indicates that it is an estimate, $\odot$ denotes the element-by-element product, $\mathbf{y}_b \equiv (\mathbf{r}'_b\ \mathbf{z}'_b)'$, $\hat{\mathbf{P}}_t^{\varphi} \equiv \prod_{i=1}^{\varphi} \hat{\mathbf{P}}_t$, and $\boldsymbol{\eta}(\mathbf{y}_{b+1})$ is the $k \times 1$ vector whose jth element gives the density of observation $\mathbf{y}_{b+1}$ in the jth state at time t_{b+1} conditional on $\hat{\boldsymbol{\theta}}_b$:

$$\boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_b) \equiv \begin{bmatrix} (2\pi)^{-N/2}|\hat{\boldsymbol{\Omega}}_1^{-1}|^{1/2} \exp[-\frac{1}{2}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{1j}\mathbf{y}_{t_b-j})'\hat{\boldsymbol{\Omega}}_1^{-1}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{1j}\mathbf{y}_{t_b-j})] \\ (2\pi)^{-N/2}|\hat{\boldsymbol{\Omega}}_2^{-1}|^{1/2} \exp[-\frac{1}{2}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{2j}\mathbf{y}_{t_b-j})'\hat{\boldsymbol{\Omega}}_2^{-1}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{2j}\mathbf{y}_{t_b-j})] \\ \vdots \\ (2\pi)^{-N/2}|\hat{\boldsymbol{\Omega}}_k^{-1}|^{1/2} \exp[-\frac{1}{2}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{kj}\mathbf{y}_{t_b-j})'\hat{\boldsymbol{\Omega}}_k^{-1}(\mathbf{y}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1}\hat{\mathbf{A}}_{kj}\mathbf{y}_{t_b-j})] \end{bmatrix}. \tag{8}$$

Our approach is consistent with the notion that investors never observe the true state. Learning effects can be important since optimal portfolio choices depend not only on future values of asset returns and predictor variables ($\mathbf{r}_b$, $\mathbf{z}_b$), but also on future perceptions of the likelihood of being in each of the unobservable regimes ($\boldsymbol{\pi}_{t_b+j}$).

Since W_b is known at time t_b, $Q(\cdot)$ simplifies to

$$Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} \mathrm{E}_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q(\mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1})\right]. \tag{9}$$

In the absence of predictor variables, $\mathbf{z}_t$, the investor's perception of the regime probabilities, $\boldsymbol{\pi}_b$, is the only state variable and the basic recursions can be written as

$$Q(\boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} \mathrm{E}_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}, t_{b+1})\right],$$

$$\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) = \frac{(\boldsymbol{\pi}'_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}'_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)]'\boldsymbol{\iota}_k}. \tag{10}$$

4.1. Solution methods

A variety of approaches have been followed in the literature on portfolio allocation under predictable returns. Barberis (2000) employs simulation methods to study a 'pure' allocation problem without interim consumption. Ang and Bekaert (2002a) solve for the optimal asset allocation using Gaussian quadrature methods. Campbell and Viceira (1999, 2001) and Campbell et al. (2003) derive approximate analytical solutions for an infinitely lived investor. Finally, some papers have derived closed-form solutions by working in continuous-time, e.g., Kim and Omberg (1996) for the case without interim consumption and Wachter (2002) for the case with interim consumption and complete markets.

Ang and Bekaert (2002a) were the first to study asset allocation under regime switching. They consider pairs of international stock market portfolios under regime switching with observable states, so the state variable simplifies to a set of dummy indicators. This setup allows them to apply quadrature methods based on a discretization scheme. Our framework is quite different since we treat the state as *unobservable* and calculate asset allocations under optimal filtering (7).

To deal with the latent state we use Monte Carlo methods for integral (expected utility) approximation. For example, for a buy-and-hold investor, we follow Barberis (2000) and Honda (2003) and approximate the integral in the expected utility functional as follows:

$$\max_{\boldsymbol{\omega}_t} N^{-1} \sum_{n=1}^{N} \left\{ \frac{[(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_3)\exp(Tr^{\mathrm{f}}) + \boldsymbol{\omega}_t' \exp(\sum_{i=1}^{T}(r^{\mathrm{f}}\boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n})]^{1-\gamma}}{1-\gamma} \right\}. \tag{11}$$

Here $\boldsymbol{\omega}_t' \exp(\sum_{i=1}^{T}(r^{\mathrm{f}}\boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n}))$ is the portfolio return in the nth Monte Carlo simulation. Each simulated path of portfolio returns is generated using draws from the model (1)–(2) that allow regimes to shift randomly as governed by the transition matrix, $\mathbf{P}$. We use $N = 30{,}000$ simulations. As pointed out by Detemple et al. (2003), numerical schemes based either on grid approximation of partial differential equations or on quadrature discretization of the state space suffer from a dimensionality curse that Monte Carlo simulation methods can help alleviate. This makes Monte Carlo methods particularly suitable to multivariate problems such as ours. Guidolin and Timmermann (2005a) and Section 5.4 provide further details on the numerical techniques employed in the solutions.

5. Asset allocation results

As a benchmark we first consider the asset allocation strategy of a buy-and-hold investor who solves the asset allocation problem once, at time t. Brennan and Xia (2002) point out that this is an interesting special case since it corresponds to the problem solved by an investor who has set aside predetermined savings for retirement and commits to a portfolio that maximizes the expected utility from consumption upon retirement. At the end of the section we introduce rebalancing. Following Aït-Sahalia and Brandt (2001) we vary the investment horizon T between 6 and 120 months in increments of 6 months. The coefficient of relative risk aversion is initially set at $\gamma = 5$.[11]

[11] Guidolin and Timmermann (2005a) show that the asset allocation results are robust to values of γ in the range [0, 20].

5.1. Optimal asset allocation in the four regimes

We found in Section 2 that the four regimes identified in the joint distribution of stock and bond returns had economic interpretations as crash, slow growth, bull and recovery states. To better understand the role of these economic states in asset allocation, Fig. 3 shows optimal asset allocations starting from each of the states, i.e. $\boldsymbol{\pi} = \mathbf{e}_j$ $(j = 1, 2, 3, 4)$, but allowing for uncertainty about future states due to randomly occurring regime shifts driven by (2). Initial state probabilities are clearly an important determinant of the portfolio weights.

State 1 is a low return state with little persistence. As the investment horizon (T) grows, investors can be reasonably certain of leaving this state and move to better ones. The weight on stocks is therefore negligible for small T but increases as T grows, producing an upward-sloping curve. Although stocks are almost completely ignored at short horizons, the low persistence of regime 1 along with the high probability of switching to the recovery state leads to a rapid increase in the optimal allocation to stocks as the investment horizon expands. Even so, the optimal allocation to stocks never exceeds 35% when starting from the crash state. The allocation to bonds grows from 0% to 30%, while the allocation to T-bills shows the opposite pattern, starting at 100% of the portfolio and declining to 40% at the 10-year horizon.

In the slow growth state (regime 2) the small firm effect is negative and the demand for small stocks is always zero while conversely that for large stocks is very high, starting at 100% at the shortest horizon before declining to a level near two-thirds of the portfolio at horizons longer than 6 months. The remainder of the portfolio is invested in bonds and T-bills. The bull state (regime 3) is associated with a sizeable small firm effect and small stocks take up 70% of the portfolio at short horizons before declining to 20% for horizons greater than 6 months. The reverse pattern is seen for large stocks that start at 30% for short horizons and grow to a level near 50% for horizons longer than 6 months. Bond and T-bill allocations are close to zero at short horizons, rising to around 10% and 15%, respectively, at long horizons.

Finally, starting from the recovery state, at short horizons 100% of the portfolio is allocated to small stocks. This proportion declines to 40% for horizons longer than 1 year, while the allocation to large stocks and bonds rise from 0% to 30% as the horizon is extended from 1 to 12 months. In this state practically nothing is invested in T-bills.

5.2. Uncertainty about the initial state

While Ang and Bekaert (2002a) use quadrature methods that require the regimes to be observable, our Monte Carlo simulation approach to computing portfolio weights allows the states to be unobservable. In fact, in our model investors have to account for revisions in future beliefs about the unknown state when determining their current asset allocation. In this sense our paper extends the rational learning exercise in Barberis (2000) to cover multivariate regime switching. Our ability to handle unknown initial and future states is important, both because states are never

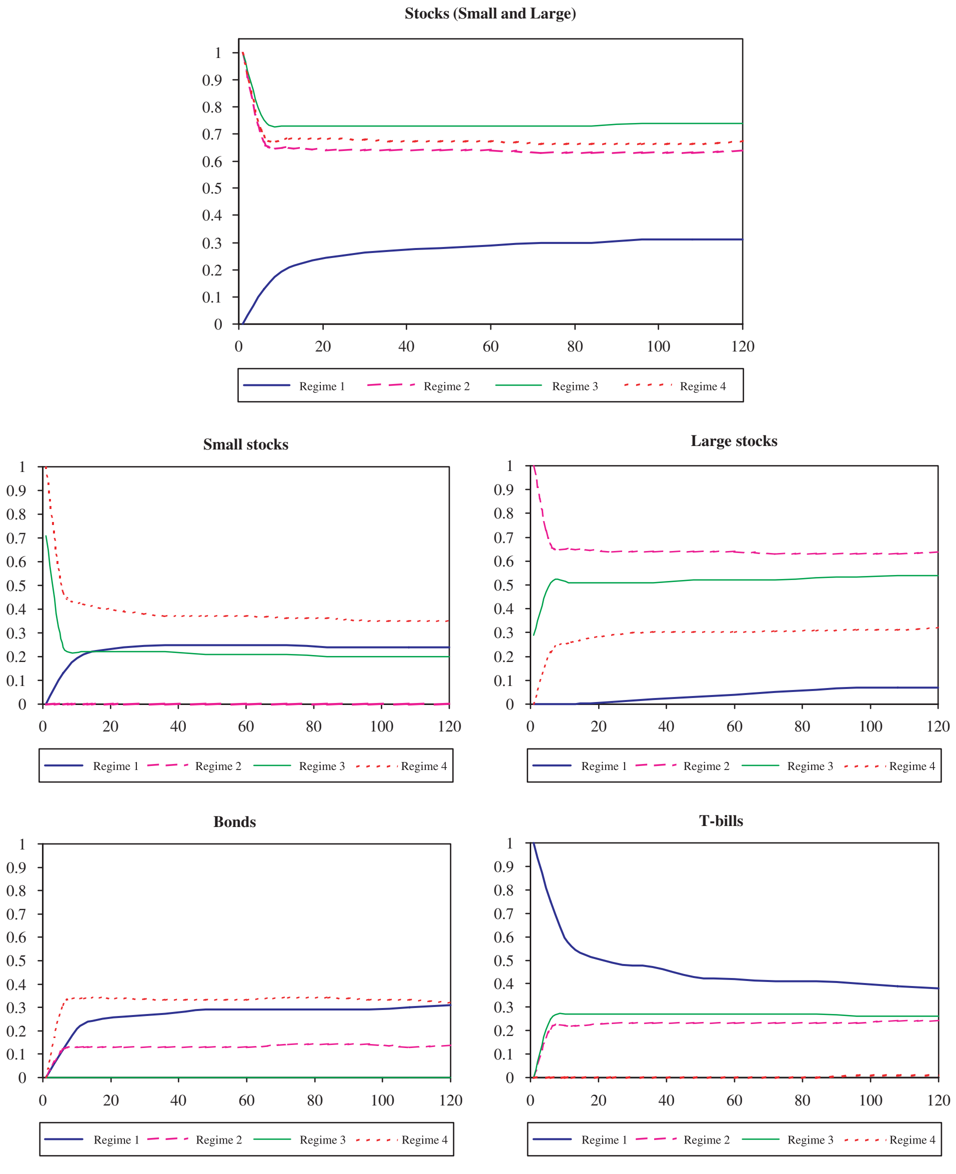

Fig. 3. Optimal buy-and-hold portfolio allocation as a function of the investment horizon: known initial states.

observable in practice and because, as shown by Veronesi (1999), uncertainty about the underlying regime may be key to understanding asset price dynamics.

We examine the asset allocation implications of uncertainty about the initial state by considering two scenarios. The first assumes that the states have the same probability (25%) while the second scenario assumes steady-state probabilities

(9%, 40%, 28% and 23% for states 1–4). The extent to which asset allocations depend on the underlying state beliefs is clear from Fig. 4: at short horizons the sign of the slope of the investment demand for stocks is reversed in the two scenarios.

These results show that uncertainty about the initial state can significantly affect portfolio decisions. The shapes of some of the investment schedules in Fig. 4 differ from all the shapes shown in Fig. 3 and are thus not merely (probability-) weighted

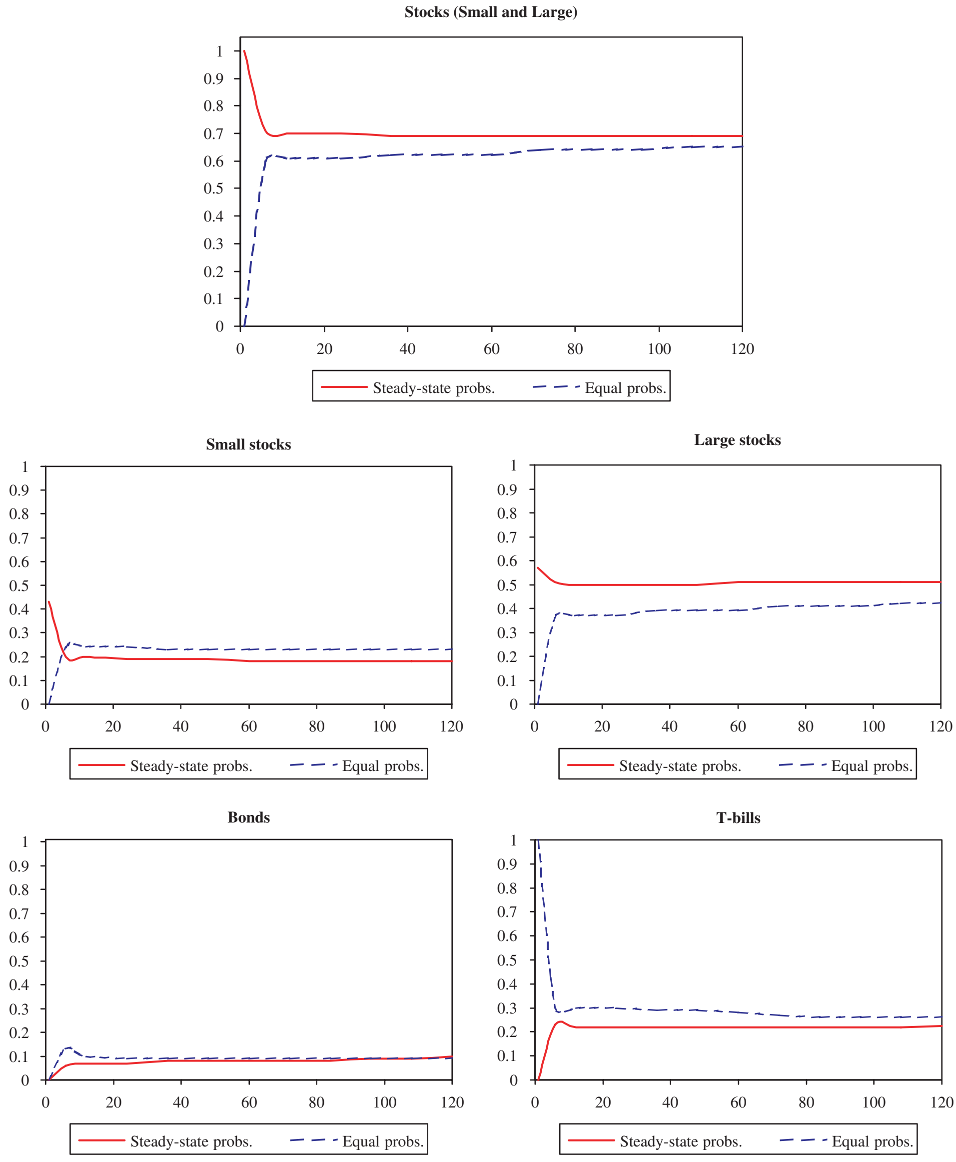

Fig. 4. Effect of uncertain states on buy-and-hold portfolio decisions.

averages of the schedules found when the initial regime is assumed to be known to the investor. Moreover, investors' perceptions of the current state probabilities are a key determinant of the relationship between the investment horizon and the optimal asset allocation, and therefore affect how an investor attempts to exploit predictability in asset returns.

5.3. Investment horizon effects

One of the key questions addressed in the literature on optimal asset allocation is how the investment horizon affects optimal portfolio weights. In the absence of predictor variables, standard models imply constant portfolio weights. In contrast, using the dividend yield as a predictor, Barberis (2000) finds that the weight on stocks should increase as a function of the investor's horizon. With reference to Figs. 3 and 4, we next show that this no longer is the case when changes in regimes can occur.

Fig. 3 shows that, in three of four states, the buy-and-hold investor gets more cautious about stocks as the investment horizon rises. Although the return on bonds is also uncertain across states, the allocation to bonds increases as a function of the horizon in three of four states. There are two reasons for this. First, the difference in mean returns across states is far smaller for bonds than for stocks. Second, the correlation between stock and bond returns is generally very low, so bond holdings diversify the risk of stock holdings independently of which state occurs.

Fig. 4 further suggests that the weight on stocks increases in the investment horizon only when the investor initially assigns a sufficiently high weight to the crash state. Even under steady-state probabilities the allocation to stocks declines as the investment horizon grows. The well-known advice of increased exposure to stocks the longer the investment horizon appears not to be robust to how predictability in returns is modeled and may even be more of an exception than the rule.

5.4. Rebalancing

The buy-and-hold results presented thus far ignore the possibility of rebalancing. However, in the presence of time-varying investment opportunities, investors should adjust their portfolio weights as new information arrives. We therefore consider the effects of periodic rebalancing on optimal asset allocations. We numerically solve the Bellman equation by discretizing the interval that defines the domain of each of the state variables on G points and use backward induction methods. Suppose that $Q(\boldsymbol{\pi}_{b+1}, t_{b+1})$ is known at all points $\boldsymbol{\pi}_{b+1} = \boldsymbol{\pi}^j_{b+1}$, $j = 1, 2, \ldots, G^{k-1}$. This will be true at time $t_B \equiv t + T$ as $Q(\boldsymbol{\pi}^j_B, t_B) = 1$ for all values of $\boldsymbol{\pi}^j_B$ on the grid. Then we can solve Eq. (4) to obtain $Q(\boldsymbol{\pi}_b, t_b)$ by choosing $\boldsymbol{\omega}_b$ to maximize

$$\mathrm{E}_{t_b}[\{(1 - \boldsymbol{\omega}'_b \boldsymbol{\iota}_3)\exp(\varphi r^{\mathrm{f}}) + \boldsymbol{\omega}'_b \exp(\mathbf{R}_{b+1,n}(s_b) + \varphi r^{\mathrm{f}} \boldsymbol{\iota}_3)\}^{1-\gamma} Q(\boldsymbol{\pi}^j_{b+1}, t_{b+1})]. \tag{12}$$

The multiple integral defining the conditional expectation is again calculated by Monte Carlo methods. For each $\boldsymbol{\pi}_b = \boldsymbol{\pi}^j_b$, $j = 1, 2, \ldots, G^{k-1}$ on the grid we draw N

samples of φ-period excess returns $\{\mathbf{R}_{b+1,n}(s_b) \equiv \sum_{i=1}^{\varphi} \mathbf{r}_{t_b+i,n}(s_b)\}_{n=1}^{N}$ from the regime switching model and approximate (12) as

$$N^{-1} \sum_{n=1}^{N} [\{(1-\boldsymbol{\omega}_b' \boldsymbol{\iota}_3)\exp(\varphi r^{\mathrm{f}}) + \boldsymbol{\omega}_b' \exp(\mathbf{R}_{b+1,n}(s_b) + \varphi r^{\mathrm{f}} \boldsymbol{\iota}_3)\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1})]. \tag{13}$$

Here $\boldsymbol{\pi}_{b+1}^{(j,n)}$ denotes the element $\boldsymbol{\pi}_{b+1}^{j}$ on the grid used to discretize the state space that – using the distance measure $\sum_{i=1}^{k-1} |\boldsymbol{\pi}_{b+1}^{j} e_i - \boldsymbol{\pi}_{b+1,n} e_i|$ – is closest to

$$\boldsymbol{\pi}_{b+1,n} = \frac{(\boldsymbol{\pi}_b' \hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1,n}; \hat{\boldsymbol{\theta}}_t)}{[(\boldsymbol{\pi}_b' \hat{P}_t^{\varphi})' \odot \boldsymbol{\eta}(\mathbf{r}_{b+1,n}; \hat{\boldsymbol{\theta}}_t)]' \boldsymbol{\iota}_k}.$$

Starting from t_{B-1}, we work backwards through the B rebalancing points until $\boldsymbol{\omega}_t \equiv \boldsymbol{\omega}_0$ is obtained.[12]

Table 2 shows optimal portfolio weights for stocks and bonds under different values of the rebalancing frequency, $\varphi = 1, 3, 6, 12, 24$ months as well as under the buy-and-hold scenario, $\varphi = T$. For a given investment horizon, T, as φ declines investors become more responsive to the current state probabilities. The smaller is φ, the shorter is the period over which the investor commits wealth to a given portfolio. As a result, the investor puts less weight on the steady-state return distribution and increasing weight on the current state, S_t. Consequently, the weight on stocks in the crash state declines as φ decreases and rebalancing becomes more frequent. For instance, when $T = 120$ and $\varphi = 1$ (monthly rebalancing), investors hold no stocks in the crash state, preferring instead to wait for an improvement in the investment opportunity set. In contrast, when φ exceeds the average duration of this regime (e.g., $\varphi = 12$), it is optimal to invest some money in stocks (40%), although the weight remains quite low. In states 2–4 investors increase their allocation to stocks as the time between rebalancing declines. In fact, when $\varphi = 1$, the optimal weight on stocks is close to 100% in these three regimes, irrespective of the investment horizon. Keeping the rebalancing frequency, φ, constant, the demand for stocks is mostly upward sloping although increasingly flat as φ declines. Once again, we find that it is not generally true that investors with longer horizons should allocate more to stocks.

As the investment horizon grows, non-monotonic patterns are observed in the allocation to bonds which in most cases first rises and then declines. Starting from the crash state the allocation to bonds is generally lower, the more frequent the rebalancing (smaller φ) since the investor does not have to account for unexpected shifts to a better state but can afford to wait for such a shift to occur. If rebalancing can occur frequently, little or nothing is invested in bonds since market timing opportunities are more significant for stocks and the remainder can be held in T-bills.

[12]See Guidolin and Timmermann (2005a) for details. Simulation experiments indicate that five grid discretization points (for rebalancing problems) and $N \geqslant 20,000$ guarantee sufficient accuracy in the calculations of optimal choices.

Table 2
Effects of rebalancing on asset allocation

Rebalancing frequency φ	Investment horizon T (in months)					
	$T=1$	$T=6$	$T=12$	$T=24$	$T=60$	$T=120$
A – optimal allocation to stocks						
Crash regime 1						
$\varphi=T$ (buy-and-hold)	0.00	0.24	0.34	0.48	0.58	0.60
$\varphi=24$ months	—	—	—	—	0.50	0.50
$\varphi=12$ months	—	—	—	0.37	0.39	0.40
$\varphi=6$ months	—	—	0.28	0.31	0.33	0.34
$\varphi=3$ months	—	0.00	0.00	0.00	0.00	0.00
$\varphi=1$ month	0.00	0.00	0.00	0.00	0.00	0.00
Slow growth regime 2						
$\varphi=T$ (buy-and-hold)	1.00	0.68	0.65	0.65	0.65	0.64
$\varphi=24$ months	—	—	—	—	0.70	0.80
$\varphi=12$ months	—	—	—	0.72	0.82	0.93
$\varphi=6$ months	—	—	0.71	0.77	0.88	0.96
$\varphi=3$ months	—	0.92	0.85	0.89	0.95	0.99
$\varphi=1$ month	1.00	1.00	1.00	1.00	1.00	1.00
Bull regime 3						
$\varphi=T$ (buy-and-hold)	1.00	0.67	0.66	0.65	0.65	0.65
$\varphi=24$ months	—	—	—	—	0.72	0.83
$\varphi=12$ months	—	—	—	0.74	0.85	0.88
$\varphi=6$ months	—	—	0.74	0.80	0.90	0.95
$\varphi=3$ months	—	0.94	0.96	0.98	1.00	1.00
$\varphi=1$ month	1.00	1.00	1.00	1.00	1.00	1.00
Recovery regime 4						
$\varphi=T$ (buy-and-hold)	1.00	0.82	0.71	0.69	0.68	0.66
$\varphi=24$ months	—	—	—	—	0.71	0.74
$\varphi=12$ months	—	—	—	0.72	0.74	0.77
$\varphi=6$ months	—	—	0.75	0.79	0.82	0.85
$\varphi=3$ months	—	0.98	1.00	1.00	1.00	1.00
$\varphi=1$ month	1.00	1.00	1.00	1.00	1.00	1.00
Steady-state probabilities						
$\varphi=T$ (buy-and-hold)	1.00	0.73	0.68	0.67	0.65	0.64
$\varphi=24$ months	—	—	—	—	0.71	0.77
$\varphi=12$ months	—	—	—	0.73	0.78	0.81
$\varphi=6$ months	—	—	0.78	0.81	0.84	0.83
$\varphi=3$ months	—	0.88	0.85	0.84	0.84	0.84
$\varphi=1$ month	1.00	0.98	0.98	0.98	0.98	0.98
B – optimal allocation to long-term bonds						
Crash regime 1						
$\varphi=T$ (buy-and-hold)	0.00	0.34	0.29	0.19	0.12	0.08
$\varphi=24$ months	—	—	—	—	0.16	0.10
$\varphi=12$ months	—	—	—	0.21	0.17	0.11
$\varphi=6$ months	—	—	0.28	0.18	0.15	0.10
$\varphi=3$ months	—	0.18	0.16	0.13	0.11	0.05
$\varphi=1$ month	0.00	0.00	0.00	0.00	0.00	0.00

Table 2 (*continued*)

Rebalancing frequency φ	Investment horizon T (in months)					
	$T=1$	$T=6$	$T=12$	$T=24$	$T=60$	$T=120$
Slow growth regime 2						
$\varphi=T$ (buy-and-hold)	0.00	0.32	0.34	0.19	0.14	0.08
$\varphi=24$ months	—	—	—	—	0.17	0.13
$\varphi=12$ months	—	—	—	0.20	0.14	0.01
$\varphi=6$ months	—	—	0.21	0.13	0.04	0.00
$\varphi=3$ months	—	0.05	0.13	0.04	0.00	0.00
$\varphi=1$ month	0.00	0.00	0.00	0.00	0.00	0.00
Bull regime 3						
$\varphi=T$ (buy-and-hold)	0.00	0.00	0.00	0.00	0.00	0.00
$\varphi=24$ months	—	—	—	—	0.05	0.00
$\varphi=12$months	—	—	—	0.06	0.03	0.00
$\varphi=6$ months	—	—	0.07	0.02	0.00	0.00
$\varphi=3$ months	—	0.02	0.00	0.00	0.00	0.00
$\varphi=1$ month	0.00	0.00	0.00	0.00	0.00	0.00
Recovery regime 4						
$\varphi=T$ (buy-and-hold)	0.00	0.17	0.12	0.10	0.08	0.08
$\varphi=24$ months	—	—	—	—	0.01	0.01
$\varphi=12$ months	—	—	—	0.00	0.00	0.00
$\varphi=6$ months	—	—	0.00	0.00	0.00	0.00
$\varphi=3$ months	—	0.00	0.00	0.00	0.00	0.00
$\varphi=1$ month	0.00	0.00	0.00	0.00	0.00	0.00
Steady-state probabilities						
$\varphi=T$ (buy-and-hold)	0.00	0.03	0.04	0.05	0.07	0.06
$\varphi=24$ months	—	—	—	—	0.10	0.12
$\varphi=12$ months	—	—	—	0.08	0.12	0.14
$\varphi=6$ months	—	—	0.07	0.10	0.16	0.17
$\varphi=3$ months	—	0.06	0.15	0.12	0.11	0.10
$\varphi=1$ month	0.00	0.02	0.02	0.02	0.02	0.02

This table reports the optimal weight on stocks (small and large) and bonds as a function of the rebalancing frequency φ for an investor with power utility and a constant relative risk aversion coefficient of 5. Excess returns are assumed to be generated by the regime switching model

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}_{s_t})$ is the vector of return innovations.

6. Asset allocation under predictability from the dividend yield

6.1. Allocations under a single state model

Asset allocation implications of linear predictability in returns from variables such as the dividend yield have been considered by Barberis (2000), Campbell and Viceira (1999), Campbell et al. (2003), and Kandel and Stambaugh (1996). It is therefore natural to compare our results to those arising from a standard VAR(1) model comprising asset returns and the dividend yield:

$$\begin{pmatrix} \mathbf{r}_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}_{dy} \end{pmatrix} + \mathbf{A} \begin{pmatrix} \mathbf{r}_{t-1} \\ dy_{t-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \varepsilon_{dy,t} \end{pmatrix}, \tag{14}$$

where $\mathbf{r}_t \equiv (r_t^l \; r_t^s \; r_t^b)'$ and $(\boldsymbol{\varepsilon}_t' \; \varepsilon_{dy,t})' \sim \mathrm{N}(0, \Omega)$. MLE estimates are as follows:

$$\begin{pmatrix} \mathbf{r}_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \underset{(0.0070)}{0.0021} \\ \underset{(0.0102)}{-0.0160} \\ \underset{(0.0036)}{-0.0032} \\ \underset{(0.0003)}{0.0004} \end{pmatrix} + \begin{bmatrix} \underset{(0.0635)}{-0.0466} & \underset{(0.0412)}{0.0370} & \underset{(0.0839)}{0.2299} & \underset{(0.2028)}{0.1261} \\ \underset{(0.0925)}{0.1236} & \underset{(0.0600)}{0.1244} & \underset{(0.1233)}{0.2624} & \underset{(0.2953)}{0.6641} \\ \underset{(0.0330)}{-0.0442} & \underset{(0.0214)}{-0.0261} & \underset{(0.0436)}{0.1070} & \underset{(0.1054)}{0.1322} \\ \underset{(0.0024)}{-0.0005} & \underset{(0.0016)}{-0.0005} & \underset{(0.0032)}{-0.0098} & \underset{(0.0077)}{0.9856} \end{bmatrix} \begin{pmatrix} \mathbf{r}_{t-1} \\ dy_{t-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \varepsilon_{dy,t} \end{pmatrix},$$

$$\hat{\boldsymbol{\Omega}}^* = \begin{bmatrix} 0.1417^{***} & 0.0018 & 0.0002 & -5.86\mathrm{e}^{-05} \\ 0.7285^{***} & 0.2063^{***} & 0.0002 & -7.10\mathrm{e}^{-05} \\ 0.2466^{*} & 0.1353 & 0.0736^{***} & -7.95\mathrm{e}^{-06} \\ -0.9243^{***} & -0.7695^{***} & -0.2413 & 0.0056^{***} \end{bmatrix},$$

where $\mathbf{r}_t \equiv [r_t^l \; r_t^s \; r_t^b]'$. Standard errors are in parenthesis below the point estimates of the conditional mean coefficients.[13] The estimate of $\hat{\mathbf{A}}^*$ suggests that a higher dividend yield forecasts higher asset returns. The dividend yield is highly persistent – its autoregressive coefficient estimate is almost 0.99 – and shocks to the dividend yield are highly negatively correlated with shocks to stock returns (−0.92 and −0.76 for large and small stocks, respectively), suggesting that time-variations in the dividend yield may induce a large hedging demand for stocks. In contrast, shocks to the dividend yield are only mildly and insignificantly correlated with shocks to bond returns (−0.24).[14]

We computed allocations to stocks and bonds under the VAR(1) for a range of values of the dividend yield.[15] Our results are comparable to earlier findings: for most values of the dividend yield the overall allocation to stocks is larger, the longer the investment horizon or the higher the initial value of the dividend yield. We also

[13]For the estimated covariance matrix, we report annualized volatilities on the main diagonal. Coefficients below the diagonal are correlation coefficients. * denotes significance at the 10%, ** at the 5%, and *** at the 1% level.

[14]Our choice of an unrestricted VAR(1) model is consistent with Campbell et al. (2003). In asset allocation problems involving investments in bonds it is important to allow for predictability from lagged bond returns to current stock returns and the zero restrictions on the VAR(1) return coefficients are strongly rejected by a likelihood ratio test.

[15]Detailed results are available in Guidolin and Timmermann (2005a).

found slightly non-monotonic investment schedules for stocks under the VAR(1) model, which is not completely surprising. Aït-Sahalia and Brandt (2001), Brandt (1999) and Barberis (2000) have found increasing equity demand, the longer the investment horizon when conditioning on a value of the dividend yield close to its sample average. However, when the dividend yield is further away from its unconditional mean, asset allocation results become more mixed and there are cases where, at short-to-intermediate investment horizons, equity demand is declining in the horizon. Medium-to-high dividend yields favor small stocks while medium–low yields increase the demand for large stocks. The reason for this is the greater sensitivity of the small stocks' returns to the dividend yield (0.66) compared with the sensitivity of the large stocks (0.12).

There is very little room for bonds in the optimal asset allocation under a VAR(1) model. This holds across all initial values of the dividend yield. When the dividend yield is either low or very low – so stocks are unattractive – short-term investors respond not by holding a larger proportion of bonds, but rather by increasing their allocation to T-bills.

6.2. Regimes and predictability from the dividend yield

We next investigate the effect of adding the dividend yield to our model. The resulting regime switching VAR model nests many of the models in the existing literature and enables the correlation between the dividend yield and asset returns to vary across different regimes. The relationship between stock returns and the dividend yield is linear within a given regime. However, since the coefficient on the dividend yield varies across regimes, as the regime probabilities change the model is capable of tracking a non-linear relationship between asset returns and the yield. This is important given the evidence of a non-linear relationship between the yield and stock returns uncovered by Ang and Bekaert (2004).[16]

Again we conducted a battery of tests to determine the best model specification. The (unreported) results suggest that a four-state VAR(1) model is supported by the data as this model passes all diagnostic tests. Unsurprisingly, given the persistence in the dividend yield, a single lag is required for this extended model. Regime 1 continues to be characterized by large, negative mean excess returns. The dividend yield is relatively high in this state (4%) and volatility is also above average. In steady state this regime occurs 15% of the time although it has an average duration of only 2 months. Regime 2 remains a slow growth state with moderate volatility. This state is highly persistent, lasting on average almost 16 months and occurring close to one-third of the time. Regime 3 continues to be a highly persistent bull state that lasts on average almost 15 months. Finally, regime 4 is again a recovery state with strong stock market rallies accompanied by substantial volatility. This state has an average duration of only 2 months. Nevertheless, at 18%, its steady-state probability is quite high.

[16]Other specifications are of course possible – for example a model that allows state transition probabilities to vary over time as a function of a set of state variables as in Perez-Quiros and Timmermann (2000) or Ang and Bekaert (2002a).

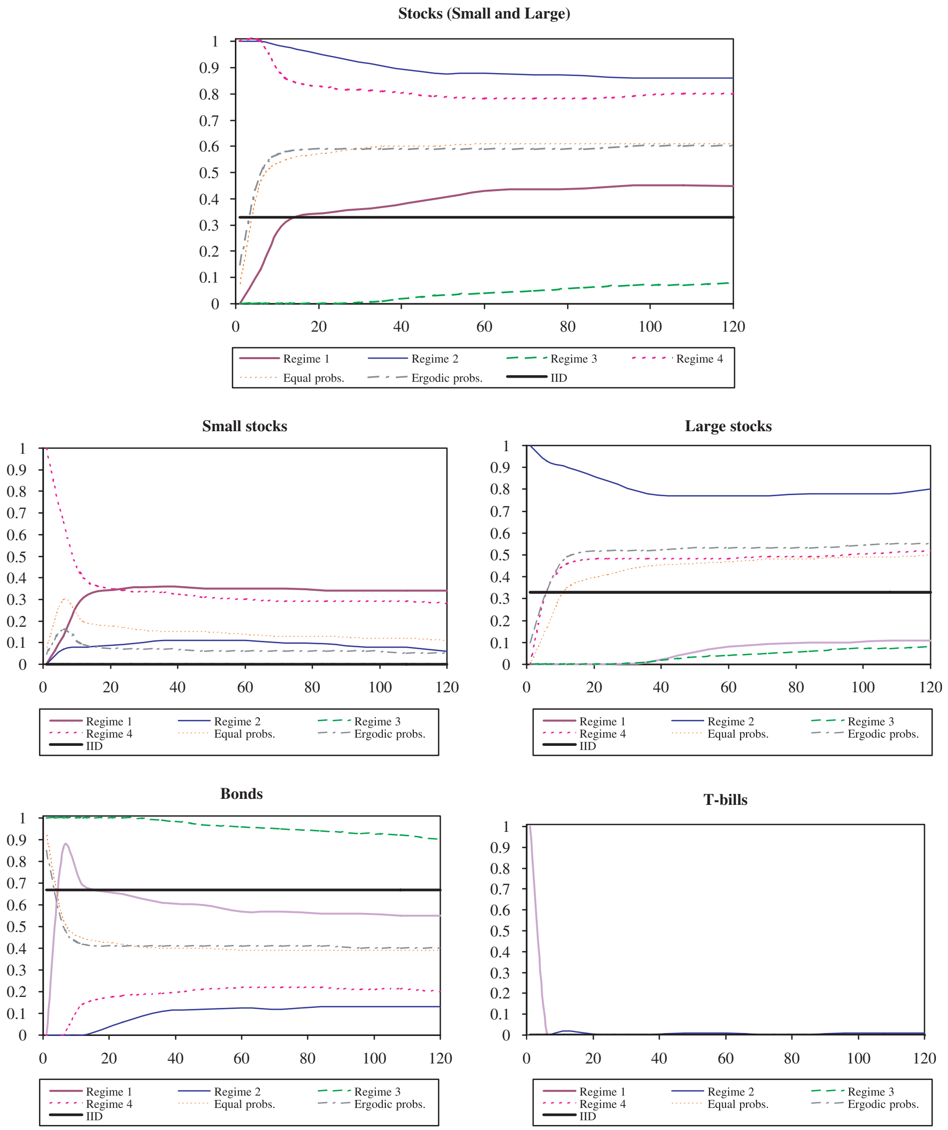

Fig. 5. Predictability from the dividend yield under regime switching: buy-and-hold results. The graphs plot the optimal allocation as a function of the investment horizon for an investor with constant relative risk aversion $\gamma = 5$ for six configurations of initial state probabilities: certainty of being in regimes 1–4, equal state probabilities, and ergodic state probabilities. In each graph, the dividend yield is set at its unconditional sample mean.

To study the asset allocation effects of regimes and predictability from the yield we report two exercises. The first, presented in Fig. 5, shows optimal allocations as a function of the horizon when the dividend yield is set at its overall sample average.

Asset allocations continue to vary significantly across the four states. Starting from state 1 the allocation to stocks (small stocks in particular) continues to rise as a function of the horizon and peaks at 40% of the portfolio at the 10-year horizon. The allocation to bonds is non-monotonic, starting from zero at the shortest horizon, rising to a level close to 90% at the 6 month horizon before declining to 60% at the longest horizon. T-bills form 100% of the portfolio at the shortest 1-month horizon but then see their allocation decline sharply to zero at horizons longer than 6 months.

The allocation to stocks continues to decline when the model starts from states two or four, although it only declines to a level near 80–85% at the 10-year horizon. The allocation to bonds makes up for the remainder and there is no demand for T-bills in these two states. In the third (bull) state the allocation to stocks is now mildly upward sloping as a function of the horizon in contrast to what we found in the model without the dividend yield shown in Fig. 3.

We also computed the effect of changing the dividend yield using a range of values spanning its mean value plus or minus three standard deviations. As expected, the higher the dividend yield, the larger the allocation to stocks. This is consistent with the common finding of a positive correlation between the yield and expected returns. The allocation to small stocks is more sensitive to the yield than that of the large stocks. When the yield is very low, the allocation to stocks is very small and the allocation to T-bills is large, but it declines as a function of the investment horizon. Irrespective of the presence of regimes, we get very sensible results for the effect of changing the dividend yield on the optimal asset allocation.

We summarize these findings as follows. First, by comparing Figs. 3 and 5, it is obvious that the dividend yield continues to have an important effect on the optimal asset allocation even in the presence of regimes. In the model extended to include the yield there is less of a role for T-bills, while conversely long-term bonds and large stocks form a larger part of the portfolio. Furthermore, irrespective of the presence of regimes, the higher the yield, the greater the typical allocation to stocks.

6.3. Understanding the term structure of reward-to-risk

As mentioned in the introduction, learning models – where investors revise their expectations of future returns upwards following positive return shocks – imply that $\beta > 0$, where

$$\frac{Var(\sum_{j=1}^{T} r_{t+j})}{Var(r_{t+1})T} = 1 + \frac{1}{2}\left(\frac{R^2}{1 - R^2}\right)\beta.$$

Constant expected return models – such as our Gaussian IID benchmark – imply that $\beta = 0$, i.e. the variance of T-period returns grows in proportion with T. Models of predictable, mean-reverting returns produce a $\beta < 0$, i.e. risk grows at a slower rate than if expected returns were constant. This simple analysis provides intuition for our basic findings. The analysis is of course complicated by the fact that under regime switching both the mean and variance of returns depend on the state

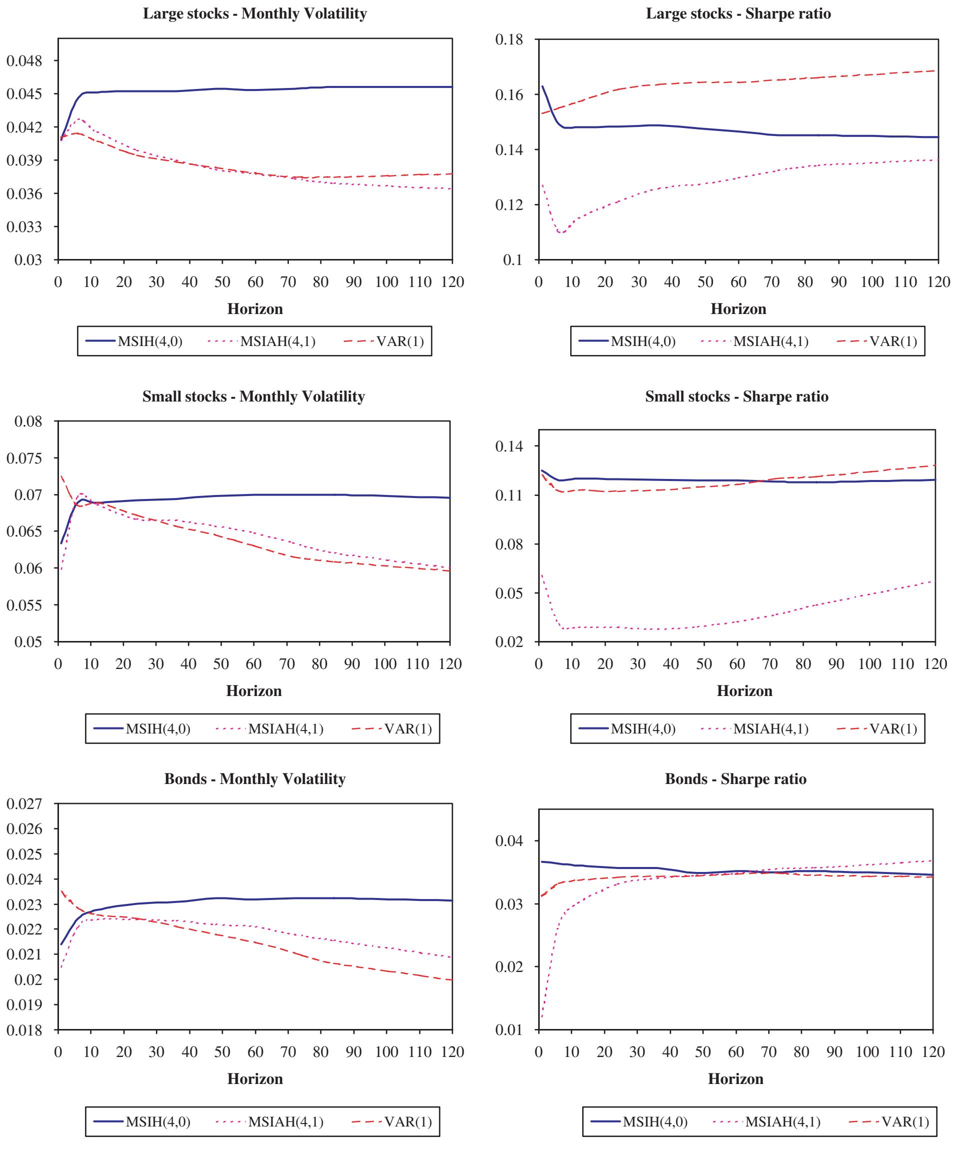

Fig. 6. Volatility and Sharpe ratios as a function of the investment horizon. These graphs plot monthly volatility and Sharpe ratios of returns on each asset class under three alternative models (four-state, MSIH(4,0), four-state VAR(1) model with predictability from the dividend yield, MSIAH(4,1), single-state model with predictability from the dividend yield, VAR(1)). State probabilities and the dividend yield are set at their steady-state values.

probabilities. The exception to this is when the state probabilities are set at their steady-state values in which case expected returns become independent of the investment horizon. For this scenario Fig. 6 plots the volatility and Sharpe ratio implied by the models considered so far (these are plots of the 'term structure' of the reward-to-risk trade-off, in the sense recently used by and Campbell and Viceira, 2005; Guidolin and Timmermann, 2006b). We only show results for small and large stocks since the effects are much smaller for bonds. The volatility and Sharpe ratio are normalized by dividing by $\sqrt{T}$ so the benchmark IID model corresponds to a straight line.

First consider the pure regime switching model. Starting from the steady state probabilities the mean return is constant whereas the volatility per month increases as a function of the investment horizon. This leads to a Sharpe ratio that declines in the investment horizon and hence to a lower allocation to risky assets. Consistent with this, Fig. 3 showed that it is only when the model starts from the crash state that the overall allocation to stocks is increasing in the horizon – the reason being that the mean return increases as a function of the investment horizon while the risk declines when starting from this state.

Next consider the VAR(1) model where the initial dividend yield is set at its unconditional mean. For this model Fig. 6 shows that the volatility decreases and hence the Sharpe ratio increases (relative to the IID benchmark) as a function of the investment horizon, leading to a greater allocation to stocks the longer the investment horizon, as we found in Section 6.1.

In the four-state model extended to include the dividend yield, learning effects – which tend to lower the allocation to stocks – compete with mean reversion effects, which tend to increase the allocation to stocks the longer the investment horizon. Which effect dominates is an empirical issue that also depends on the initial values assumed for the dividend yield and the state probabilities. In practice, it seems that learning effects are stronger at short horizons, so that Sharpe ratios tend to decrease in T. However, at horizons in excess of 1 or 2 years, learning effects become weaker as the predictive distribution of returns converges to the steady state distribution, so mean-reversion effects captured by linear predictability from the dividend yield eventually lead to increasing Sharpe ratios.

7. Economic importance of regimes

So far we established that the presence of regime switching in the joint distribution of asset returns can affect portfolio decisions substantially. This evidence does not imply, however, that investors are necessarily better off by accounting for regimes in the return distribution. In this section we therefore assess the economic importance of regimes. We do so by computing the certainty equivalent of ignoring regimes (Section 7.1), by checking whether differences in portfolio weights from a regime switching versus a time-invariant model of the return distribution are mostly due to parameter uncertainty (Section 7.2), and by evaluating the out-of-sample performance of alternative investment strategies (Sections 7.3–7.4). Section 7.5 finally asks the more specific question whether there is any added value to selecting a four-state model over a simpler two-state model. In all cases we focus on buy-and-hold results for an investor with $\gamma = 5$.

7.1. Utility cost calculations

It is natural to report a measure of the economic value of accounting for regimes in investors' asset allocation decisions. We obtain an estimate of this by comparing the investor's expected utility under the regime switching model to that assuming the investor is constrained to choose optimal portfolio weights $\boldsymbol{\omega}_t^{\mathrm{IID}}$ under the assumption that asset returns follow a simple IID process. In the latter case the portfolio choice is independent of the investment horizon and the value function for the constrained investor is

$$J_t^{\mathrm{IID}} \equiv \frac{1}{1-\gamma}\sum_{b=0}^{B}\beta^b \mathrm{E}_t[W_b^{1-\gamma}],$$
$$W_b = W_{b-1}[(1-(\boldsymbol{\omega}_t^{\mathrm{IID}})'\boldsymbol{\iota}_3)\exp(\varphi r^{\mathrm{f}}) + \boldsymbol{\omega}_b' \exp(\mathbf{R}_{b+1} + \varphi r^{\mathrm{f}}\boldsymbol{\iota}_3)]. \tag{15}$$

The assumption of IID returns is a constrained special case of the model with regime switching, so

$$J_t^{\mathrm{IID}} \leqslant J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\pi}_t, t),$$

where $J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\pi}_t, t)$ is the value function for the four-state model. We compute the compensatory premium, η_t^{IID}, that an investor would be willing to pay to obtain the same expected utility from the constrained and unconstrained consumption and asset allocation problems:

$$\eta_t^{\mathrm{IID}} = \left\{\frac{Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b)}{(1-\psi_t^{\mathrm{IID}})^{1-\gamma}\sum_{b=0}^{B}\beta^b \mathrm{E}_t[(W_b)^{1-\gamma}]}\right\}^{1/(1-\gamma)} - 1. \tag{16}$$

Fig. 7 plots the annualized compensation rate, $100 \times [(1+\eta_t^{\mathrm{IID}})^{12/T} - 1]$, needed to make a buy-and-hold investor indifferent between implementing strategies that

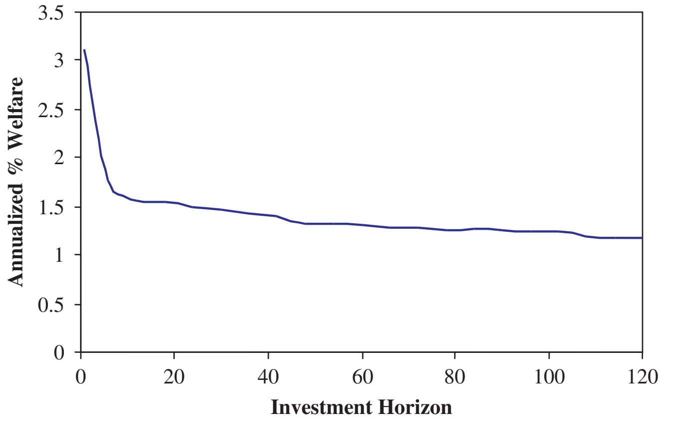

Fig. 7. Utility costs from ignoring regimes. The graph plots the compensation (as an annualized percentage) required to persuade a buy-and-hold investor with power utility (and $\gamma = 5$) to be willing to ignore regimes in asset returns, starting from steady-state probabilities.

exploit the presence of regimes and using the IID portfolio when the current regime probabilities are set at their steady-state values. The utility cost of ignoring regimes is as high as 3% at short horizons – where investors can exploit market timing more aggressively – while, at the longest horizons, the compensating rate is around 130 basis points per annum.

7.2. Parameter uncertainty

The presence of four regimes complicates parameter estimation so we consider the effect of parameter estimation errors on our results. Since we use Monte Carlo methods to derive optimal portfolio weights we can exploit that, in large samples,

$$\sqrt{T}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \stackrel{A}{\rightarrow} \mathrm{N}(\mathbf{0}, \mathbf{V}_{\theta}).$$

This allows us to set up the following bootstrap procedure. In the qth iteration we draw a vector of parameters, $\widehat{\hat{\boldsymbol{\theta}}}^{q}$, from $\mathrm{N}(\widehat{\boldsymbol{\theta}}, T^{-1}\hat{\mathbf{V}}_{\theta})$ where $\hat{\mathbf{V}}_{\theta}$ is a consistent estimator of $\mathbf{V}_{\theta}$. Conditional on this draw, $\widehat{\hat{\boldsymbol{\theta}}}^{q}$, we solve (4) to obtain a new vector of portfolio weights $\widehat{\hat{\boldsymbol{\omega}}}^{q}$. We repeat this process Q times. Confidence intervals for the optimal asset allocation $\hat{\boldsymbol{\omega}}_t$ can then be derived from the distribution for $\widehat{\hat{\boldsymbol{\omega}}}^{q}$, $q = 1, 2, \ldots, Q$. This approach is computationally intensive as (4) must be solved numerically so we restrict the number of bootstrap trials to $Q = 1,000$. Table 3 shows the optimal asset allocation plus or minus one standard deviation of the bootstrapped distribution. Figures in bold indicate that this band does not include the IID asset allocation, which represents a formal test of the difference between portfolio weights with and without regimes. Standard error bands are wide, but sufficiently narrow to confirm the validity of our conclusions concerning the optimal shape of equity investment schedules as a function of the investment horizon. The allocation to stocks is upward sloping only in the crash state. In regimes 2–4, however, the equity investment schedules are downward sloping, as their bands decline from a maximum of [0.7, 1] at $T = 1$ to [0.4, 0.8] for long investment horizons.

These methods also allow us to consider the joint effect of parameter estimation uncertainty and uncertainty about the underlying state. We do so by calculating the compensating variation $\eta_t^{\mathrm{IID},q}$ 1,000 times using parameter estimates $\widehat{\hat{\boldsymbol{\theta}}}^{q}$ drawn from their asymptotic distribution. Fig. 8 shows confidence intervals under steady-state probabilities. The null hypothesis of zero welfare loss implies that such intervals should include zero for all Ts. Evidence that the lower bound of the interval is positive suggests that ignoring regime switching in asset allocation problems leads to a significant reduction in expected utility. The null of no significant welfare cost from ignoring regime switching is strongly rejected. The lower bound of the confidence band is positive and also economically significant. At longer horizons the lower bound attains levels of 7–8%, which is a considerable fraction of wealth. Using a misspecified model in asset allocation decisions may thus be quite costly.

Table 3
Effect of parameter estimation uncertainty on asset allocation

		Investment horizon T						
		$T=1$	$T=6$	$T=24$	$T=48$	$T=72$	$T=96$	$T=120$
A: allocation to small stocks								
Crash	Mean + 1*SD	0.000	**0.319**	**0.393**	**0.392**	**0.395**	**0.390**	**0.394**
regime 1	Mean	0.000	**0.173**	**0.230**	**0.228**	**0.228**	**0.225**	**0.226**
	Mean − 1*SD	0.000	**0.028**	**0.067**	**0.063**	**0.061**	**0.060**	**0.058**
Slow growth	Mean + 1*SD	0.211	0.277	**0.357**	**0.375**	**0.385**	**0.383**	**0.383**
regime 2	Mean	0.061	0.127	**0.197**	**0.212**	**0.217**	**0.218**	**0.217**
	Mean − 1*SD	0.000	0.000	**0.037**	**0.049**	**0.050**	**0.053**	**0.052**
Bull	Mean + 1*SD	**0.915**	**0.530**	**0.432**	**0.410**	**0.404**	**0.403**	**0.401**
regime 3	Mean	**0.632**	**0.313**	**0.258**	**0.242**	**0.235**	**0.233**	**0.231**
	Mean − 1*SD	**0.349**	**0.096**	**0.083**	**0.073**	**0.067**	**0.064**	**0.060**
Recovery	Mean + 1*SD	**1.000**	**0.607**	**0.457**	**0.424**	**0.417**	**0.410**	**0.411**
regime 4	Mean	**0.890**	**0.406**	**0.279**	**0.252**	**0.245**	**0.238**	**0.236**
	Mean − 1*SD	**0.706**	**0.205**	**0.101**	**0.080**	**0.073**	**0.066**	**0.061**
Steady-state	Mean + 1*SD	**1.000**	**0.573**	**0.447**	**0.418**	**0.407**	**0.405**	**0.401**
	Mean	**0.827**	**0.361**	**0.270**	**0.247**	**0.238**	**0.235**	**0.231**
	Mean − 1*SD	**0.634**	**0.149**	**0.092**	**0.076**	**0.069**	**0.065**	**0.061**
B: allocation to large stocks								
Crash	Mean + 1*SD	**0.050**	**0.290**	0.497	0.553	0.573	0.579	0.590
regime 1	Mean	**0.005**	**0.114**	0.275	0.323	0.341	0.347	0.355
	Mean − 1*SD	**0.000**	**0.000**	0.053	0.093	0.109	0.116	0.119
Slow growth	Mean + 1*SD	**1.000**	0.709	0.629	0.616	0.613	0.611	0.613
regime 2	Mean	**0.834**	0.470	0.395	0.384	0.380	0.379	0.380
	Mean − 1*SD	**0.621**	0.232	0.161	0.151	0.148	0.147	0.147
Bull	Mean + 1*SD	0.630	0.703	0.632	0.620	0.616	0.619	0.616
regime 3	Mean	0.351	0.441	0.393	0.384	0.382	0.384	0.381
	Mean − 1*SD	0.073	0.179	0.154	0.148	0.147	0.148	0.146
Recovery	Mean + 1*SD	**0.275**	0.500	0.570	0.591	0.592	0.603	0.604
regime 4	Mean	**0.101**	0.268	0.336	0.356	0.360	0.368	0.369
	Mean − 1*SD	**0.000**	0.039	0.102	0.122	0.128	0.132	0.135
Steady-state	Mean + 1*SD	0.724	0.648	0.611	0.608	0.609	0.610	0.608
	Mean	0.174	0.406	0.386	0.381	0.380	0.378	0.380
	Mean − 1*SD	0.195	0.145	0.135	0.137	0.139	0.139	0.140
C: allocation to bonds								
Crash	Mean + 1*SD	**0.033**	**0.481**	**0.406**	**0.375**	**0.363**	**0.360**	**0.356**
regime 1	Mean	**0.000**	**0.264**	**0.221**	**0.200**	**0.190**	**0.190**	**0.186**
	Mean − 1*SD	**0.000**	**0.047**	**0.036**	**0.024**	**0.018**	**0.019**	**0.015**
Slow growth	Mean + 1*SD	**0.229**	**0.383**	**0.359**	**0.348**	**0.345**	**0.343**	**0.343**
regime 2	Mean	**0.084**	**0.206**	**0.191**	**0.183**	**0.180**	**0.179**	**0.178**
	Mean − 1*SD	**0.000**	**0.028**	**0.025**	**0.019**	**0.015**	**0.014**	**0.012**
Bull	Mean + 1*SD	**0.000**	**0.043**	**0.221**	**0.276**	**0.296**	**0.307**	**0.313**
regime 3	Mean	**0.000**	**0.010**	**0.095**	**0.130**	**0.143**	**0.151**	**0.156**
	Mean − 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
Recovery	Mean + 1*SD	**0.037**	**0.401**	**0.371**	**0.357**	**0.350**	**0.346**	**0.347**
regime 4	Mean	**0.006**	**0.230**	**0.203**	**0.191**	**0.185**	**0.180**	**0.182**
	Mean − 1*SD	**0.000**	**0.059**	**0.036**	**0.024**	**0.021**	**0.014**	**0.017**

Table 3 (*continued*)

		Investment horizon T						
		$T=1$	$T=6$	$T=24$	$T=48$	$T=72$	$T=96$	$T=120$
Steady-state	Mean + 1*SD	**0.000**	**0.125**	**0.255**	**0.295**	**0.309**	**0.318**	**0.321**
	Mean	**0.000**	**0.043**	**0.117**	**0.143**	**0.152**	**0.158**	**0.161**
	Mean − 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.001**
D: allocation to T-bills								
Crash regime 1	Mean + 1*SD	**1.000**	**0.607**	**0.489**	**0.453**	**0.442**	**0.438**	**0.433**
	Mean	**0.996**	**0.349**	**0.275**	**0.250**	**0.240**	**0.238**	**0.233**
	Mean − 1*SD	**0.966**	**0.091**	**0.060**	**0.046**	**0.039**	**0.038**	**0.034**
Slow growth regime 2	Mean + 1*SD	0.083	**0.391**	**0.408**	**0.413**	**0.415**	**0.416**	**0.418**
	Mean	0.024	**0.202**	**0.217**	**0.221**	**0.223**	**0.224**	**0.225**
	Mean − 1*SD	0.000	**0.012**	**0.027**	**0.030**	**0.031**	**0.032**	**0.032**
Bull regime 3	Mean + 1*SD	0.000	**0.392**	**0.435**	**0.431**	**0.430**	**0.424**	**0.423**
	Mean	0.000	**0.225**	**0.249**	**0.240**	**0.237**	**0.229**	**0.229**
	Mean − 1*SD	0.000	**0.059**	**0.064**	**0.049**	**0.044**	**0.035**	**0.035**
Recovery regime 4	Mean + 1*SD	0.000	0.222	0.356	**0.385**	**0.396**	**0.401**	**0.402**
	Mean	0.000	0.090	0.178	**0.198**	**0.207**	**0.211**	**0.211**
	Mean − 1*SD	0.000	0.000	0.000	**0.012**	**0.019**	**0.022**	**0.019**
Steady-state	Mean + 1*SD	0.000	**0.347**	**0.410**	**0.418**	**0.421**	**0.420**	**0.419**
	Mean	0.000	**0.188**	**0.226**	**0.228**	**0.228**	**0.227**	**0.226**
	Mean − 1*SD	0.000	**0.030**	**0.043**	**0.038**	**0.036**	**0.033**	**0.033**
E: overall allocation to stocks (small and large)								
Crash regime 1	Mean + 1*SD	**0.000**	0.478	0.701	**0.745**	**0.766**	**0.769**	**0.779**
	Mean	**0.000**	0.284	0.500	**0.545**	**0.565**	**0.569**	**0.576**
	Mean − 1*SD	**0.000**	0.091	0.299	**0.346**	**0.363**	**0.369**	**0.374**
Slow growth regime 2	Mean + 1*SD	**1.000**	**0.794**	**0.781**	**0.786**	**0.790**	**0.789**	**0.792**
	Mean	**0.893**	**0.590**	**0.586**	**0.591**	**0.593**	**0.592**	**0.593**
	Mean − 1*SD	**0.736**	**0.387**	**0.392**	**0.396**	**0.396**	**0.395**	**0.394**
Bull regime 3	Mean + 1*SD	**1.000**	**0.925**	**0.836**	**0.816**	**0.810**	**0.814**	**0.809**
	Mean	**1.000**	**0.760**	**0.651**	**0.625**	**0.616**	**0.617**	**0.611**
	Mean − 1*SD	**1.000**	**0.595**	**0.468**	**0.434**	**0.423**	**0.418**	**0.412**
Recovery regime 4	Mean + 1*SD	**1.000**	**0.872**	**0.808**	**0.802**	**0.799**	**0.804**	**0.805**
	Mean	**0.994**	**0.676**	**0.614**	**0.607**	**0.603**	**0.604**	**0.604**
	Mean − 1*SD	**0.962**	**0.481**	**0.421**	**0.411**	**0.407**	**0.404**	**0.403**
Steady-state	Mean + 1*SD	**1.000**	**0.926**	**0.839**	**0.817**	**0.809**	**0.808**	**0.807**
	Mean	**1.000**	**0.764**	**0.652**	**0.625**	**0.615**	**0.611**	**0.610**
	Mean − 1*SD	**1.000**	**0.602**	**0.466**	**0.433**	**0.420**	**0.414**	**0.411**

This table reports confidence bands for a buy-and-hold investor's optimal portfolio weights at different investment horizons, T, assuming power utility and a constant relative risk aversion coefficient of 5. Portfolio returns under regime switching are assumed to be generated by the model

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\varepsilon_{2t}\varepsilon_{3t}]' \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}_{s_t})$ is the vector of return innovations. In the IID case, $k = 1$. Boldfaced blocks of cells indicate a portfolio weight confidence interval that fails to include the IID weight.

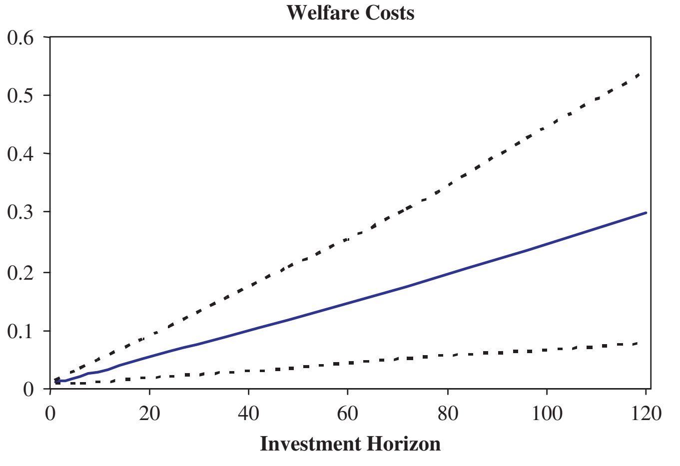

Fig. 8. Ninty percent bootstrapped confidence bands for utility costs from ignoring regimes. The graphs plot means and bootstrap confidence intervals for the compensation (as a fraction of initial wealth) required to persuade a buy-and-hold investor with power utility (and $\gamma = 5$) to be willing to ignore regimes in asset returns. State probabilities are set at their steady-state values.

7.3. Out-of-sample performance

A legitimate concern about the results so far is that although the regime switching model leads to sensible portfolio choice recommendations at the end of our sample, it may be difficult to use in 'real time' due to parameter estimation errors which could translate into implausible time-variations in the portfolio weights.

This concern is related to the prediction model's out-of-sample performance. To get a sufficiently long evaluation sample, we first perform a 'pseudo-' real time asset allocation exercise for the period 1980:01–1999:12, a total of 240 months. We focus on the buy-and-hold asset allocation problem for three horizons, $T = 1$, 12, and 120 months. We compare the performance of a four-state regime switching model, the VAR(1) model (14), a four-state regime switching model that includes predictability from the dividend yield, and a simple IID model with constant means and covariance matrix. As additional benchmarks, we also report results for a minimum-variance portfolio and a static, mean–variance tangency portfolio.[17] We preclude the investor from having any benefit of hindsight. For instance, the four-state regime switching model is estimated for 1954:01–1979:12 and the estimates and state probabilities as of 1979:12 are used to calculate portfolio performance for 1980:01. Next period the sample is extended to 1954:01–1980:01 and estimation and portfolio optimization is repeated, and so forth.

For this exercise Fig. 1 showed the optimal portfolio weights under the four state (panels (b) and (c)) and IID (panel (d)) models. Interestingly, the turnover in the

[17]Minimum-variance and tangency portfolios are calculated using sample moments for T-period returns. For $T = 120$ we use overlapping returns to have enough sample observations to be able to calculate the required moments.

equity portfolio was found to be smaller under pure regime switching than under the VAR(1) model. Once the dividend yield is included in the regime switching model, the volatility of the equity weights increases and becomes comparable to that under the VAR(1) benchmark. Regime switching increases the overall demand for stocks (approximately 55%) relative to the benchmark VAR(1) model (40%) because the models that include the dividend yield as a predictor shift out of stocks during parts of the 1990s. Regimes also have a strong effect on the average demand for small stocks. The weight on these stocks is approximately 35% under regime switching, less than 25% when both regimes and the dividend yield are included, and only 10% under the VAR(1) model.[18] Bonds receive a substantial weight under regime switching – between 25% and 40%, depending on T and irrespective of whether the dividend yield is included as a predictor. Conversely, the VAR(1) model puts a large weight on cash investments (in excess of 50%). This suggests that the presence of regimes is important in understanding the demand for long-term bonds.

We next calculated realized utility under the different models, each associated with a particular portfolio weight $\hat{\boldsymbol{\omega}}_t^T$ and hence a different realized utility:

$$V_t^T \equiv \frac{[W_T(\hat{\boldsymbol{\omega}}_t^T)]^{1-\gamma}}{1-\gamma} = \frac{[(1-(\hat{\boldsymbol{\omega}}_t^T)'\boldsymbol{\iota}_3)\exp(Tr^{\mathrm{f}}) + (\hat{\boldsymbol{\omega}}_t^T)'\exp(\sum_{j=1}^T \mathbf{r}_{t+j} + Tr^{\mathrm{f}}\boldsymbol{\iota}_3)]^{1-\gamma}}{1-\gamma}.$$

Here $\gamma = 5$, $T = 1$, 12 and 120 months and $\{\mathbf{r}_{t+\tau}\}_{\tau=1}^T$ are the realized (excess) asset returns between $t+1$ and $t+T$. The period-t weights, $\hat{\boldsymbol{\omega}}_t^T$, are computed by maximizing the objective $\mathrm{E}_t[W_T^{1-\gamma}/(1-\gamma)]$ so that for each investment horizon, T, and each asset allocation model we obtain a time series $\{V_\tau^T\}$, $\tau = 1980{:}01, \ldots, 1999{:}12\text{-}T$ of realized utilities. Panels A and B of Table 4 reports summary statistics for the distribution of $\{-V_\tau^T\}$ with smaller values indicating higher welfare. Following Guidolin and Timmermann (2006b), we use a block bootstrap (with 50,000 independent trials) for the empirical distribution of $-V_\tau^T$ to account for the fact that realized utility levels are likely to be serially dependent as time-variations in the conditional distribution of asset returns may translate into dependencies in the portfolio weights and hence in realized utilities.

The VAR(1) model performs best over the shortest investment horizon ($T = 1$) although the 5% and 10% confidence intervals for the realized utility overlap under the VAR(1) and regime switching models suggesting that their performances are statistically indistinguishable. For the longer horizons, $T = 12$, 120 months, the pure regime switching model produces the highest mean realized utility. At the 12 month horizon the out-of-sample forecasting performance of this model is sufficiently good to be statistically significant against three of the five alternative models.

[18] Once the dividend yield is included as a predictor, the demand for stocks is close to zero between 1993 and 1997. This is consistent with the real time results reported by Campbell and Viceira (1999, 2001) and is explained by the low value of the dividend yield after 1993 (less than 2.5% vs. an unconditional sample mean of 3.4%).

Table 4
Real-time out-of-sample performance of predictability models

	MSIAH(4,0)			VAR(1)			MSIAH(4,1)			IID/Myopic			Minimum variance portfolio			Tangency portfolio		
	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$
A – (Pseudo) Out-of-sample (1980:01–1999:12) realized power utility																		
Mean	0.248	**0.196**	**0.009**	**0.244**	0.198	0.021	0.247	0.209	0.034	0.246	0.212	0.028	0.246	0.207	0.012	0.245	0.197	0.011
St. deviation	0.048	0.091	0.004	0.032	0.083	0.015	0.038	0.082	0.028	0.026	0.087	0.011	0.024	0.074	0.005	0.047	0.106	0.006
5% c.i.-lower	0.243	0.168	0.007	0.240	0.174	0.017	0.243	0.185	0.017	0.243	0.183	0.022	0.243	0.180	0.010	0.239	0.161	0.008
5%c.i.-upper	0.255	0.225	0.011	0.248	0.223	0.025	0.252	0.233	0.051	0.249	0.241	0.034	0.249	0.231	0.015	0.251	0.231	0.014
10% c.i.-lower	0.243	0.173	0.007	0.241	0.178	0.018	0.243	0.189	0.019	0.243	0.187	0.023	0.244	0.184	0.010	0.240	0.166	0.008
10% c.i.-upper	0.253	0.220	0.011	0.248	0.218	0.025	0.251	0.230	0.049	0.249	0.236	0.033	0.249	0.227	0.015	0.250	0.225	0.014
B – 100 × Differences in out-of-sample realized power utility vs. four-state regime switching model (1980:01–1999:12)																		
Mean	NA	NA	NA	−0.381	0.017	1.203	−0.115	1.270	2.331	−0.104	1. 460	1.764	−0.183	0.991	0.029	−0.288	0.058	0.036
St. deviation	NA	NA	NA	0.930	0.602	1.408	0.438	0.511	2.435	0.326	0.353	0.077	0.393	0.402	0.028	0.423	0.057	0.033
t-stat	NA	NA	NA	0.410	0.116	0.854	0.262	**2.489**	0.957	0.319	**4.115**	**2.306**	0.466	**2.479**	1.049	0.681	1.023	1.070
C – Out-of-sample (2000:01–2003:12) realized power utility																		
Mean	**0.247**	0.420	NA	0.247	0.223	NA	0.247	**0.207**	NA	0.250	0.208	NA	0.249	0.241	NA	0.253	0.370	NA
St. deviation	0.046	0.236	NA	0.036	0.031	NA	0.039	0.053	NA	0.023	0.040	NA	0.018	0.034	NA	0.042	0.221	NA
5% c.i.-lower	0.243	0.377	NA	0.240	0.209	NA	0.243	0.197	NA	0.243	0.189	NA	0.243	0.217	NA	0.239	0.298	NA
5% c.i.-upper	0.255	0.616	NA	0.248	0.251	NA	0.252	0.220	NA	0.249	0.227	NA	0.249	0.256	NA	0.251	0.548	NA
10% c.i.-lower	0.243	0.400	NA	0.241	0.211	NA	0.243	0.199	NA	0.243	0.192	NA	0.244	0.219	NA	0.240	0.317	NA
10% c.i.-upper	0.253	0.602	NA	0.248	0.248	NA	0.251	0.218	NA	0.249	0.225	NA	0.249	0.252	NA	0.250	0.529	NA
D – 100 × Differences in out-of-sample realized power utility vs. four-state regime switching model (2000:01–2003:12)																		
Mean	NA	NA	NA	0.000	−0.197	NA	0.001	−0.212	NA	0.003	−0.211	NA	0.001	−0.179	NA	0.006	−0.050	NA
St. deviation	NA	NA	NA	0.044	0.229	NA	0.038	0.234	NA	0.038	0.210	NA	0.048	0.243	NA	0.022	0.146	NA
t-stat	NA	NA	NA	0.002	−0.858	NA	0.010	−0.910	NA	0.083	−1.005	NA	0.025	−0.735	NA	0.253	−0.344	NA

This table reports out-of-sample performance measures for portfolio choices under alternative return prediction models and for three investment horizons: 1, 12, and 120 months. The monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The predictor is the dividend yield. For realized power utility ($\gamma = 5$), we report the negative of the calculated values. Investors aim at minimizing such values. In panels A and C, 'c.i.' stands for confidence interval. In panels B and D, positive differences reflect higher realized ex post utilities for the MSIH(4,0) model. Panels A and B refer to the (pseudo) out-of-sample period, 1980:01–1999:12; panels C and D include the genuine out-of-sample period 2000:01–2003:12. In the table, MSIAH(k,p) denotes a k-state multivariate regime switching (MS) model, with shifts in intercepts (I), covariance matrices (H), and p autoregressive (A) lags.

7.4. Performance during 2000–2003

Although our pseudo-out-of-sample results for the period 1980–1999 do not use any data for parameter estimation that was unavailable at the time of the forecast, the choice of model specification could itself have benefited from full-sample information that only became available in 1999. To address this concern and to see how the various models performed during 2000–2003, we compute asset allocations and realized utilities over this post-sample period.

Results from this experiment are reported in panels C and D of Table 4. All models generally suggest a more cautious asset allocation over this period, as reflected by an increase in the demand for T-bills and bonds. At the shortest investment horizon ($T = 1$) the myopic IID, VAR(1) and regime switching model extended by the dividend yield produce almost identical realized utilities. At the intermediate ($T = 12$) horizon, the pure regime switching model performs somewhat worse due to its continued high investment in stocks (70% on average) which stands in contrast to the models that include the dividend yield as a regressor. Since the dividend yield was below its unconditional mean during this sample, both the VAR(1) and regime switching model with the yield included lead to far smaller portions (less than 20%) invested in stocks over this period.

Viewed over the entire out-of-sample period 1980–2003 – and hence averaging across the lengthy bull and bear states of recent years – the four-state model continues to produce the best average realized utility performance at the 1- and 120-month horizons, while the VAR(1) model generates the best out-of-sample results for the interim 12-month horizon.

7.5. Do we need four states?

Throughout our analysis we maintained a four-state model for the joint distribution of stock and bond returns rather than a more common two-state model (e.g., Ang and Bekaert, 2002a). Readers might be concerned that the four-state model overfits the data and, because of imprecisely estimated parameters, performs worse out-of-sample than the simpler two-state model.

To explore these issues, we generated 2,000 independent samples (each containing 600 periods to reproduce the sample 1954:01–2003:12) based on the four-state model whose parameters are reported in Table 1. For each sample we conducted the out-of-sample tests of Section 7.3 by recursively estimating three alternative models – a (correctly specified) four-state regime switching model, a misspecified two-state regime switching model, and a Gaussian IID model. The first 312 observations (corresponding to 1954:01–1980:01) were used to estimate the initial set of model parameters. The estimation window was then expanded recursively and the portfolio weights were computed at each point in time for an investor with $\gamma = 5$. Results are reported by means of the realized portfolio return, Sharpe ratio and realized utility averaged over the 288 out-of-sample data points.

Table 5 reports the results from these simulations. If keeping four states is economically important, we should expect to see a decline in the portfolio

Table 5
Comparison of out-of-sample performance in simulated samples

		Four-state model MSIH(4,0)			Two-state model MSIH(2,0)			IID/myopic		
		$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$	$T=1$	$T=12$	$T=120$
A – portfolio weights (average over 288 simulated out-of-sample periods)										
Small stocks	Mean	0.424	0.219	0.174	**0.248**	**0.256**	**0.248**	**0.261**	**0.261**	**0.261**
	10% ci-l	0.363	0.195	0.168	0.215	0.247	0.243	0.245	0.245	0.245
	10% ci-u	0.486	0.243	0.183	0.279	0.268	0.255	0.278	0.278	0.278
Large stocks	Mean	0.408	0.445	0.474	**0.509**	0.500	**0.505**	**0.537**	**0.537**	**0.537**
	10% ci-l	0.351	0.414	0.464	0.456	0.473	0.497	0.513	0.513	0.513
	10% ci-u	0.463	0.474	0.483	0.561	0.526	0.514	0.559	0.559	0.559
Bonds	Mean	0.029	0.131	0.119	**0.146**	0.120	0.107	**0.196**	**0.196**	**0.196**
	10% ci-l	0.014	0.112	0.109	0.110	0.100	0.100	0.177	0.177	0.177
	10% ci-u	0.046	0.152	0.127	0.180	0.137	0.113	0.211	0.211	0.211
1-month T-bills	Mean	0.139	0.205	0.233	0.097	**0.124**	**0.140**	**0.006**	**0.006**	**0.006**
	10% ci-l	0.094	0.169	0.224	0.054	0.098	0.125	0.000	0.000	0.000
	10% ci-u	0.181	0.244	0.239	0.129	0.137	0.156	0.059	0.059	0.059
B – realized power utility (average over 288-T out-of-sample simulated periods)										
Mean		−0.234	−0.177	−0.036	**−0.245**	**−0.209**	−0.052	**−0.247**	**−0.259**	−0.056
10% ci-lower		−0.240	−0.194	−0.130	−0.252	−0.236	−0.156	−0.253	−0.319	−0.080
10% ci-upper		−0.229	−0.158	−0.009	−0.240	−0.183	−0.013	−0.241	−0.220	−0.045
C – realized portfolio returns (average over 288-T out-of-sample simulated periods)										
Total return (%)	Mean	1.182	9.757	232.6	0.837	9.825	234.2	0.688	7.742	137.1
	10% ci-l	0.572	1.837	150.7	0.292	1.341	135.1	0.130	1.420	34.49
	10% ci-u	1.791	17.68	314.5	1.383	18.31	333.4	1.247	12.42	221.7
Sharpe ratio	Mean	0.162	0.123	0.209	**0.111**	0.122	0.198	**0.069**	**0.062**	**0.098**
	10% ci-l	0.152	0.074	0.072	0.099	0.057	−0.021	0.055	0.039	0.054
	10% ci-u	0.173	0.176	0.314	0.122	0.191	0.443	0.089	0.084	0.129
D – annual (%) compensatory variation (average over 288-T out-of-sample simulated periods)										
Mean		3.360	0.846	1.016	**2.182**	**0.300**	0.412	NA	NA	NA
10% ci-lower		2.532	0.546	0.147	2.033	0.113	0.283	NA	NA	NA
10% ci-upper		4.189	1.146	2.178	2.330	0.488	0.541	NA	NA	NA

This table reports portfolio weights and measures of out-of-sample portfolio performance under three alternative return models, a four-state MSIH(4,0), a two-state MSIH(2,0), and a simple Gaussian IID benchmark. The statistics are computed over 2,000 independent, 600-month long samples drawn from the four-state regime switching model reported in Table 1. For each sample, the three models are estimated recursively, and portfolio weights are computed (for three alternative horizons) on the basis of the updated vector of parameter estimates. In particular, the statistics are based on averages over the $(288-T)$-month long out-of-sample periods. Realized portfolio returns and utility are computed for the last $288-T$ months in each of the simulated samples. In the table, '10% ci-l' and '10% ci-u' indicate the lower and upper limits of the 10% confidence bound based on the simulated distribution of each of the statistics across the 2,000 independent trials. For the two-state and Gaussian IID models, boldfaced statistics indicate that they fall outside the 10% simulated confidence interval obtained for the four-state model.

performance and realized utility when using the simpler (misspecified) models.[19] In fact, the table reports clear (and statistically significant) evidence that correctly specifying the number of regimes is important in portfolio choice applications. Panel A shows that the portfolio weights are very different under the two- and four-state models. At short horizons allocations under the two-state model assign too little weight to small caps, while at long horizons this model puts too big a weight on such stocks. The two-state model also puts far too much weight on long-term bonds – which appear to be less risky than they truly are – and too little weight on T-bills. Most of these differences in portfolio weights are significant at the 10% level in the sense that the confidence bounds fail to overlap. Conversely, the two-state model generates weights that, on average, are not particularly different from the myopic IID weights. The only discernible difference concerns the relative weights of bonds and T-bills.

Panel B of Table 5 reveals that using a two-state model when the data support a four-state model generates statistically significant welfare costs, as measured by the out-of-sample realized utility. In particular, for both $T = 1$ and 12, the realized utility is significantly higher under the four-state model than under the other models. Panels C and D show that a four-state model also delivers higher realized portfolio returns, higher Sharpe ratios, and higher compensatory variation of ignoring regimes. In the latter case, the differences are also statistically significant for $T = 1$ and 12 months.[20]

8. Conclusion

This paper explored the asset allocation implications of the presence of regimes in the joint distribution of stock and bond returns. Our model captures predictability not just in the conditional mean of returns (which most of the existing literature has focused on) but in the full (joint) return distribution. While two states were transitory (the crash and recovery state), the slow growth and bull state are persistent with average durations of several months. This means that the regime switching model captures both short-term and long-term variations in investment opportunities.

We found that the optimal asset allocation varies significantly across regimes as the weights on the various asset classes strongly depend on which state the economy is perceived to be in. Asset allocations therefore vary significantly over time even in the absence of 'outside' predictor variables such as the dividend yield. Stock allocations were found to be monotonically increasing as the investment horizon gets longer in only one of the four regimes. In the other regimes we observed a downward sloping allocation to stocks. The common investment advice of allocating more

[19] In unreported results we found that the two-state model provides a poor fit to our data. Instead of detecting bull and bear states (as one would expect) the two regimes isolate low and high volatility periods, with considerable persistence but weakly significant risk premia.

[20] We verified that these conclusions – higher realized power utility and superior portfolio performance (in particular, Sharpe ratio) from a four-state vs. a two-state model – hold for our original data set.

money to stocks the longer the investment horizon should therefore be made conditional on the underlying state.

Our framework can be extended in several ways. We evaluated the effects on the optimal asset allocation of modeling asset returns as a data-driven mixture of distributions that can vary significantly across regimes. In related research Brennan and Xia (2002) and Pástor (2000) propose a Bayesian framework to address optimal portfolio choice when investors face model uncertainty and asset return distributions take the form of mixtures over a range of theoretical and data-driven models. Such extensions are likely to deepen our understanding of the effects of multiple regimes on asset allocation. Finally, one could extend our framework to jointly modeling regimes in equity and bond returns as well as in short-term interest rates which are well-known to incorporate strong non-linearities (see e.g., Ang and Bekaert, 2002b; Detemple et al., 2003).

Acknowledgments

We are grateful to John Campbell, Wouter den Haan, and two anonymous referees for comments and also thank seminar participants at Caltech, the Innovations in Financial Econometrics conference at NYU, European Econometric Society meetings in Stockholm, European Finance Association meetings in Maastricht, Federal Reserve Bank of St. Louis, North American Summer meetings of the Econometric Society in Evanston, Australasian Econometric Society meetings in Melbourne, University of Houston, Rice, USC, Arizona State University, University of Rochester, and University of Turin - CERP.

References

Aït-Sahalia, Y., Brandt, M., 2001. Variable selection for portfolio choice. Journal of Finance 56, 1297–1351.

Andersen, T., Bollerslev, T., Diebold, F.X., Vega, C., 2004. Real-time price discovery in stock, bond and foreign exchange markets. Mimeo, University of Pennsylvania.

Ang, A., Bekaert, G., 2002a. International asset allocation with regime shifts. Review of Financial Studies 15, 1137–1187.

Ang, A., Bekaert, G., 2002b. Regime switches in interest rates. Journal of Business and Economic Statistics 20, 163–182.

Ang, A., Bekaert, G., 2004. Stock return predictability. Is it there? Manuscript. Columbia University.

Ang, A., Chen, J., 2002. Asymmetric correlations of equity portfolios. Journal of Financial Economics 63, 443–494.

Barberis, N., 2000. Investing for the long run when returns are predictable. Journal of Finance 55, 225–264.

Berkowitz, J., 2001. Testing density forecasts with applications to risk management. Journal of Business and Economic Statistics 19, 465–474.

Bielecki, T., Pliska, S., Sherris, M., 2000. Risk sensitive asset allocation. Journal of Economic Dynamics and Control 24, 1145–1177.

Bossaerts, P., Hillion, P., 1999. Implementing statistical criteria to select return forecasting models: what do we learn? Review of Financial Studies 12, 405–428.

Brandt, M., 1999. Estimating portfolio and consumption choice: a conditional Euler equations approach. Journal of Finance 54, 1609–1645.

Brennan, M., Xia, Y., 2002. Dynamic asset allocation under inflation. Journal of Finance 41, 1201–1238.

Brennan, M., Schwartz, E., Lagnado, R., 1997. Strategic asset allocation. Journal of Economic Dynamics and Control 21, 1377–1403.

Calvet, L., Fisher, A., 2005. Multifrequency jump-diffusions: an equilibrium approach. Working Paper, Harvard University and University of British Columbia.

Campbell, J., Chan, L., Viceira, L., 2003. A multivariate model of strategic asset allocation. Journal of Financial Economics 67, 41–80.

Campbell, J., Shiller, R., 1988. The dividend price ratio and expectations of future dividends and discount factors. Review of Financial Studies 1, 195–228.

Campbell, J., Viceira, L., 1999. Consumption and portfolio decisions when expected returns are time varying. Quarterly Journal of Economics 114, 433–495.

Campbell, J., Viceira, L., 2001. Who should buy long-term bonds? American Economic Review 91, 99–127.

Campbell, J., Viceira, L., 2002. Strategic Asset Allocation: Portfolio Choice for Long-Term Investors. Oxford University Press, Oxford.

Campbell, J., Viceira, L., 2005. The term structure of the risk-return trade-off. Financial Analysts Journal 61, 34–44.

David, A., 1997. Fluctuating confidence in stock markets: implications for returns and volatility. Journal of Financial and Quantitative Analysis 32, 427–462.

Detemple, J., Garcia, R., Rindisbacher, M., 2003. A Monte Carlo method for optimal portfolios. Journal of Finance 58, 401–446.

Garcia, R., Perron, P., 1996. An analysis of the real interest rate under regime shifts. Review of Economics and Statistics 78, 111–125.

Gray, S., 1996. Modeling the conditional distribution of interest rates as regime-switching process. Journal of Financial Economics 42, 27–62.

Guidolin, M., Timmermann, A., 2005a. Strategic asset allocation and consumption decisions under multivariate regime switching. Federal Reserve Bank of St. Louis Working Paper No. 2005-002A.

Guidolin, M., Timmermann, A., 2005b. Size and value anomalies under regime switching. Federal Reserve Bank of St. Louis Working Paper No. 2005-007A.

Guidolin, M., Timmermann, A., 2006a. An econometric model of nonlinear dynamics in the joint distribution of stock and bond returns. Journal of Applied Econometrics 21, 1–22.

Guidolin, M., Timmermann, A., 2006b. Term structure of risk under alternative econometric specifications. Journal of Econometrics 131, 285–308.

Guidolin, M., Timmermann, A., 2006c. International asset allocation under regime switching, skew and kurtosis preferences. Federal Reserve Bank of St. Louis Working Paper No. 2005-018A.

Hamilton, J., 1989. A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica 57, 357–384.

Honda, T., 2003. Optimal portfolio choice for unobservable and regime-switching mean returns. Journal of Economic Dynamics and Control 28, 45–78.

Kandel, S., Stambaugh, R., 1996. On the predictability of stock returns: an asset allocation perspective. Journal of Finance 51, 385–424.

Kim, T.S., Omberg, E., 1996. Dynamic nonmyopic portfolio behavior. Review of Financial Studies 9, 141–161.

Lettau, M., Ludvigsson, S., Wachter, J., 2005. The declining equity premium: what role does macroeconomic risk play? Working Paper, New York University and University of Pennsylvania.

Pástor, L., 2000. Portfolio selection and asset pricing models. Journal of Finance 55, 179–223.

Perez-Quiros, G., Timmermann, A., 2000. Firm size and cyclical variations in stock returns. Journal of Finance 55, 1229–1262.

Timmermann, A., 2000. Moments of Markov switching models. Journal of Econometrics 96, 75–111.

Turner, C., Startz, R., Nelson, C., 1989. A Markov model of heteroskedasticity, risk, and learning in the stock market. Journal of Financial Economics 25, 3–22.

Veronesi, P., 1999. Stock market overreaction to bad news in good times: a rational expectations equilibrium model. Review of Financial Studies 12, 975–1007.
Wachter, J., 2002. Portfolio and consumption decisions under mean-reverting returns: an exact solution for complete markets. Journal of Financial and Quantitative Analysis 37, 63–91.
Whitelaw, R., 2001. Stock market risk and return: an equilibrium approach. Review of Financial Studies 13, 521–548.