تمرین ۴ ۹۸۲۲۳۰۳ : سوال ۱ :

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=-2}^{2} a_k e^{jk\omega_0 n} = a_{-2} e^{-2j\omega_0 n} + a_{-1} e^{-j\omega_0 n}$$

$$+ a_0 + a_1 e^{j\omega_0 n} + a_2 e^{2j\omega_0 n} =$$

$$\omega_0 = \frac{2\pi}{5}$$

$$a_{-3} = a_{-3+5} = a_2 = 3e^{j\frac{\pi}{4}}, \quad a_4 = a_{4-5} = a_{-1} = 2e^{j\frac{\pi}{3}}, \quad a_1 = a_{-1}^* = 2e^{-j\frac{\pi}{3}}$$

$$a_{-2} = a_2^* = 3e^{-j\frac{\pi}{4}}$$

$$x(n) = 3e^{-j\frac{\pi}{4}} e^{-2j\omega_0 n} + \underbrace{2e^{j\frac{\pi}{3}} e^{-j\omega_0 n} + 2 + 2e^{-j\frac{\pi}{3}} e^{j\omega_0 n}}_{2\cos\left(\frac{2\pi}{5}n - \frac{\pi}{3}\right)} + 3e^{j\frac{\pi}{4}} e^{2j\omega_0 n} =$$

$$3\cos\left(\frac{4\pi}{5}n + \frac{\pi}{4}\right)$$

$$\boxed{2 + 6\cos\left(\frac{4\pi}{5}n + \frac{\pi}{4}\right) + 4\cos\left(\frac{2\pi}{5}n - \frac{\pi}{3}\right)} = x(n)$$

سوال ۳ :

طبق فرض : $a_k = a_{k-4} \to \frac{1}{N}\sum_{\langle N\rangle} x(n)\, e^{-jk\omega_0 n} \Rightarrow -\frac{1}{N}\sum_{\langle N\rangle} x(n)\, e^{-j(k-4)\omega_0 n}$

$\to \cancel{\sum_{\langle N\rangle} x(n)\, e^{-jk\omega_0 n}} = \sum_{\langle N\rangle} -x(n)\, e^{-jk\omega_0 n} e^{4j\omega_0 n} = -e^{j4\omega_0 n} \cancel{\sum_{\langle N\rangle} x(n)\, e^{-jk\omega_0 n}}$

$\to 1 = -e^{j4\times\frac{2\pi}{8}n} \to 1 = -e^{j\pi n} \xrightarrow[x(n)]{\text{ضرب طرفین در}} x(n) = -x(n)\, e^{j\pi n} = -(-1)^n x(n)$

$e^{j\pi n} = \cos\pi n + j\sin\pi n = (-1)^n$

$= \begin{cases} -x(n) & \text{n زوج} \\ x(n) & \text{n فرد} \end{cases}$

if n زوج $\Rightarrow x(n) = -x(n) = 0$

پس در واقع $y(n)$ همان شیفت یافته $x(n)$ است.

$y(n) = \begin{cases} x(n-1) & \text{n زوج} \\ 0 & \text{n فرد} \end{cases} = x(n-1)$

$b_k = e^{-jk\omega_0} a_k \Rightarrow \boxed{f(k) = e^{-jk\frac{\pi}{4}}}$

$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

سوال ۳ - الف) :

می‌دانیم $\sin\alpha\cos\beta = \dfrac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$

$$\sin\left(\frac{2\pi n}{3}\right)\cos\left(\frac{\pi n}{2}\right) = \frac{\sin\left(\frac{2\pi n}{3}-\frac{\pi n}{2}\right)+\sin\left(\frac{2\pi n}{3}+\frac{\pi n}{2}\right)}{2} =$$

$$\frac{\sin\left(\frac{4\pi n-3\pi n}{6}\right)+\sin\left(\frac{4\pi n+3\pi n}{6}\right)}{2} = \frac{1}{2}\sin\frac{\pi}{6}n + \frac{1}{2}\sin\frac{7\pi}{6}n$$

$$\frac{1}{2}\sin\frac{\pi}{6}n = \frac{1}{2}\left[\frac{1}{2j}e^{j\frac{\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{6}n}\right] = \frac{1}{4j}e^{j\frac{\pi}{6}n} - \frac{1}{4j}e^{-j\frac{\pi}{6}n}$$

$$\frac{1}{2}\sin\frac{7\pi}{6}n = \frac{1}{2}\left[\frac{1}{2j}e^{j\frac{7\pi}{6}n} - \frac{1}{2j}e^{-j\frac{7\pi}{6}n}\right] = \frac{1}{4j}e^{j\frac{7\pi}{6}n} - \frac{1}{4j}e^{-j\frac{7\pi}{6}n}$$

$$\sin\frac{\pi}{6}n \rightarrow \frac{2\pi}{T} = \frac{\pi}{6} \rightarrow T_1 = 12, \quad \sin\frac{7\pi}{6}n \rightarrow \omega_0 = \frac{2\pi}{T}$$

$$\rightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{7\pi}{6}} = \frac{2\pi\times 6}{7\pi} = \frac{12}{7} \rightarrow T_2 = 12$$

پس کل حاصل جمع دو سیگنال با دوره تناوب ۱۲ برابر با $lcm(12,12) = 12$ است.

$$x(n) = \frac{1}{4j}e^{j\frac{\pi}{6}n} - \frac{1}{4j}e^{-j\frac{\pi}{6}n} + \frac{1}{4j}e^{j\frac{7\pi}{6}n} - \frac{1}{4j}e^{-j\frac{7\pi}{6}n}$$

$$T_{x(n)} = 12 \rightarrow \omega_{0\,x(n)} = \frac{2\pi}{12} = \frac{\pi}{6}, \quad x(n) \xrightarrow{FS} a_k$$

$$a_1 = \frac{1}{4j} \qquad a_{-1} = \frac{-1}{4j} \qquad a_7 = \frac{1}{4j} \qquad a_{-7} = \frac{-1}{4j}$$

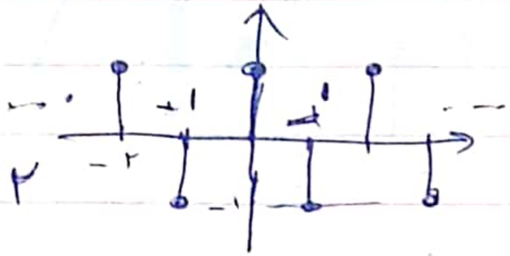
$$a_{7-12} = a_{-5} = \frac{1}{4j} \qquad a_{-7+12} = a_5 = \frac{-1}{4j} \qquad a_{-1+12} = a_{11} = \frac{-1}{4j}$$

ادامه سوال ۳ ـ الف) : پس اگر یک دوره تناوب $[0, 11]$ را فرض کنیم ،

$$a_1 = a_7 = \frac{1}{4j} \quad , \quad a_5 = a_{11} = -\frac{1}{4j} \quad , \text{other } k \quad a_k = 0$$

سوال ۳ - ب) :

$\cos\frac{\pi}{4}n$: $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4} \to T_1 = 8$, $(-1)^n$ و $T_2 = 2$



حاصل جمع سیگنال متناوب با دوره تناوب ⇐ $lcm(8,2) = 8$ است. $\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\frac{\pi}{4}n} = \frac{1}{8}\sum_{n=-3}^{4} x(n)\, e^{-jk\frac{\pi}{4}n}$$

$x(-3) = \cos -\frac{3\pi}{4} + \overset{-1}{\cancel{(-1)^{-3}}} = -\frac{\sqrt{2}}{2} - 1$, $x(-2) = \cos -\frac{\pi}{2} + \overset{1}{\cancel{(-1)^{-2}}} = 1$

$x(-1) = \cos -\frac{\pi}{4} + \overset{-1}{\cancel{(-1)^{-1}}} = \frac{\sqrt{2}}{2} - 1$, $x(0) = \overset{1}{\cancel{\cos 0}} + \overset{1}{\cancel{(-1)^{0}}} = 2$

$x(2) = \cos\frac{\pi}{2} + \overset{1}{\cancel{(-1)^{2}}} = 1$, $x(3) = \cos\frac{3\pi}{4} + \overset{-1}{\cancel{(-1)^{3}}} = -\frac{\sqrt{2}}{2} - 1$

$x(4) = \overset{-1}{\cancel{\cos\pi}} + \overset{1}{\cancel{(-1)^{4}}} = 0$

ادامه سوال ۳-ب :

$$\frac{1}{8}\Big[\left(-\frac{\sqrt{2}}{2}-1\right)e^{jk\frac{3\pi}{4}}+e^{-jk\frac{\pi}{4}}+\left(\frac{\sqrt{2}}{2}-1\right)e^{jk\frac{\pi}{4}}+2+e^{jk\frac{\pi}{2}}+\left(-\frac{\sqrt{2}}{2}-1\right)e^{-jk\frac{3\pi}{4}}+0\Big]=\frac{1}{8}\Big[\left(-\frac{\sqrt{2}}{2}-1\right)\left(\cos\frac{3\pi}{4}k+j\sin\frac{3\pi}{4}k\right)+$$

$$\cos\frac{\pi}{4}k-j\sin\frac{\pi}{4}k+\left(\frac{\sqrt{2}}{2}-1\right)\left(\cos\frac{k\pi}{4}+j\sin\frac{k\pi}{4}\right)+2+\cos\frac{k\pi}{2}+j\sin\frac{k\pi}{2}$$

$$+\left(-\frac{\sqrt{2}}{2}-1\right)\left(\cos\frac{3\pi}{4}k-j\sin\frac{3\pi}{4}k\right)\Big]=\frac{1}{8}\Big[\cancel{-\frac{\sqrt{2}}{2}\cos\frac{3\pi}{4}k-\frac{\sqrt{2}}{2}\cos\frac{3\pi}{4}k}\ \ -\sqrt{2}\cos\frac{3\pi}{4}k$$

$$-2\cos\frac{3\pi}{4}k+\cancel{\cos\frac{\pi}{4}k}-j\sin\frac{\pi}{4}k+\frac{\sqrt{2}}{2}\cos\frac{k\pi}{4}+j\frac{\sqrt{2}}{2}\sin\frac{k\pi}{4}\cancel{-\cos\frac{k\pi}{4}}$$

$$-j\sin\frac{k\pi}{4}+2+\cos\frac{k\pi}{2}+j\sin\frac{k\pi}{2}\Big]=\frac{1}{8}\Big[-(2+\sqrt{2})\cos\frac{3\pi}{4}k+j\left(-2+\frac{\sqrt{2}}{2}\right)\sin\frac{k\pi}{4}$$

$$+\frac{\sqrt{2}}{2}\cos\frac{k\pi}{4}+2+\cos\frac{k\pi}{2}+j\sin\frac{k\pi}{2}\Big]=a_k$$

$k=-3,-2,\dots,4$

سوال ۳-ج :

$$a_k=\frac{1}{7}\sum_{n=\langle N\rangle}x(n)e^{-jk\omega_0 n}=\frac{1}{7}\sum_{n=-1}^{5}x(n)e^{-jk\omega_0 n}=\frac{1}{7}\Big[2e^{jk\omega_0}+2+$$

$$2e^{-jk\omega_0}+2e^{-2jk\omega_0}+2e^{-3jk\omega_0}\Big]=\frac{1}{7}\Big[2e^{jk\frac{2\pi}{7}}+2+2e^{-jk\frac{2\pi}{7}}$$

$$+2e^{-jk\frac{4\pi}{7}}+2e^{-jk\frac{6\pi}{7}}\Big]=\frac{2}{7}\Big[\left(\cos\frac{2\pi k}{7}+j\sin\frac{2\pi}{7}k\right)+1+\left(\cos\frac{2\pi}{7}k-j\sin\frac{2\pi}{7}k\right)$$

$$+\left(\cos\frac{4\pi}{7}k-j\sin\frac{4\pi}{7}k\right)+\left(\cos\frac{6\pi}{7}k-j\sin\frac{6\pi}{7}k\right)\Big]=\frac{2}{7}\Big[2\cos\frac{2\pi}{7}k+1+$$

$$\cos\frac{4\pi}{7}k-j\sin\frac{4\pi}{7}k+\cos\frac{6\pi}{7}k-j\sin\frac{6\pi}{7}k\Big]$$

سوال ۴- الف) : مقدار متوسط سیگنال صفر است $\Rightarrow a_0 = 0$

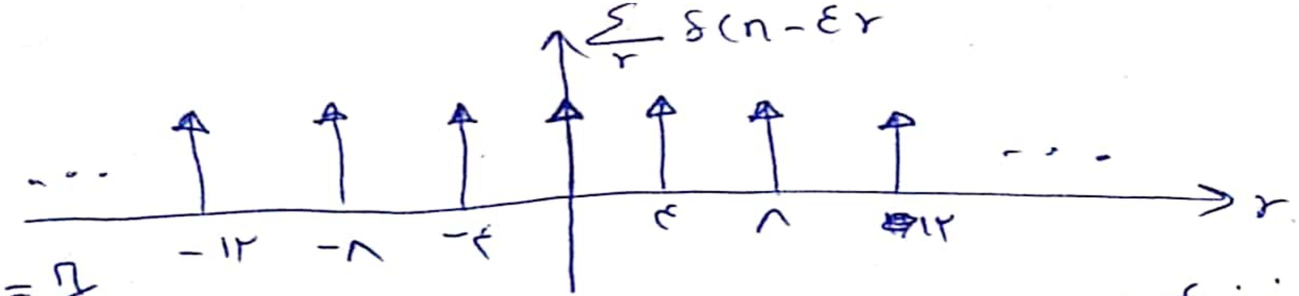
سیگنال حقیقی است $\Rightarrow a_k = a_{-k}^* \Rightarrow a_1 = a_{-1}^* \Rightarrow a_{-1} = a_1^* = (2 - j/2)^* = 2 + j/2$

توان متوسط سیگنال دیگر تغییر نکرد $= \frac{1}{N}\sum_{n=\langle N\rangle} |x(n)|^2 \overset{\text{قضیه پارسوال}}{=} \sum_{k=\langle N\rangle} |a_k|^2 =$

$$\sum_{k=-1}^{2} |a_k|^2 = |2 + j/2|^2 + |2 - j/2|^2 + 1^2 = 4 + \frac{j^2}{4} + \cancel{2j} + 4 + \frac{j^2}{4} - \cancel{2j}$$

$$+1 = 9 + \frac{j^2}{2} = 9 - \frac{1}{2} = \frac{17}{2} = \boxed{8.5}$$

سوال ۴ - پ) :



دوره تناوب = ۴

دوره تناوب $x(n)$ نیز ۴ است پس دوره تناوب حاصل ضرب این دو سیگنال ۴ است $lcm(4,4)=4$.

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$y(n) \xrightarrow{FS} b_k \qquad b_k = \frac{1}{N}\sum_{n=\langle N\rangle}\left[x(n)\sum_r \delta(n-4r)\right]e^{-jk\frac{\pi}{2}n} =$$

$$\frac{1}{4}\sum_{n=-2}^{2}\underbrace{x(n)\sum_r \delta(n-4r)}_{}\,e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\left[x(0)\cancel{\delta(0)}\right] = \frac{x(0)}{4}$$

در این بازه فقط در نقطه صفر مقدار دارند.

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=-1}^{2} a_k e^{jk\omega_0 n} \Rightarrow x(0) = \sum_{k=-1}^{2} a_k =$$

$$a_{-1}+a_0+a_1+a_2 = 2+\cancel{\tfrac{j}{2}}+0+2-\cancel{\tfrac{j}{2}}+1 = \boxed{5}$$

$$b_k = \frac{x(0)}{4} = \frac{5}{4}, \quad k \in [0, N]$$

سوال ۵ :

$x(n) \xrightarrow{Fs} a_k \quad , \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{9}$

$a_k = \frac{1}{9} \sum_{n=\langle N \rangle} x(n) e^{-jk\omega_0 n} = \frac{1}{9} \sum_{n=-4}^{4} x(n) e^{-jk\frac{2\pi}{9}n} =$

$\frac{1}{9}\left[\cancel{x(-1)} e^{jk\frac{2\pi}{9}} + \cancel{x(0)} + \cancel{x(1)} e^{-jk\frac{2\pi}{9}}\right] = \frac{1}{9}\left[\underbrace{e^{jk\frac{2\pi}{9}} + 1 + e^{-jk\frac{2\pi}{9}}}_{2\cos k\frac{2\pi}{9}}\right]$

$\boxed{\frac{2}{9}\cos \frac{2\pi}{9} k + \frac{1}{9}}$

طبق صورت سوال : $H(e^{jk\omega_0}) = \begin{cases} 1 & |k\omega_0| < \frac{\pi}{4} \\ 0 & \text{———} \end{cases}$

$-\frac{\pi}{4} \le k\omega_0 \le \frac{\pi}{4} \Rightarrow -\frac{\cancel{\pi}}{4} \le \frac{2\cancel{\pi}}{9} k \le \frac{\cancel{\pi}}{4} \rightarrow -\frac{1}{4} \le \frac{2}{9} k \le \frac{1}{4} \rightarrow$

$-\frac{9}{8} \le k \le \frac{9}{8} \rightarrow k = 0, \pm 1$

پس فقط در نقاط $k = 0, \pm 1$ ضرایب سری فوریه خروجی برابر a_k و در سایر نقاط برابر صفر است.

ضرایب سری فوریه خروجی $= a_k \underline{H(e^{jk\omega_0})}$

$y(n) \xrightarrow{Fs} b_k \qquad b_k = \begin{cases} a_k = \frac{2}{9}\cos\frac{2\pi}{9}k + \frac{1}{9} & k = 0, \pm 1 \\ 0 & \text{other} \end{cases}$

$y(n) = \sum_{k=\langle N \rangle} b_k e^{jk\omega_0 n} = \sum_{k=-1}^{1} b_k e^{jk\frac{2\pi}{9}n} = b_{-1} e^{-j\frac{2\pi}{9}n} + b_0 + b_1 e^{j\frac{2\pi}{9}n} =$

$\left(\frac{2}{9}\cos\frac{2\pi}{9} + \frac{1}{9}\right) e^{-j\frac{2\pi}{9}n} + \frac{1}{3} + \left(\frac{2}{9}\cos\frac{2\pi}{9} + \frac{1}{9}\right) e^{j\frac{2\pi}{9}n} =$

$\left(\frac{2}{9}\cos\frac{2\pi}{9} + \frac{1}{9}\right)\Big[\underbrace{\cos\frac{2\pi}{9}n \cancel{- j\sin\frac{2\pi}{9}n} + \cos\frac{2\pi}{9}n \cancel{+ j\sin\frac{2\pi}{9}n}}_{2\cos\frac{2\pi}{9}n}\Big] + \frac{1}{3}$

جواب نهایی سؤال ۵:

$$\boxed{y(n) = \left(\frac{۲}{۹}\cos\frac{۲\pi}{۹}n + \frac{۱}{۹}\right)\left(۲\cos\frac{۲\pi}{۹}n\right) + \frac{۱}{۳}}$$