

# Situation Theory

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# 1 Introduction

This chapter is meant as a self-contained introduction to the main themes and technical contributions of situation theory. This is a body of work which is concerned with several foundational points having to do with *information*. We are interested in modeling the entities which bear information, and then in seeing how logic is related to them.

In writing this chapter, we had the conflicting goals of presenting the existing literature on the subject, and reorganizing it. As we see matters, a basic problem in the existing literature is that it has not yet produced a single unified theory. It is more a collection of technical contributions, some quite sophisticated, that have not been put together. However, we decided not to simply survey this material, but rather to re-present situation theory, so that readers unfamiliar with it could hope to learn what it is about. In effect, we reconstruct the basic “ontological categories” of the subject in stages: beginning with infons and related structures (Section 2), and then later building situation structures (Section 4).

In between these parts, we have a new technical development a theory of *structural relations* introduced in Section 3. This work is a generalisation of the notion of bisimulation to more general structures. It also leads to treatments of partiality, parameters, restriction, and abstraction. We believe that it simplifies and unifies some existing ideas in the area.

This chapter is intended to serve as an introduction to situation theory, not as a final say on the subject. In writing it, we have tried to emphasize the overall conceptual points of the subject, and to present a unified theoretical account. As a result, we place in the background many technical questions which deserve much more attention.

A final word of introduction is in order for a chapter in a Handbook of Logic in *Linguistics*. Please do not confuse *situation theory* with *situation semantics*. We are most concerned with situation theory; situation semantics is introduced only toward the end of the chapter, in Section 4.8. As we see things, the former subject got started as a search for foundations of the latter. However, at this point in time situation theory can be motivated as a study of information, and linguistic information is not the only kind of information. Still, some of the issues in semantics have left their mark on what we do (for example on our work on appropriateness of assignments). It is for this reason that the subject might be of interest to formal semanticists. Furthermore, since we are starting from scratch and not heading toward any particular linguistic analyses, the overall framework here might be useful to semanticists working in different frameworks.

## 2 The Structure of Information

Situation Theory has its fair share of neologisms, the most inspired of which is surely the term *infon*, coined by Keith Devlin. An infon is an item of information. The term is intended to be as neutral as possible with regard to the form in which the information is represented. The common item of information separately conveyed by the statements made in the following little conversation is an example of an infon.

Raymond: (to Paul, displaying an omelette) I have cooked this for you.

Paul: (proudly, to Delia) Raymond cooked this omelette for me.

Delia: (to Albert) The chef cooked the omelette for Paul.

Of course, much more information can be gleaned about the situation from each of these statements, but that it is one in which the two people, Raymond and Paul, and an omelette stand in a certain relationship, that of Raymond having cooked the omelette for Paul, is information expressed by all three. The very same information may also be represented by other means, in a film, or a cartoon-strip, for example, or in the minds of Raymond, Paul, Delia and even Albert.

The first goal of this chapter is to show how information of this simple form is modelled. To specify the structure of the infon in the example, we must say which individuals are involved and how they are related. We shall abbreviate 'Raymond' and 'Paul' to 'R' and 'P', and introduce the name 'O' to refer to the omelette. We use 'cooked' to name the relation which holds between a cook, a guest and a dish just in case the cook has cooked the dish for the guest. The infon of our example, which we name 'I,' is the information that R, P, and O stand in the relation *cooked*. The individuals R, P, and O are called *arguments* of the infon I, and the relation *cooked*, is called the *relation* of I.

By specifying the relation and arguments of an infon we have done a lot toward characterizing everything of informational significance, but not quite all. Care must be taken to distinguish the information that Raymond cooked the omelette for Paul from the information that Paul cooked the omelette for Raymond. For this purpose, we require the concept of a *role*. In any infon involving the relation *cooked* there are three roles to be filled: that of the cook, the guest and the dish. In the present case, the cook is R, the guest is P, and the dish is O, but if it were claimed that Paul cooked the omelette for Raymond, then the roles of cook and guest would be reversed. We say that an individual is an *argument* of an infon if it fills one of its roles. For example, the omelette O is an argument of the infon I by virtue of filling the *dish* role of I's relation *cooked*.

We may capture these intuitions by supposing that there are two basic ingredients determining the structure of  $I$ : that of **cooked** being the relation of  $I$ , and that of the three individuals  $R$ ,  $P$ , and  $O$  filling the three roles of  $I$ , which we name **cook**, **guest**, and **dish**, respectively. We introduce the predicates **Rel** and **Arg** to express these structural relations, writing  $\text{Rel}(\text{cooked}, I)$  to mean that **cooked** is the relation of  $I$ , and  $\text{Arg}(O, \text{dish}, I)$  to mean that  $O$ , the omelette, fills the **dish** role of our infon  $I$ .

More generally, we may say that simple infons of the kind we have been considering are ‘structured’ by the two relations, **Rel** and **Arg**. The first of these, **Rel**, determines the relation of the infon and the other, **Arg**, determines which arguments fill which roles. In fact, we *define* a ‘simple infon’ to be anything which has a relation or an argument. Similarly, a ‘relation’ is anything which is a relation *of* some infon, and a ‘role’ is anything which is filled by an argument in an infon. In other words,

$$\begin{aligned} \sigma \text{ is a (simple) infon} & \quad \text{if } \exists r \text{ Rel}(r, \sigma) \text{ or } \exists a, i \text{ Arg}(a, i, \sigma). \\ r \text{ is a relation} & \quad \text{if } \exists \sigma \text{ Rel}(r, \sigma). \\ r \text{ is a role} & \quad \text{if } \exists a, \sigma \text{ Arg}(a, i, \sigma). \end{aligned}$$

These definitions form the basis of the abstract account of the structure of information that we will develop throughout this section. The approach is ‘abstract’ because we consider any structure in which the predicates ‘**Rel**’ and ‘**Arg**’ are defined to be a candidate model for the theory of information, with the derived predicates ‘relation’, ‘role’ and ‘infon’ interpreted according to the above definitions. **Our approach will be to isolate various axioms which describe the central properties of the concept of an ‘infon’, as it has been formulated in Situation Theory, and so home in on those models that have been proposed in the literature.**

## 2.1 Relational Structures

In all foundational theories one must be very careful to distinguish between the terminologies of the meta-theory and object-theory. Situation Theory is no exception, and special difficulties arises with the word ‘relation’. Before going further it will prove useful to make some preliminary definitions concerning relational structures. The following is slightly non-standard but enables us to avoid certain ambiguities with our use of ‘relation’ and similar words.

A *(relational) signature* is a function  $\nu: S \rightarrow N$  from a finite set  $S$  to the set  $N$  of positive integers. The elements of  $S$  are called *primitive relations* and the number  $\nu(R)$  associated with a primitive relation  $R$  is called its *arity*. A *relational structure*  $\mathfrak{A}$  of signature  $\nu$  consists of a class  $|A|$ , called the *universe* of  $\mathfrak{A}$ , and for each primitive relation  $R$ , a class  $[[R]]^{\mathfrak{A}}$  of finite sequences of elements of  $|A|$ , each of length  $\nu(R)$ , called the *extension* of  $R$  in  $\mathfrak{A}$ .<sup>1</sup>

<sup>1</sup>We allow relational structures whose universe is a proper class because models of Situation Theory are often very large. The exact meaning of the word ‘set’ will be discussed below, but for the moment the reader may assume that we are working with any theory of sets which

For example, we may model the situation of Raymond having cooked an omelette for Paul as a relational structure  $\mathfrak{N}$  of signature  $\nu: \{\text{cooked}\} \rightarrow N$  defined by  $\nu(\text{cooked}) = 3$ , with universe  $|\mathfrak{N}| = \{R, P, O\}$  and  $\llbracket \text{cooked} \rrbracket^{\mathfrak{N}}$  containing, at least, the sequence ROP.

In constructing  $\mathfrak{N}$  we will be forced to determine the complete extension of **cooked** and so include more information that was conveyed by the conversation between Raymond, Paul, Delia and Albert. Presumably, none of the sequences containing **O** in the first or third position are included in the extension of **cooked** on grounds of absurdity, but if, for example, the sequence ROR is included then the situation modelled is one in which Raymond also cooked the omelette for himself—in which case **O** is clearly a two-person omelette. If, on the other hand, the sequence ROP is omitted from the extension of **cooked** then Paul has the feast to himself. The necessary overdetermination of this rather crude approach to modelling information is one of the deficiencies which Situation Theory was designed to overcome.

In discussing relational structures, we shall make use of the following notational shorthand. The statement

$$\mathfrak{A} : [A, R_1^{\nu_1}, \dots, R_n^{\nu_n}]$$

(which may be read as ‘ $\mathfrak{A}$  is of type  $[A, R_1^{\nu_1}, \dots, R_n^{\nu_n}]$ ’) means that  $\mathfrak{A}$  is a relational structure with universe  $A$  and primitive relations  $R_1, \dots, R_n$ , with arities  $\nu_1, \dots, \nu_n$  respectively. For example,  $\mathfrak{N} : [\{R, P, O\}, \text{cooked}^3]$ . Also, we use names for the primitive relations of a structure to form sentences about the structure as if they were predicates in a formal language. Given elements  $a_1, \dots, a_{\nu_i}$  of  $|\mathfrak{A}|$ , the sentence

$$R_i(a_1, \dots, a_{\nu_i})$$

when used to describe the structure  $\mathfrak{A}$ , is true if and only if the finite sequence  $a_1, \dots, a_{\nu_i}$  lies in  $\llbracket R_i \rrbracket^{\mathfrak{A}}$ . For example, the sentence ‘**cooked**( $R, O, P$ )’ provides yet another way of saying that Raymond cooked the omelette for Paul.

We also need to talk about specific elements of the universe of a relational structure. Given a set  $C$ , a relational structure *with constants*  $C$  is a relational structure  $\mathfrak{A}$  together with an element  $\llbracket c \rrbracket^{\mathfrak{A}}$  of  $|\mathfrak{A}|$  for each  $c \in C$ . The effect of including constants in the definition of a relational structure is a subtle one. It makes no difference to the truth or falsity of descriptions of the structure, but plays a significant role in comparing two ostensibly different structures to see if they are structurally identical, or ‘isomorphic’.

Relational structures  $\mathfrak{A}$  and  $\mathfrak{B}$  both of signature  $\nu: S \rightarrow N$  and with constants  $C$  are *isomorphic* if there is a bijective function  $f: |\mathfrak{A}| \rightarrow |\mathfrak{B}|$  such that

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accounts for their mundane properties, supports a distinction between ‘set’ and ‘class’, and allows collections of ordinary objects to be called ‘sets’.

1. for each  $R \in S$  and  $a_1, \dots, a_{\nu(R)} \in |\mathfrak{A}|$ ,  
 $R(a_1, \dots, a_{\nu(R)})$  in  $\mathfrak{A}$  iff  $R(f(a_1), \dots, f(a_{\nu(R)}))$  in  $\mathfrak{B}$
2. for each  $c \in C$ ,  $\llbracket c \rrbracket^{\mathfrak{B}} = f(\llbracket c \rrbracket^{\mathfrak{A}})$ .

## 2.2 Simple Information Structures

We are interested in relational structures in two quite different ways. First, as in the example of  $\mathfrak{N}$ , we use relational structure to model the facts of a situation, and as a first attempt at providing a mathematical model of the structure of the information it contains. Second, and more significantly, we our abstract approach to Situation Theory proceeds by defining successive classes of relational structures, which may be regarded as approximations to a suitable model for the theory as a whole. An example of the second use is given by our first substantial definition.

**Definition 2.1** A relational structure  $\mathfrak{A}$  of type  $[A, \text{Rel}^2, \text{Arg}^3]$  (possibly with constants) is called a (*simple*) *information structure* if it satisfies the following conditions:

- A 1** If  $\text{Rel}(r, \sigma)$  and  $\text{Rel}(r', \sigma)$  then  $r = r'$ .
- A 2** If  $\text{Arg}(a, i, \sigma)$  and  $\text{Arg}(b, i, \sigma)$  then  $a = b$ .
- A 3** No relation is a role.
- A 4** No role is an infon.
- A 5** No infon is a relation.

Axioms A1 to A5 are not particularly stringent; they merely impose some discipline on the domain. Together, they say nothing more than that the relations, roles and infons form disjoint classes, and that infons may have at most one relation and at most one argument filling each role.

The previously mentioned ambiguity in the word ‘relation’ can be seen clearly with the help of this definition.<sup>2</sup> The first use occurs when we use it to talk about the relations of any relational structure. For example, the theoretical predicates ‘Rel’ and ‘Arg’ refer to the primitive *relations* of simple information structures. Relations of this kind will be dubbed ‘*external*’. Any other relation between elements of a relational structure whose extension is determined by the meta-theory is also called ‘external’. Examples include relations defined from primitive relations, such as the property of being a simple infon.<sup>3</sup>

<sup>2</sup>Other words we use, such as ‘role’ and ‘argument’, have a similar ambiguity.

<sup>3</sup>Typically, an  $n$ -ary external relations is identified with its *graph*, the set, or class, of sequences  $\langle x_1, \dots, x_n \rangle$  of objects which stand in the relation. We make no such assumption here. A meta-theory in which intensional distinctions are made between relations—as they are in our object-theory—is quite compatible with our approach.

The second use of ‘relation’ is governed by the extension of  $\text{Rel}$  in a specific information structure. Any element  $r$  of the universe of a simple information structure  $\mathfrak{A}$  is an *internal* relation of  $\mathfrak{A}$  if  $\text{Rel}(r, \sigma)$  for some infon  $\sigma$ . The ambiguity, once understood, should not cause the reader any difficulty. Where necessary the sense will be made clear using the adjectives ‘internal’ and ‘external’.

Aside from illustrating ambiguities, simple information structures provide our first, crude approximation to the models of Situation Theory. Their principal use is to show how ordinary relational structures, of any signature, can be used to model simple infons. Suppose, for example, we use the structure  $\mathfrak{N}$  to model the situation discussed by Raymond and his friends. There is a sense in which  $\mathfrak{N}$  gives us all we need to know about the common informational content of the three culinary pronouncements. The relation of the infon, *cooked*, is included as a primitive relation of  $\mathfrak{N}$ ; the three roles of this relation, *cook*, *guest*, and *dish*, are modelled by the first, second, and third positions in the sequences in the extension of *cooked*; and the arguments—Raymond, Paul and the omelette—are also directly included in the universe of  $\mathfrak{N}$ . The thought that we have thereby obtained a simple information structure is made more precise in the following set-theoretic construction.

**Construction 2.2** Given a relational structure  $\mathfrak{M}$  of type  $[M, R_1^{\nu_1}, \dots, R_n^{\nu_n}]$ , possibly with constants, we construct an information structure  $\text{SInf}(\mathfrak{M})$  as follows. First, let

$$\begin{aligned} A_1 &= \{1\} \times \{R_1, \dots, R_n\} \\ A_2 &= \{2\} \times \bigcup_{1 \leq i \leq n} \{1, \dots, \nu_i\} \\ A_3 &= \{\langle 3, R_i, \alpha \rangle \mid 1 \leq i \leq n \text{ and } \alpha: \{1, \dots, \nu_i\} \rightarrow M\} \end{aligned}$$

The basic idea is to model simple infons as pairs of the form  $\langle R_i, \alpha \rangle$  in which  $\alpha$  is a function from  $\{1, \dots, \nu_i\}$  to the universe of  $\mathfrak{M}$ . But we must be careful to keep relations, roles and infons distinct, and so we also include the prefix indicating the sort of object modelled: 1 for relations, 2 for roles, and 3 for simple infons. We first define the information structure  $\text{SInf}^-(\mathfrak{M})$  to have universe  $M \cup A_1 \cup A_2 \cup A_3$  and with  $\text{Rel}$  and  $\text{Arg}$  defined as follows:

$$\begin{aligned} \text{Rel}(r, \sigma) &\quad \text{iff} \quad \exists R, \alpha \text{ such that } r = \langle 1, R \rangle \text{ and } \sigma = \langle 3, R, \alpha \rangle, \\ \text{Arg}(a, i, \sigma) &\quad \text{iff} \quad \exists j, R, \alpha \text{ such that } i = \langle 2, j \rangle, \\ &\quad \sigma = \langle 3, R, \alpha \rangle \text{ and } \alpha(j) = a. \end{aligned}$$

The constants of  $\text{SInf}^-(\mathfrak{M})$  include all the constants of  $\mathfrak{M}$ . In addition, we take each primitive relation  $R$  of  $\mathfrak{M}$  to be a constant denoting  $\langle 1, R \rangle$ . It is a simple matter to check that the resulting structure  $\mathfrak{A}$  satisfies axioms A1 to A5.

A minor complication arises because of the need to ensure that isomorphic relational structures give rise to isomorphic information structures—we don’t want accidental features of the representation to encode more information than



there is in the original structure. We would get more if, for example,  $M$  happens to contain an element  $m = \langle 1, R \rangle$  in which  $R$  is one of the primitive relations of  $\mathfrak{M}$ . In the structure  $\text{SInf}^-(\mathfrak{M})$ , this pair plays two parts: that object  $m$  of  $\mathfrak{M}$  and that of the (internal) relation denoted by the constant  $R$ .

We overcome this difficulty by first making a copy of  $\mathfrak{M}$ . Let  $\mathfrak{M}_0$  be the isomorphic copy of  $\mathfrak{M}$  in which each element  $m$  of  $M$  is replaced by the pair  $\langle 0, m \rangle$ . Now we define  $\text{SInf}(\mathfrak{M}) = \text{SInf}^-(\mathfrak{M}_0)$ . Unlike  $\text{SInf}^-(\mathfrak{M})$ , the structure  $\text{SInf}(\mathfrak{M})$  keeps the old elements distinct from the newly constructed elements, and so our construction has the desired property of preserving isomorphism: if  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are isomorphic, so are  $\text{SInf}(\mathfrak{M}_1)$  and  $\text{SInf}(\mathfrak{M}_2)$ .<sup>4</sup>

By way of example, consider the model  $\mathfrak{N}$  of omelette cooking, defined earlier. The information structure  $\text{SInf}(\mathfrak{N})$  has individuals  $\langle 0, R \rangle$ ,  $\langle 0, P \rangle$ , and  $\langle 0, O \rangle$ , copied from  $\mathfrak{N}$ , a single relation  $\langle 1, \text{cooking} \rangle$ , three roles  $\langle 2, 1 \rangle$ ,  $\langle 2, 2 \rangle$ , and  $\langle 2, 3 \rangle$ , and infons of the form  $\langle 3, \langle \text{cooked}, \alpha \rangle \rangle$ , where  $\alpha$  is a function with domain  $\{1, 2, 3\}$ , taking values from  $\{\langle 0, R \rangle, \langle 0, P \rangle, \langle 0, O \rangle\}$ . The structure also has a constant  $\text{cooking}$  which denotes the (internal) relation  $\langle 1, \text{cooking} \rangle$ . The information that Raymond cooked the omelette  $O$  for Paul is modelled by the element  $\langle 3, \langle \text{cooked}, \alpha \rangle \rangle$  in which  $\alpha$  is the function with domain  $\{1, 2, 3\}$ , with  $\alpha(1) = \langle 0, R \rangle$ ,  $\alpha(2) = \langle 0, P \rangle$ , and  $\alpha(3) = \langle 0, O \rangle$ .

Various aspects of the construction of simple information structures from relation structures are artifacts of the set-theoretic tools used to build them. For example, the choice of the numbers 0 to 3 as indicators of the sort of object modelled is quite arbitrary. In making theoretical claims about the properties of these structures we must therefore take care to focus only on the essential aspects of the construction. This is easily achieved by widening our study to include all information structures isomorphic to one constructed from a relational structure.

**Definition 2.3** A simple information structure is *standard* if it is isomorphic to  $\text{SInf}(\mathfrak{M})$  for some relational structure  $\mathfrak{M}$ .

In the remainder of this section we shall study the standard structures in some detail. We shall see that some properties of the standard structures reflect central intuitions about the structure of information, but others reflect only the limitations of the construction. By the end of the section we shall have a list of axioms characterizing the class of standard structures, and a number of alternative axioms designed to overcome their limitations.<sup>5</sup>

<sup>4</sup>Note also that the construction depends only on the type of the relational structure. If  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are isomorphic relational structures of the same type then indeed  $\text{SInf}(\mathfrak{M}_1) = \text{SInf}(\mathfrak{M}_2)$ .

<sup>5</sup>The axioms for standard information structures are labelled **A1**, **A2**, and so on. The alternatives to axiom **An** are labelled **An.1**, **An.2**, etc. Some of the alternatives are weaker than the corresponding standard axioms, generalizing some aspect of standard structures. Others are stronger, asserting the existence of objects not present in standard structures.

## 2.3 Roles

Suppose  $\mathfrak{N}'$  is a relational structure of type  $[N', \text{stir-fried}^2, \text{braised}^2]$  and that  $R$  and  $P$  are elements of  $M$ . Furthermore, let  $\alpha_1$  and  $\alpha_2$  be functions with domain  $\{1, 2\}$  and such that  $\alpha_1(1) = R$  and  $\alpha_2(1) = P$ . The information structure  $\text{SInf}(\mathfrak{N}')$  contains the simple infons  $\sigma_1 = \langle 3, \text{stir-fried}^2, \alpha_1 \rangle$  and  $\sigma_2 = \langle 3, \text{braised}^2, \alpha_2 \rangle$ . Thus, in  $\text{SInf}(\mathfrak{N}')$ , we have that

$$\text{Arg}(\langle 0, R \rangle, \langle 2, 1 \rangle, \sigma_1) \text{ and } \text{Arg}(\langle 0, P \rangle, \langle 2, 1 \rangle, \sigma_2).$$

The elements  $\langle 0, R \rangle$  and  $\langle 0, P \rangle$  have the common property of filling the role  $\langle 2, 1 \rangle$ . What, if anything, does this signify?

In information structures constructed from relational structures in the above way, roles are simply pairs of the form  $\langle 2, n \rangle$ , where  $n$  is a positive integer. These roles are simply indices recording the relative position of arguments in original relational structure, and it is difficult to see how the identity of roles—mere indices—in different infons can be given much import.

But we must not be lead astray by the peculiarities of the coding used in our constructions. The statement that Raymond stir-fried the frogs' legs and Paul braised the monkfish tells us, among many other things, that both Raymond and Paul are cooks; at least, it tells us that they played the role of cook in the events described. It may be that we can regard the role  $\langle 2, 1 \rangle$  as one which is filled only by cooks. No such information present in to the underlying structure  $\mathfrak{N}'$ , and so it would be a mistake to give this interpretation to the role  $\langle 2, 1 \rangle$  of  $\text{SInf}(\mathfrak{N}')$  simply because it happens to be filled only by cooks. Nonetheless, the possibility of giving roles some independent significance has been opened up.

However, the behaviour of roles in standard information structures is severely constrained. The roles of **stir-fried** and **braised** in  $\mathfrak{N}'$  are the same two,  $\langle 2, 1 \rangle$  and  $\langle 2, 2 \rangle$ , not because of any culinary subtleties, but simply because both are binary relations, and so use the same numbers, 1 and 2, to index their arguments. This rather artificial limitations of standard structures is captured by the following rather artificial axiom.

**A 6** *If  $\sigma$  has a role which is not a role of  $\tau$ , then every role of  $\tau$  is a role of  $\sigma$ .*

In effect, axiom A6 says that the sets of roles of infons are linearly ordered by inclusion. In moving away from standard structures, this axiom is sure to be dropped. Theoretical considerations concerning roles have been studied in connection with the linguistic notion of a ‘thematic role’ by Engdahl (1991). Another restriction obeyed in the standard structures is that

**A 7** *each infon has only finitely many roles.*

Relations with infinite arities are conceivable, and perhaps of some theoretical use, but it is important to be able to focus on those items of information which are finitely expressible. Indeed, it has been proposed (Devlin 1991) that

the finiteness of infons is an essential characteristic. A much less reasonable restriction is that

**A 8** *There are only finitely many relations.*

This is satisfied by all standard information structures, but it should be dropped when we generalize.

## 2.4 Identity

In all standard information structures, the following criterion of identity is satisfied.

**A 9** *Suppose  $\sigma$  and  $\tau$  are infons such that for all  $r$ ,  $a$ , and  $i$ ,*

1.  $\text{Rel}(r, \sigma)$  iff  $\text{Rel}(r, \tau)$ , and
2.  $\text{Arg}(a, i, \sigma)$  iff  $\text{Arg}(a, i, \tau)$ .

*Then  $\sigma = \tau$ .*

In many presentations of Situation Theory there is a further condition, relating to the ‘polarity’ of an infon. The intuition is that the information expressed by ‘Dave is short’ and ‘Dave is not short’ are to be considered on an equal footing, and the latter is to be distinguished from the information expressed by the denial of the first and the assertion of ‘It is not the case that Dave is short’. The two infons have the same relation (being short) and the same argument (Dave) filling the same role.

In most treatments of Situation Theory, this problem has been solved by introducing an extra attribute of infons, called *polarity*, which is either positive or negative. We could incorporate this proposal into the present account in various ways. For example, we could introduce a new unary relation **Pos** which holds of the positive infons only, and axioms to give the modified identity conditions and to ensure that for every positive infon there is a corresponding negative infon. The approach adopted here is keep our basic definitions uncluttered by matters of polarity and to leave the proper treatment of negation to the logic of infons, which is discussed in Section 4.2.

Considerations of polarity aside, axiom A9 has found wide support. Moreover, it allows us to introduce a convenient *functional notation*. For any finite infon  $\sigma$  there is a relation  $r$  and a finite sequence  $\langle i_1, a_1 \rangle, \dots, \langle i_n, a_n \rangle$  of role-argument pairs, such that  $\text{Rel}(r, \sigma)$  and  $\text{Arg}(a_j, i_j, \sigma)$  for  $1 \leq j \leq n$  (and no other pair  $\langle i, a \rangle$  is such that  $\text{Arg}(a, i, \sigma)$ ). Given axiom A9, the identity of  $\sigma$  is completely determined by this information, and so we may write  $\sigma$  unambiguously as

$$\langle\langle r; i_1: a_1, \dots, i_n: a_n \rangle\rangle$$

The set of role-argument pairs in a basic infon is called an *assignment*. Although confined by the linearity of text, the elements of the assignment are intended to be unordered, so that, for example,  $\langle\langle r; i : a, j : b \rangle\rangle$  and  $\langle\langle r; j : b, i : a \rangle\rangle$  denote the same infon—this follows from axiom A9, of course.

The functional notation and its variants is widely used. Indeed, it is tempting to base the theoretical account on an infon-building *function* which maps each pair consisting of a relation  $r$  and an assignment  $\langle\langle i_1, a_1 \rangle, \dots, \langle i_n, a_n \rangle\rangle$  to the infon  $\langle\langle r; i_1 : a_1, \dots, i_n : a_n \rangle\rangle$ . The problem with the functional approach is that the infon-building function is not always properly defined. Even in the standard structures, it is not the case that every relation-assignment pair determines an infon; an infon will result only if the length of assignment is the same as the arity of the relation.

Faced with partiality there are three natural responses. The first (Plotkin 1991), is to have an infon-building function which is total but does not always deliver an infon. The advantage is an uncomplicated model; the disadvantage is the existence of many ‘pseudo-objects’ in the universe which serve no real purpose. The second response (ref? Plotkin, Barwise and Cooper) is to accept that the infon-building function is partial, and make do with a partial language (and logic) for describing situation-theoretic objects. The third response (ref? Muskens, Cooper, and this paper) is to base the theory on relations instead of functions. The advantage is that we may retain the services of classical logic and our models remain uncomplicated. The disadvantage is that we must abandon the functional notation, at least for theoretical purposes.

In less formal discussions we shall still use terms of the form ‘ $\langle\langle r; i : a, j : b \rangle\rangle$ ’, with the presupposition that there is an infon to which the term refers. When the roles are clear from the context, or are simply numerical indices in the canonical order ( $i = 1, j = 2$ ), we use the abbreviated form ‘ $\langle\langle r; a, b \rangle\rangle$ ’.

## 2.5 Arguments

Standard information structures have the property that

**A 10** *no argument is an infon, relation or role.*

In other words, the information modelled is of a purely ‘non-reflexive’ kind: no information *about* infons, relations, or roles is included. This is a severe limitation whose removal has motivated a number of important developments in Situation Theory.

One reason for rejecting axiom A10 in favour of a more liberal framework is that there are a number of linguistic expressions that are often taken to express ‘higher-order’ information—information about infons. Ascriptions of propositional attitudes and conditionals are the obvious examples. For example, if  $\sigma$  is the information that Albert is replete, then the information that Delia knows that Albert is replete may be thought to involve some relation between Delia and  $\sigma$  (and possibly other things). A first candidate for modelling this would

be an infon of the form  $\langle\langle \text{knows}; \text{Delia}, \sigma \rangle\rangle$ . Likewise, if  $\tau$  is the information that Albert is snoring, then we might regard the conditional ‘if Albert is snoring then he is replete’ as expressing the information  $\langle\langle \text{if}, \sigma, \tau \rangle\rangle$ .

A second reason for dropping axiom A10 is to allow the modelling of information about situation-theoretic objects, such as relations and infons, within the theory itself. For example, we may model the information that  $\text{Rel}(r, \sigma)$  (i.e., that  $r$  is a relation of  $\sigma$ ) by the infon  $\langle\langle \text{Rel}; r, \sigma \rangle\rangle$ .

Recalling Construction 2.2, it is easy to find the source of compliance with axiom A10. If  $\mathfrak{M}$  is a relational structure, the arguments of infons in  $\text{SInf}(\mathfrak{M})$  are all of the form  $\langle 0, m \rangle$  for some element  $m$  of  $|\mathfrak{M}|$ , but any infon, relation or role is of the form  $\langle i, a \rangle$  for some  $i \neq 0$ . These sortal restrictions are built in to the construction to ensure that isomorphic relational structures give rise to isomorphic information structures.

The crucial step enforcing axiom A10 is the copying of  $\mathfrak{M}$  to form  $\mathfrak{M}_0$  before applying the operation  $\text{SInf}^-$ . If we construct  $\text{SInf}^-(\mathfrak{M})$  directly, the restriction on arguments is removed and we may obtain non-standard information structures.

To see how this can give us higher-order infons, suppose  $\mathfrak{M}$  is of type  $[M, R_1^1, R_2^2]$  and  $M$  happens to contain an element  $\sigma = \langle 3, R_2, \alpha \rangle$ , in which  $\alpha$  is a function from  $\{1, 2\}$  to  $M$ . Then  $\sigma$  is also included in the universe of  $\text{SInf}^-(\mathfrak{M})$  and is at the same time deemed to be an infon of that structure. But it is still an element of  $M$  and so may also appear as an argument to another infon. In particular, if  $\beta$  is the function with domain  $\{1\}$  and  $\beta(1) = \sigma$  then  $\tau = \langle 3, R_1, \beta \rangle$  is such a ‘higher-order’ infon. (In the functional notation,  $\tau = \langle\langle R_1; \sigma \rangle\rangle$ .) The best we can do in  $\text{SInf}(\mathfrak{M})$  is the infon  $\langle\langle R_1; \langle 0, \sigma \rangle \rangle\rangle$ , which is not the same thing at all.

It is also worthwhile to iterate the  $\text{SInf}^-(\mathfrak{M})$  construction.

**Construction 2.4** Given a relational structure  $\mathfrak{M}$ , we define an infinite sequence  $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$  of information structures as follows. Let  $\mathfrak{A}_0$  be  $\mathfrak{M}$ , and for each integer  $n$ , let  $\mathfrak{A}_{n+1}$  be  $\text{SInf}^-(\mathfrak{A}_n)$ . Finally, let  $\mathfrak{A}$  be the structure whose universe is the union of the universes  $|\mathfrak{A}_n|$  for positive integers  $n$ , and with  $\llbracket \text{Rel} \rrbracket^{\mathfrak{A}}$  and  $\llbracket \text{Arg} \rrbracket^{\mathfrak{A}}$  defined to be the unions, for all  $n$ , of  $\llbracket \text{Rel} \rrbracket^{\mathfrak{A}_n}$  and  $\llbracket \text{Arg} \rrbracket^{\mathfrak{A}_n}$ , respectively. It is simple to check that  $\mathfrak{A}$  is an information structure. Moreover, it is a fixed point of the construction:  $\text{SInf}^-(\mathfrak{A}) = \mathfrak{A}$ .

The construction of ‘iterated’ information structures is respectable — isomorphic relational structures yield isomorphic information structures — yet it generates non-standard structures in which axiom A10 fails. In addition to the infons generated from  $\mathfrak{M}$ , the iterated structures contain many ‘situation-theoretic’ infons of the forms  $\langle\langle \text{Rel}; r, \sigma \rangle\rangle$  and  $\langle\langle \text{Arg}; a, i, \sigma \rangle\rangle$ . To include other ‘higher-order’ infons, we would need to add further clauses to the iterated operation. For example, we could incorporate ascriptions of knowledge and conditions by adding the following clauses to the definition of  $\text{SInf}^-(\mathfrak{M})$ :

Key Construction

$$\begin{aligned}
A_4 &= \{ \langle 4, \text{knows}, \alpha \rangle \mid \alpha: \{1, 2\} \rightarrow M \text{ and } \exists x, y \alpha(1) = \langle 0, x \rangle, \alpha(2) = \langle 3, y \rangle \} \\
A_5 &= \{ \langle 5, \text{if}, \alpha \rangle \mid \alpha: \{1, 2\} \rightarrow M \text{ and } \exists x, y \alpha(1) = \langle 3, x \rangle, \alpha(2) = \langle 3, y \rangle \}
\end{aligned}$$

## 2.6 Circularity

Despite escaping axiom A10, the arguments of infons in iterated information structures are not entirely without restriction. The following axiom is satisfied by all iterated information structures, as well as all standard ones.

**A 10.1** *There is no infinite sequence  $a_0, a_1, a_2, \dots$  such that, for each integer  $n$ ,  $a_{n+1}$  is an argument of  $a_n$ .*

Axiom A10.1 is a weakening of axiom A10, which disallows any such sequences, even ones of the form  $a_0, a_1, a_2$ . The Axiom of Foundation (in Set Theory) implies that Axiom A10.1 holds in standard structures. Any such infinite sequence would be an infinitely descending sequence in the transitive closure of the membership relation, contradicting Foundation.

Axiom A10.1 prevents a direct model of ‘circular’ information. For example, it is natural to take the information expressed by

‘The information expressed by this sentence is expressible in English’

to be a simple infon  $\sigma$  described as follows. The relation of  $\sigma$  should be *expressible-in-English*. There should be a role  $i$  so that  $\text{Arg}(\sigma, i, \sigma)$ , and  $\sigma$  should have no other arguments. Indeed, it might be even assumed that we have an *equation*

$$\sigma = \langle \langle \text{expressible-in-English}; \sigma \rangle \rangle \quad (1)$$

In any information structure containing this infon  $\sigma$ , the infinite path  $\sigma, \sigma, \sigma, \dots$  would be of kind forbidden by Axiom A10.1.

Once again, at this point we know that an infon like  $\sigma$  cannot exist in any standard structure. However, it is quite easy to build non-standard structures with circular infons, and we discuss this below. In line with our axiomatic approach, we want an axiom which guarantees the existence of infons like  $\sigma$ .

**Definition 2.5** Let  $I$  be the set of roles of an information structure  $\mathfrak{A}$ , and let  $Re$  be the set of relations. An *infon description system* for  $\mathfrak{A}$  is a triple  $\mathcal{E} = \langle X, r, e \rangle$ , where  $X$  is a set,  $r$  is a partial function from  $X$  to  $Re$ , and  $e$  is a partial function from  $X \times I$  to  $|\mathfrak{A}| \cup X$ . A *solution* to  $\mathcal{E}$  is a (total) function  $s$  from  $X$  to the infons of  $\mathfrak{A}$  such that for all  $x \in X$ ,

1.  $r(x)$  is defined iff  $\text{Rel}(r(x), s(x))$ .
2. for all roles  $i$  such that  $e(x, i)$  is defined and belongs to  $X$ ,

$$\text{Arg}(s(e(x, i)), i, s(x)).$$

3. for all roles  $i$  such that  $e(x, i)$  is defined and belongs to  $X - |\mathfrak{A}|$ ,

$$\text{Arg}(e(x, i), i, s(x)).$$

4. If  $\text{Arg}(a, i, s(x))$ , then  $a$  comes from either clause (2) or (3).

As an example,  $\mathcal{E}$  might be an arbitrary singleton  $\{x\}$  together with functions  $r$  and  $e$  given by  $r(x) = \text{expressible-in-English}$  and  $e(x, i) = x$ . Then a solution to  $\mathcal{E}$  would be a function  $s$  defined on  $\{x\}$  giving an infon  $s(x)$  which we will write as  $\sigma$ . This infon  $\sigma$  has relation  $r(x)$  so that  $\text{Arg}(\sigma, i, \sigma)$  and  $\sigma$  has no other roles.

At this point we can state an axiom:

**A 10.2** *Every infon description system has a solution.*

We digress to discuss the uniqueness of solutions to infon description systems. The easiest way to approach this is to consider what would happen if we had two solutions to the system above. This would give two distinct infons  $\sigma$  and  $\tau$  with all of the properties above. The same intuitions which justify axiom A9 suggest that  $\sigma$  and  $\tau$  must be the same. After all, it is difficult to see how one would choose between them when modelling the information expressed in English, above. To capture these intuitions in full generality, we need the following definition.

**Definition 2.6** Given an information structure  $\mathfrak{A}$ , a binary relation  $R$  on  $A$ , is a *bisimulation* iff for all  $\sigma, \tau$  in  $A$ , if  $R(\sigma, \tau)$  then

1. if either  $\sigma$  or  $\tau$  are not infons then  $\sigma = \tau$ ,
2. if  $\sigma$  and  $\tau$  are infons then they have the same relation and roles,
3.  $\forall i, a, b$  if  $\text{Arg}(a, i, \sigma)$  and  $\text{Arg}(b, i, \tau)$  then  $R(a, b)$ , and
4.  $R(\tau, \sigma)$ .

For any elements  $a, b$  in  $A$ , we say that  $a$  is *bisimilar* to  $b$  iff there is a bisimulation  $R$  such that  $R(a, b)$ . It is easy to show that the relation of being bisimilar is itself a bisimulation, the largest bisimulation in  $\mathfrak{A}$ .

This discussion leads to the following axiom:

**A 10.3** *If  $a$  is bisimilar to  $b$  then  $a = b$ .*

Further, if we adopt axiom A10.3, then it is natural to strengthen axiom A10.2 to

**A 10.4** *Every infon description system has a unique solution.*

We now turn to the matter of getting models satisfying axioms like A10.4. As we have seen, the most natural models are of the form  $\text{SInf}^-(\mathfrak{M})$  generated from a relational structure  $\mathfrak{M}$ . However, by the Axiom of Foundation in set theory, each such structure satisfies axiom A10.1 and hence falsifies axiom A10.4. However, suppose we drop the Axiom of Foundation from  $ZFC$  and call the result ‘ $ZFC^-$ ’. There is no longer any guarantee that the information structure  $\text{SInf}^-(\mathfrak{M})$  generated from a relational structure  $\mathfrak{M}$  satisfies axiom A10.1. Moreover, it is possible to replace the Axiom of Foundation by other axioms which ensure that many such structures exist. **In this setting, the most attractive candidate is the Anti-foundation Axiom (AFA).** We defer a statement of AFA to Section 3.2 because we don’t need it at this point. For more on it, see also Aczel (1988), or Barwise and Moss (1996).

The key point is that assuming AFA, we can get standard structures satisfying axiom A10.4. Indeed, we can get structures with the stronger property that the infons literally *are* sets satisfying equations like (1) above. We dub these *hyperinfons*, and the sets in a universe of set theory where AFA is assumed are called *hypersets*. Note that we have made a rather radical move. Faced with a problem in our object-theory (how to obtain models with circular infons) we change part of our meta-theory — in this case, the theory of sets. Put this way, the strategy seems a bit like cheating, at least without some independent motivation for hypersets. From a purely formal point of view there is no problem at all. Aczel (1988) has shown that  $ZFC^- + \text{AFA}$  is consistent relative to  $ZFC^-$ . Nonetheless, certain conceptual worries may remain, especially because the existence of hypersets flies in the face of the cumulative (hierarchical) conception of sets which most of us were taught to be a conceptual pillar of set theory.

There are other ways of thinking about sets, discussed for example in Barwise and Etchemendy (1987), which may give us reason to change our meta-theory. Indeed, these other ways are special cases of the “structural view” that we develop in Section 3 of this chapter. The same view is behind the adoption of AFA in a number of areas involving circularity (see Barwise (1989), and also Barwise and Moss 1996) for examples). At this point, we want to make two comments on the issue:

First, AFA is only needed to get standard models with circular infons. It is possible to get other models directly. However, when one starts to build such models, AFA comes out fairly naturally. This shows that any ontological qualms one might have about hypersets are quite irrelevant to the development of Situation Theory.

Second, AFA not only makes the construction of such models easier, it allows for the construction of models of other axioms of interest in this chapter. We adopt AFA in our meta-theory for this reason, and also because its conceptual background fits in with our theory of structural relations.



## 2.7 Appropriateness

Let's say that an object is *ordinary* if it is neither a role, nor an infon, nor a relation. In standard structures, for each relation  $r$  of arity  $n$  and each sequence  $a_1 \dots a_n$  of ordinary objects, the infon  $\langle\langle r; 1:a_1, \dots, n:a_n \rangle\rangle$  exists. We call this property of standard structures 'generality'. To state it precisely, we say that infons  $\sigma$  and  $\tau$  each with role  $i$  are *i-variants* if for each role  $j \neq i$  and each  $a$ ,  $\text{Arg}(a, j, \sigma)$  iff  $\text{Arg}(a, j, \tau)$ . With this terminology, the principle of generality, which is respected by all standard information structures, may be stated as follows.

**A 11** *For each infon  $\sigma$  with role  $i$  and each ordinary object  $a$ , there is an  $i$ -variant  $\tau$  of  $\sigma$  such that  $\text{Arg}(a, i, \sigma)$ .*

There are reasons for thinking that Axiom A11 is both too weak and too strong. It is too weak because it says nothing about the appropriateness of arguments that are not ordinary. Removing the restriction to ordinary objects, we get the following:

**A 11.1** *For each infon  $\sigma$  and with role  $i$ , and each object  $a$ , there is an  $i$ -variant  $\tau$  of  $\sigma$  such that  $\text{Arg}(a, i, \sigma)$ .*

Axiom A11.1 is not satisfied by standard information structures. However, the iterated structures of Construction 2.4 obey a version of the axiom when restricted to infons with relation  $\text{Rel}$  or  $\text{Arg}$ . We shall see more examples of structures satisfying A11.1 in Section ???

The reason for thinking that axioms A11 and A11.1 are both too strong is that they do not permit sortal restrictions. For example, the information expressed by 'Albert tasted the *crème brûlée*' presupposes that Albert is an agent, perhaps even that he is subject of conscious experience. Good evidence for this is that the presupposition accompanies the claim that Albert did not taste the *crème brûlée*, and we would be surprised to the point of incomprehension if we were told later that Albert is a Moulinex food processor. Sortal presuppositions like these may be explained by supposing that the argument roles of infons carry certain restrictions on what can fill them. The omelette  $O$  of our original example simply cannot fill the role of *cook* played by Raymond, and vice versa. This may be captured by ensuring that there is no infon  $\tau$  such that  $\text{Arg}(O, \text{cook}, \tau)$ . Since there is an infon  $\sigma$  such that  $\text{Arg}(R, \text{cook}, \tau)$  and  $O$  is an ordinary object, this is inconsistent with axiom A11.

We say that an object  $a$  is an *appropriate filler* of role  $i$  just in case there is an infon  $\sigma$  such that  $\text{Arg}(a, i, \sigma)$ . To allow for role-linked sortal restrictions, we may modify axiom A11 to the principle of 'sortal generality':

**A 11.2** *for each infon  $\sigma$  with role  $i$  and each appropriate filler  $a$  of  $i$ , there is an  $i$ -variant  $\tau$  of  $\sigma$  such that  $\text{Arg}(a, i, \sigma)$ .*

This is a great improvement. In effect, we have associated a domain of objects with each role—their appropriate fillers—and allowed unrestricted replacement of one argument by any other argument in the domain. In standard information structures, the domain of appropriate fillers for each role is just the class of ordinary objects, but other structures may have different domains for different roles. The iterated structures of Construction 2.4 all satisfy axiom A11.2. They have two domains: roles of the original structure may be filled by any ordinary object, and roles of infons whose relation is *Arg* of *Rel* may be filled by any object at all.

A theory of appropriateness, stating in general terms which arguments may be fill which roles, has not yet been given. Axiom A11.2 accords with the way roles are usually thought to restrict their fillers, but there may be restrictions on the formation of infons that are not attributable to a single role. For example, consider a sketch of two people standing side-by-side, and call the depicted relationship between the people ‘next-to’. It is arguable that *next-to* is not the same as the relation expressed by English preposition ‘next to’ because any picture of two people standing side-by-side presupposes that there are *two* people rather than one. We may analyze the difference by saying that the two roles of *next-to* place a joint restriction on their fillers, namely, that they cannot be filled by the same argument. This kind of restriction is not permitted by axiom A11 or by axiom A11.2.

## 2.8 Partiality

In standard information structures every infon has a relation and a set of roles determined by that relation. This is captured by the following two axioms.

**A 12** *For each infon  $\sigma$  there is a relation  $r$  such that  $\text{Rel}(r, \sigma)$ .*

**A 13** *If  $\text{Rel}(r, \sigma)$  and  $\text{Rel}(r, \tau)$  and  $\text{Arg}(a, i, \sigma)$  then for some  $b$ ,  $\text{Arg}(b, i, \tau)$ .*

One reason for dropping these axioms is to allow infons to be *unsaturated*. For example, suppose you overhear someone saying ‘Mary saw Elvis in Tokyo’ but a distracting noise prevents you from hearing the word ‘Elvis’. The information conveyed is unsaturated because the filler of one of the roles is missing. We may represent the unsaturated infon as  $\langle\langle \text{saw; seer: Mary, location: Tokyo} \rangle\rangle$ , to be contrasted with the infon

$\langle\langle \text{saw; seer: Mary, seen: Elvis, location: Tokyo} \rangle\rangle$

which would have been conveyed had you heard the whole utterance. The coexistence of these infons is forbidden by axiom A13, which must therefore be dropped if they are to be modelled successfully. Another way in which an infon can be unsaturated is by lacking a relation—consider, for example, the

information conveyed if the word ‘saw’ had been obscured. This counts against axiom A12.

In the absence of the constraints of axioms A12 and A13, it is useful to define an ordering of infons that captures the degree to which they are saturated.

**Definition 2.7** Infon  $\tau$  is *part of* infon  $\sigma$ , written  $\sigma \sqsubseteq \tau$ , if

1. for all  $r$ , if  $\text{Rel}(r, \sigma)$  then  $\text{Rel}(r, \tau)$ , and
2. for each role  $i$  and object  $a$ , if  $\text{Arg}(a, i, \sigma)$  then  $\text{Arg}(a, i, \tau)$ .

An infon  $\sigma$  is *unsaturated* if there is another infon  $\tau$  such that  $\sigma \sqsubseteq \tau$ . If there is no such infon,  $\sigma$  is said to be *saturated*.

Despite our need to have unsaturated infons, there is something to be said for the intuition that an infons *should* have a relation which determines its roles. We can recover the force of this idea by restricting axioms A12 and A13 to apply only to saturated infons.

**A 13.1** *Every saturated infon has a relation.*

**A 13.2** *For saturated infons,  $\sigma$  and  $\tau$ , having the same relation, and for each object  $a$  and role  $i$ , if  $\text{Arg}(a, i, \sigma)$  then for some  $b$ ,  $\text{Arg}(b, i, \tau)$ .*

In this way, we see that the standard structures are a limiting special case. Axioms A12 and A13 follow from A13.1 and A13.2 given the additional assumption that every infon is saturated.

The ordering  $\sqsubseteq$  is clearly a preorder of infons (reflexive and transitive). In information structures satisfying axiom A9 it is a partial order (reflexive, transitive, and antisymmetric), and in standard structures it is trivial ( $\sigma \sqsubseteq \tau$  iff  $\sigma = \tau$ ) because every infon is saturated. The properties of the information ordering in non-standard information structures may be constrained by the assumptions we make concerning the existence of upper bounds.

**Definition 2.8** Infons  $\sigma$  and  $\tau$  are *compatible* if they have the same relation and for each role  $i$  and objects  $a$  and  $b$ , if  $\text{Arg}(a, i, \sigma)$  and  $\text{Arg}(b, i, \tau)$  then  $a = b$ . They are *unifiable* if they possess a least upper bound in the  $\sqsubseteq$  ordering.<sup>6</sup>

For example, the information that Mary saw Elvis is compatible with the information that Elvis was seen in Tokyo. Moreover, it seems reasonable to suppose that these two infons may be unified to produce the information that Mary saw Elvis in Tokyo.<sup>7</sup> This suggests the following unification axiom.

**A 13.3** *Every strictly compatible pair of infons is unifiable.*

<sup>6</sup>I.e., if  $\exists x$  such that  $\sigma \sqsubseteq x$ ,  $\tau \sqsubseteq x$ , and  $\forall y$ , if  $\sigma \sqsubseteq y$  and  $\tau \sqsubseteq y$ , then  $x \sqsubseteq y$ .

<sup>7</sup>There is no requirement that the unification of two infons is ‘entailed’ by the infons being unified; just that the unification of the two infons is the least infon containing them.

Questions concerning the order of infons become a lot more difficult to answer if the arguments of infons are also partial objects. This is unavoidable if we wish to model ‘higher-order’ infons, as discussed in Section 2.5. There are a variety of ways of extending the order between infons by taking the order of arguments into account, but none of them is entirely convincing. Consider, for example, the idea that if  $a \sqsubseteq a'$  then the infon  $\langle\langle r; i: a, j: b \rangle\rangle$  is part of the infon  $\langle\langle r; i: a', j: b \rangle\rangle$ . This seems plausible at first, until one recalls the idea that the information expressed by a conditional is an infon of the form  $\langle\langle \text{if}; \sigma, \tau \rangle\rangle$ . If  $\sigma \neq \sigma'$  then we would *not* wish to say that  $\langle\langle \text{if}; \sigma, \tau \rangle\rangle$  is part of  $\langle\langle \text{if}; \sigma', \tau \rangle\rangle$  even if  $\sigma \sqsubseteq \sigma'$ . Consequently, we shall stick to the above definition, even for infons with other infons as arguments.

With discussions of unification, comparison with the literature on feature structures (cf. Rounds’ chapter in this volume) is unavoidable. There is a strong similarity between feature structures and information structures. Indeed, it is easy to see how one can construct information structures using features structures to model the infons. Conversely, one can use information structures to model feature structures, by identifying features with roles and values with arguments. The order  $\sqsubseteq$  defined for infons is relatively uninteresting when applied to feature structures, precisely because it does not take into account the order of arguments. Rounds (1991) discusses the relationship between the two subjects.

## 2.9 Representation Theorems

In addition to the axioms discussed above, we need one more axiom to account for the behaviour of constants in standard information structures.

**A 14** *Every relation but no role and no infon is denoted by a constant.*

This axiom is entirely artificial, and of little consequence to the general theory.

**Theorem 2.9** *A simple information structure is standard if and only if it satisfies the axioms A1 to A14.*

PROOF: We have already seen that each of these axioms is satisfied by all standard information structures. For the converse, let  $\mathfrak{A}$  be an information structure satisfying the axioms, with constants  $C$ . We shall construct a relational structure  $\mathfrak{M}$  and an isomorphism  $f: |\mathfrak{A}| \rightarrow |\text{SInf}(\mathfrak{M})|$ .

Let  $R$  be the class of relations in  $\mathfrak{A}$ ,  $I$  the class of roles,  $\Sigma$  the class of infons, and  $M$  the class of ordinary objects. By axioms A3 to A5, these four classes are pairwise disjoint and exhaust the universe of  $\mathfrak{A}$ .

The class  $R$  of relations is finite by axiom A8. For each  $r$  in  $R$ , let

$$I_r = \{i \mid \exists a, \sigma \text{ Rel}(r, \sigma) \text{ and Arg}(a, i, \sigma)\}.$$

We shall show that  $I_r$  is finite. For each infon  $\sigma$  of  $\mathfrak{A}$ , let  $I_\sigma = \{i \mid \exists a \text{ Arg}(a, i, \sigma)\}$ . By axiom A13, for all infons  $\sigma$  and  $\tau$  such that  $\text{Rel}(r, \sigma)$  and  $\text{Rel}(r, \tau)$ ,  $I_\sigma = I_\tau$ . Thus for each  $\sigma$  having relation  $r$ ,  $I_r = I_\sigma$ . Moreover, by axiom A7,  $I_\sigma$  is finite for each infon  $\sigma$ , and so  $I_r$  is finite also.

Let  $S \subseteq C$  be the class of those constants that denote relations in  $\mathfrak{A}$ . By axiom A14,  $S$  is the same size as  $R$ . This enables us to define a relational signature  $\nu$  with primitive relations  $S$  and such that for each  $R'$  in  $S$ ,  $\nu(R')$  is equal to the size of  $I_{\llbracket R' \rrbracket^{\mathfrak{A}}}$ . Let  $\mathfrak{M}$  be any relational structure with signature  $\nu$ , universe  $M$ , and constants  $C - S$ , each with the same denotation it has in  $\mathfrak{A}$  (which is possible, because these constants denote ordinary objects, by axiom A14). We shall construct an isomorphism  $f$  from  $\mathfrak{A}$  to  $\text{SInf}(\mathfrak{M})$ .

First, we must enumerate the set  $I$  of roles of  $\mathfrak{A}$ . By axioms A1 and A12, for each infon  $\sigma$  of  $\mathfrak{A}$  there is a unique relation  $r_\sigma$  such that  $\text{Rel}(r_\sigma, \sigma)$ . By the above,  $I_\sigma = I_{r_\sigma}$ . But there are only a finite number of relations (axiom A8), and so there are only a finite number of sets of the form  $I_\sigma$ , even if the number of infons is infinite. Moreover, by axiom A6, these sets are linearly ordered by inclusion. Consequently, there is an enumeration  $r_1, \dots, r_n$  of  $R$  such that  $I_{r_k} \subseteq I_{r_{k+1}}$ , for  $1 \leq k \leq n-1$ . Now

$$I = I_{r_1} \cup \dots \cup I_{r_n} = I_{r_1} \cup \dots \cup (I_{r_{k+1}} - I_{r_k}) \cup \dots \cup (I_{r_n} - I_{r_{n-1}}).$$

and so we may enumerate  $I$  by enumerating each of the sets  $(I_{r_{k+1}} - I_{r_k})$  in order, and without repetition. Let  $i_1, \dots, i_N$  be such an enumeration. It has the property that for  $1 \leq k \leq n$ ,  $I_{r_k} = \{i_1, \dots, i_{\nu(r_k)}\}$ .

Next, we must consider the arguments of infons. For each infon  $\sigma$ , we have seen that

$$I_\sigma = I_{r_\sigma} = \{i_1, \dots, i_{\nu(r_\sigma)}\}$$

and by axiom A2, for  $1 \leq k \leq n$ , there is a unique object  $a_k$  such that  $\text{Arg}(a_k, i_k, \sigma)$ . By axiom A10,  $a_k$  is ordinary and so is in  $M$ . Define the function  $\alpha_\sigma: \{1, \dots, \nu(r_\sigma)\} \rightarrow \{0\} \times M$  by  $\alpha_\sigma(k) = \langle 0, a_k \rangle$ .

Now we are ready to define the function  $f$  from  $|\mathfrak{A}|$  to  $|\text{SInf}(\mathfrak{M})|$ .

$$f(x) = \begin{cases} \langle 0, x \rangle & \text{if } x \text{ is in } M \\ \langle 1, x \rangle & \text{if } x \text{ is in } R \\ \langle 2, k \rangle & \text{if } x \text{ is in } I \text{ and } i_k = x \\ \langle 3, \langle r_x, \alpha_x \rangle \rangle & \text{if } x \text{ is in } \Sigma \end{cases}.$$

This is a good definition, because every element of  $|\mathfrak{A}|$  is in exactly one of the classes  $M$ ,  $R$ ,  $I$  or  $\Sigma$ , and for each  $x$  in  $I$  there is a unique  $k$  such that  $i_k = x$ . To show that  $f$  is one-one, suppose that  $f(x) = f(y)$ . We can show that  $x = y$  by cases according to whether  $x$  and  $y$  are in  $M$ ,  $R$ ,  $I$  or  $\Sigma$ . The only non-trivial case is that in which  $x$  and  $y$  are infons. Then  $f(x) = \langle 3, \langle r_x, \alpha_x \rangle \rangle$  and  $f(y) = \langle 3, \langle r_y, \alpha_y \rangle \rangle$ . Thus,  $r_x = r_y$  and  $\alpha_x = \alpha_y$ , and so by axiom A9,  $x = y$ , as required.

To show that  $f$  is onto the universe of  $\text{SInf}(\mathfrak{M})$ , consider any element  $\langle j, x \rangle$  of that universe. We need to find a  $y$  such that  $f(y) = \langle j, x \rangle$ . The only non-trivial case is that in which  $j = 3$ . Then  $x = \langle r, \alpha \rangle$  for some relation  $r$  and function  $\alpha: \{1, \dots, \nu(r)\} \rightarrow \{0\} \times M$ . For  $1 \leq k \leq \nu(r)$ , let  $m_k$  be the element of  $M$  for which  $\alpha(k) = \langle 0, m_k \rangle$ . Now,  $r$  is a relation, so there is some infon  $\sigma$  of  $\mathfrak{A}$  such that  $\text{Rel}(r, \sigma)$ . The roles of  $\sigma$  are those in the set  $I_\sigma = \{i_1, \dots, i_{\nu(r)}\}$  and so, applying axiom A11 repeatedly ( $\nu(r)$  times), we obtain an infon  $\sigma'$  such that  $\text{Arg}(m_k, i_k, \sigma')$  for  $1 \leq k \leq \nu(r)$ . Spelling out the definition of  $\alpha_{\sigma'}$  we see that this function is just  $\alpha$ , so  $\langle r_{\sigma'}, \alpha_{\sigma'} \rangle = \langle r, \alpha \rangle = x$ , and so  $f(\sigma') = \langle 3, \langle r_{\sigma'}, \alpha_{\sigma'} \rangle \rangle = \langle 3, x \rangle$ , as required.

Finally, we must show that  $f$  preserves the structure of  $\mathfrak{A}$ . This follows from the following two chains of equivalences. Firstly for  $\text{Rel}$ :

$$\begin{aligned} & \text{Rel}(f(r), f(\sigma)) \text{ in } \text{SInf}(\mathfrak{M}) \\ & \text{Rel}(\langle 1, r \rangle, \langle 3, \langle r_\sigma, \alpha_\sigma \rangle \rangle) \text{ in } \text{SInf}(\mathfrak{M}) \\ & \quad r = r_\sigma \\ & \text{Rel}(r, \sigma) \text{ in } \mathfrak{A}. \end{aligned}$$

And then for  $\text{Arg}$ :

$$\begin{aligned} & \text{Arg}(f(m), f(i), f(\sigma)) \text{ in } \text{SInf}(\mathfrak{M}) \\ & \text{Arg}(\langle 0, m \rangle, \langle 2, k \rangle, \langle 3, \langle r_\sigma, \alpha_\sigma \rangle \rangle) \text{ in } \text{SInf}(\mathfrak{M}) \text{ and } i_k = i \\ & \quad \alpha_\sigma(k) = \langle 0, m \rangle \text{ and } i_k = i \\ & \text{Arg}(m, i, \sigma) \text{ in } \mathfrak{A}. \end{aligned}$$

That  $f$  preserves the denotation of the constants in  $C$  follows from axiom A14 and the definition of  $\text{SInf}(\mathfrak{M})$ . QED

We close this section with representation result for another class of structures.

**Definition 2.10** A *generalized standard information structure* is a structure  $\mathfrak{M}$  which satisfies axioms A10.3 and A11.2.

Note that the standard structures satisfy both of these conditions; the importance of such generalized structures will emerge from our discussions in the sequel. We focus on these now to highlight the importance of A10.3 and A11.2 and to introduce work on bisimulations that will also be important.

**Theorem 2.11** *Let  $R$ ,  $I$ , and  $\text{Ord}$  be sets or classes. Then there is a fixed generalized information structure  $\mathfrak{M}$  with the following properties:*

1.  $R$ ,  $I$ , and  $\text{Ord}$  are the relations, roles, and ordinary objects of  $\mathfrak{M}$ , respectively.
2. If  $\mathfrak{N}$  has the property of (1), then  $\mathfrak{N}$  is isomorphic to a substructure of  $\mathfrak{M}$ .

Before turning to the proof, we have two remarks. First, the result is a representation theorem: every  $\mathfrak{N}$  using the pre-specified collections  $R$ ,  $I$ , and  $Ord$  is a substructure of a big structure  $\mathfrak{M}$ . Second, the ideas behind the proof are more important than the result itself.

PROOF: We sketch the details. We need only construct the infons of  $\mathfrak{M}$ . For this, we use *infol systems*. These are systems of equations of the following form:

$$x = \langle r, \{ \langle i_1, p_1 \rangle, \dots, \langle i_k, p_k \rangle, \dots \} \rangle.$$

Here  $r$  is a relation; the  $i$ 's are roles, and the  $x$  on the left is a fresh object which we call a variable. Each  $p_j$  must either be an ordinary object which is an appropriate filler for the corresponding role  $i_j$ , or again a variable, say  $y$  which also occurs on the left side of some equation in our system. An infon system is a collection of such equations.

A *pointed infon system* is a pair  $\langle \mathcal{S}, x \rangle$ , where  $\mathcal{S}$  is an infon system and  $x$  is one of the variables of  $\mathcal{S}$ . It is not hard to define a relation of bisimulation on the collection of all pointed systems. One then considers pointed systems modulo bisimulation. This collection serves as the infons of  $\mathfrak{M}$ ; the rest of the structure is easy to come by, as is the verification that we have a generalized standard structure. To check the universality property fix some appropriate structure  $\mathfrak{N}$ . Each infon  $\sigma$  of  $\mathfrak{N}$  gives rise to a pointed system, hence to an element of  $\mathfrak{M}$ . This association is one-to-one, using the assumption that  $\mathfrak{N}$  satisfies axiom A10.3. QED

### 3 A Theory of Structural Relations

Information processed, inferred, conveyed, expressed, or otherwise represented need not be of the simple kind considered in the previous section; it may come in larger, more complex chunks.

Logical complexes present an obvious example: we need to be able to model conjunctions, disjunctions, and perhaps negations of infons. However, even with the first item on the list there is a potential problem. The identity condition for (well-founded) basic infons was very clear:  $\sigma$  and  $\tau$  are identical if and only if they have the same relation, and the same arguments filling the same roles. It is much less clear what should be said about conjunctions. A strong condition, analogous to the one for basic infons, would be that  $\sigma_1 \wedge \tau_1$  is identical to  $\sigma_2 \wedge \tau_2$  if and only if  $\sigma_1$  is identical to  $\sigma_2$  and  $\tau_1$  is identical to  $\tau_2$ . The problem arises if we combine this condition with natural logically-motivated requirements on conjunction, such as commutativity, associativity and idempotence. Then the condition becomes too strong. For example, consider an arbitrary conjunction  $\sigma \wedge \tau$ . By idempotence,  $\sigma \wedge \tau = (\sigma \wedge \tau) \wedge (\sigma \wedge \tau)$ , and so, by the above condition,  $\sigma = \sigma \wedge \tau = \tau$ .

The heart of the problem is an inherent tension in the concept of information. On the one hand, information is *representation-independent*: the same information may be represented in many different ways. On the other hand, information is *fine-grained*: two pieces of information may be logically equivalent without being identical. Consequently, the identity conditions for information represented by complex signs must lie somewhere between those for the syntactic form of the sign and those for its semantic content. Striking the right balance is, in general, very difficult.

Another problem is that there has been little agreement as to which combinations of infons are needed. Finite conjunction and disjunction are commonly adopted; infinite conjunctions and disjunction, quantified infons (with variable binding), various negations and conditionals, have also been proposed; and for applications in computer science other forms may be found useful—for example, is not clear that the information stored as a list is simply a conjunction of the items of information stored in each cell, and even if it is some kind of conjunction, it is not clear that it is the same conjunction as expressed by an unordered set of the same items.

In view of the above, it would seem sensible to pursue a fairly open-minded policy about complex infons. The approach adopted here is intended to cover various proposals made in the literature, as well as offering a framework in which other approaches may be tried.



### 3.1 Extensional Structures

Our idea is to generalize the notion of bisimulation to apply in a wider context. In the sequel, we will have (relational) structures of type

$$[A, S_1, \dots, S_m; R_1, \dots, R_n]$$

The relations in the first group,  $S_1, \dots, S_m$ , are called *structural relations* because they capture the structure of elements of the domain. A structural relation  $S_i$  of arity  $n+1$  is to be thought of as relating a list of  $n$  objects to a single structured object. If  $S_i(x_1, \dots, x_n, y)$  then  $y$  is a structured object with components  $x_1, \dots, x_n$ , which may or may not be structured themselves.

More generally, we say that  $b$  is a *component* of  $a$  in  $\mathfrak{A}$  if there is a structural relation  $S_i$  of arity  $n+1$  and elements  $x_1, \dots, x_n$  of  $A$  such that  $S_i(x_1, \dots, x_n, a)$  and  $x_j = b$  for some  $1 \leq j \leq n$ . For technical reasons, we require that the number of components of any is not a proper class—that is to say, the class of all components of a given object can be placed in one-to-one correspondence with some set. An object  $a$  is an *atom* of  $\mathfrak{A}$  if it has no components.

In an information structure, the only structured objects are the infons. Relations, roles and ordinary objects are all atomic, but infons have a component structure captured by the relations **Rel** and **Arg**. These are the structural relations of information structures. Information structures have no other primitive relations, but the defined relation  $\sqsubseteq$  is an example of a non-structural relation—albeit one whose extension is determined by the structure of infons. The important distinction between **Rel** and **Arg**, on the one hand, and  $\sqsubseteq$ , on the other, is that the identity of infons is determined by the former, by virtue of adherence to axiom A10.3.

But now consider an arbitrary relational structure. What conditions must the a relation satisfy to qualify as a structural relation? Our answer is based on the following definition.

**Definition 3.1** Given a relational structure  $\mathfrak{A}$  of type

$$[A, S_1, \dots, S_m; R_1, \dots, R_n],$$

a binary relation  $E$  on  $A$  is said to be a *bisimulation* on  $\mathfrak{A}$  if for all  $a, b \in A$ , if  $E(a, b)$  then the following three conditions hold:

1. if  $a$  is atomic then  $a = b$ ,
2. for  $1 \leq i \leq m$ , if  $\nu_i = k$ , then for all  $y_1, \dots, y_k$  such that  $S_i(y_1, \dots, y_k, a)$ , there are  $z_1, \dots, z_k$  such that  $S_i(z_1, \dots, z_k, b)$ , and  $E(y_j, z_j)$  for  $1 \leq j \leq k$ , and
3.  $E(b, a)$

$a$  is *bisimilar* to  $b$  in  $\mathfrak{A}$  iff there is a bisimulation  $E$  of  $\mathfrak{A}$  such that  $E(a, b)$ . The structure  $\mathfrak{A}$  is *extensional* if it has no distinct, bisimilar objects, i.e. if  $a$  is bisimilar to  $b$  in  $\mathfrak{A}$  then  $a = b$ .

If two objects are bisimilar then they are structurally equivalent, so there must be something else that distinguishes them. But if they are structural objects, then there can be nothing else to distinguish them. In this way, the extensionality condition enforces structural conditions of identity on the non-atomic objects of the domain, with the structure of an object determined solely by the relations structural relations. The non-structural relations,  $R_1, \dots, R_n$ , on the other hand, are not constrained in any way. They are used to model additional facts about the objects of the domain, which need not be dependent on their structure alone.

Extensional structures will be used throughout the rest of this chapter to model a variety of situation-theoretic objects. Our strategy will be to define different kinds of extensional structures to model different parts of the situation-theoretic universe, but to do so in a modular fashion. The modularity is facilitated by the notion of a ‘sort’.

**Definition 3.2** If  $\mathfrak{A}$  is an extensional structure, then for each structural relation  $S$  of  $\mathfrak{A}$ , let  $S^*$  be the class of those objects  $a$  such that  $S(\vec{x}, a)$  in  $\mathfrak{A}$  for some sequence  $\vec{x}$  (possibly empty) of elements of  $|\mathfrak{A}|$ . In other words,  $S^*$  is the projection of  $S$  along its last co-ordinate. We call these classes the (*structural*) *sorts* of  $\mathfrak{A}$ .

Every generalized information structure is an extensional structure with structural relations **Rel** and **Arg**. This is given by axiom A10.3. The standard structures and even the iterated structures are also well-founded, by axiom A10.1. In a standard information structure  $\text{Rel}^* = \text{Arg}^*$  is the class of infons, the only structured objects. In generalized information structures,  $\text{Rel}^*$  and  $\text{Arg}^*$  may be slightly different if there are infons with a relation but no arguments or with arguments but no relation. In any case, the class of infons is  $\text{Rel}^* \cup \text{Arg}^*$ . The relations, roles and ordinary objects are all atoms.

Another, familiar example of an extensional structure is the structure  $\mathfrak{V} = \langle V, \in^2; \rangle$  where  $V$  is the class of all sets, and  $\in$  is the binary relation of membership. The class  $\in^*$  consists of all non-empty sets. Extensionality is assured by the set-theoretic axiom of the same name, together with the axiom of Foundation or Anti-foundation, depending on whether  $V$  is assumed to satisfy the axioms of well-founded or anti-founded set theory.

**Definition 3.3** An extensional structure  $\mathfrak{A}$  of type  $[A, \text{Set}^1, \in^2; ]$  is called a *set structure* if  $\in^* \subseteq \text{Set}^*$  and  $\mathfrak{A}$  satisfy the axioms of  $\text{ZFC}^-$  with quantifiers restricted to  $\text{Set}^*$ .

The ambiguity in our use of the word ‘set’ and the predicate ‘ $\in$ ’ is of the now familiar kind. Internal sets are those elements of the structure in  $\text{Set}^*$ , which

may or may not be (external) sets. But for any internal set  $a$ , we may define a corresponding external set  $a^* = \{b \mid \in (b, a)\}$ .

Our third example of an extensional structure presents functions as structured objects whose components are their arguments and values.

**Definition 3.4** An extensional structure  $\mathfrak{A}$  of type  $[A, \mathbf{App}^3, \mathbf{Fun}^1; ]$  is a *function structure* if  $\mathbf{App}^* \subseteq \mathbf{Fun}^*$  and for all  $\alpha, x, y$  and  $i$  in  $A$ , if  $\mathbf{App}(i, x, \alpha)$  and  $\mathbf{App}(i, y, \alpha)$ , then  $x = y$ .

If  $\alpha \in \mathbf{App}^*$  then  $\alpha$  is an (internal) function of  $\mathfrak{A}$ . We associate it with an external function  $\alpha^*$  whose domain consists of those elements  $i$  of  $A$  for which there is an  $x$  such that  $\mathbf{App}(i, x, \alpha)$ . For each such  $i$ ,  $\alpha^*(i) = x$ , which is uniquely specified, by the defining condition of function structures.

What is the import of the extensionality requirement on function structures? It amounts to the condition that different internal functions represent different external functions. For if  $a^* = b^*$ , then  $I_A \cup \{\langle a, b \rangle, \langle b, a \rangle\}$  would be a bisimulation relating  $a$  and  $b$ . If the structure is *flat* (i.e., if no function is also an argument), then extensionality is equivalent to this condition. In other structures, extensionality is a stronger condition.

### 3.2 Structural Types

In any extensional structure, one can identify a non-atomic object by listing its structural relationships with other objects. Suppose  $\mathfrak{A}$  is an extensional structure and  $x$  is an object not necessarily in  $|\mathfrak{A}|$ , which we call a *parameter*. A *unary structural type* in  $\mathfrak{A}$  is a pair  $\langle x, T_x \rangle$  where  $T_x$  is a set of tuples of the form  $\langle S, b_1, \dots, b_k \rangle$  such that  $S$  is a structural relation of  $\mathfrak{A}$  of arity  $k + 1$  and  $b_1, \dots, b_k$  are elements of  $|\mathfrak{A}| \cup \{x\}$ .

Now, if  $T = \langle x, T_x \rangle$  is a unary structural type and  $a$  is an object of  $\mathfrak{A}$ , let  $a.T$  be the set of tuples of the form  $\langle S, b'_1, \dots, b'_k \rangle$  such that  $\langle S, b_1, \dots, b_k \rangle$  is in  $T_x$  and for  $1 \leq j \leq k$ ,  $b'_j = a$  if  $b_j = x$  and  $b'_j = b_j$  if  $b_j \neq x$ . An object  $a$  of  $\mathfrak{A}$  is *of type*  $T$  if for each structural relation  $S$  of arity  $k + 1$ , and each sequence  $b_1, \dots, b_k$  in  $|\mathfrak{A}|$ ,

$$S(b_1, \dots, b_k, a) \text{ in } \mathfrak{A} \quad \text{iff} \quad \langle S, b_1, \dots, b_k \rangle \text{ is in } a.T$$

For example, in an information structure the infon  $\sigma = \langle\langle r; i : \sigma, j : a \rangle\rangle$  is of type

$$T = \langle x, \{\langle \text{Rel}, r \rangle, \langle \text{Arg}, i, x \rangle, \langle \text{Arg}, j, a \rangle\} \rangle$$

because  $\sigma.T = \{\langle \text{Rel}, r \rangle, \langle \text{Arg}, i, \sigma \rangle, \langle \text{Arg}, j, a \rangle\}$  and  $\text{Rel}(r, \sigma)$ ,  $\text{Arg}(i, \sigma, \sigma)$ , and  $\text{Arg}(j, a, \sigma)$ ; and these are all the structural relationships in which  $\sigma$  participates.

Every object  $a$  is of some type, for example, the type  $\langle a, T_a \rangle$  where  $T_a$  is the set of tuples of the form  $\langle S, b_1, \dots, b_k \rangle$  such that  $S(b_1, \dots, b_k, a)$  in  $\mathfrak{A}$ .<sup>1</sup> Also, by

<sup>1</sup> $T_a$  cannot be a proper class because we have assumed that every object has only a set-sized class of components.

extensionality, no two objects are of the same type. So the type of an object serves as its unique identifier. However, there are many more types than there are objects of those types; many structural types are uninstantiated. We may use this idea to give some measure of how rich our ontology is, and to express principles of existence.

**Definition 3.5** Suppose  $\mathfrak{A}$  is an extensional structure. A *structural type*  $T$  in  $\mathfrak{A}$  is an indexed family  $\{T_x\}_{x \in X}$  of sets  $T_x$  of tuples of the form  $\langle S, b_1, \dots, b_k \rangle$  such that  $S$  is a structural relation of  $\mathfrak{A}$  of arity  $k+1$  and  $b_1, \dots, b_k$  are elements of  $|\mathfrak{A}| \cup X$ . The elements of the index class  $X$  are called *parameters*. Given a mapping  $s$  from  $X$  to  $|\mathfrak{A}|$  let  $s.T$  be the family of sets  $s.T_x$  of tuples of the form  $\langle S, b'_1, \dots, b'_k \rangle$  such that  $\langle S, b_1, \dots, b_k \rangle$  is in  $T_x$  and for  $1 \leq j \leq k$ ,  $b'_j = s(b_j)$  if  $b_j$  is in  $X$  and  $b'_j = b_j$  if  $b_j$  is not in  $X$ . The mapping  $s$  *instantiates*  $T$  if for each  $x$  in  $X$ ,  $s(x)$  is of the unary type  $\langle s(x), s.T_x \rangle$ .<sup>2</sup>

To see how this definition works, it is helpful to consider the special case of set structures. A set  $a$  is of unary type  $T = \langle x, T_x \rangle$  iff  $T_a = \{ \langle \in, b \rangle \mid b \text{ in } a^* \} \cup \{ \langle \text{Set} \rangle \}$ . Thus if we define  $T_a^* = \{ b \mid \langle \in, b \rangle \text{ in } T_a \}$ , then  $a$  is of type  $T$  iff  $a^* = T_a^*$ . Likewise, if  $T$  is a (polyadic) structural type, then  $s$  instantiates  $T$  iff  $s(x)^* = (s.T_x)^*$  for each parameter  $x$  of  $T$ . We can therefore think of a structural type in a set structure as a system of simultaneous equations, which is instantiated by a mapping of parameters to sets that *solves* the equations. For example, consider the system of equations shown on the left below.

$$\begin{array}{lcl} x & = & \{y, z\} \\ y & = & \emptyset \\ z & = & \{x, a\} \end{array} \quad \left| \quad \begin{array}{lcl} T_x & = & \{ \langle \in, y \rangle, \langle \in, z \rangle, \langle \text{Set} \rangle \} \\ T_y & = & \{ \langle \text{Set} \rangle \} \\ T_z & = & \{ \langle \in, x \rangle, \langle \in, a \rangle, \langle \text{Set} \rangle \} \end{array}$$

Here  $a$  is some arbitrary object (perhaps not a set). Now this system corresponds to the structural type shown on the right. Suppose that  $s$  instantiates that type. Then  $s(x)$ ,  $s(y)$ , and  $s(z)$  are sets, and

$$\begin{array}{lcl} s(x) & = & \{s(y), s(z)\} \\ s(y) & = & \emptyset \\ s(z) & = & \{s(x), a\} \end{array}$$

In other words,  $s$  is a *solution* to the system of equations.

Now we can see why the Foundation Axiom precludes the existence of instantiations of structural types: if  $s$  exists, then we get an infinite descending sequence of arguments:

$$s(x) \ni s(z) \ni \dots \ni s(x) \ni s(z) \ni \dots$$

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<sup>2</sup>For typographical convenience, we sometimes indicate applications by subscripts (as in  $T_x$ ) and sometimes by using parentheses.

**Definition 3.6**  $\mathfrak{A}$  is *well-founded* if there is no infinite sequence  $a_1, a_2, \dots$  of elements of  $A$  such that  $a_{n+1}$  is a component of  $a_n$  for each positive integer  $n$ .

$\mathfrak{A}$  is *structurally saturated* if every structural type in  $\mathfrak{A}$  is instantiated.

Recall that we mentioned AFA in Section 2.6. We did not state AFA at the time, but we can do so now.

**Definition 3.7** AFA is the statement that the universe of sets  $\mathfrak{V} = \langle V, \in^2, \text{Set}^1 \rangle$  is an structurally saturated, extensional set structure.

An equivalent (and more standard) formulation of AFA is that all systems of equations involving sets have unique solutions. Such systems are given by a set  $X$  together with a map  $e$  from  $X$  to  $X \cup V$ . A solution to a system is an instantiation of the corresponding structural type. As it happens, AFA implies that more general systems of equations have solutions. Consider  $X = \{x, y\}$  and

$$\begin{aligned} e(x) &= \{\langle x, y \rangle\} \\ e(y) &= \{x, y, \emptyset\} \end{aligned}$$

This system is unlike the ones we've seen so far because  $e(x)$  contains the *ordered pair*  $\langle x, y \rangle$ . However, in set theory, this pair is standardly taken to be  $\{\{x\}, \{x, y\}\}$ . So when we want to solve a system involving pairs, we add more variables to  $X$  and more equations. Here we would take  $X' = \{x, y, z_0, z_1, z_2\}$  and

$$\begin{aligned} e(x) &= \{z_0\} \\ e(y) &= \{x, y, \emptyset\} \\ e(z_0) &= \{z_1, z_2\} \\ e(z_1) &= \{x\} \\ e(z_2) &= \{x, y\} \end{aligned}$$

A solution to the expanded system gives us a solution to the original system. This would be a map  $s$  defined on  $X$  with the property that  $s(x) = \{\langle s(x), s(y) \rangle\}$  and  $s(y) = \{s(x), s(y), \emptyset\}$ .

**Theorem 3.8** *Every extensional structure  $\mathfrak{A}$  is isomorphically embedded in a structurally saturated extensional structure  $\mathfrak{B}$ .*

PROOF: Our proof here uses AFA. With each structural relation  $S$  of  $\mathfrak{A}$ , we associate some urelement  $z_s$ . We also may associate an urelement  $z_a$  with every atom of  $\mathfrak{A}$ . We assume that the association is one-to-one, and also that none of these  $z$ 's occurs in the support of  $\mathfrak{A}$ . Let  $U$  be the collection of all these new urelements.

For each structural type  $T$  we associate a system of equations  $\mathcal{E}(T)$  as follows: The set  $X$  of parameters of  $T$  serves as the set of variables for  $T$ . For each  $x \in X$ ,  $e_x$  is  $T_x$ , with each structural relation  $S$  replaced by the corresponding  $z_s$ . Each  $\mathcal{E}(T)$  is a system of equations involving tuples, something close to what we have seen above.

By AFA, each  $\mathcal{E}(T)$  has a unique instantiation, say  $s(T)$ . Let

$$B = \{s(T)_x : T \text{ a structural type and } x \text{ is a parameter of } T\}.$$

We'll see how to interpret the relations of  $\mathfrak{A}$  on this set to get the desired structure  $\mathfrak{B}$ . But first, we need a subsidiary fact on structural types.

Recall that each  $a \in A$  satisfies a type  $T_a$  in one variable  $x$ . Let  $i : A \rightarrow B$  be given by  $i(a) = s(T_a)_x$  for  $a$  a non-atom of  $\mathfrak{A}$ , and  $i(a) = z_a$  if  $a$  is an atom. Then  $i$  is one-to-one: if  $s(T_a)_x = s(T_b)_x$ , then we could get a bisimulation relation on  $\mathfrak{A}$  relating  $a$  and  $b$ . Since  $\mathfrak{A}$  is extensional, we would have  $a = b$ . And if  $a$  is a non-atom, then  $i(a)$  is a set, so it cannot equal  $i(b) = z_b$  for  $b$  a non-atom.

Now we can see how to make  $B$  into a structure  $\mathfrak{B}$  of the same type as  $\mathfrak{A}$ . Let  $S$  be a structural relation. We interpret it by

$$S_{\mathfrak{B}}(c_1, \dots, c_n, c) \text{ iff } \langle z_S, c_1, \dots, c_n \rangle \in c.$$

The definition of the principal types  $T_a$  implies that

$$S_{\mathfrak{A}}(b_1, \dots, b_k, a) \text{ iff } S_{\mathfrak{A}}(i(b_1), \dots, i(b_k), i(a)).$$

The non-structural relations extend to the image of  $\mathfrak{A}$  via the isomorphism; we'll say that on tuples outside the image of  $i$ , each non-structural relation is false.

At this point we have  $\mathfrak{B}$ . This structure is extensional by AFA. A bisimulation of structures relating  $c$  and  $c'$  would give us a bisimulation of *sets* doing the same thing. And according to AFA, bisimilar sets are equal.

Finally, we sketch a proof that  $\mathfrak{B}$  is structurally saturated. Before we do this, we need to recall that for each  $c \in B$  there is a structural type  $T_c$  over the original  $\mathfrak{A}$  and a parameter  $x$  of  $T$  such that  $s(T)_x = c$ . Let  $T'$  be a structural type in  $\mathfrak{B}$ . Now each  $T_y$  may contain elements  $c \in |\mathfrak{B}|$ . However, we may simply replace each such  $c$  by a new urelement (say  $z_c$ ) and then adjoin all of the structural types  $T_c$  to  $T$ . Of course, we must use new parameters to do this. In this way,  $T'$  is equivalent to a structural type in the original  $\mathfrak{A}$ . So by definition of  $\mathfrak{B}$ , there is some  $a \in |\mathfrak{B}|$  instantiating  $T'$ . QED

**Corollary 3.9** *Every extensional simple information structure can be extended to an extensional structure in which axiom A10.4 holds.*

### 3.3 Partiality and order

We have seen how infons may be ordered according by their arguments and relations, enabling us to model the partiality of information. In fact, the definition of  $\sqsubseteq$  is quite general.

**Definition 3.10** Given an extensional structure  $\mathfrak{A} : [A, S_1, \dots, S_m; R_1, \dots, R_n]$ , we define a relation  $\sqsubseteq$  on the non-atomic elements of  $A$  as follows:  $a \sqsubseteq b$  if for  $1 \leq i \leq m$ , and each sequence  $\vec{x}$  of appropriate length, if  $S_i(\vec{x}, a)$  then  $S_i(\vec{x}, b)$ .

For information structures, the definition of  $\sqsubseteq$  agrees with our previous definition. For set structures,

$$a \sqsubseteq b \text{ iff } a^* \subseteq b^*$$

and for function structures, it is the usual order on partial functions:

$$f \sqsubseteq g \text{ iff each } i \text{ in the domain of } f^* \text{ belongs to the domain of } g^* \text{ and } f^*(i) = g^*(i).$$

In these examples, and in the general case,  $\sqsubseteq$  is a partial order on structured (non-atomic) objects. It is clearly reflexive and transitive, and anti-symmetry follows from extensionality.<sup>3</sup>

Other ordering of structured objects are possible and have been studied extensively. The most important of these is the ‘hereditary’ order, which takes into account the order of components. Roughly, an object  $a$  is a hereditary part of  $b$  if for every component  $a'$  of  $a$  there is a component  $b'$  of  $b$  such that either  $a' = b'$  or  $a'$  is a hereditary component of  $b'$ . Stated thus, the definition is circular. In some cases we can rectify this problem using an inductive layering of the domain. Say that an object is of order 0 if it is atomic, and of order  $n+1$  if it has a component of order  $n$  but no components of order greater than  $n$ . This enables us to give a recursive definition of the hereditary order on objects of order  $n$  in terms of the hereditary order of objects of order less than  $n$ .

This strategy works for well-founded structures, in which every object has an order. But in structures with a ‘circular’ component relation there are objects which do not have an order. This is a ubiquitous problem in the study of circular phenomena. The solution is as follows. First, we regard the ‘definition’ of the order as an operation on binary relations. Given one binary relation  $R$ , we let  $R^+$  be defined by

$$R^+(a, b) \text{ if for every component } a' \text{ of } a \text{ there is a component } b' \text{ of } b \text{ such that either } a' = b' \text{ or } R(a', b').$$

Now, the operation taking  $R$  to  $R^+$  is *monotone*: if  $R \subseteq S$  then  $R^+ \subseteq S^+$ . A basic result of the theory of order is that every monotone operation has a *greatest fixed point*. In other words, there is an  $R$  such that  $R^+ = R$  and for any  $S$  such that  $S^+ = S$ ,  $S \subseteq R$ .<sup>4</sup> This observation enables to define the hereditary order in the general case as follows.

**Definition 3.11** The *hereditary order*,  $\sqsubseteq^h$ , on  $\mathfrak{A} : [A, S_1, \dots, S_m; R_1, \dots, R_n]$  is the largest binary relation between non-atomic objects of  $\mathfrak{A}$  such that if  $a \sqsubseteq^h b$  then for  $1 \leq i \leq m$ , if  $S_i(x_1, \dots, x_k, a)$  then there are  $y_1, \dots, y_k$  such that  $S_i(y_1, \dots, y_k, b)$  and  $x_j \sqsubseteq^h y_j$  for each  $1 \leq j \leq k$ .

<sup>3</sup>If  $a \sqsubseteq b$  and  $b \sqsubseteq a$  then the union of  $\{\langle a, b \rangle, \langle b, a \rangle\}$  and the identity relation is a bisimulation of  $\mathfrak{A}$ , and so  $a = b$ .

<sup>4</sup>This observation generalizes to any preorder (reflexive and transitive relation). If  $<$  is a preorder and  $F$  is a monotone operation on  $<$  (if  $x < y$  then  $F(x) < F(y)$ ) then there is an  $x$  such that  $F(x) = x$  and for all  $y$  if  $F(y) = y$  then  $y < x$ .

The hereditary order is also a partial order on non-atomic objects for the same reason that  $\sqsubseteq$  is. Other orders may be obtained by defining different monotone operations.

It is important to realize that both  $\sqsubseteq$  and  $\sqsubseteq^h$  are purely structural concepts. In the case of simple infons, if  $\sigma \sqsubseteq \tau$  then it is possible to go further and interpret the order in terms of epistemological advantage: knowing  $\sigma$  is not as valuable as knowing  $\tau$ . However, this gloss is not available in the general case. For example, in Section 3.4 we shall show how to obtain a disjunction  $\sigma_1 \vee \sigma_2$  whose components are  $\sigma_1$  and  $\sigma_2$ , and such that  $\sigma_1 \vee \sigma_2 \sqsubseteq \sigma_1 \vee \sigma_2 \vee \sigma_3$ .

The discussion of unification in information structures may also be generalized, although no uniform definition of compatibility is available. Instead, we may introduce the following abstract version.

**Definition 3.12** A relational structure

$$\mathfrak{A} : [A, S_1, \dots, S_m; R_1, \dots, R_n, C^2, P^2]$$

is a *unification structure* with order  $P$  and *compatibility relation*  $C$  if

1.  $\mathfrak{A}$  is extensional,
2.  $C$  is symmetric and reflexive on non-atoms,
3.  $P$  is a partial order of non-atoms, and
4. If  $C(a, b)$  then  $a$  and  $b$  have a least upper bound in  $P$ .

For example, taking  $C(\sigma, \tau)$  iff  $\sigma$  and  $\tau$  are compatible simple infons, and  $P(\sigma, \tau)$  iff  $\sigma \sqsubseteq \tau$ , we see that generalized information structures are unification structures; for set structures, take  $P$  to be  $\sqsubseteq$  and  $C$  to be the universal relation on  $\text{Set}^*$ —the unification of any two sets is just their union; and for function structures with  $P$  taken to be the  $\sqsubseteq$  order, let  $C(f, g)$  iff there is no  $i$  in the domains of both  $f^*$  and  $g^*$  and such that  $f^*(i) \neq g^*(i)$ . The hereditary orders may also be used to define unification structures with weaker, hereditary compatibility relations.

### 3.4 Complex Infons

We have seen that infons, sets and functions may all be modelled using extensional structures. In each of these cases, the identity conditions are relatively straightforward. Now we shall show how structured objects with more complicated identity conditions can also be modelled.

Suppose we wish to have a binary operation of conjunction with the following properties:

$$(\text{commutativity}) \quad \sigma \wedge \tau = \tau \wedge \sigma,$$



(associativity)  $\sigma \wedge (\tau_1 \wedge \tau_2) = (\sigma \wedge \tau_1) \wedge \tau_2$ , and

(idempotence)  $\sigma \wedge \sigma = \sigma$ .

We shall suppose that we already have a stock of infons  $\Sigma$  in some information structure  $\mathfrak{A}$  from which to form conjunctions. Our task is to extend  $\mathfrak{A}$  with new elements and relations designed to capture the structure of the conjunctions.

The solution is to define a new binary structural relation **ConjunctOf** which holds between conjuncts and conjunction. Using this relation we can define the operation of conjunction as follows:

given infons  $\sigma$  and  $\tau$ , the conjunction  $\sigma \wedge \tau$  is the least  $x$  in the  $\sqsubseteq$  order such that **ConjunctOf**( $\sigma, x$ ) and **ConjunctOf**( $\tau, x$ ).

In other words,  $\sigma \wedge \tau$  is the structurally smallest conjunction containing  $\sigma$  and  $\tau$  as conjuncts. Clearly any such conjunction is commutative. Additional properties of  $\wedge$  may be imposed by careful construction of **ConjunctOf**. By way of example, we shall show that it is possible to extend an information structure with a conjunction which is associative and idempotent, and another which has neither of these properties.

**Construction 3.13** Let  $\mathfrak{A}$  be an extensional structure with a class  $\Sigma$  of non-atomic objects, which we shall call ‘infons’. We shall define an extensional structure  $\mathfrak{B}$  extending  $\mathfrak{A}$  with new elements and a new binary structural relation **ConjunctOf** such that the conjunction  $\wedge$  defined, as above, on objects in  $\Sigma$  is both associative and idempotent. Let

$$\begin{aligned} B_0 &= 0 \times (|\mathfrak{A}| - \Sigma) \\ B_1 &= 1 \times \text{pow}(\Sigma) \end{aligned}$$

and for each  $a \in |\mathfrak{A}|$  let

$$a^* = \begin{cases} \langle 0, a \rangle & \text{if } a \notin \Sigma \\ \langle 1, \{a\} \rangle & \text{if } a \in \Sigma \end{cases}.$$

Now let  $\mathfrak{B}$  be the structure with universe  $B_0 \cup B_1$ , and with

$$R(a_1^*, \dots, a_n^*) \text{ in } \mathfrak{B} \text{ iff } R(a_1, \dots, a_n) \text{ in } \mathfrak{A}$$

for each primitive relation  $R$  of  $\mathfrak{A}$  and

$$\text{ConjunctOf}(\langle 1, x \rangle, \langle 1, y \rangle) \text{ iff } x \subseteq y.$$

The new relation **ConjunctOf** is treated as a structural relation of  $\mathfrak{B}$  so we must check that  $\mathfrak{B}$  is extensional.<sup>5</sup> The binary conjunction  $\wedge$  is defined, as above, for all ‘infons’, which in this case refers to those objects in  $B_1$ . Now for any  $x, y \subseteq \Sigma$ , it is easily checked that  $\langle 1, x \rangle \wedge \langle 1, y \rangle = \langle 1, x \cup y \rangle$ . Thus conjunction is both idempotent and associative in  $\mathfrak{B}$ .

<sup>5</sup>If  $a$  is bisimilar to  $b$  then  $a = \langle i, a_0 \rangle$ ,  $b = \langle j, b_0 \rangle$  for some  $a_0$  and  $b_0$ . First observe that  $i = j$ . For if  $i \neq j$ , say  $i = 0$  and  $j = 1$ , then  $b_0$  is a subset of  $\Sigma$ , and so **ConjunctOf**( $\langle 1, \emptyset \rangle, b$ )

**Construction 3.14** For the second construction, let  $\mathfrak{A}$  and  $\Sigma$  be as before. Let  $\Sigma_1 = \{1\} \times \text{pow}(\{0\} \times \Sigma)$ ,  $\Sigma_{(n+1)} = \Sigma_n \cup \{1\} \times \text{pow} \Sigma_n$ , and  $\Sigma^* = \bigcup_{n=1}^{\omega} \Sigma_n$ . Now let  $\mathfrak{B}$  be the structure with universe  $\Sigma^* \cup \{0\} \times |\mathfrak{A}|$ , and with

$$R(\langle 0, a_1 \rangle, \dots, \langle 0, a_n \rangle) \text{ in } \mathfrak{B} \text{ iff } R(a_1, \dots, a_n) \text{ in } \mathfrak{A}$$

for each primitive relation  $R$  of  $\mathfrak{A}$  and

$$\text{ConjunctOf}(x, \langle 1, y \rangle) \text{ iff } x \in y$$

Again we must check that  $\mathfrak{B}$  is extensional.<sup>6</sup> The ‘infons’ of  $\mathfrak{B}$  are those objects of non-zero rank together with those objects of the form  $\langle 0, x \rangle$  for which  $x \in \Sigma$ . It is easy to check that  $a \wedge b = \langle 1, \{a, b\} \rangle$  and so conjunction is neither idempotent nor associative.

Both constructions give a commutative conjunction. In fact, commutativity follows from the definition of  $\wedge$  from  $\text{ConjunctOf}$ . To get a non-commutative conjunction, we would need use different structural relations. For a non-associative, non-commutative conjunction we could use two binary structural relations, say,  $\text{ConjunctOf}_1$  and  $\text{ConjunctOf}_2$ , with  $\wedge$  defined such that  $\text{ConjunctOf}_1(\sigma, \sigma \wedge \tau)$  and  $\text{ConjunctOf}_2(\tau, \sigma \wedge \tau)$ . For an associative, non-commutative conjunction, we could use a ternary structural relation  $\text{ConjunctOf}_*$ , with integer roles, and  $\wedge$  defined so that

$$\text{ConjunctOf}_*(i, \sigma_i, (\sigma_1 \wedge \dots \wedge \sigma_n)).$$

The goal in defining a structural notion of conjunction is to capture those properties of conjunctive information which are immediately available to anyone

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whereas there is no  $x$  such that  $\text{ConjunctOf}(x, a)$ , and this contradicts the assumption that  $a$  is bisimilar to  $b$ . In the case that  $i = j = 0$ ,  $a_0$  is bisimilar to  $b_0$  in  $\mathfrak{A}$ , so  $a_0 = b_0$  by the extensionality of  $\mathfrak{A}$ , and so  $a = b$ . And if  $i = j = 1$  then for every  $x \in a_0$ , there is a  $b_1 \subseteq b_0$  such that  $\langle 1, \{x\} \rangle$  is bisimilar to  $\langle 1, b_1 \rangle$ . Then for each  $y \in b_1$ ,  $\langle 1, \{y\} \rangle$  is bisimilar to  $\langle 1, \{x\} \rangle$ —it cannot be bisimilar to  $\langle 1, \emptyset \rangle$ , the only other conjunct of  $\langle 1, \{x\} \rangle$ . But then either  $x$  and  $y$  are atoms of  $\mathfrak{A}$ , which is ruled out because  $x, y \in \Sigma$ , or  $x$  is bisimilar to  $y$  in  $\mathfrak{A}$  and so  $x = y$  by the extensionality of  $\mathfrak{A}$ . Thus  $b_1 = \{x\}$  and so  $a_0 \subseteq b_0$ . By a symmetrical argument, it can be shown that  $b_0 \subseteq a_0$ , so  $a_0 = b_0$  and so  $a = b$ .

<sup>6</sup>The proof requires the notion of the ‘rank’ of an element of  $|\mathfrak{B}|$ . An element  $b \in |\mathfrak{B}|$  is of rank 0 if  $b = \langle 0, a \rangle$  for some  $a \in |\mathfrak{A}|$  and is of rank  $n > 0$  if  $b \in \Sigma_n$  but  $b \notin \Sigma_m$  for each  $m < n$ . We show that if  $a$  is bisimilar to  $b$  then  $a = b$ , by induction on the maximum of the ranks of  $a$  and  $b$ . Suppose that  $a$  is bisimilar to  $b$ . There are three cases: (i) both  $x$  and  $y$  have rank 0; (ii) one of  $x$  or  $y$  has rank 0 and the other has rank greater than 0; and (iii) both  $x$  and  $y$  have ranks greater than 0. Case (i) is the base of the induction. In this case,  $a = \langle 0, x \rangle$  and  $b = \langle 0, y \rangle$  for some  $x$  and  $y$ , so  $x$  is bisimilar to  $y$  in  $\mathfrak{A}$ , so  $x = y$  by the extensionality of  $\mathfrak{A}$ , and so  $a = b$ . Case (ii) is impossible. Suppose, for example, that  $a = \langle 0, x \rangle$  and  $b = \langle 1, y \rangle$ . Then either  $y$  is empty, making  $b$  an atom of  $\mathfrak{B}$  which is only bisimilar to itself, or  $y$  has an element  $y'$  and so  $\text{ConjunctOf}(y', b)$ , whereas there is no  $x'$  such that  $\text{ConjunctOf}(x', a)$ , contradicting the assumption that  $a$  is bisimilar to  $b$ . Case (iii) is the inductive case. We may suppose that  $a = \langle 1, x \rangle$  and  $b = \langle 1, y \rangle$  for some  $x$  and  $y$ . Then for each  $x' \in x$  there is a  $y' \in y$  such that  $x'$  is bisimilar to  $y'$ . But the ranks of  $x'$  and  $y'$  are strictly less than those of  $x$  and  $y$  respectively, and so by the inductive hypothesis,  $x' = y'$ . Thus  $x \subseteq y$ , and by a similar argument  $y \subseteq x$ , so  $x = y$ , and so  $a = b$ .

receiving the information, without further reflection on the significance of the information itself. For this reason, it may be that conjunctive information represented in different forms may be best modelled by different conjunctions. For example, information about the relative heights of Indiana's Hoosiers may be conveyed in a list of sentences of the form ' $x$  is taller than  $y$ ', or by a team photograph. In the former case, the information conveyed may be best represented by a non-commutative, associative conjunction; in the latter case, a commutative, associative conjunction would be preferable.

Similar issues affect the choice of structural relations for modelling other complex infons, such as disjunctions and conditionals. Negations may be modelled as complex infons, or as basic infons, as indicated in Section 2.4. For disjunctions, constructions almost identical to Constructions 3.13 and 3.14 may be used. We simply re-name 'ConjunctOf' as 'DisjunctOf' and ' $\wedge$ ' as ' $\vee$ ' throughout. This further underlines the point that the issues pertaining to the modelling of complex infons are structural, not logical. Definitions of several kinds of structures having complex infons are given in Section 3.6, below.

The question of how to represent quantifiers is even more vexed (Robin Cooper (1991) and Richard Cooper (1991)). One approach is to use infinitary conjunctions and disjunctions. In both structures constructed above we may define an infinitary conjunction as follows:

given a set  $X$  of infons, the conjunction  $\bigwedge X$  is the least  $x$  in the  $\sqsubseteq$  order such that  $\text{ConjunctOf}(\sigma, x)$  for each  $\sigma \in X$ .

This operation has similar properties to the corresponding binary conjunction. In the structure defined in Construction 3.13 it is both associative and idempotent, but in the structure defined in Construction 3.14 it is neither. Infinitary conjunctions are always commutative.

Another approach is to model a quantified infon as a pair  $\langle Q, \lambda x.\sigma \rangle$ , in which  $Q$  is a quantifier and  $\lambda x.\sigma$  is a 'infon-abstract'. We shall discuss the question of how to model abstracts in Section 3.7.

### 3.5 Substitution

In any extensional structure there is a well-defined operation of substitution.

**Definition 3.15** Let  $\mathfrak{A} : [A, S_1, \dots, S_m; R_1, \dots, R_n]$ . A function  $f$  is a *substitution* in  $\mathfrak{A}$  if its arguments and values are all elements of  $A$ . Given a substitution  $f$ , a binary relation  $E$  on  $A$  is an  *$f$ -simulation* if for all  $a, b \in A$ , if  $E(a, b)$  then  $b = f(a)$  if  $a$  is in the domain of  $f$ , and if  $a$  is not in the domain of  $f$  then

1. if  $a$  is atomic then  $a = b$
2. for  $1 \leq i \leq m$ , if  $S_i$  is of arity  $k$  then for all  $x_1, \dots, x_k$  such that  $S_i(x_1, \dots, x_k, a)$ ,

there are  $y_1, \dots, y_k$  such that  $S_i(y_1, \dots, y_k, b)$ , and  $E(x_j, y_j)$  for  $1 \leq j \leq k$ , and

3. for  $1 \leq i \leq m$ , if  $S_i$  is of arity  $k$  then for all  $y_1, \dots, y_k$  such that  $S_i(y_1, \dots, y_k, b)$ , there are  $x_1, \dots, x_k$  such that  $S_i(x_1, \dots, x_k, a)$ , and  $E(x_j, y_j)$  for  $1 \leq j \leq k$ .

$a$  is *f-similar* to  $b$  in  $\mathfrak{A}$  if there is an  $f$ -simulation  $E$  of  $\mathfrak{A}$  such that  $E(a, b)$ . For a given  $f$  and  $a$ , there need not be an element of  $\mathfrak{A}$  that is  $f$ -similar to  $a$ , but if there is one then there is only one, by the extensionality of  $\mathfrak{A}$ . We write  $f.a$  for the unique element  $f$ -similar to  $a$ , should it exist. An extensional structure is a *substitution structure* if  $f.a$  exists for every  $f$  and  $a$ .

For example, suppose that there are solutions to the equations

$$\begin{aligned} v &= \langle\langle r; a, b \rangle\rangle \\ w &= \langle\langle r; b, b \rangle\rangle \\ x &= \langle\langle r; y, a \rangle\rangle \\ y &= \langle\langle r; x, b \rangle\rangle \\ z &= \langle\langle r; z, b \rangle\rangle \end{aligned}$$

in an extensional information structure  $\mathfrak{A}$ . Let  $[a \mapsto b]$  be the function with domain  $\{a\}$  such that  $[a \mapsto b](a) = b$ , and let  $E$  be the relation with graph

$$\{\langle v, w \rangle, \langle x, z \rangle, \langle y, z \rangle, \langle a, b \rangle\} \cup I_{\{b, r, z\}},$$

where  $I_{\{b, r, z\}}$  is the identity relation on  $\{b, r, z\}$ . It is easy to check that  $E$  is an  $[a \mapsto b]$ -simulation on  $\mathfrak{A}$ , and so  $[a \mapsto b].v = w$  and  $[a \mapsto b].x = [a \mapsto b].y = z$ .

Set structures are substitution structures, by the set-theoretic axiom of Replacement, together with Foundation or Anti-foundation, as the case may be; but many of the structures in which we are interested are not. For example, any simple information structure satisfying axioms A7 (finite roles) and A10.3 (extensionality) is a substitution structure if and only if it satisfies the generality principle given in axiom A11.1. No room is made for ‘legitimate’ grounds for the non-existence of  $f.a$ , such as the sortal restrictions considered in Section 2.7. We would like to generalize the principle of sortal generality, given by axiom A11.2, but it refers to specific features of the structure of simple infons (their roles), and so cannot be generalized to arbitrary extensional structures. Instead, we rest content with the fact that all extensional structures are at least *partial* substitution structures, and interpret the non-existence of  $f.a$  as the inappropriateness of the substitution  $f$  for  $a$ .<sup>7</sup>

We next want to prove a result that states that every extensional structure may be embedded in a substitution structure.

<sup>7</sup>An interesting, weaker, substitutional property of extensional structures is as follows: if  $f.a$  and  $g.a$  both exist and  $f$  and  $g$  are compatible functions then  $(f \sqcup g).a$  also exists. This is implied by sortal generality if the domains of  $f$  and  $g$  do not contain roles.

**Proposition 3.16** *Let  $\mathfrak{A} : [A, S_1, \dots, S_m; R_1, \dots, R_n]$  be a structurally saturated extensional structure. then  $\mathfrak{A}$  is a substitution structure.*

PROOF: Let  $f$  be a substitution in  $\mathfrak{A}$ . Let  $X = \{x_a : a \text{ a non-atom of } \mathfrak{A}\}$ . We consider a structural type  $T = \{T(x)\}_{x \in X}$ , using  $x$  as the set of parameters.

For a non-atom  $a$ ,  $T(x_a)$  is obtained from the canonical type  $T_a$  as follows: For each  $\langle S, b_1, \dots, b_k \rangle$  which holds in  $\mathfrak{A}$ , let  $y_i = f(b_i)$  if  $b_i$  is in the domain of  $f$   $y_i = x_{b_i}$  if  $b_i$  is not in the domain of  $f$ . We put the corresponding tuple  $\langle S, y_1, \dots, y_k \rangle$  into  $T(x_a)$ . This defines the structural type  $T$ .

Since  $\mathfrak{A}$  is structurally saturated, let  $s$  instantiate  $T$  in  $\mathfrak{A}$ . For each  $a \in \mathfrak{A}$ , the construction has arranged that each  $s(x_a)$  is  $f$ -similar to  $a$ . This proves that  $f.a$  exists. QED

As an immediate corollary of Proposition 3.16 and Theorem 3.8, we see that every extensional  $\mathfrak{A}$  may be embedded in an extensional substitution structure.

Our notion of a substitution structure is inspired by the work of Aczel (1990) and Aczel and Lunnon (1991). It might be noted that their work was motivated by the foundational needs of situation theory. At the same time, the methods seem applicable to other areas.

### 3.6 Infon Structures

We are now in a position to improve our model of infons in a way that allows for the smooth incorporation of complex infons and a uniform treatment of abstraction (to be given in Section 3.7). An advantage of the models used in Section 2—the ‘simple information structures’—is their simplicity, with only two structural relations. However, there are disadvantages. First, because ‘infon’, as applied to these structures, is a defined term that applies only to simple infons, the move to structures containing complex infons is tricky. Second, the functions associating objects to roles in infons are not internal objects. This complicates the treatment of substitution and abstraction.

aaieeee — list hack

Now that we know how to represent the structure of functions (in function structures) in the framework of the theory of structural relations, we can solve this problem. The basic idea is to represent the infon  $\sigma = \langle\langle r; i : a, j : b, k : c \rangle\rangle$  by as a pair consisting of the relation  $r$  and an internal function  $\alpha$  with domain  $\{i, j, k\}$  and such that  $\alpha^*(i) = a$ ,  $\alpha^*(j) = b$ , and  $\alpha^*(k) = c$ . The function  $\alpha$  is called the *assignment* of  $\sigma$ . We adapt the functional notation for assignments and write

$$\alpha = [i : a, j : b, k : c] \text{ and } \sigma = \langle\langle r; \alpha \rangle\rangle.$$

The structure of infons is now represented using three structural relations,  $\text{Rel}^2$ ,  $\text{Ass}^2$  and  $\text{Inf}^1$ , and one non-structural relation,  $\text{Approp}^1$ .  $\text{Inf}(\sigma)$  iff  $\sigma$  is an infon;  $\text{Rel}(r, \sigma)$  and  $\text{Ass}(\alpha, \sigma)$  iff  $\sigma = \langle\langle r; \alpha \rangle\rangle$ . An assignment  $\alpha = [i : a, j : b, k : c]$  is represented as an internal function, so that  $\text{Fun}(\alpha)$  and  $\text{App}(i, a, \alpha)$ , and  $\alpha$  is appropriate iff  $\text{Approp}(\alpha)$ .

**Definition 3.17** A function structure of type  $[\mathfrak{A}, \text{Ass}^2, \text{Rel}^2, \text{Inf}^1; \text{Approp}^1]$  is an *infon structure* if it satisfies the following conditions:

1. (sorts)  $\text{Ass}^* \subseteq \text{Inf}^*$ ,  $\text{Rel}^* \subseteq \text{Inf}^*$ , and  $\text{Fun}^* \cap \text{Inf}^* = \emptyset$ ,
2. (basic infons) if  $\text{Rel}(r, \sigma)$  then  $\exists \alpha \text{ Ass}(\alpha, \sigma)$ ,
3. (appropriateness)  $\text{Approp}(\alpha)$  iff  $\exists \sigma \text{ Ass}(\alpha, \sigma)$ , and
4. (substitution) if  $\text{Ass}(\alpha, \sigma)$  and if  $f$  is substitution such that  $f.\alpha$  exists and  $\text{Approp}(f.\alpha)$ , then  $f.\sigma$  also exists.

An infon  $\sigma$  is *basic* if there is an  $\alpha$  such that  $\text{Ass}(\alpha, \sigma)$ . An infon structure is a *basic-infon structure* if every infon is basic.

Clause 3 (appropriateness) links the appropriateness of an assignment to the existence of an infon with that assignment. If, for example,  $a$  is not an appropriate filler for the role  $i$  then there is no  $\alpha$  in  $\text{Fun}^*$  such that  $\text{Approp}(\alpha)$  and  $\alpha^*(i) = a$ . This is a generalization of the treatment of appropriateness in information structures. Clause 4 (substitution) ensures that the only issue governing substitution of infons is the appropriateness of the resulting assignment. This is an abstract form of the various principles of generality discussed in Section 2.7. To go further, we must look at the unification structure of appropriate assignments.

**Definition 3.18** A unification structure is an *infon unification structure* if it is an infon structure and

1.  $P(x, y)$  iff  $x \sqsubseteq y$ ,
2. (function compatibility) if  $\text{Fun}(\alpha_1)$  and  $\text{Fun}(\alpha_2)$ , then  $C(\alpha_1, \alpha_2)$  iff for every  $i$ ,  $a$  and  $b$ , if  $\text{App}(i, a, \alpha_1)$  and  $\text{App}(i, b, \alpha_2)$  then  $a = b$ ,
3. (infon compatibility) Suppose that  $\text{Ass}(\alpha_1, \sigma_1)$ ,  $\text{Ass}(\alpha_2, \sigma_2)$ ,  $\text{Rel}(r_1, \sigma_1)$ , and  $\text{Rel}(r_1, \sigma_1)$ . Then  $C(\sigma_1, \sigma_2)$  iff  $r_1 = r_2$  and  $C(\alpha_1, \alpha_2)$ , and
4. (unification of appropriate assignments) if  $C(\alpha_1, \alpha_2)$ ,  $\text{Approp}(\alpha_1)$  and  $\text{Approp}(\alpha_2)$ , then  $\text{Approp}(\alpha_1 \sqcup \alpha_2)$ .<sup>8</sup>

Infon unification structures give us our best theory of appropriateness. Part 4 ensures that the appropriateness of assignments depends only on ‘local’ issues: if  $\alpha_1$  and  $\alpha_2$  are compatible appropriate assignments, then so is  $\alpha_1 \sqcup \alpha_2$ ; no additional grounds for inappropriateness can be introduced. Moreover, by clause 4 (substitution) of the definition of infon structures, substitutions which result in appropriate assignments may always be used to obtain new infons. For example, if

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<sup>8</sup>In effect, both  $\mathfrak{A}$  and the structure in which  $\text{Fun}^*$  is restricted to appropriate assignments are required to be function unification structures.

$\sigma = \langle\langle r; \alpha \rangle\rangle$  and  $\alpha = [i : a, j : b, k : c]$ ,

$f$  is the function with domain  $\{a, b\}$  such that  $f(a) = x$  and  $f(b) = y$ , and

$g$  is the function with domain  $\{b, c\}$  such that  $g(b) = y$  and  $g(c) = z$ ,

and both  $f.\alpha$  and  $g.\alpha$  exist and are appropriate—that is,

$$\text{Approp}([i : x, j : y, k : c]) \quad \text{and} \quad \text{Approp}([i : a, j : y, k : z]),$$

then not only is  $[i : x, j : y, k : z]$  an assignment (by function unification), and appropriate (by unification of appropriate assignments) but the infon  $f.\sigma = \langle\langle r; i : x, j : y, k : z \rangle\rangle$  exists (by substitution). This is still weaker than the principle of sortal generality, according to which appropriateness is dependent only on restrictions applying to individual roles. In particular, in infon unification structures there may be restrictions on related roles, as countenanced in the example of *next-to* at the end of Section 2.7. If *next-to* has roles  $\text{nt}_1$  and  $\text{nt}_2$ , then we may deem assignments of the form  $[\text{nt}_1 : x, \text{nt}_2 : x]$  inappropriate, without jeopardizing the unification principle.<sup>9</sup>

The second feature of infon structures not present in information structures, is the unary structural relation  $\text{Inf}$ . We say that  $\sigma$  is a *basic infon* if  $\text{Ass}(\alpha, \sigma)$  for some assignment  $\alpha$ —by clause 2 (basic infons) all infons having a relation are required to have an assignment, possibly the empty assignment  $[]$ . In simple information structures, by definition of the term ‘infon’, all infons are basic, but in infon structures we may have non-basic infons. Of course, by extensionality and clause 1 (sorts), if there are no more structural relations in  $\mathfrak{A}$  then there is at most one non-basic infon. If there are more structural relations, such as  $\text{ConjunctOf}$ , then we can add the axiom that  $\text{ConjunctOf}^* \subseteq \text{Inf}^*$  without any conflict with the axioms for infon structures.

This is a good example of how the ‘holistic’ nature of the extensionality property, which quantifies over *all* structural relations, allows a modular approach to the theory of structured objects. Another example is the treatment of polarity, discussed earlier in Section 2.4. We need only add new structural relations to deal with polarity and the extensionality condition takes care of the rest.

**Definition 3.19** An infon structure of type  $[\mathfrak{A}, \text{Pos}^1, \text{Neg}^1; ]$  is *bi-polar* if it satisfies the following conditions:

1.  $\sigma$  is a basic infon iff either  $\text{Pos}(\sigma)$  or  $\text{Neg}(\sigma)$ , and
2. if  $\text{Pos}(\sigma)$  then not  $\text{Neg}(\sigma)$ .

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<sup>9</sup>This would imply that either  $[\text{nt}_1 : x]$  or  $[\text{nt}_2 : x]$  is also inappropriate, which may appear counterintuitive until one realizes that this does not prevent  $[\text{nt}_1 : x, \text{nt}_2 : y]$  from being appropriate when  $x \neq y$ .

Infons in  $\mathbf{Pos}^*$  are *positive* and those in  $\mathbf{Neg}^*$  are *negative*.<sup>10</sup>

Finite conjunctions and disjunctions are introduced effortlessly, as follows:

**Definition 3.20** An infon structure of type  $[\mathfrak{A}, \text{ConjunctOf}^2, \text{DisjunctOf}^2;]$  is  $\wedge\vee$ -closed if it satisfies the following conditions:

1.  $\text{ConjunctOf}^* \subseteq \text{Inf}^*$  and  $\text{DisjunctOf}^* \subseteq \text{Inf}^*$ ,
2. if  $\text{Inf}(\sigma)$  and  $\text{Inf}(\tau)$  then there is a  $\sqsubseteq$ -minimal object  $\sigma \wedge \tau$  such that  $\text{ConjunctOf}(\sigma, \sigma \wedge \tau)$  and  $\text{ConjunctOf}(\tau, \sigma \wedge \tau)$
3. if  $\text{Inf}(\sigma)$  and  $\text{Inf}(\tau)$  then there is a  $\sqsubseteq$ -minimal object  $\sigma \vee \tau$  such that  $\text{DisjunctOf}(\sigma, \sigma \vee \tau)$  and  $\text{DisjunctOf}(\tau, \sigma \vee \tau)$ .

The principles of existence for infinite conjunctions and disjunctions are similar. We would like to say that for each set  $X$  of infons there is a conjunction  $\bigwedge X$  and a disjunction  $\bigvee X$ . In stating these we must be careful because the properties of (external) sets are determined by the metatheory, which—as we have seen—is not fixed. It is a good idea to bring recognize this indeterminacy explicitly by using internal sets instead, leaving open the question of how many internal sets there are. Thus we define:

**Definition 3.21** An infon structure of type

$$[\mathfrak{A}, \text{Set}^1, \in^2, \text{ConjunctOf}^2, \text{DisjunctOf}^2;]$$

is  $\bigwedge\bigvee$ -closed if it is also a set structure, and

1.  $\text{ConjunctOf}^* \subseteq \text{Inf}^*$ ,  $\text{DisjunctOf}^* \subseteq \text{Inf}^*$ , and  $\text{NegOf}^* \subseteq \text{Inf}^*$
2. if  $\text{Set}(X)$  and  $\text{Inf}(\sigma)$  for each  $\sigma$  such that  $\in(\sigma, X)$ , then there is a  $\sqsubseteq$ -minimal object  $\bigwedge X$  such that  $\text{ConjunctOf}(\sigma, \bigwedge X)$  for each  $\sigma$  such that  $\in(\sigma, X)$ ,
3. if  $\text{Set}(X)$  and  $\text{Inf}(\sigma)$  for each  $\sigma$  such that  $\in(\sigma, X)$ , then there is a  $\sqsubseteq$ -minimal object  $\bigvee X$  such that  $\text{DisjunctOf}(\sigma, \bigvee X)$  for each  $\sigma$  such that  $\in(\sigma, X)$ .<sup>11</sup>

Negations may be modelled using a structural relation  $\text{NegOf}^2$  with the following existence condition: if  $\text{Inf}(\sigma)$  then there is a unique object  $\neg\sigma$  such that  $\text{NegOf}(\sigma, \neg\sigma)$ . Another approach is to base our model on a bi-polar infon structure and treat  $\text{NegOf}^2$  as a *non*-structural relation defined as follows:

<sup>10</sup>For *bi-polar infon unification structures* we must modify clause 3 of the definition of infon unification structures to take polarity into account:

3'. if  $\text{Ass}(\alpha_1, \sigma_1)$ ,  $\text{Ass}(\alpha_2, \sigma_2)$ ,  $\text{Rel}(r_1, \sigma_1)$ , and  $\text{Rel}(r_1, \sigma_1)$ , then  $C(\sigma_1, \sigma_2)$  iff  $r_1 = r_2$ ,  $C(\alpha_1, \alpha_2)$ , and  $\text{Pos}(\sigma_1) \leftrightarrow \text{Pos}(\sigma_2)$ .

<sup>11</sup>Every  $\bigwedge\bigvee$ -closed infon structure is also  $\wedge\vee$ -closed because for infons  $\sigma$  and  $\tau$  there is an internal set  $X$  with  $X^* = \{\sigma, \tau\}$  and so  $\sigma \wedge \tau = \bigwedge X$ ; and similarly for disjunction.



**Definition 3.22** A bi-polar infon structure of type  $[\mathfrak{A}; \text{NegOf}]$  is a *de Morgan infon structure* if it is  $\wedge \vee$ -closed and satisfies the following conditions:

1. if  $\text{NegOf}(\sigma, \tau)$  then  $\text{Inf}(\sigma)$  and  $\text{Inf}(\tau)$ ,
2. if  $\text{NegOf}(\sigma, \tau)$  and  $\text{NegOf}(\sigma, \tau')$  then  $\tau = \tau'$ ,
3. if  $\text{NegOf}(\sigma, \tau)$  then  $\text{NegOf}(\tau, \sigma)$ ,
4. if  $\text{Rel}(r, \sigma)$  and  $\text{Arg}(\alpha, \sigma)$  then there is an infon  $\neg\sigma \neq \sigma$  such that  $\text{NegOf}(\sigma, \neg\sigma)$ ,  $\text{Rel}(r, \neg\sigma)$  and  $\text{Arg}(\alpha, \neg\sigma)$ , and
5. if  $\text{Set}(X)$ ,  $\text{Set}(Y)$  and
  - (a) for each  $\sigma \in X^*$  there is a  $\tau \in Y^*$  such that  $\text{NegOf}(\sigma, \tau)$
  - (b) for each  $\tau \in Y^*$  there is a  $\sigma \in X^*$  such that  $\text{NegOf}(\sigma, \tau)$

then  $\text{NegOf}(\bigwedge X, \bigvee Y)$ .<sup>12</sup>

A remaining question is how to distinguish between saturated and unsaturated infons in infon structures. In basic-infon structures, the answer is clear from the discussion in Section 2.8: saturated basic infons are those that are maximal in the  $\sqsubseteq$ -ordering. For complex infons, it is not so easy. In a  $\wedge \vee$ -closed infon structure, if  $\sigma$  and  $\tau$  are distinct basic infon then  $\sigma \sqsubseteq \sigma \wedge \tau$  and so  $\sigma$  is not maximal, no matter how many roles are filled. We decide the matter as follows.

**Definition 3.23** A basic infon  $\sigma$  is *unsaturated* if there is a distinct basic infon  $\tau$  such that  $\sigma \sqsubseteq \tau$ . A non-basic infon is *unsaturated* if it has a component that is unsaturated. An infon is *saturated* if it is not unsaturated.<sup>13</sup>

By now, the reader will have an impression of some of the possibilities for modelling basic and complex infons using the theory of structural relations. Observers of the development of Situation Theory have been somewhat frustrated because of the lack of consensus on which structures—or even on which kinds of structures, obeying which axioms—are to be chosen for the theory (see Barwise 1989). We will see other reasons for diversity in Section 4, but the basic reason is clear from the above: logical and structural properties of infons are separate but interdependent, and striking the right balance is a difficult matter.

<sup>12</sup>No mention is made of polarities here, but from the definition of bi-polar infon structures we know that if  $\sigma$  is a basic infon then it is either positive or negative (but not both), and from clause 4,  $\sigma \neq \neg\sigma$ ; so if  $\sigma$  is positive then  $\neg\sigma$  is negative, and vice versa.

<sup>13</sup>This is not a complete definition. In particular, it does not legislate on the saturation of a non-basic infon that is a component of itself but all of whose other components are saturated. In all such cases we say that the infon is saturated.

### 3.7 Abstraction

Abstraction is a natural operation in semantics. By abstracting Paul from the information  $\sigma$  that Raymond cooked the omelette for Paul, we obtain the property of being someone for whom Raymond has cooked the omelette. By abstracting both Paul and the omelette from  $\sigma$ , we obtain the relation which holds between  $x$  and  $y$  just in case Raymond cooked  $x$  for  $y$ . This relation and the previous property differ from the relations and properties we have considered so far in that they are clearly not atomic. In this section we see how the infon structures can be used to model these complex relations.

There are two parts to abstraction over a structural object: the object after abstraction containing holes where the abstracted object used to be, and something indicating where the holes are so that they may subsequently be filled again—we call these the ‘abstract’ and the ‘pointer’, respectively.<sup>14</sup> The relationship between abstract and pointer may be captured by a binary structural relation **Abs** with  $\text{Abs}(a, x)$  meaning that  $x$  is a pointer to the abstract  $a$ .

**Definition 3.24** An *abstraction structure*  $\mathfrak{A}$  is an extensional structure with structural relation **Abs**<sup>2</sup> such that if  $\text{Abs}(a, x)$  and  $\text{Abs}(a', x)$  then  $a = a'$ . The objects of sort **Abs**<sup>\*</sup> are called *pointers*. If  $\text{Abs}(a, x)$  then  $x$  is said to *point* to the *abstract*  $a$ .

The objects of interest in abstraction structures are the abstracts, but they do not form a structural sort; rather, it is the pointers that are structurally determined. The intention is that each pointer  $x$  of an abstract  $a$  is a hereditary component of  $a$ . We obtain the results of ‘applying’ an abstract  $a$  to some argument  $b$  by substituting  $b$  for a pointer  $x$  in  $a$ . To see how this achieves the desired effect we must consider how abstracts arise by abstraction from other objects in the domain.

Suppose, for example, that we want to abstract  $b$  from the infon  $\sigma = \langle\langle r; i : b, j : c \rangle\rangle$ . We would expect to obtain an abstract  $a$  with exactly one pointer  $x$  such that the result of substituting  $b$  for  $x$  in  $a$  is just  $\sigma$ . Furthermore, the result of substituting  $b'$  for  $x$  in  $a$  should be the infon  $\langle\langle r; i : b', j : c \rangle\rangle$ , if this infon exists. This is captured in the following definition.

**Definition 3.25** Suppose  $\mathfrak{A}$  is an abstraction structure. Given elements  $a$  and  $b$  in  $|\mathfrak{A}|$ , an abstract  $\lambda b.a$  is *the abstraction of  $b$  from  $a$*  if there is an  $x$  such that

1.  $[b \mapsto x].a = \lambda b.a$ ,
2.  $\text{Abs}(\lambda b.a, x)$ , and
3. the only sort of  $x$  is **Abs**<sup>\*</sup>.

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<sup>14</sup>Analogues of pointer in the literature are ‘indeterminates’ and ‘parameters’.

The definite article is justified in the above definition, because if  $\text{Abs}([b \mapsto x].a, x)$  and  $\text{Abs}([b \mapsto y].a, y)$  then, by extensionality,  $x = y$ . For example, if  $\sigma = \langle\langle r; i : b, j : c \rangle\rangle$  and  $\lambda b.\sigma$  exists, then there is an  $x$  such that  $\text{Abs}(\langle\langle r; i : x, j : c \rangle\rangle, x)$ . But suppose there was a  $y$  such that  $\text{Abs}(\langle\langle r; i : y, j : c \rangle\rangle, y)$ . Since this is these are the only structural relations determining  $x$  and  $y$ , they are obviously bisimilar, and so equal. A similar argument shows that

$$\lambda b.\langle\langle r; i : b, j : c \rangle\rangle = \lambda b'.\langle\langle r; i : b', j : c \rangle\rangle,$$

assuming that  $b$  and  $b'$  are not hereditary components of  $r$ ,  $i$ ,  $j$ , or  $c$ . This is the expected principle of  $\alpha$ -equivalence.

Notice also that under similar assumptions the order of abstraction is insignificant. If, for example,  $b$  and  $c$  are both atomic then

$$\lambda c.\lambda b.\langle\langle r; i : b, j : c \rangle\rangle = \lambda b.\lambda c.\langle\langle r; i : b, j : c \rangle\rangle.$$

In both cases, the abstract obtained has two pointers, pointing to the positions previously occupied by  $b$  and  $c$  respectively, but there nothing to distinguish the pointers apart from that, and so no trace of the order in which the abstraction was performed. This motivates a slightly more general notion of abstraction.

**Definition 3.26** Suppose  $\mathfrak{A}$  is an abstraction structure. Given an element  $a$  of  $|\mathfrak{A}|$  and a set  $B$  of elements of  $|\mathfrak{A}|$ , an abstract  $\Lambda B.a$  is *the simultaneous abstraction of  $B$  from  $a$*  if there is a function  $\pi : B \rightarrow |\mathfrak{A}|$  such that

1.  $\pi.a = \Lambda B.a$ ,
2.  $\text{Abs}(\pi.a, \pi(b))$  for each  $b \in B$ , and
3. for each  $b \in B$ , the only sort of  $\pi(b)$  is  $\text{Abs}^*$ .

Despite the previous observation about commuting lambdas, one should be careful to distinguish simultaneous abstraction from successive abstractions. When the elements of  $B$  are structurally dependent, the results obtained may differ (see Aczel and Lunnon (1991), Ruhrberg (1995)). However, it is clear that simultaneous abstraction is a generalization of abstraction because  $\lambda b.a = \Lambda\{b\}.a$ .

We have already observed that abstraction obeys an identity principle analogous to  $\alpha$ -equivalence in the  $\lambda$ -calculus, at least in special cases. To state the general result observe that if  $\sigma = \langle\langle r; b, c \rangle\rangle$  and  $\sigma' = \langle\langle r; b', c \rangle\rangle$  and  $b$  and  $b'$  are structurally independent then  $[b \mapsto b'].\sigma = \sigma$  and  $[b' \mapsto b].\sigma' = \sigma$ . These are the conditions needed to show that the two abstracts are bisimilar and so justify the claim that  $\lambda b.\sigma = \lambda b'.\sigma'$ . We state, without proof, the generalization of this result.

**Theorem 3.27 ( $\alpha$ -identity)** *If  $\Lambda B.a$  and  $\Lambda B'.a'$  exist in an abstraction structure and there is a one-to-one correspondence  $f$  between  $B$  and  $B'$  such that  $f.a = a'$  and  $f^{-1}.a' = a$  then  $\Lambda B.a = \Lambda B'.a'$ .*

We now show that extensional structures with abstraction really exist.

**Definition 3.28** An abstraction structure  $\mathfrak{A}$  is a *Lambda structure* if for every element  $a$  of  $|\mathfrak{A}|$  and a set  $B$  of elements of  $|\mathfrak{A}|$ , the simultaneous abstraction  $\Lambda B.a$  exists.

**Theorem 3.29** *Every extensional structure can be extended to a Lambda structure.*

PROOF: Given an extensional structure  $\mathfrak{A}$ , extend it trivially to an abstraction structure by adding the structural relation  $\text{Abs}^2$  with an empty extension. By Theorem 3.8 this can be extended to a structurally saturated structure  $\mathfrak{B}_0$ . Throw away all pointers in this structure that point to more than one abstract to obtain an abstraction structure  $\mathfrak{B}$ . We show that  $\mathfrak{B}$  contains a Lambda structure extending  $\mathfrak{A}$ . Suppose  $a$  is an element of  $|\mathfrak{B}|$  and  $B$  is a set of elements of  $|\mathfrak{B}|$ . We exhibit a structural type whose satisfier gives the needed element  $\Lambda B.a$ . For each  $b \in B$ , let  $T_b = \{\langle \text{Abs}, a \rangle\}$ . Let  $H$  be the set of hereditary components of  $a$  that are not in  $B$ , and for each  $c \in H$ , let  $T_c$  be the canonical type of  $c$  in  $\mathfrak{B}$ . Now the structural type we want is  $T = \{T_x\}_{x \in B \cup H}$ . Since  $\mathfrak{B}$  is contained in a structurally saturated structure, there is a function  $s$  mapping into  $\mathfrak{B}_0$  that instantiates  $T$ . But, clearly, the range of  $s$  does not contain any of the elements we threw away, so  $s$  maps into  $\mathfrak{B}$ . Thus for each  $b$  in  $B$ ,  $s(b)$  is of type  $s.T_b = \{\langle \text{Abs}, s.a \rangle\}$  and so  $\text{Abs}(s.a, s(b))$ . Moreover,  $s(b)$  is only of sort  $\text{Abs}^*$ , and so  $s.a$  is the required abstract  $\Lambda B.a$ . QED

A short digression on the ontology of abstracts is in order. If  $\sigma$  is an infon with argument  $b$  then  $\lambda b.\sigma$  is also of sort  $\text{Inf}^*$ . But, intuitively, it is a property not an infon. Moreover, it has a pointer as an argument, filling the role left by  $b$ , and so interfering with whatever implicit appropriateness conditions there may have been on this role. It is therefore useful to distinguish between genuine infons and these new entities, which we may call *infon abstracts*. Infon abstracts are just properties (or relations if they have more than one pointer) and so ontologically acceptable. That is not quite the end of the matter. Consider the infon  $\tau = \langle\langle r, \sigma \rangle\rangle$  and the abstract  $\lambda b.\tau$  with pointer  $x$ . The abstract  $\lambda b.\tau$  is an infon abstract, not a genuine infon. But it has an argument  $x.\sigma$  which is also of sort  $\text{Inf}^*$  but neither an infon abstract nor, intuitively, a genuine infon. The only reasonable position to adopt is that  $x.\sigma$  is a *part* of an abstract. Fortunately, it is easily determined which abstract it is a part of, because the pointer  $x$  points to it. In this way, we may satisfy ourselves that no unexpected ontological categories have been created.<sup>15</sup>

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<sup>15</sup>There has been much discussion of the ontological status of ‘parametric objects’, which are closely related to our abstracts and their parts. See, for example, Westerståhl (1990).

### 3.8 Application

In the previous section we saw that an extensional structure may be extended with abstracts of the form  $\Lambda B.a$ , obtained by abstracting the elements of  $B$  from the object  $a$ . The purpose of these abstracts is to model generalizations across a class of structured objects. In particular, infon abstracts may be used to model properties and relations. To see how this works, we must first give an account of application.

**Definition 3.30** Suppose  $\mathfrak{A}$  is an abstraction structure. If  $a$  is an abstract then a function  $f: X \rightarrow |\mathfrak{A}|$  is an (*external*) *assignment for  $a$*  if every  $x$  in  $X$  is a pointer of  $a$ . It is *appropriate for  $a$*  if  $f.a$  exists. If  $f$  is appropriate for  $a$  then *the application of  $a$  to  $f$* , written  $a^*(f)$ , is  $f.a$ . In this way, every abstract  $a$  is associated with a (second-order) function  $a^*$  mapping appropriate assignments to elements of  $|\mathfrak{A}|$ .

**Theorem 3.31** ( $\beta$ -identity) *if  $\Lambda B.a$  exists and  $f$  is an appropriate assignment for  $\Lambda B.a$  then  $(\Lambda B.a)^*(f) = f.a$ .*

This follows directly from the definitions of simultaneous abstraction and application.

With the notion of application in place we can return to the question of how to incorporate complex properties and relations into infon structures. The key point is that the appropriateness conditions of infons having complex relations should be determined by those of basic infons.

**Definition 3.32** Let  $\mathfrak{A}$  be an infon structure that is also an abstraction structure.  $\mathfrak{A}$  has *coherent appropriateness conditions* if for each infon  $\sigma = \langle\langle r; f \rangle\rangle$  in  $|\mathfrak{A}|$ , if  $r$  is an infon abstract then  $f^*$  is an appropriate assignment for  $r$ .  $\mathfrak{A}$  is *relation-closed* if for each infon  $\sigma$  and each set  $B$  of elements of  $\mathfrak{A}$ , the infon abstract  $\Lambda B.\sigma$  exists and for each appropriate assignment  $f$  for  $\Lambda B.\sigma$ , the infon  $\langle\langle \Lambda B.\sigma; f \rangle\rangle$  also exists.

We state without proof the following result:

**Theorem 3.33** *Every infon structure is extendible to a relation-closed infon structure.*

By way of closing this section, we note that application has only been modelled in an ‘external’ way. We may ask what the properties of an ‘internal’ application operation should be. For more on this, see Aczel and Lunnon (1990), and Lunnon (199?).

## 4 Truth and Circumstance

Notoriously, ‘information’ is ambiguous. Traditionally, the possession of information that Smith is an anarchist, like the knowledge of the same, implies that Smith *is* an anarchist. With the advent of mechanisms for the gathering, storing and retrieval of large amounts of information, the word has taken on a more neutral meaning. According to more modern usage, which will be adopted here, the information that Smith is an anarchist may be stored on a computer file in ASCII code, transmitted across the globe, converted to speech and heard over a loud speaker, without Smith’s ever having had a subversive political thought in his life.

A problem with the modern usage is that a new word is required to separate genuine items of information, in the traditional, truth-implying sense, from mere infons. Provisionally, we will use the adjective ‘factual’ for this purpose. Thus, the infon  $\langle\langle \text{anarchist; Smith} \rangle\rangle$  is *factual* just in case Smith is indeed an anarchist; otherwise it is non-factual, but information nonetheless.

Our use of ‘factual’ is provisional because the distinction on which it rests is an important contention in situation theory. No one would deny of the statement that Smith is an anarchist, that it is true if and only if Smith is an anarchist. It is less clear that this biconditional may be used to determine whether the corresponding infon is factual. To see the difficulty, let us suppose that Smith had a rebellious youth, but has now settled down into conformist middle age. An utterance of ‘Smith is an anarchist’ during Smith’s early period would have made a true statement, but an utterance of the same sentence today would not. On the assumption that both utterances express the infon  $\langle\langle \text{anarchist, Smith} \rangle\rangle$ , we arrive at an impasse: which statement do we use to decide whether the infon is factual?

Of course, the problem can be resolved by denying the assumption that both utterances express the same information. We may say that the infons expressed by the two utterances are distinguished by a temporal role which is filled by the time of each utterance. Instead of one infon, we have two:

$\langle\langle \text{anarchist; subject: Smith, time: Friday 15th May, 1970} \rangle\rangle$  and  
 $\langle\langle \text{anarchist; subject: Smith, time: Monday 18th July, 1994} \rangle\rangle$ .

The first infon is presumed to be factual, because of the truth of the statement made in 1970, and the second non-factual, because of the falsity of the statement made today. On this account, which we will style the *hidden parameter* account,  $\langle\langle \text{anarchist; Smith} \rangle\rangle$  is either excused from considerations of factuality because it is unsaturated, or supposed not to exist at all.

A quite different resolution of the difficulty is to take the truth or falsity of a statement to depend on more than just the information expressed. Following

Austin (1961), two components of a statement are distinguished: the situation, event or state of affairs which the statement is about, and what is said about it. On this account, which we will style the *referential* account, the difference between the two statements arises not because of a difference in the information they express, but because different situations are described: the utterance in 1970 correctly describes Smith’s attitudes at that time, whereas today’s utterance falsely describes his present state of bourgeois conformity. The infon  $\langle\langle \text{anarchist}; \text{Smith} \rangle\rangle$  is factual if taken to be about Smith’s beliefs and actions in 1970, but not if it is taken to be about his current views.

Proponents of both accounts agree that the common information expressed by the two utterances is neither factual nor straightforwardly non-factual. They differ on the reason for this failure. According to the hidden-parameter account, the infon  $\langle\langle \text{anarchist}; \text{Smith} \rangle\rangle$ , if it exists at all, is merely the unsaturated common part of the distinct, saturated infons expressed by the two utterances. On the referential account all claims to factuality, if intelligible at all, are relative to the situation the information is about. The infon  $\langle\langle \text{anarchist}; \text{Smith} \rangle\rangle$  is factual if it is taken to be information about the situation described by the first utterance, but not if it is taken to be about the situation described by the second.

On matters of detail, there is room for interplay between the two positions. For example, the referentialist may agree that the two statements express different information by virtue of the indexical nature of tense, but still maintain that the factual status of the information expressed is determined by the described situation. Moreover, the debate may be conducted in different areas. For example, instead of focusing on tense, the reference theorist may claim an implicit relativity to a perspective from which the assessment of Smith’s political views is to be made—it may be Smith’s own, his fellow citizens’, ‘the authorities’, or even that of the person making the statement. The proponent of the hidden-parameter account may go on to insist either that the term ‘anarchist’ is ambiguous, or that the infon expressed has an extra role for the perspective from which the property of being an anarchist is judged.

In the next few sections we shall consider the hidden parameter account in more detail, before returning to the referential account in Section 4.5.

## 4.1 Fact Structures

From one perspective, the hidden parameter account is the more straightforward of the two. The information expressed by an unambiguous statement is modelled as a saturated infon, which may or may not be factual. If it is factual then the statement is true; if not, it is false. Those roles of the infon that are not filled by arguments given explicitly by linguistic features of a statement must be determined by other features of the context in which the statement is made. These additional roles are called ‘hidden parameters’. For example, the temporal ‘parameter’ of the statement that Smith is an anarchist is filled by the

time at which the statement is made. If there are further hidden parameters, such as the perspective from which this assessment is made, then the statement is either ambiguous or the additional role must be filled by some other aspect of the context.

This account can be modelled using a ‘fact structures’, defined as follows.

**Definition 4.1** An infon structure  $\mathfrak{F} = \langle \mathfrak{A} ; \mathbf{Fact} \rangle$  is a *fact structure* if for each  $\sigma \in A$ , if  $\mathbf{Fact}(\sigma)$  then  $\sigma$  is a saturated infon. An infon  $\sigma$  is a *fact* if  $\mathbf{Fact}(\sigma)$ .  $\mathfrak{F}$  is *trivial* if every saturated infon is a fact.

Here is our main example of a fact structure: If  $\mathfrak{M} = \langle M, R_1, \dots, R_n \rangle$  is a relational structure and  $\mathbf{SInf}(\mathfrak{M})$  information structure constructed from it in Construction 2.2, then let  $F(\mathfrak{M})$  be the fact structure extending  $\mathbf{SInf}(\mathfrak{M})$  with

$$\mathbf{Fact}(\langle 3, R_i, \alpha \rangle) \text{ iff in } \mathfrak{M}_0, R_i(\alpha(1), \dots, \alpha(\nu(i)))$$

for each  $i \leq n$  and each  $\alpha: \{1, \dots, \nu_i\} \rightarrow M$ . Every infon in a standard structure is saturated, so there is nothing more to check. We call  $F(\mathfrak{M})$  the *standard* fact structure generated by  $\mathfrak{M}$ .<sup>1</sup>

The main problem with the hidden parameter account is that there are many statements which would not usually be called ambiguous, but for which hidden parameters remain unfilled. These are statements which, when all contextual factors are considered, still only express unsaturated information. These are more common than one might suspect. Suppose that in the course of telling the story of Smith’s involvement in a riot, we say that he panicked and ran. Superficially, the content of the statement is the information that Smith panicked and ran, which might be modelled by the infon

$$\langle\langle \text{panic}; \text{Smith} \rangle\rangle \wedge \langle\langle \text{run}; \text{Smith} \rangle\rangle.$$

This infon is unsaturated because panicking and running are events, and so occur at a particular time, which has not yet been specified. Unlike in the previous example, the tense of the statement (simple past) is not sufficient to determine a saturated content. Unless there are aspects of the circumstances in which the statement is made which somehow provide a way of saturating the infon, it is difficult to see how the truth value of the statement may be determined.

There are various ways in which the missing information may be provided. The date of Smith’s alleged actions may be given explicitly at some earlier point in the narrative (“On the 15th May, 1970, Smith was involved in a riot...”) or

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<sup>1</sup>Unlike the construction of standard basic-infon structures, the construction of standard basic-fact structure is one-one: every relational structure is uniquely represented as a basic-fact structure by the above construction. Conversely, by Theorem 2.9, every fact structure satisfying axioms  $(T_0)$  is isomorphic to a standard fact structure; but in this direction the representation is not unique, because of the arbitrary ordering of relations in a relational structure.



provided by some other means: the date of the riot may be common knowledge, or the narrator may have special access to the facts, perhaps by being witness to the original event.

The hunt for contextual factors that determine the content of statements is ultimately restricted by the lack of precision in ordinary language. A threat to the hidden parameter view is that suggestion that ordinary language is *indefinitely imprecise*: there are always ways of making a given statement more precise, with successive approximations to the ‘missing’ information, and no definite level of precision beyond which such improvements cannot be made.

For example, it there is no way of finding the exact date of the riot from the context—if for example, the context only specifies that it occurred in May, 1970—then we would be forced to conclude that the statement is ambiguous. But even if the date is supplied by context, we may still be in trouble. Supposing the date to be determined to be 15 May, 1970, the content of the statement would be the infon

⟨⟨ panic; subject : Smith, date : 5/15/70 ⟩⟩ ∧  
 ⟨⟨ run, subject : Smith, date : 5/15/70 ⟩⟩.

Unfortunately, this infon is still unsaturated, as we can see by supposing that Smith was involved in two separate clashes with the police on the same day. On the first occasion, he bravely stood his ground; on the second he panicked and ran. The mere possibility of two clashes is sufficient to show that the infon is unsaturated and so, on the hidden parameter account, we are left with ambiguity, even if there was only one clash.

In an effort to avoid the conclusion that there is widespread ambiguity in our use of ordinary language, we may search for more elaborate ways in which the context fills hidden parameters. If the exact time is not supplied in the linguistic context or by common knowledge, it may be determined by a causally related chain of events starting with Smith’s actions in the riot.

The danger in this move is that information expressed by a statement may be unknown, even to the person who makes the statement. The narrator may well remember the day of the riot, and even whether it occurred in the morning or afternoon, but not the exact time. If this time is to be a component of the infon expressed by the statement then this is information which the narrator does not possess.

If, on the other hand, we are to resist the move to considering factors which go beyond the knowledge of the narrator, there are other problems. If we take the narrator’s beliefs to determine the temporal parameter, then falsehoods are liable to appear in surprising places. Suppose, for example, that the narrator falsely believed the incident took place at around noon on the 15th May, 1970; in fact, it occurred at about 11 a.m. It is awkward to maintain that this false belief, which is otherwise irrelevant to the narrative, makes his statement that Smith panicked and ran false.

Finally, there is a problem in making the demand for ever-greater precision compatible with our somewhat roughshod ontology of events and processes. Although we think of the event of Smith panicking and running as having duration, it is not the sort of thing which we can measure in seconds.

We may hope that for each kind of description there is a fixed tolerance with which information about the values of hidden parameters must be specified. In this way, we could avoid the problems of requiring perfect precision. In many cases, the truth of a statement in which a time is not mentioned explicitly is fairly robust: small variations do not alter the truth-value. But there are circumstances in which small changes do matter—Smith’s two clashes with the police may have happened within half-an-hour of each other—and so, the precision required for the saturation of a given piece of information is indefinite, depending on particular facts about the circumstances of the original event, and not just about the later circumstances in which the statement is made.

A way out of this web of difficulty is to embrace the conclusion that many, if not all, of the statements we make are ambiguous; or, better, to cut the link between lack of a truth value and ambiguity. We may say that a statement is unambiguous if it expresses a determinate item of information, even if this information is unsaturated and the statement neither true nor false.

A quite different approach is to claim that an unsaturated infon is factual if there is some saturated fact of which it is a part. By quantifying over the possible ways of saturating the infon, we arrive at the rather charitable position that a statement is true just in case, in the circumstances, there is some way of construing it as expressing a saturated fact. This is reminiscent of Davidson’s treatment (1967) of the same problem. There are two main difficulties with this solution. Firstly, an unrestricted quantification over possible arguments is clearly too generous. If in the course of discussing the riot in 1970 we state that Smith panicked and ran, then this statement cannot be made true by Smith’s panicking and running last week. Thus, we must appeal to the context again, this time to provide suitable restrictions to the range of quantification. Secondly, the restrictions on quantifiers are often interdependent. For example, if in interpreting the text ‘Smith panicked. He ran.’ we may use of the quantificational strategy, then the restrictions on the possibly ways of saturating the two unsaturated infons  $\langle\langle \text{panic}; \text{Smith} \rangle\rangle$  and  $\langle\langle \text{run}; \text{Smith} \rangle\rangle$  must be related: if  $\langle\langle \text{panic}; \text{Smith}, \text{time: } t_1, \dots \rangle\rangle$  and  $\langle\langle \text{run}; \text{Smith}, \text{time: } t_2, \dots \rangle\rangle$  are among the permissible saturations then  $t_1$  must be before  $t_2$ . Such technical obstacles are not insuperable (see the chapter on dynamics in this volume) but introduce complications which we shall not go into here.

## 4.2 Logic in Fact Structures

In Section 3.4 we showed how to model compound infons using the theory of structural relations. There we were concerned only with the structural properties of those objects; now we shall examine their logical properties. For example,

any reasonable model of conjunctive facts should satisfy the condition that

$$\mathbf{Fact}(\sigma \wedge \tau) \text{ iff } \mathbf{Fact}(\sigma) \text{ and } \mathbf{Fact}(\tau).$$

Certain special cases of this condition may be determined by structural properties of conjunction. For example, in the infon structure of Construction 3.13 conjunction is idempotent, and so in any fact structure extending this,  $\mathbf{Fact}(\sigma \wedge \sigma)$  iff  $\mathbf{Fact}(\sigma)$ . But structural properties alone will not usually suffice to determine every instance of the above condition, and so we shall need to impose additional axioms. For example, a de Morgan infon structure (from Section 3.6)  $\mathfrak{F}$  may be expected to satisfy the following conditions:

1.  $\mathbf{Fact}(\bigwedge \Sigma)$  iff  $\mathbf{Fact}(\sigma)$  for each  $\sigma \in \Sigma$ , and
2.  $\mathbf{Fact}(\bigvee \Sigma)$  iff  $\mathbf{Fact}(\sigma)$  for some  $\sigma \in \Sigma$ .
3.  $\mathbf{Fact}(\neg\sigma)$  iff  $\mathbf{Inf}(\sigma)$  and not  $\mathbf{Fact}(\sigma)$ .

Let us say that a fact structure  $\mathfrak{F}$  is *classical* if it does satisfy the above.

We could go on to define a host of different kinds of fact structure, catalogued by their logical properties. However, there are several good reasons for not doing this. Firstly, it is a historical fact that logicians have failed to arrive at a consensus on the logical properties of even the most fundamental compounds. Indeed, there is much wisdom in treating logic as an applied subject, for which the logical properties of compounds are determined by the application. The upshot is that any list would be tediously long and necessarily incomplete. Secondly, we wish the approach developed here to apply not just to the multitude of existing logics whose compounds have the familiar syntactic structure of (formal) sentences, but to future logics acting on more intricate structures—circular structures, infinite structures, and anything else that can be modelled using the theory of structural relations.

Instead of giving a list of logics on fact structures, we give a general method for categorizing the logical properties of facts.

**Definition 4.2** A *consequence relation*  $\vdash$  on an infon structure is a binary relation between classes of infons such that  $\Gamma \vdash \Delta$  iff  $\Gamma' \vdash \Delta'$  for every partition  $\langle \Gamma', \Delta' \rangle$  of  $\mathbf{Inf}^*$  such that  $\Gamma \subseteq \Gamma'$  and  $\Delta \subseteq \Delta'$ .<sup>2</sup> Given a fact structure  $\mathfrak{F}$ , a consequence relation  $\vdash$  on  $\mathfrak{F}$  is *sound* if whenever  $\Gamma \vdash \Delta$  and every infon in  $\Gamma$  is a fact then some infon in  $\Delta$  is a fact.

Consequence relations have the following familiar properties:

(identity)  $\sigma \vdash \sigma$ ,

(weakening) if  $\Gamma \vdash \Delta$  then  $\Gamma', \Gamma \vdash \Delta, \Delta'$ , and

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<sup>2</sup> $\langle X, Y \rangle$  is a *partition* of  $Z$  if  $X \cup Y = Z$  and  $X \cap Y = \emptyset$ .

(cut) if  $\Gamma \vdash \Delta, \sigma$  and  $\sigma, \Gamma \vdash \Delta$  then  $\Gamma \vdash \Delta$ .<sup>3</sup>

Moreover, if  $\vdash$  is compact then these three conditions are jointly sufficient to ensure that  $\vdash$  is a consequence relation (Barwise and Seligman 199?).<sup>4</sup> Also note that a consequence relation is entirely determined by the class of partitions that lie in it. It is easy to see that a consequence relation is sound in a fact structure just in case the partition of  $\text{Inf}^*$  into facts and non-facts does not lie in the consequence relation.

In any de Morgan infon structure  $\mathfrak{F}$ , if  $\vdash$  is a sound consequence relation on  $\mathfrak{F}$  and  $\mathfrak{F}$  is classical then  $\vdash$  must satisfy the following conditions:

1.  $\bigwedge \Sigma, \Gamma \vdash \Delta$  if  $\Sigma, \Gamma \vdash \Delta$ ,
2.  $\Gamma \vdash \Delta, \bigwedge \Sigma$  if  $\Gamma \vdash \Delta, \sigma$  for each  $\sigma$  in  $\Sigma$ ,
3.  $\Gamma \vdash \Delta, \bigvee \Sigma$  if  $\Gamma \vdash \Delta, \Sigma$ ,
4.  $\bigvee \Sigma, \Gamma \vdash \Delta$  if  $\sigma, \Gamma \vdash \Delta$  for each  $\sigma$  in  $\Sigma$ ,
5.  $\Gamma, \neg \sigma \vdash \Delta$  if  $\Gamma \vdash \sigma \Delta$ ,
6.  $\Gamma \vdash \neg \sigma \Delta$  if  $\Gamma, \sigma \vdash \Delta$ .

The conditions are analogues of the standard inference rules of the sequent calculus for (infinitary) classical propositional logic; let us say that  $\vdash$  is *classical* if it satisfies all these conditions. Then it is easy to check that if  $\mathfrak{F}$  is non-trivial and has a classical consequence relation then  $\mathfrak{F}$  must be classical. These observations amount to the following characterization of de Morgan infon structures: a non-trivial de Morgan infon structure is classical iff it has a sound classical consequence relation.<sup>5</sup>

In this way, we may study the logical properties of complex infons by studying their consequence relations, without any explicit reference to fact structures. A multitude of consequence relations have already been catalogued by logicians, and so we can import whichever logic we wish, defining the corresponding class of fact structures on which the consequence relation is sound.

### 4.3 Restriction

In Section 3.8 we saw that properties defined by means of abstraction inherit the appropriateness conditions of the infons over which they are defined. For example, the property  $\lambda x. \langle \langle \text{fed}; \text{Martha}, x \rangle \rangle$  of being eaten by Martha is appropriate for just those objects  $a$  for which  $\langle \langle \text{fed}; \text{Martha}, a \rangle \rangle$  is an infon.

<sup>3</sup>We use the usual abbreviations, e.g.,  $\Delta, \sigma$  stands for  $\Gamma \cup \{\sigma\}$  when an argument of  $\vdash$ .

<sup>4</sup> $\vdash$  is *compact* if whenever  $\Gamma \vdash \Delta$  there are finite sets  $\Gamma_0 \subseteq \Gamma$  and  $\Delta_0 \subseteq \Delta$  such that  $\Gamma_0 \vdash \Delta_0$ .

<sup>5</sup>Furthermore, there is a smallest classical consequence relation  $\vdash_c$  on any infon structure  $\mathfrak{M}$ , and so a non-trivial de Morgan infon structure  $\mathfrak{M}'$  extending  $\mathfrak{M}$  is classical iff  $\vdash_c$  is sound on  $\mathfrak{M}'$ .

For semantics, it has proved useful to introduce a more flexible notion of abstraction by which a much wider range of restrictions may be placed on appropriate assignments. On some semantic analyses, the information expressed by the sentence ‘Martha fed him’ is that the referent of ‘him’ has the property of being fed by Martha, modelled by the abstract  $\lambda x. \langle\langle \text{fed}; \text{Martha}, x \rangle\rangle$ . But the sentence also conveys the information that the person fed by Martha is male. One way of incorporating this information is to maintain that the abstract is restricted so that it may only be applied to males. In other words, we want a property  $p$  such that  $\langle\langle p; a \rangle\rangle$  is

1. an infon iff  $a$  is male and  $\langle\langle \text{fed}; \text{Martha}, a \rangle\rangle$  is an infon
2. factual iff  $a$  is male and  $\langle\langle \text{fed}; \text{Martha}, a \rangle\rangle$  is factual

In some formulations, the restriction that  $a$  be male attaches to the abstracted variable (or ‘parameter’)  $x$ , and a theory of *restricted parameters* is developed (Gawron and Peters 1990, Fernando (1991); in others, the situation-theoretic universe is extended to include *restricted objects* quite generally: such objects are identical to their unrestricted counterparts when the (often contingent) restricting condition is met, and are otherwise “undefined” or of some other ontologically inferior status. Plotkin (1990) adapts Curry’s “illative” approach to logic, treating restriction by means of a connective  $\upharpoonright$  whose formation rules require that the restriction be met: if  $\sigma$  is an infon and  $\tau$  is a fact then  $\sigma \upharpoonright \tau$  is an infon; otherwise the expression ‘ $\sigma \upharpoonright \tau$ ’ is not even well-formed.

Barwise and Cooper (refs???) introduce an elegant graphical notation for restrictions and abstracts called Extended Kamp Notation, in honour of Kamp’s Discourse Representation Structures (see the chapter on DRS in this volume). They write situation theoretic objects as boxes. A box directly to the right of another box acts as a restriction, and a box directly above lists abstracted objects. For example, the restricted property  $p$  mentioned above would be written as

|  |  |
|--|--|
| $a$  |  |
| $\langle\langle \text{fed}; \text{Martha}, a \rangle\rangle$ | $\langle\langle \text{male}; a \rangle\rangle$ |

The approach adopted here is to extend the theory of abstraction and application developed in Sections 3.7 and 3.8.

**Definition 4.3** A fact structure  $\mathfrak{F}$  has *restricted abstraction* if it is also an abstraction structure and has an additional structural relation  $\text{Res}^2$ , such that if  $\text{Res}(\sigma, a)$  then  $a$  is an abstract and for each assignment  $f$  for  $a$ ,  $f.\sigma$  is an infon. We say that  $a$  is a *restricted abstract* and that  $\sigma$  is a *restriction* of  $a$ . For each object  $a$ , set  $B$  of objects in  $|\mathfrak{F}|$ , and set  $\Sigma$  of infons in  $|\mathfrak{F}|$ , an object  $\Lambda B[\Sigma].a$  is the *abstraction of  $B$  from  $a$  restricted by  $\Sigma$*  if there is a function  $\pi: B \rightarrow |\mathfrak{A}|$  such that

1.  $\text{Abs}(\Lambda B[\Sigma].a, \pi(b))$  for each  $b$  in  $B$ ,
2. for each  $b$  in  $B$ ,  $\pi(b)$  is only of sort  $\text{Abs}^*$ , and
3.  $\Lambda B[\Sigma].a$  is of type  $\langle \Lambda B[\Sigma].a, T_{\pi.a} \cup \{ \langle \text{Res}, \pi.\sigma \rangle \mid \sigma \in \Sigma \} \rangle$ .<sup>6</sup>

$\mathfrak{F}$  is a *restricted-Lambda structure* if  $\Lambda B[\Sigma].a$  exists for each  $a$ ,  $B$ , and  $\Sigma$ .

The key definition: an assignment  $f$  for  $a$  is *appropriate for  $a$*  if  $f.a$  exists and  $f.\sigma$  is a fact.

As before, if  $f$  is appropriate for  $a$  then *the application of  $a$  to  $f$* , written  $a^*(f)$ , is just  $f.a$ .  $\mathfrak{F}$  has *coherent appropriateness conditions* if for each infon  $\sigma = \langle \langle r; f \rangle \rangle$ , if  $r$  is a (restricted) infon abstract then  $f^*$  is an appropriate assignment for  $r$ .  $\mathfrak{F}$  is *restricted-relation-closed* if for each infon  $\sigma$ , each set  $B$  of elements of  $\mathfrak{A}$ , and each set  $\Sigma$  of infons, the infon abstract  $\Lambda B[\Sigma].\sigma$  exists and for each appropriate assignment  $f$  for  $\Lambda B[\Sigma].\sigma$ , the infon  $\langle \langle \Lambda B[\Sigma].\sigma; f \rangle \rangle$  also exists.

Technically there are no difficulties here. Restricted Lambda structures and restricted-relation-closed structures may be constructed using methods similar to those used in Theorems 3.29 and 3.33.

#### 4.4 Internal Definability

We have already seen how an internal function  $f$  represents an external function  $f^*$  in a function structure, and how in an abstraction structure an abstract  $a$  represents a second-order function  $a^*$  mapping assignments to objects of the domain. In fact structures, internal relations represent external relations. Suppose  $\mathfrak{F}$  is a fact structure and  $R$  is an  $n$ -ary relation on  $|\mathfrak{F}|$ . The basic idea is that an internal relation  $r$  of  $\mathfrak{F}$  represents  $R$  if for each  $a_1, \dots, a_n$  in  $|\mathfrak{F}|$ , there is a saturated infon  $\langle \langle r; a_1, \dots, a_n \rangle \rangle$  in  $\mathfrak{F}$  and

$$R(a_1, \dots, a_n) \text{ iff } \text{Fact}(\langle \langle r; a_1, \dots, a_n \rangle \rangle).$$

However, there are several complications. Firstly, our use of the functional notation for infons hides an assumed correlation between the roles of  $r$  and the integers  $1, \dots, n$ ; this must be made explicit. Secondly, the condition requires that the infon  $\langle \langle r; a_1, \dots, a_n \rangle \rangle$  exists for every sequence  $a_1, \dots, a_n$  of elements of  $|\mathfrak{F}|$ . This will rarely be satisfied because most internal relations have non-trivial appropriateness conditions—it fails even in standard fact structures. We shall have to revise the definition so that the domain of relation  $R$  is restricted in some way.

**Definition 4.4** Given a fact structure  $\mathfrak{F}$ , an  $n$ -ary external relation  $R$  on  $|\mathfrak{F}|$  is *represented by* elements  $r, i_1, \dots, i_n$  of  $|\mathfrak{F}|$  on the domain  $A \subseteq |\mathfrak{F}|$  if

<sup>6</sup>This clause is slightly more complicated than the corresponding clause for unrestricted abstraction. It says that the restricted abstract  $\Lambda B[\Sigma].a$  has the component structure of  $\pi.a$ , together with the additional structural relationships  $\text{Res}(\pi.\sigma, \Lambda B[\Sigma].a)$  for each  $\sigma$  in  $\Sigma$ .

1.  $f$  is appropriate for  $r$  iff  $f^*: \{i_1, \dots, i_n\} \rightarrow A$ , and
2. if  $f$  is appropriate for  $r$  then  $\text{Fact}(\langle\langle r; f \rangle\rangle)$  iff  $R(f^*(i_1), \dots, f^*(i_n))$ .

In standard fact structures, every relation  $r$  has a fixed finite arity,  $n$  say, and (by the generality principle) the infon  $\langle\langle r; a_1, \dots, a_n \rangle\rangle$  exists for each sequence  $a_1, \dots, a_n$  of ordinary objects. Thus every internal relation of a standard fact structure represents an external relation on the domain of ordinary objects.

In the general case, however, internal relations may fail to represent for a variety of reasons: they may have infinite or variable arity; they may generate unsaturated infons; or they may have sortal—or even more complex—appropriateness conditions. A simple generalization of the above definition handles the last of these cases.

**Definition 4.5** Given a fact structure  $\mathfrak{F}$ , an  $n$ -ary external relation  $R$  on  $|\mathfrak{F}|$  is *represented by* elements  $r, i_1, \dots, i_n$  of  $|\mathfrak{F}|$  relative to another  $n$ -ary relation  $D$  on  $|\mathfrak{F}|$  if

1.  $f$  is appropriate for  $r$  iff  $D(f^*(i_1), \dots, f^*(i_n))$ , and
2. if  $f$  is appropriate for  $r$  then  $\text{Fact}(\langle\langle r; f \rangle\rangle)$  iff  $R(f^*(i_1), \dots, f^*(i_n))$ .

Effectively, we regard  $r$  as representing the relation  $R$  only in the context of  $D$ ; outside of  $D$  we do not care which sequences are in  $R$  and this is reflected by the failure of appropriateness for the corresponding assignments for  $r$ . In other words, there are three possible judgements concerning a sequence  $\vec{a}$ :  $\vec{a}$  in  $D$  and  $R$ ;  $\vec{a}$  in  $D$  but not in  $R$ ; and  $\vec{a}$  not in  $D$ . These are represented by the corresponding judgements for an assignment  $f$ :  $\text{Fact}(\langle\langle r; f \rangle\rangle)$ ;  $\text{not Fact}(\langle\langle r; f \rangle\rangle)$ ; and no infon of the form  $\langle\langle r; f \rangle\rangle$ .

Another approach is to say that, in general, internal relations represent *partial* relations on  $|\mathfrak{F}|$ . A partial relation has an extension and an anti-extension, which are disjoint but which need not exhaust the domain. Facts with relation  $r$  represent sequences in the extension of  $R$ , and infons that are not facts represent sequences in the anti-extension of  $R$ .

The issue of infinite aries and variable arities can be handled with similar generalizations of the our metatheoretic concept of ‘relation’. Relations that have appropriate assignments generating unsaturated infons in an irregular way are beyond the scope of these methods. There is nothing about external relations that might correspond to this features of internal relations. The reason for this is that internal relations may capture intensional differences of real relations that are lost in our rather crude characterisation of external relations as a collections of sequences.

Whichever characterization of representation is adopted, it is natural to ask which external relations are represented in a given fact structure. For example, we may wonder if any of the structural relations  $\text{Arg}$ ,  $\text{Rel}$ ,  $\text{Inf}$  and  $\text{Fact}$  are represented.

Plotkin (1990) has various negative results. Suppose that **Fact** is represented by  $r$  in a lambda structure with negation, and let  $p$  be the abstract  $\lambda x. \neg \langle\langle r; x \rangle\rangle$ , which is sure to exist so long as there is any infon of the form  $\neg \langle\langle r; \sigma \rangle\rangle$ . A simple argument shows that  $p$  cannot have a fixed point: if  $\sigma$  was such then **Fact**( $\sigma$ ) iff **Fact**( $\neg \langle\langle r; \sigma \rangle\rangle$ ) iff  $\neg \mathbf{Fact}(\langle\langle r; \sigma \rangle\rangle)$ , by the properties of abstraction and negation; and **Fact**( $\langle\langle r; \sigma \rangle\rangle$ ) iff **Fact**( $\sigma$ ) under the assumption that  $r$  represents **Fact**; so **Fact**( $\sigma$ ) iff  $\neg \mathbf{Fact}(\sigma)$ , a contradiction.

In the structures used by Plotkin, every abstract has a fixed point, and so **Fact** cannot be represented. Plotkin's uses Frege structures (Aczel 1980), which are constructed from models of the untyped  $\lambda$ -calculus in which fixed point combinators such as  $Y$  exist. Similar problems arise in structures with internalized application, because they also contain fixed point combinators. Even without such combinators fixed points will exist in any structurally saturated fact structure. For example, the type  $\langle x, \neg \langle\langle r; x \rangle\rangle \rangle$  is instantiated in any such structure, and this solution is a fixed point of  $p$ .

These considerations illustrate an important trade-off in the construction of models for Situation Theory. There are two ways in which we may measure the power of our structures as modelling tools. On the one hand, we may see which operations are possible, desiring closure under such operations as abstraction, infon-formation, and restricted abstraction. On the other hand, we may see which external entities are internalized, which functions and relations are represented, whether application can be internalized, and so on. We have seen in Section 3.2 that structures in which many types are instantiated—in particular, structurally saturated structures—are very useful for ensuring closure properties under structural operations. However, the above results show that if many types that are instantiated, then there are limits on what can be internalized. The boundary between the goals of structural closure and internalization are still far from clear.

## 4.5 Situation Structures

Sections 4.1 to 4.4 have been directed toward an articulation of what we have called the hidden-parameter approach to the relationship between information and truth. Now we turn to the alternative, referential approach.

Consider again the example of telling a story about Smith's involvement in a riot during which he panicked and ran. The straightforward approach is to say that the statement that Smith panicked and ran conveys the information

$$\langle\langle \text{panic}; \text{Smith} \rangle\rangle \wedge \langle\langle \text{run}; \text{Smith} \rangle\rangle.$$

The puzzle was to determine the criterion for this (unsaturated) infon to be factual.

On the hidden-parameter account, for the statement to be true it must express a saturated infon whose additional roles are filled by various contextually determined parameters. Common usage dictates that the statement is true, or



at least *could* be true, and so that there are such hidden parameters determining the fully saturated information expressed by any unambiguous statement. The hidden-parameter approach to semantics therefore revolves around the identifying and cataloguing of contextual parameters.<sup>7</sup>

The referential account is quite different, attributing the alleged gap between information and truth to the neglect of a fundamental component of descriptive statements: the situation being described. In the above example, the narrator is describing a certain situation, a course of events twenty-six years ago, in which (we may suppose) Smith panicked and ran. Consequently, the description is correct and the narrator's statement is true. However, the very same sentence, 'Smith panicked and ran', may be used to describe a quite different situation in which Smith was engaged in some activity incompatible with panicking and running, such as standing and fighting. In this case, the statement—necessarily a different statement than the previous one—would be false. A third possibility is that the described situation is one which encompasses many different events, making the statement genuinely ambiguous.

The range of outcomes shows why it is difficult for the reference theorist to sort infons into facts and non-facts. Instead, a new predicate, 'supports', is introduced to describe the relationship between a situation and the information expressed by a correct description of it. In other words, a statement describing a situation is true if the situation it describes supports the information it expresses; otherwise it is false. Consequently, a statement describing an event *e* using the sentence 'Smith panicked and ran' may be false, even if there is another event *e'* which would be correctly described by the same sentence. Moreover, it is possible for two statements to be made using this sentence but about the two different events *e* and *e'*, and for the statements to express the *same information*, even though one is true and the other false. The difference in truth-value is explained as a difference in reference, not in the information expressed.

Criticism of the referential account (for example Seligman (199?)) has centered on two related problems. Firstly, it is difficult to account for just how people refer to situations when they make statements. Various proposals are akin to corresponding proposal for the reference of proper names: the intentions of the person making the statement may be involved; there may be some kind of causal relationship between the statement and described situation; and conventions within a linguistic community may be appealed to. In addition, the context of a statement has considerable importance. A past tense sentence, such as the one we have been discussing, may well be referentially ambiguous when uttered out of context, attaining a definite reference only in the context of a

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<sup>7</sup>An alternative is to abandon the common sense idea that much of what is said is really true (or false) and try to explain semantic relationships in terms of information conditions. This is summed up nicely by Barwise's slogan 'Information conditions not truth conditions' and has become a core idea in Dynamic Semantics (see the chapter on dynamic semantics in this volume)

discourse or narrative text (see Seligman and ter Meulen (1993)). Nonetheless, no there is no general account of how reference is achieved and this remains a serious lacuna.

The first problem is compounded by the second: that it is unclear what is referred to when reference is achieved. The word ‘situation’ is intended to be neutral between ‘event’ and ‘state’, so that it makes sense to speak both of the situation in Bosnia, meaning the war (an event or course of events), and the situation on Capitol Hill, meaning (the state of) deadlock over the budget.<sup>8</sup> The theoretical uniformity of this terminology is bought at a price. Questions about the identity of situations appear much more pressing than the traditional (and equally baffling) metaphysical questions about the identity of material objects, persons and members of a host of other ontological categories.

A short, bold answer to the question of the identity of situations is that distinct situations support different infons. Contraposing we get the

*Principle of Extensionality:* if for each infon  $\sigma$ ,  $s$  supports  $\sigma$  iff  $s'$  supports  $\sigma$ , then  $s = s'$ .

If we accept this principle then we may think of situations and model them as structured objects whose structure is determined by the infons they support. This is the basis of a model of situations using the theory of structural relations. We introduce the structural relation  $\text{Sit}^1$  of being a situation, and  $\text{HoldsIn}^2$ , which holds between an infon and a situation just in case the situation supports the infon.

**Definition 4.6** An infon structure  $\mathfrak{S}$  with structural relations  $\text{Sit}^1$  and  $\text{HoldsIn}^2$  is a *situation structure* if

1. if  $x$  is an infon, a relation, or a role then not  $\text{Sit}(x)$ , and
2. if  $\text{HoldsIn}(\sigma, s)$  then  $\text{Inf}(\sigma)$  and  $\text{Sit}(s)$ .

The elements of  $\text{Sit}^*$  are called *situations* and we say that  $s$  *supports*  $\sigma$ , written  $s \models \sigma$ , if  $\text{HoldsIn}(\sigma, s)$ .

Any fact structure  $\mathfrak{F}$  may be used to define a situation structure  $\mathfrak{F}^\#$  by adding a new element  $s_\Sigma$  for each set  $\Sigma$  of facts, and defining:

$\text{HoldsIn}(\sigma, s_\Sigma)$  iff  $\sigma$  is in  $\Sigma$ .

We say that a situation structure is *standard* if it is isomorphic to  $\mathfrak{F}^\#$ , for some standard fact structure  $\mathfrak{F}$ .

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<sup>8</sup>Indeed, the flexibility of the words ‘situation’ and ‘situated’ has become almost comical: any traditional theory—of language, cognition, and even sociology—gives birth to a new ‘situated’ theory, sometimes with little more than the liberal use of this adjective together with a strong, if vague, commitment to the idea that everything is relative to contextual factors.

## 4.6 Parts of the World

The relationship of smaller situations to larger situations is perhaps the most controversial aspect of Situation Theory. To see why, consider the following two situations: Let  $s_{TW}$  be the situation in Taiwan, or some relevant piece of that situation. Let's suppose that  $s_{TW}$  supports each infon of form  $\langle\langle \text{ma}; t \rangle\rangle$  that could be expressed by a sentence of form 'There was a motorscooter accident at time  $t$ ', used to describe correctly events occurring in Taiwan. Let  $s_W$  be the situation on Earth as a whole. We assume that  $s_W$  supports each infon of the form  $\langle\langle \text{ma}; t, z \rangle\rangle$  that could be expressed by a sentence of form 'There was a motorscooter accident at time  $t$ , time zone  $z$ ', used to describe correctly events occurring anywhere in the world. On an intuitive level,  $s_{TW}$  is a part of  $s_W$ .

However, this assumption is not correctly modelled in our theory as it stands. The reason is that the only order we have around is the 'flat' order  $\sqsubseteq$  between situations obtained as an instance of our theory of partiality in Section 3.3. On this,  $s_1 \sqsubseteq s_2$  iff all infons holding in  $s_1$  situation also hold in  $s_2$ . According to this definition,  $s_{TW} \not\sqsubseteq s_W$  since the infons supported by  $s_{TW}$  are *not* supported by  $s_W$ .

On closer inspection, it appears that there are two reasonable ways of saying what it means for one situation to be contained in another. The first is  $\sqsubseteq$  as above. The second way is that  $s_1$  is part of  $s_2$  if all the infons holding in  $s_1$  are *unsaturated versions* of the infons holding in  $s_2$ . For example, let  $s_{TW^*}$  be the situation that supports each infon of the form  $\langle\langle \text{ma}; t, z \rangle\rangle$  that could be expressed by a sentence of form 'There was a motorscooter accident at time  $t$ , time zone  $z$ ', used to describe correctly events occurring in Taiwan. It is in this second sense that  $s_{TW}$  is part of  $s_{TW^*}$ . Also  $s_{TW^*} \sqsubseteq s_W$ . Combining the two ways, we get the following definition:

**Definition 4.7**  $s_1$  is a part of  $s_2$ , written  $s_1 \trianglelefteq s_2$ , if:

$$\forall \sigma \text{ if } s_1 \models \sigma \text{ then } \exists \tau (s_2 \models \tau \text{ and } \sigma \sqsubseteq \tau).$$

Two situations are *compatible* if they have an upper bound in the  $\trianglelefteq$ -ordering. A situation  $s$  is *maximal* if for each situation  $s'$ , if  $s \trianglelefteq s'$  then  $s = s'$ . A set of situations may be *joined* if they have a least upper bound in the  $\trianglelefteq$ -ordering, called the *join* of  $S$ .

**Proposition 4.8** *Every standard situation structure satisfies the following conditions:*

- S1** *for every situation  $s$ , there is a maximal situation of which it is a part,*
- S2** *any two situations are compatible,*
- S3** *there is a unique maximal situation of which every situation is a part,*
- S4** *every pairwise compatible set of situations may be joined, and*

**S5** *every set of situations may be merged.*

These properties concern the global properties of the universe. In standard situation structures, there is a unique maximal situation, call it  $w$ , which is given concretely by the set of facts from the underlying fact structure. This gives is a simple relationship between facts and situations:

$$\text{Fact}(\sigma) \quad \text{iff} \quad \exists s. s \models \sigma \quad \text{iff} \quad w \models \sigma$$

As we move away from standard situation structures this relationship will break down. But first, we should consider what it would mean for the conditions listed above to fail. How are we to make sense of incompatible situations in the general case? And how can there be two distinct maximal situations?

Maximal situations are completely saturated, in the sense that information about the filling of roles in infons is specified, and there is no compatible way of adding further infons. This suggests the interpretation of maximal situations as possible worlds. If condition S1 is satisfied then every situation is part of some possible world, and so compatibility becomes compossibility: two situations are compossible iff they are part of the same possible world. We call this the *compossibility interpretation*.

Under this interpretation, one maximal situation  $s_a$  must be distinguished as the *actual world*, and the other incompatible maximal situations are ways the world might have been. There is a sharp metaphysical distinction between those situations which are part of  $s_a$  and those which are not; the former are *actual situations*, the latter are merely *possible situations*.

The compossibility interpretation allows one to introduce many of the tools of the theory of possible worlds. For example, standard analyses of modality and conditionals may be given. However, such analyses tend to run into difficulties due to the partiality of situations; we shall not go into the details here.<sup>9</sup>

On the *relativist interpretation*, incompatible situations are regarded as embodiments of different perspectives on the same, actual world. For example, suppose we are facing each other across the dinner table, so that for you the salt is to the left of the pepper. The situation  $s_1$  concerning the arrangement of objects on the table from your perspective supports the infon  $\langle\langle \text{LeftOf}; \text{salt}, \text{pepper} \rangle\rangle$ , whereas the situation  $s_2$  from my perspective supports  $\langle\langle \text{LeftOf}; \text{pepper}, \text{salt} \rangle\rangle$ . On a relativist conception, these situations may be incompatible because there is no situation that supports both of these infons.

Even on the relativist conception, the two perspectives may be made compatible by appealing to a hidden parameter. If there are infons

$$\langle\langle \text{LeftOf}; \text{salt}, \text{pepper}, \text{me} \rangle\rangle \text{ and } \langle\langle \text{LeftOf}; \text{salt}, \text{pepper}, \text{you} \rangle\rangle,$$

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<sup>9</sup>Our construction of standard situation structures may be modified in a fairly obvious way to build standard ‘modal’ situation structures from first-order Kripke structures, by generating situations from sets of compossible facts.

in which a role for an egocentric frame of reference is filled by *me* and *you* respectively, then there could be a situation  $s_3$  supporting both of these infons without conflict. Moreover, other things being equal,  $s_1 \trianglelefteq s_3$  and  $s_2 \trianglelefteq s_3$ , and so  $s_1$  and  $s_2$  are compatible after all. In this way, we can account for the fact that there are different perspectives in this example ( $s_1$  and  $s_2$  have no upper bound in the  $\sqsubseteq$ -ordering), while showing how the two perspectives can be joined using hidden parameters. On the relativist conception, situations are genuinely incompatible, only if there is no way of adding a hidden parameter to resolve the conflict.<sup>10</sup>

Closely related to but strictly stronger than Extensionality, is the

*(Principle of Anti-symmetry) if  $s_1 \trianglelefteq s_2$  and  $s_2 \trianglelefteq s_1$  then  $s_1 = s_2$ .*

In every saturated situation structure, Anti-symmetry is satisfied, making  $\trianglelefteq$  a partial order; but this is not so in general. For example, consider the infons

$$\sigma_1 = \langle\langle \text{eat}; \text{Kara} \rangle\rangle \text{ and } \sigma_2 = \langle\langle \text{eat}; \text{Kara, trout} \rangle\rangle$$

and suppose that  $s_2$  is a (necessarily unique) situation which supports only the information  $\sigma_2$  that Kara eats trout, and that  $s_1$  is a situation which also supports the information  $\sigma_1$  that Kara eats. Then  $s_1 \trianglelefteq s_2$  and  $s_2 \trianglelefteq s_1$  but  $s_1 \neq s_2$ .

One might think that such examples should not arise. This is a consequence of the widely held

*(Principle of Persistence) if  $s_1 \trianglelefteq s_2$  and  $s_1 \models \sigma$  then  $s_2 \models \sigma$*

Extensionality ensures that Anti-symmetry follows from Persistence. Indeed, Extensionality ensures that situations are completely determined by the information they support. Persistence goes a step further and identifies the  $\trianglelefteq$ -order on situations with the  $\sqsubseteq$ -order; this order is always anti-symmetric.

We might also want axioms positing the existence of situations which support infons which are known to be supported by other situations:

**B 1** *If  $\sigma \sqsubseteq \tau$  and  $t \models \tau$  then  $\exists s \trianglelefteq t$  such that  $s \models \sigma$ .*

**B 2** *Let  $t$  be such that for each  $\sigma \in \Sigma$ ,  $t \models \sigma$ . Then  $\exists s \trianglelefteq t$  such that ( $s \models \sigma$  iff  $\sigma \in \Sigma$ ).*

**B 3** *Let  $t$  be such that for each  $\sigma \in \Sigma$ , there is a  $\tau$  such that  $\sigma \sqsubseteq \tau$  and  $t \models \tau$ . Then  $\exists s \trianglelefteq t$  ( $s \models \sigma$  iff  $\sigma \in \Sigma$ ).*

Each of these existence axioms is satisfied by every standard situation structure.

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<sup>10</sup>For more discussion of perspectives, see Barwise (1989b) and Seligman (1991).

## 4.7 Logic in Situation Structures

One can see the move from fact structures to situation structures as a generalization in two directions. First, the non-relational property **Fact** of being factual is replaced by the relational property **HoldsIn** of holding in a situation. Second, this property may be had by infons which are unsaturated. In other words, situations are *partial* in two respects: they need not support all of the facts, and the infons they do support may be only unsaturated parts of facts.

More precisely, we may relate fact structures and situation structures as follows. Say that a situation  $s$  is *saturated* if every infon it supports is saturated. For each situation  $s$ , let  $\mathbf{Fact}_s$  be the property of holding in  $s$ , i.e.,  $\mathbf{Fact}_s(\sigma)$  iff  $s \models \sigma$ . If  $\mathfrak{S} = \langle \mathfrak{A}, \text{Inf}, \text{Sit}, \text{HoldsIn} ; \models, \sqsubseteq \rangle$  is a situation structure, then for each saturated situation  $s$  in  $\mathfrak{S}$ , the structure  $\mathfrak{F}_s = \langle \mathfrak{A}, \text{Inf}, ; \mathbf{Fact}_s, \sqsubseteq \rangle$  is a fact structure.

This construction suggests a way of extending our discussion of logic on fact structures to the present setting. We say that a situation structure  $\mathfrak{S}$  is *locally classical* if for each situation  $s$  in  $\mathfrak{S}$ , the fact structure  $\mathfrak{F}_s$  is a classical fact structure. By the previous discussion, we know that this is equivalent to the following: for each situation  $s$  in  $\mathfrak{S}$ ,

**B 4**  $s \models \bigwedge \Sigma$  iff  $s \models \sigma$  for each  $\sigma \in \Sigma$

**B 5**  $s \models \bigvee \Sigma$  iff  $s \models \sigma$  for some  $\sigma \in \Sigma$

**B 6**  $s \models \neg\sigma$  iff  $s \not\models \sigma$

This is not the only way of understanding entailment in situation structures, and may not be the best. One potentially undesirable feature of locally classical situation structures is that the set of infons supported by any given situation is bound to be infinite. The set of facts in a fact structure is closed under logical consequence, and so will be infinite, given a few assumptions about the structure of compound infons. The requirement that the set of infons supported by a situation form a fact structure therefore ensures that this set is infinite.

Whether or not this is acceptable depends on which theoretical role situations are intended to play. As objective truth-makers they may be taken to support infinitely many infons without causing alarm; but if situations are to represent the information state of a finite agent, some might argue that supporting infinitely many infons is an unfortunate consequence of the theory.

Alternatives may be found by looking at different ways in which the notion of a fact is to be related to that of a situation. For example, given a situation structure, we may construct a fact structure by defining

$\mathbf{Fact}(\sigma)$  iff  $\sigma$  is saturated and  $\exists s \ s \models \sigma$ .

Say that a situation structure is *globally classical (intuitionistic)* if the fact structure so-defined is classical (intuitionistic). Clearly every locally classical

structure is globally classical; indeed, it seems likely that this is a minimal requirement for a situation structure in which classical reasoning makes any sense at all.

Intermediate positions may be taken by linking factuality to the  $\sqsubseteq$ -order between situations. For example, define the *eventual* fact structure of each situation  $s$  by

$$\text{Fact}(s) \text{ iff } \sigma \text{ is saturated and } \forall s' \sqsubseteq s \exists s'' \sqsubseteq s' s'' \models \sigma.$$

A situation structure is *eventually classical* if the eventual fact structure of each situation is classical. A situation structure may be both locally intuitionistic and eventually classical.<sup>11</sup>

## 4.8 Propositions

On the referential account the informativeness of an infon depends on which situation it is taken to be about. The information that Melanie has Jerry's car keys may be correct information about some situations and incorrect information about others.

It is natural to ask: what has happened to the concept of truth on this account? The short answer is that truth is a property of *propositions*, which will be modelled later in this section. The longer answer is best given in the context of providing an analysis of the use of language to make statements.

Suppose that a sentence  $S$  is uttered. The utterance occurs as part of a situation  $u$  involving the speaker, the hearer, and perhaps many other things which play a part in determining what is said;  $u$  is called the *utterance situation*. Among the many facts supported by  $u$  only some are linguistically relevant. Let  $\sigma_u$  be some linguistically relevant information supported by  $u$ . Typically,  $\sigma_u$

<sup>11</sup>Indeed, this is an attractive combination. Take an extensional Kripke-model for intuitionistic predicate logic, consisting of a partially ordered set of points, a domain monotonically associated with each point, and an interpretation of each atomic predicate in each domain (monotonically, again), with the property that for points  $p_1$  and  $p_2$ , if the domain and extensions of predicates in  $p_1$  are included in those of  $p_2$  then  $p_1 \leq p_2$ . Now construct a situation structure by defining basic infons to be tuples  $\langle r, a_1, \dots, a_n \rangle$  for each  $n$ -ary predicate  $r$  and  $a_1, \dots, a_n$  elements of some domain. Compound infons are terms built from the basic infons using  $\bigwedge$ ,  $\bigvee$ ,  $\supset$  and  $\neg$ . Situations are points and the supports relation is defined inductively by

1.  $\text{HoldsIn}(\langle r, a_1, \dots, a_n \rangle, p)$  iff  $\langle a_1, \dots, a_n \rangle$  is in the extension of  $r$  in  $p$
2.  $\text{HoldsIn}(\bigwedge \Sigma, p)$  iff  $\text{HoldsIn}(\sigma, p)$  for each  $\sigma$  in  $\Sigma$
3.  $\text{HoldsIn}(\bigvee \Sigma, p)$  iff  $\text{HoldsIn}(\sigma, p)$  for some  $\sigma$  in  $\Sigma$
4.  $\text{HoldsIn}(\neg \sigma, p)$  iff for no  $p' \geq p$   $\text{HoldsIn}(\sigma, p')$
5.  $\text{HoldsIn}(\sigma \supset \tau, p)$  iff for each  $p' \geq p$  if  $\text{HoldsIn}(\sigma, p')$  then  $\text{HoldsIn}(\tau, p')$

The resulting structure is locally intuitionistic, and may not be locally classical, as there is no guarantee that either  $\sigma$  or  $\neg \sigma$  holds at a given  $p$ . However, it is eventually classical because if  $\sigma$  is not an eventual fact of  $p$  ( $\neg \forall p' \geq p \exists p'' \geq p' p'' \models \sigma$ ) then there is a  $p' \geq p$  such that  $\forall p'' \geq p' p'' \not\models \sigma$ , and so  $p' \models \neg \sigma$ , and so  $\neg \sigma$  is an eventual fact of  $p$ .

might contain information about the syntactic composition of  $S$ , the referents of indexicals and anaphors in  $S$ , and so on.

Suppose further that by uttering  $S$  the speaker is making a statement, rather than asking a question, giving a command, or any of the many other things one can do with words. For example, suppose  $S$  is the sentence ‘No one is wearing shorts’, said by Alex to Nicholas while looking out of the window at people walking along the street. In such a case, the conventions of language use and the particular circumstances of  $u$ , including perhaps the intentions of the speaker, the discourse context, the demonstrative actions of the speaker, and so on, all go to determine what the speaker is talking about. More specifically, they determine which situation is being described. In this case, the *described situation*  $d$  is the goings-on in the street outside Alex’s apartment.

Other conventions of language use, especially those fixing the meanings of those words occurring in  $S$  and their mode of syntactic combination, go to determine what is being said about  $d$ . In our example, Alex said, about the situation  $d$ , that it is one in which no one is wearing shorts. The information (or misinformation)  $\tau_u$  that no one is wearing shorts is the *descriptive content* of Alex’s utterance.

Now we are in a position to say what is true and what is not. The primary candidate for truth is Alex’s statement; it is true iff the described situation  $d$  supports the descriptive content  $\tau_u$ . But it is far from clear what sort of thing Alex’s statement is and how it is related to the various other components of our analysis: the utterance situation, the described situation, the linguistically relevant aspects of the utterance and the descriptive content. All these things are aspects of Alex’s statement, but its truth is determined just by  $d$  and  $\tau_u$ .

Situation theorists and situation semanticists have appropriated the word ‘proposition’ to refer to an abstract entity ( $s \models \sigma$ ) consisting of the combination of a situation  $s$  and an infon  $\sigma$ . On this modelling, propositions are either true or false: ( $s \models \sigma$ ) is true iff  $s \models \sigma$ .

The *propositional content* of Alex’s statement is defined to be the proposition ( $d \models \tau_u$ ) consisting of the described situation  $d$  and the descriptive content  $\tau_u$ . Thus her statement is true iff it has a true propositional content.

We see that there are two different “contents” to a given utterance  $u$ . First, there is the purely descriptive content  $\tau_u$ . This is shared by a host of other statements. In this case, assuming that there is no ambiguity involved, the descriptive content of Alex’s statement is the same as that of any other utterance of the sentence  $S$ .<sup>12</sup> Second, there is the propositional content ( $d \models \tau_u$ ). Not all utterances of  $S$  have the same propositional content because they may be used to describe different situations. For example, if the shutters were closed and Alex was whispering  $S$  to Nicholas as their guests arrived at the apartment, the situation  $d'$  described would be quite different, and so the proposition content

<sup>12</sup>If Alex had used a sentence containing indexicals or demonstratives then this would not be the case. For example, Nicholas’s utterance of the sentence ‘You are wearing shorts’ does not have the same descriptive content as an utterance of the same sentence by Alex.



$(d' \models \tau_u)$  would also be different. Indeed, it could be that  $(d' \models \tau_u)$  is true, but  $(d \models \tau_u)$  is false.

For more on situation semantics, a good place to look are the conference proceedings entitled *Situation Theory and its Applications*, published by CSLI press. (These are the name of a biannual conference. The name of the conference has now changed to “Information-Theoretic Approaches to Logic, Language, and Computation.”) We have listed in the references a number of papers from these books. See also Barwise (1989), Barwise and Perry (1983), Barwise and Etchemendy (1987) Cooper (1991), Devlin (1991), Fenstad et al (1987), Gawron and Peters (1990), and ter Meulen (1995).

## 4.9 Types and Proposition Structures

A basic difference between infons and propositions is that propositions are true or false absolutely, whereas infons may only be determined as factual or not relative to a given situation. Once this perspective is adopted, it becomes tempting to consider a wider class of propositions. For example, whether or not a given object  $\sigma$  is an infon does not vary from situation to situation, and so we might also consider propositions determined by an object  $\sigma$  and a *type* of object, say the type  $\text{Inf}^*$  of infons, which is true just in case  $\sigma$  is an infon. We write such a proposition as  $(\sigma: \text{Inf}^*)$ .

Extending this idea to other structural relations, we write  $R^*$  to denote the *type* corresponding to the structural relation  $R$ , and  $(\alpha: R^*)$  for the proposition that the objects in the sequence  $\alpha$  stand in the (structural) relation  $R$ . This move allows us to make do with one kind of proposition. Propositions of the form  $(s \models \sigma)$  may be identified with those of the form  $(s, \sigma: \text{HoldsIn}^*)$ .

Moreover, it is now easy to see how to use our present framework to study propositions and types. We need structural relations **Seq**, **Type**, **Prop**, and a non-structural relation **True** with the following interpretations:

|                            |  |
|----------------------------|--|
| <b>Seq</b> ( $\alpha, p$ ) | $\alpha$ is a sequence of objects in the basic proposition $p$ |
| <b>Type</b> ( $T, p$ )     | $T$ is the type in the basic proposition $p$                   |
| <b>Prop</b> ( $p$ )        | $p$ is a proposition   |
| <b>True</b> ( $p$ )        | $p$ is true.   |

The key is to observe that the axioms governing the structure of propositions are exactly the same as those governing facts, with the following relabelling of predicates:

|             |               |             |
|-------------|---------------|-------------|
| <b>Ass</b>  | $\rightarrow$ | <b>Seq</b>  |
| <b>Rel</b>  | $\rightarrow$ | <b>Type</b> |
| <b>Inf</b>  | $\rightarrow$ | <b>Prop</b> |
| <b>Fact</b> | $\rightarrow$ | <b>True</b> |

With this in mind, we make the following definition:

**Definition 4.9** A *proposition structure*  $\mathfrak{P} = [\mathfrak{A}, \text{Fun}, \text{Seq}, \text{Type}, \text{Prop}; \text{True}, \sqsubseteq]$  is a fact structure in which each structural relation in  $\mathfrak{A}$  is represented.

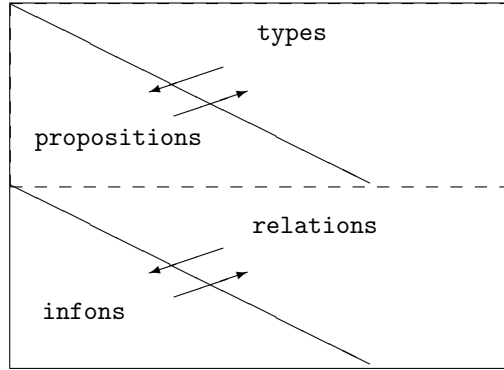
The requirement that each structural relation is represented is exactly what is needed. For example, if  $\mathfrak{A}$  is a situation structure then it will have a binary structural relation **HoldsIn**, and so there must be a type **HoldsIn**<sup>\*</sup>*A*, roles **HoldsIn**<sub>1</sub><sup>\*</sup> and **HoldsIn**<sub>2</sub><sup>\*</sup>, and for each situation *s* and infon  $\sigma$ , a function *f* such that  $\text{Fun}(\text{HoldsIn}_1^*, \sigma, f)$  and  $\text{Fun}(\text{HoldsIn}_2^*, s, f)$ , and a proposition *p* such that  $\text{Seq}(f, p)$  and  $\text{Type}(\text{HoldsIn}^*, p)$ , and

$$\text{True}(p) \text{ iff } \text{HoldsIn}(\sigma, s)$$

A proposition is *basic* if it has a type. Compound propositions, such as conjunctions, disjunctions and negations may be modelled in the same way as compound infons. Typically, situation theorists have proposed that the logic of propositions is classical. This is compatible with any logic on infons, but a favourite combination is to take a Kleene situation structure with a classical proposition structure. To see that this is a coherent combination, note that the proposition  $(s \models \sigma)$  is either true or not, and so in a classical proposition structure one of the propositions  $(s \models \sigma)$  or  $\neg(s \models \sigma)$  is true. If a situation *s* fails to support  $\sigma$  then  $\neg(s \models \sigma)$  is true, but this does not imply that *s* supports  $\neg\sigma$  unless the underlying situation structure is also (locally) classical.

Likewise, we may build models in which *abstract types* of the form  $\lambda x.p(x)$  exist (where  $p(x)$  is a proposition for some range of values of *x*), just as we did for abstract relations.

The account of propositions developed by situation theorists, and summarised here, leads to a two-story residence for logical structure.



On the ground floor there are infons. Basic infons are made up of assignments and relations; compound infons are structured objects constructed from basic infons; and more complex relations are formed from infons by abstraction.

On the upper floor there are propositions. Basic propositions are made up of assignments and types; compound propositions are structured objects constructed from basic propositions; and more complex types are formed from propositions by abstraction.

The two floors may differ in matters of truth and factuality. Typically, the ground floor is *situated*. An infon needs a situation to determine if it is factual or not; likewise, the extension of a relation may vary from situation to situation. The upper floor is *absolute*, the lower *relative*. Propositions are either true or not, and types have a fixed extensions. The floors are linked by the relationship between a structural relation  $R$  on the ground floor and the type  $R^*$  which represents it on the upper floor.

Like all metaphors involving a duality, the two-story picture raises a question of redundancy: can we make do with just one of the two, and if so, which one? We do not have any final answers on this question, but we believe that the conceptual and technical work of this chapter might help sort out the relationship between the absolute and the relative.

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