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Associative algebra

In mathematics, an **associative algebra** is an algebraic structure with compatible operations of addition, multiplication (assumed to be associative), and a scalar multiplication by elements in some field. The addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a vector space over K. In this article we will also use the term K-algebra to mean an associative algebra over the field K. A standard first example of a K-algebra is a ring of square matrices over a field K, with the usual matrix multiplication.

In this article associative algebras are assumed to have a multiplicative unit, denoted 1; they are sometimes called **unital associative algebras** for clarification. In some areas of mathematics this assumption is not made, and we will call such structures non-unital associative algebras. We will also assume that all rings are unital, and all ring homomorphisms are unital.

Many authors consider the more general concept of an associative algebra over a commutative ring R, instead of a field: An ***R*-algebra** is an R-module with an associative R-bilinear binary operation, which also contains a multiplicative identity. For examples of this concept, if S is any ring with center C, then S is an associative C-algebra.

Contents

Definition

Let R be a fixed commutative ring (so R could be a field). An **associative *R*-algebra** (or more simply, an ***R*-algebra**) is an additive abelian group A which has the structure of both a ring and an R-module in such a way that the scalar multiplication satisfies

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$$

for all $r \in R$ and $x, y \in A$. Furthermore, A is assumed to be unital, which is to say it contains an element 1 such that

$$1x = x = x1$$

for all $x \in A$. Note that such an element 1 must be unique.

In other words, A is an R-module together with (1) an R-bilinear map $A \times A \to A$, called the multiplication, and (2) the multiplicative identity, such that the multiplication is associative:

$$x(yz) = (xy)z$$

for all x, y, and z in A. (Technical note: the multiplicative identity is a datum,[1] while associativity is a property. By the uniqueness of the multiplicative identity, "unitarity" is often treated like a property.) If one drops the requirement for the associativity, then one obtains a non-associative algebra.

If A itself is commutative (as a ring) then it is called a **commutative *R*-algebra**.

As a monoid object in the category of modules

The definition is equivalent to saying that a unital associative R-algebra is a monoid object in ***R*-Mod** (the monoidal category of R-modules). By definition, a ring is a monoid object in the category of abelian groups; thus, the notion of an associative algebra is obtained by replacing the category of abelian groups with the category of modules.

Pushing this idea further, some authors have introduced a "generalized ring" as a monoid object in some other category that behaves like the category of modules. Indeed, this reinterpretation allows one to avoid making an explicit reference to elements of an algebra A. For example, the associativity can be expressed as follows. By the universal property of a tensor product of modules, the multiplication (the R-bilinear map) corresponds to a unique R-linear map

$$m : A \otimes_R A \to A.$$

The associativity then refers to the identity:

$$m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}).$$

From ring homomorphisms

An associative algebra amounts to a ring homomorphism whose image lies in the center. Indeed, starting with a ring A and a ring homomorphism $\eta \colon R \to A$ whose image lies in the center of A, we can make A an R-algebra by defining

$$r \cdot x = \eta(r)x$$

for all $r \in R$ and $x \in A$. If A is an R-algebra, taking $x = 1$, the same formula in turn defines a ring homomorphism $\eta\colon R \to A$ whose image lies in the center.

If A is commutative then the center of A is equal to A, so that a commutative R-algebra can be defined simply as a homomorphism $\eta\colon R \to A$ of commutative rings.

The ring homomorphism η appearing in the above is often called a structure map. In the commutative case, one can consider the category whose objects are ring homomorphisms $R \to A$; i.e., commutative R-algebras and whose morphisms are ring homomorphisms $A \to A'$ that are under R; i.e., $R \to A \to A'$ is $R \to A'$ (i.e., the coslice category of the category of commutative rings under R.) The prime spectrum functor Spec then determines an anti-equivalence of this category to the category of affine schemes over Spec R.

How to weaken the commutativity assumption is a subject matter of noncommutative algebraic geometry and, more recently, of derived algebraic geometry. See also: generic matrix ring.

Algebra homomorphisms

A homomorphism between two R-algebras is an R-linear ring homomorphism. Explicitly, $\varphi : A_1 \to A_2$ is an **associative algebra homomorphism** if

$$\begin{aligned}
&\varphi(r \cdot x) = r \cdot \varphi(x)\\
&\varphi(x + y) = \varphi(x) + \varphi(y)\\
&\varphi(xy) = \varphi(x)\varphi(y)\\
&\varphi(1) = 1
\end{aligned}$$

The class of all R-algebras together with algebra homomorphisms between them form a category, sometimes denoted ***R*-Alg**.

The subcategory of commutative R-algebras can be characterized as the coslice category R/**CRing** where **CRing** is the category of commutative rings.

Examples

The most basic example is a ring itself; it is an algebra over its center or any subring lying in the center. In particular, any commutative ring is an algebra over any of its subrings. Other examples abound both from algebra and other fields of mathematics.

Algebra

- Any ring *A* can be considered as a **Z**-algebra. The unique ring homomorphism from **Z** to *A* is determined by the fact that it must send 1 to the identity in A. Therefore, rings and **Z**-algebras are equivalent concepts, in the same way that abelian groups and **Z**-modules are equivalent.

- Any ring of characteristic n is a ($\mathbf{Z}/n\mathbf{Z}$)-algebra in the same way.
- Given an R-module M, the endomorphism ring of M, denoted $\mathrm{End}_R(M)$ is an R-algebra by defining $(r\cdot\phi)(x) = r\cdot\phi(x)$.
- Any ring of matrices with coefficients in a commutative ring R forms an R-algebra under matrix addition and multiplication. This coincides with the previous example when M is a finitely-generated, free R-module.
- The square n-by-n matrices with entries from the field K form an associative algebra over K. In particular, the 2 × 2 real matrices form an associative algebra useful in plane mapping.
- The complex numbers form a 2-dimensional associative algebra over the real numbers.
- The quaternions form a 4-dimensional associative algebra over the reals (but not an algebra over the complex numbers, since the complex numbers are not in the center of the quaternions).
- The polynomials with real coefficients form an associative algebra over the reals.
- Every polynomial ring $R[x_1, ..., x_n]$ is a commutative R-algebra. In fact, this is the free commutative R-algebra on the set $\{x_1, ..., x_n\}$.
- The free R-algebra on a set E is an algebra of polynomials with coefficients in R and noncommuting indeterminates taken from the set E.
- The tensor algebra of an R-module is naturally an R-algebra. The same is true for quotients such as the exterior and symmetric algebras. Categorically speaking, the functor which maps an R-module to its tensor algebra is left adjoint to the functor which sends an R-algebra to its underlying R-module (forgetting the ring structure).
- Given a commutative ring R and any ring A the tensor product $R\otimes_{\mathbf{Z}}A$ can be given the structure of an R-algebra by defining $r\cdot(s\otimes a) = (rs\otimes a)$. The functor which sends A to $R\otimes_{\mathbf{Z}}A$ is left adjoint to the functor which sends an R-algebra to its underlying ring (forgetting the module structure).

Representation theory

- The universal enveloping algebra of a Lie algebra is an associative algebra that can be used to study the given Lie algebra.
- If G is a group and R is a commutative ring, the set of all functions from G to R with finite support form an R-algebra with the convolution as multiplication. It is called the group algebra of G. The construction is the starting point for the application to the study of (discrete) groups.
- If G is an algebraic group (e.g., semisimple complex Lie group), then the coordinate ring of G is the Hopf algebra A corresponding to G. Many structures of G translate to those of A.

Analysis

- Given any Banach space X, the continuous linear operators $A : X \to X$ form an associative algebra (using composition of operators as multiplication); this is a Banach algebra.
- Given any topological space X, the continuous real- or complex-valued functions on X form a real or complex associative algebra; here the functions are added and multiplied pointwise.
- The set of semimartingales defined on the filtered probability space $(\Omega,F,(F_t)_{t\geq 0},\mathrm{P})$ forms a ring under stochastic integration.
- The Weyl algebra

Geometry and combinatorics

- The Clifford algebras, which are useful in geometry and physics.
- Incidence algebras of locally finite partially ordered sets are associative algebras considered in combinatorics.

Constructions

Subalgebras

A subalgebra of an *R*-algebra *A* is a subset of *A* which is both a subring and a submodule of *A*. That is, it must be closed under addition, ring multiplication, scalar multiplication, and it must contain the identity element of *A*.

Quotient algebras

Let *A* be an *R*-algebra. Any ring-theoretic ideal *I* in *A* is automatically an *R*-module since $r \cdot x = (r1_A)x$. This gives the quotient ring *A*/*I* the structure of an *R*-module and, in fact, an *R*-algebra. It follows that any ring homomorphic image of *A* is also an *R*-algebra.

Direct products

The direct product of a family of *R*-algebras is the ring-theoretic direct product. This becomes an *R*-algebra with the obvious scalar multiplication.

Free products

One can form a free product of *R*-algebras in a manner similar to the free product of groups. The free product is the coproduct in the category of *R*-algebras.

Tensor products

The tensor product of two *R*-algebras is also an *R*-algebra in a natural way. See tensor product of algebras for more details.

Coalgebras

An associative algebra over *K* is given by a *K*-vector space *A* endowed with a bilinear map $A \times A \to A$ having 2 inputs (multiplicator and multiplicand) and one output (product), as well as a morphism $K \to A$ identifying the scalar multiples of the multiplicative identity. If the bilinear map $A \times A \to A$ is reinterpreted as a linear map (i. e., morphism in the category of *K*-vector spaces) $A \otimes A \to A$ (by the universal property of the tensor product), then we can view an associative algebra over *K* as a *K*-vector space *A* endowed with two morphisms (one of the form $A \otimes A \to A$ and one of the form $K \to A$) satisfying certain conditions which boil down to the algebra axioms. These two morphisms can be dualized using categorial duality by reversing all arrows in the commutative diagrams which describe the algebra axioms; this defines the structure of a coalgebra.

There is also an abstract notion of F-coalgebra, where *F* is a functor. This is vaguely related to the notion of coalgebra discussed above.

Representations

A representation of an algebra *A* is an algebra homomorphism $\rho: A \to \mathrm{End}(V)$ from *A* to the endomorphism algebra of some vector space (or module) *V*. The property of ρ being an algebra homomorphism means that ρ preserves the multiplicative operation (that is, $\rho(xy)=\rho(x)\rho(y)$ for all *x* and *y* in *A*), and that ρ sends the unity of *A* to the unity of End(*V*) (that is, to the identity endomorphism of *V*).

If *A* and *B* are two algebras, and $\rho: A \to \mathrm{End}(V)$ and $\tau: B \to \mathrm{End}(W)$ are two representations, then there is a (canonical) representation $A \otimes B \to \mathrm{End}(V \otimes W)$ of the tensor product algebra $A \otimes B$ on the vector space $V \otimes W$. However, there is no natural way of defining a tensor product of two representations of a single associative algebra in such a way that the result is still a representation of that same algebra (not of its tensor product with itself), without somehow imposing additional conditions. Here, by *tensor product of representations*, the usual meaning is intended: the result should be a linear representation of the same algebra on the product vector space. Imposing such additional structure typically leads to the idea of a Hopf algebra or a Lie algebra, as demonstrated below.

Motivation for a Hopf algebra

Consider, for example, two representations $\sigma : A \to \mathrm{End}(V)$ and $\tau : A \to \mathrm{End}(W)$. One might try to form a tensor product representation $\rho : x \mapsto \sigma(x) \otimes \tau(x)$ according to how it acts on the product vector space, so that

$$\rho(x)(v \otimes w) = (\sigma(x)(v)) \otimes (\tau(x)(w)).$$

However, such a map would not be linear, since one would have

$$\rho(kx) = \sigma(kx) \otimes \tau(kx) = k\sigma(x) \otimes k\tau(x) = k^2(\sigma(x) \otimes \tau(x)) = k^2\rho(x)$$

for $k \in K$. One can rescue this attempt and restore linearity by imposing additional structure, by defining an algebra homomorphism $\Delta: A \to A \otimes A$, and defining the tensor product representation as

$$\rho = (\sigma \otimes \tau) \circ \Delta.$$

Such a homomorphism Δ is called a comultiplication if it satisfies certain axioms. The resulting structure is called a bialgebra. To be consistent with the definitions of the associative algebra, the coalgebra must be co-associative, and, if the algebra is unital, then the co-algebra must be co-unital as well. A Hopf algebra is a bialgebra with an additional piece of structure (the so-called antipode), which allows not only to define the tensor product of two representations, but also the Hom module of two representations (again, similarly to how it is done in the representation theory of groups).

Motivation for a Lie algebra

One can try to be more clever in defining a tensor product. Consider, for example,

$$x \mapsto \rho(x) = \sigma(x) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \tau(x)$$

so that the action on the tensor product space is given by

$$\rho(x)(v \otimes w) = (\sigma(x)v) \otimes w + v \otimes (\tau(x)w) \ .$$

This map is clearly linear in x, and so it does not have the problem of the earlier definition. However, it fails to preserve multiplication:

$$\rho(xy) = \sigma(x)\sigma(y) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \tau(x)\tau(y) \ .$$

But, in general, this does not equal

$$\rho(x)\rho(y) = \sigma(x)\sigma(y) \otimes \mathrm{Id}_W + \sigma(x) \otimes \tau(y) + \sigma(y) \otimes \tau(x) + \mathrm{Id}_V \otimes \tau(x)\tau(y) \quad .$$

This shows that this definition of a tensor product is too naive; the obvious fix is to define it such that it is antisymmetric, so that the middle two terms cancel. This leads to the concept of a Lie algebra.

Non-unital algebras

Some authors use the term "associative algebra" to refer to structures which do not necessarily have a multiplicative identity, and hence consider homomorphisms which are not necessarily unital.

An example of a non-unital associative algebra is given by the set of all functions f: $\mathbf{R} \to \mathbf{R}$ whose limit as x nears infinity is zero.

See also

- Abstract algebra
- Algebraic structure
- Algebra over a field

Notes

1. Put in another way, there is the forgetful functor from the category of unital associative algebras to the category of possibly non-unital associative algebras.

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