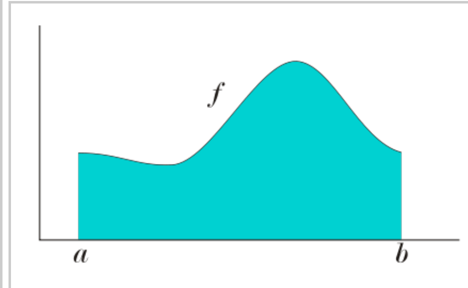
Lebesgue integration

From Wikipedia, the free encyclopedia

In mathematics, the integral of a non-negative function of a single variable can be regarded, in the simplest case, as the area between the graph of that function and the x-axis. The **Lebesgue integral** extends the integral to a larger class of functions. It also extends the domains on which these functions can be defined.



The integral of a positive function can be interpreted as the area under a curve.

Mathematicians had long understood that for non-negative functions with a smooth enough graph—such as continuous functions on closed bounded intervals—the *area under the curve* could be defined as the integral, and computed using approximation techniques on the region by polygons. However, as the need to consider more irregular functions arose—e.g., as a result of the limiting processes of mathematical analysis and the mathematical theory of probability—it became clear that more careful approximation techniques were needed to define a suitable integral. Also, one might wish to integrate on spaces more general than the real line. The Lebesgue integral provides the right abstractions needed to do this important job.

The Lebesgue integral plays an important role in probability theory, in the branch of mathematics called real analysis and in many other fields in the mathematical sciences. It is named after Henri Lebesgue (1875–1941), who introduced the integral (Lebesgue 1904). It is also a pivotal part of the axiomatic theory of probability.

The term *Lebesgue integration* can mean either the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue, or the specific case of integration of a function defined on a sub-domain of the real line with respect to Lebesgue measure.

Contents

Introduction

The integral of a function f between limits a and b can be interpreted as the area under the graph of f. This is easy to understand for familiar functions such as polynomials, but what does it mean for more exotic functions? In general, for which class of functions does "area under the curve" make sense? The answer to this question has great theoretical and practical importance.

As part of a general movement toward rigor in mathematics in the nineteenth century, mathematicians attempted to put integral calculus on a firm foundation. The Riemann integral—proposed by Bernhard Riemann (1826–1866)—is a broadly successful attempt to provide such a foundation. Riemann's definition starts with the construction of a sequence of easily calculated areas that converge to the integral of a given function. This definition is successful in the sense that it gives the expected answer for many already-solved problems, and gives useful results for many other problems.

However, Riemann integration does not interact well with taking limits of sequences of functions, making such limiting processes difficult to analyze. This is important, for instance, in the study of Fourier series, Fourier transforms, and other topics. The Lebesgue integral is better able to describe how and when it is possible to take limits under the integral sign (via the powerful monotone convergence theorem and dominated convergence theorem).

While the Riemann integral considers the area under a curve as made out of vertical rectangles, the Lebesgue definition considers horizontal slabs that are not necessarily just rectangles, and so it is more flexible. For this reason, the Lebesgue definition makes it possible to calculate integrals for a broader class of functions. For example, the Dirichlet function, which is 0 where its argument is irrational and 1 otherwise, has a Lebesgue integral, but does not have a Riemann integral. Furthermore, the Lebesgue integral of this function is zero, which agrees with the intuition that picking a real number uniformly at random from the unit interval, the probability of picking a rational number should be zero.

Lebesgue summarized his approach to integration in a letter to Paul Montel:

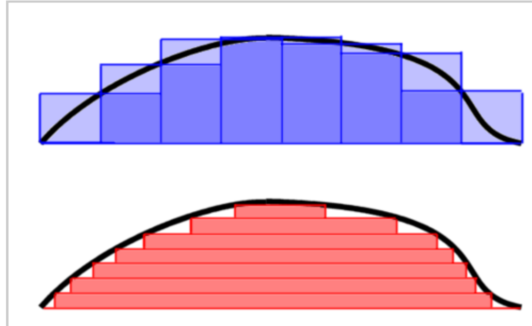
> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

— *Source*: (Siegmund-Schultze 2008)

The insight is that one should be able to rearrange the values of a function freely, while preserving the value of the integral. This process of rearrangement can convert a very pathological function into one that is "nice" from the point of view of integration, and thus let such pathological functions be integrated.

Intuitive interpretation

To get some intuition about the different approaches to integration, let us imagine that we want to find a mountain's volume (above sea level).



Riemann-Darboux's integration (in blue [min. approx.] and light blue [max. approx.]) and Lebesgue integration (in red).

The Riemann–Darboux approach

Divide the base of the mountain into two grids of 1 meter square each; one grid where each square ends above the mountain except at one point (the max approximation), and another grid where each square ends below the mountain except at one point (the min approximation). The volume of any single grid square is exactly 1 m^2 × (that square's altitude). The total volume is approximated from above and below by each respective grid, and converges (under given conditions) to their average value as each grid area is reduced.

The Lebesgue approach

Draw a contour map of the mountain, where adjacent contours are 1 meter of altitude apart. The volume of earth that a single contour slab contains is exactly 1 m × (that contour's area). The total volume is approximated by the sum of these volumes and converges (under given conditions) to a unique sum as the altitude of each slab is reduced.

Folland summarizes the difference between the Riemann and Lebesgue approaches thus: "to compute the Riemann integral of f, one partitions the domain $[a, b]$ into subintervals", while in the Lebesgue integral, "one is in effect partitioning the range of f ."[1]

Towards a formal definition

To define the Lebesgue integral requires the formal notion of a measure that, roughly, associates to each set A of real numbers a nonnegative number $\mu(A)$ representing the "size" of A. This notion of "size" should agree with the usual length of an interval or disjoint union of intervals. Suppose that $f : \mathbb{R} \to \mathbb{R}^+$ is a non-negative real-valued function. Using the "partitioning the range of f" philosophy, the integral of f should be the sum over t of the elementary area contained in the thin horizontal strip between $y = t$ and $y = t + dt$. This elementary area is just

$$\mu(\{x \mid f(x) > t\})\ dt.$$

Let

$$f^*(t) = \mu\left(\{x \mid f(x) > t\}\right).$$

The Lebesgue integral of f is then defined by[2]

$$\int f\,d\mu = \int_0^\infty f^*(t)\,dt$$

where the integral on the right is an ordinary improper Riemann integral (note that f^* is a non-negative decreasing function, and therefore has a well-defined improper Riemann integral). For a suitable class of functions (the measurable functions), this defines the Lebesgue integral.

A general (not necessarily positive) function f is Lebesgue integrable if the area between the graph of f and the x-axis is finite:

$$\int |f|\,d\mu < +\infty.$$

In that case, the integral is, as in the Riemannian case, the difference between the area above the x-axis and the area below the x-axis:

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu$$

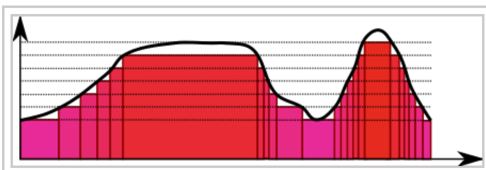
where $f = f^+ - f^-$ is the unique decomposition of f into the difference of two non-negative functions, given explicitly by

$$\begin{aligned} f^+(x) &= \max\{f(x), 0\} &&= \begin{cases} f(x), & \text{if } f(x) > 0, \\ 0, & \text{otherwise} \end{cases} \\ f^-(x) &= \max\{-f(x), 0\} &&= \begin{cases} -f(x), & \text{if } f(x) < 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Construction

The discussion that follows parallels the most common expository approach to the Lebesgue integral. In this approach, the theory of integration has two distinct parts:

1. A theory of measurable sets and measures on these sets
2. A theory of measurable functions and integrals on these functions



Approximating a function by simple functions.

The function whose integral is to be found is then approximated by certain so-called simple functions, whose integrals can be written in terms of the measure. The integral of the original function is then the limit of the integral of the simple functions.

Measure theory

Measure theory was initially created to provide a useful abstraction of the notion of length of subsets of the real line—and, more generally, area and volume of subsets of Euclidean spaces. In particular, it provided a systematic answer to the question of which subsets of $\mathbb{R}$ have a length. As later set theory developments showed (see non-measurable set), it is actually impossible to assign a length to all subsets of $\mathbb{R}$ in a way that preserves some natural additivity and translation invariance properties. This suggests that picking out a suitable class of *measurable* subsets is an essential prerequisite.

The Riemann integral uses the notion of length explicitly. Indeed, the element of calculation for the Riemann integral is the rectangle $[a, b] \times [c, d]$, whose area is calculated to be $(b-a)(d-c)$. The quantity $b-a$ is the length of the base of the rectangle and $d-c$ is the height of the rectangle. Riemann could only use planar rectangles to approximate the area under the curve, because there was no adequate theory for measuring more general sets.

In the development of the theory in most modern textbooks (after 1950), the approach to measure and integration is *axiomatic*. This means that a measure is any function μ defined on a certain class X of subsets of a set E, which satisfies a certain list of properties. These properties can be shown to hold in many different cases.

Integration

We start with a measure space (E, X, μ) where E is a set, X is a σ-algebra of subsets of E, and μ is a (non-negative) measure on E defined on the sets of X.

For example, E can be Euclidean n-space $\mathbb{R}^n$ or some Lebesgue measurable subset of it, X is the σ-algebra of all Lebesgue measurable subsets of E, and μ is the Lebesgue measure. In the mathematical theory of probability, we confine our study to a probability measure μ, which satisfies $\mu(E) = 1$.

Lebesgue's theory defines integrals for a class of functions called measurable functions. A real-valued function f on E is measurable if the pre-image of every interval of the form (t, ∞) is in X:

$$\{x \mid f(x) > t\} \in X \quad \forall t \in \mathbb{R}.$$

We can show that this is equivalent to requiring that the pre-image of any Borel subset of $\mathbb{R}$ be in X. The set of measurable functions is closed under algebraic operations, but more importantly it is closed under various kinds of point-wise sequential limits:

$$\sup_{k\in\mathbb{N}} f_k, \quad \liminf_{k\in\mathbb{N}} f_k, \quad \limsup_{k\in\mathbb{N}} f_k$$

are measurable if the original sequence $(f_k)_k$, where $k \in \mathbb{N}$, consists of measurable functions.

We build up an integral

$$\int_E f\,d\mu = \int_E f(x)\ \mu(dx)$$

for measurable real-valued functions f defined on E in stages:

Indicator functions: To assign a value to the integral of the indicator function 1_S of a measurable set S consistent with the given measure μ, the only reasonable choice is to set:

$$\int 1_S\,d\mu = \mu(S).$$

Notice that the result may be equal to $+\infty$, unless μ is a *finite* measure.

Simple functions: A finite linear combination of indicator functions

$$\sum_k a_k 1_{S_k}$$

where the coefficients a_k are real numbers and the sets S_k are measurable, is called a measurable simple function. We extend the integral by linearity to *non-negative* measurable simple functions. When the coefficients a_k are non-negative, we set

$$\int \left(\sum_k a_k 1_{S_k}\right) d\mu = \sum_k a_k \int 1_{S_k}\,d\mu = \sum_k a_k\,\mu(S_k).$$

The convention $0 \times \infty = 0$ must be used, and the result may be infinite. Even if a simple function can be written in many ways as a linear combination of indicator functions, the integral is always the same. This can be shown using the additivity property of measures.

Some care is needed when defining the integral of a *real-valued* simple function, to avoid the undefined expression $\infty - \infty$: one assumes that the representation

$$f = \sum_k a_k 1_{S_k}$$

is such that $\mu(S_k) < \infty$ whenever $a_k \neq 0$. Then the above formula for the integral of f makes sense, and the result does not depend upon the particular representation of f satisfying the assumptions.

If B is a measurable subset of E and s is a measurable simple function one defines

$$\int_B s\,d\mu = \int 1_B\,s\,d\mu = \sum_k a_k\,\mu(S_k \cap B).$$

Non-negative functions: Let f be a non-negative measurable function on E, which we allow to attain the value $+\infty$, in other words, f takes non-negative values in the extended real number line. We define

$$\int_E f\,d\mu = \sup\left\{\int_E s\,d\mu : 0 \leq s \leq f,\ s \text{ simple}\right\}.$$

We need to show this integral coincides with the preceding one, defined on the set of simple functions, when E is a segment $[a, b]$. There is also the question of whether this corresponds in any way to a Riemann notion of integration. It is possible to prove that the answer to both questions is yes.

We have defined the integral of f for any non-negative extended real-valued measurable function on E. For some functions, this integral $\int_E f\,d\mu$ is infinite.

Signed functions: To handle signed functions, we need a few more definitions. If f is a measurable function of the set E to the reals (including $\pm\infty$), then we can write

$$f = f^+ - f^-,$$

where

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that both f^+ and f^- are non-negative measurable functions. Also note that

$$|f| = f^+ + f^-.$$

We say that the Lebesgue integral of the measurable function f *exists*, or *is defined* if at least one of $\int f^+\,d\mu$ and $\int f^-\,d\mu$ is finite:

$$\min\left(\int f^+\,d\mu, \int f^-\,d\mu\right) < \infty.$$

In this case we *define*

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu.$$

If

$$\int |f| \, d\mu < \infty,$$

we say that *f* is *Lebesgue integrable*.

It turns out that this definition gives the desirable properties of the integral.

Complex valued functions can be similarly integrated, by considering the real part and the imaginary part separately.

If $h=f+ig$ for real-valued integrable functions f, g, then the integral of h is defined by

$$\int h \, d\mu = \int f \, d\mu + i \int g \, d\mu.$$

The function is Lebesgue integrable if and only if its absolute value is Lebesgue integrable (see Absolutely integrable function).

Example

Consider the indicator function of the rational numbers, $1_{\mathbf{Q}}$. This function is nowhere continuous.

- **$1_{\mathbf{Q}}$ is not Riemann-integrable on $[0, 1]$**: No matter how the set $[0, 1]$ is partitioned into subintervals, each partition contains at least one rational and at least one irrational number, because rationals and irrationals are both dense in the reals. Thus the upper Darboux sums are all one, and the lower Darboux sums are all zero.
- **$1_{\mathbf{Q}}$ is Lebesgue-integrable on $[0, 1]$** using the Lebesgue measure: Indeed, it is the indicator function of the rationals so by definition

  $$\int_{[0,1]} 1_{\mathbf{Q}} \, d\mu = \mu(\mathbf{Q} \cap [0,1]) = 0,$$

  because $\mathbf{Q}$ is countable.

Domain of integration

A technical issue in Lebesgue integration is that the domain of integration is defined as a *set* (a subset of a measure space), with no notion of orientation. In elementary calculus, one defines integration with respect to an orientation:

$$\int_b^a f := -\int_a^b f.$$

Generalizing this to higher dimensions yields integration of differential forms. By contrast, Lebesgue integration provides an alternative generalization, integrating over subsets with respect to a measure; this can be notated as

$$\int_A f\,d\mu = \int_{[a,b]} f\,d\mu$$

to indicate integration over a subset A. For details on the relation between these generalizations, see Differential form § Relation with measures.

Limitations of the Riemann integral

Here we discuss the limitations of the Riemann integral and the greater scope offered by the Lebesgue integral. This discussion presumes a working understanding of the Riemann integral.

With the advent of Fourier series, many analytical problems involving integrals came up whose satisfactory solution required interchanging limit processes and integral signs. However, the conditions under which the integrals

$$\sum_k \int f_k(x)dx, \quad \int \left[\sum_k f_k(x)\right] dx$$

are equal proved quite elusive in the Riemann framework. There are some other technical difficulties with the Riemann integral. These are linked with the limit-taking difficulty discussed above.

Failure of monotone convergence. As shown above, the indicator function $1_{\mathbf{Q}}$ on the rationals is not Riemann integrable. In particular, the Monotone convergence theorem fails. To see why, let $\{a_k\}$ be an enumeration of all the rational numbers in $[0, 1]$ (they are countable so this can be done.) Then let

$$g_k(x) = \begin{cases} 1 & \text{if } x = a_j, j \leq k \\ 0 & \text{otherwise} \end{cases}$$

The function g_k is zero everywhere, except on a finite set of points. Hence its Riemann integral is zero. Each g_k is non-negative, and this sequence of functions is monotonically increasing, but its limit as $k \to \infty$ is $1_{\mathbf{Q}}$, which is not Riemann integrable.

Unsuitability for unbounded intervals. The Riemann integral can only integrate functions on a bounded interval. It can however be extended to unbounded intervals by taking limits, so long as this doesn't yield an answer such as $\infty - \infty$.

Integrating on structures other than Euclidean space. The Riemann integral is inextricably linked to the order structure of the real line.

Basic theorems of the Lebesgue integral

The Lebesgue integral does not distinguish between functions that differ only on a set of μ-measure zero. To make this precise, functions f and g are said to be equal almost everywhere (a.e.) if

$$\mu(\{x \in E : f(x) \neq g(x)\}) = 0.$$

- If f, g are non-negative measurable functions (possibly assuming the value $+\infty$) such that $f = g$ almost everywhere, then

$$\int f \, d\mu = \int g \, d\mu.$$

To wit, the integral respects the equivalence relation of almost-everywhere equality.

- If f, g are functions such that $f = g$ almost everywhere, then f is Lebesgue integrable if and only if g is Lebesgue integrable, and the integrals of f and g are the same if they exist.

The Lebesgue integral has the following properties:

Linearity: If f and g are Lebesgue integrable functions and a and b are real numbers, then $af + bg$ is Lebesgue integrable and

$$\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu.$$

Monotonicity: If $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu.$$

Monotone convergence theorem: Suppose $\{f_k\}_{k \in \mathbb{N}}$ is a sequence of non-negative measurable functions such that

$$f_k(x) \leq f_{k+1}(x) \quad \forall k \in \mathbb{N}, \forall x \in E.$$

Then, the pointwise limit f of f_k is Lebesgue measurable and

$$\lim_k \int f_k \, d\mu = \int f \, d\mu.$$

The value of any of the integrals is allowed to be infinite.

Fatou's lemma: If $\{f_k\}_{k \in \mathbf{N}}$ is a sequence of non-negative measurable functions, then

$$\int \liminf_k f_k \, d\mu \le \liminf_k \int f_k \, d\mu.$$

Again, the value of any of the integrals may be infinite.

Dominated convergence theorem: Suppose $\{f_k\}_{k \in \mathbf{N}}$ is a sequence of complex measurable functions with pointwise limit f, and there is a Lebesgue integrable function g (i.e., g belongs to the space L^1) such that $|f_k| \le g$ for all k.

Then, f is Lebesgue integrable and

$$\lim_k \int f_k \, d\mu = \int f \, d\mu.$$

Proof techniques

To illustrate some of the proof techniques used in Lebesgue integration theory, we sketch a proof of the above-mentioned Lebesgue monotone convergence theorem. Let $\{f_k\}_{k \in \mathbf{N}}$ be a non-decreasing sequence of non-negative measurable functions and put

$$f = \sup_{k \in \mathbb{N}} f_k = \lim_{k \in \mathbb{N}} f_k.$$

By the monotonicity property of the integral, it is immediate that:

$$\int f \, d\mu \ge \lim_k \int f_k \, d\mu$$

and the limit on the right exists, because the sequence is monotonic. We now prove the inequality in the other direction. It follows from the definition of integral that there is a non-decreasing sequence (g_n) of non-negative simple functions such that $g_n \le f$ and

$$\lim_n \int g_n \, d\mu = \int f \, d\mu.$$

Therefore, it suffices to prove that for each $n \in \mathbb{N}$,

$$\int g_n \, d\mu \le \lim_k \int f_k \, d\mu.$$

We will show that if g is a simple function and

$$\lim_k f_k(x) \ge g(x)$$

almost everywhere, then

$$\lim_k \int f_k \, d\mu \geq \int g \, d\mu.$$

By breaking up the function g into its constant value parts, this reduces to the case in which g is the indicator function of a set. The result we have to prove is then

> Suppose A is a measurable set and $\{f_k\}_{k \in \mathbb{N}}$ is a nondecreasing sequence of non-negative measurable functions on E such that
>
> $$\lim_k f_k(x) \geq 1$$
>
> for almost all $x \in A$. Then
>
> $$\lim_k \int f_k \, d\mu \geq \mu(A).$$

To prove this result, fix $\varepsilon > 0$ and define the sequence of measurable sets

$$B_k = \{x \in A : f_k(x) \geq 1 - \varepsilon\}.$$

By monotonicity of the integral, it follows that for any $k \in \mathbb{N}$,

$$(1 - \varepsilon)\mu(B_k) = \int (1 - \varepsilon) 1_{B_k} \, d\mu \leq \int f_k \, d\mu$$

Because almost every x is in B_k for large enough k, we have

$$\bigcup_k B_k = A,$$

up to a set of measure 0. Thus by countable additivity of μ, and because B_k increases with k,

$$\mu(A) = \lim_k \mu(B_k) \leq \lim_k (1 - \varepsilon)^{-1} \int f_k \, d\mu.$$

As this is true for any positive ε the result follows.

For another Proof of the Monotone Convergence Theorem, we follow:[1]

Let (X, M, μ) be a measure space.

$\{f_n\}$ is an increasing sequence of numbers, therefore its limit exists, even if it is equal to ∞. We know that

$$\int f_n \leq \int f$$

for all n, so that

$$\lim_{n\to\infty} \int f_n \leq \int f.$$

Now we need to establish the reverse inequality. Fix $\alpha \in (0, 1)$, let ϕ be a simple function with $0 \leq \phi \leq f$ and let

$$E_n = \{x : f_n(x) \geq \alpha\phi(x)\}.$$

Then $\{\mathrm{E_n}\}$ is an increasing sequence of measurable sets with $\bigcup^{\infty} E_n = X$. We know that

$$\int f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \phi.$$

This is true for all n, including the limit:

$$\lim \int_{E_n} \phi = \int \phi.$$

Hence,

$$\lim \int f_n \geq \alpha \int \phi.$$

This was true for all $\alpha \in (0, 1)$, so it remains true for $\alpha = 1$, and taking the supremum over simple $\phi \leq f$ by the definition of integration in L^+,

$$\lim \int f_n \geq \int f.$$

Now we have both inequalities, so we've shown the Monotone Convergence theorem:

$$\lim \int f_n = \int f$$

for $f_{\{n+1\}} \geq f_n$, and $f_\mathrm{n} \to f$ pointwise, $\{f_\mathrm{n}\} \in L^+$, the set of positive measurable functions from $X \to [0, \infty]$.

Alternative formulations

It is possible to develop the integral with respect to the Lebesgue measure without relying on the full machinery of measure theory. One such approach is provided by the Daniell integral.

There is also an alternative approach to developing the theory of integration via methods of functional analysis. The Riemann integral exists for any continuous function f of compact support defined on $\mathbb{R}^n$ (or a fixed open subset). Integrals of more general functions can be built starting from these integrals.

Let C_c be the space of all real-valued compactly supported continuous functions of $\mathbb{R}$. Define a norm on C_c by

$$\|f\| = \int |f(x)|\, dx.$$

Then C_c is a normed vector space (and in particular, it is a metric space.) All metric spaces have Hausdorff completions, so let L^1 be its completion. This space is isomorphic to the space of Lebesgue integrable functions modulo the subspace of functions with integral zero. Furthermore, the Riemann integral $\int$ is a uniformly continuous functional with respect to the norm on C_c, which is dense in L^1. Hence $\int$ has a unique extension to all of L^1. This integral is precisely the Lebesgue integral.

More generally, when the measure space on which the functions are defined is also a locally compact topological space (as is the case with the real numbers $\mathbb{R}$), measures compatible with the topology in a suitable sense (Radon measures, of which the Lebesgue measure is an example) an integral with respect to them can be defined in the same manner, starting from the integrals of continuous functions with compact support. More precisely, the compactly supported functions form a vector space that carries a natural topology, and a (Radon) measure is defined as a continuous linear functional on this space. The value of a measure at a compactly supported function is then also by definition the integral of the function. One then proceeds to expand the measure (the integral) to more general functions by continuity, and defines the measure of a set as the integral of its indicator function. This is the approach taken by Bourbaki (2004) and a certain number of other authors. For details see Radon measures.

Limitations of Lebesgue integral

The main purpose of Lebesgue integral is to provide an integral notion where limits of integrals hold under mild assumptions. There is no guarantee that every function is Lebesgue integrable. But it may happen that improper integrals exist for functions that are not Lebesgue integrable. One example would be

$$\frac{\sin(x)}{x}$$

over the entire real line. This function is not Lebesgue integrable, as

$$\int_{-\infty}^{\infty} \left| \frac{\sin(x)}{x} \right| dx = \infty.$$

On the other hand, $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$ exists as an improper integral and can be computed to be finite; it is twice the Dirichlet integral.

See also

- Henri Lebesgue, for a non-technical description of Lebesgue integration
- Null set
- Integration
- Measure
- Sigma-algebra
- Lebesgue space
- Lebesgue–Stieltjes integration
- Henstock–Kurzweil integral

Notes

1. Folland, Gerald B. (1984). *Real Analysis: Modern Techniques and Their Applications*. Wiley. p. 56.
2. Lieb & Loss 2001

References

- Bartle, Robert G. (1995). *The elements of integration and Lebesgue measure*. Wiley Classics Library. New York: John Wiley & Sons Inc. xii+179. ISBN 0-471-04222-6. MR 1312157.

- Bauer, Heinz (2001). *Measure and Integration Theory*. De Gruyter Studies in Mathematics 26. Berlin: De Gruyter. 236. ISBN 978-3-11-016719-1.

- Bourbaki, Nicolas (2004). *Integration. I. Chapters 1–6. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian*. Elements of Mathematics (Berlin). Berlin: Springer-Verlag. xvi+472. ISBN 3-540-41129-1. MR 2018901.

- Dudley, Richard M. (1989). *Real analysis and probability*. The Wadsworth & Brooks/Cole Mathematics Series. Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software. xii+436. ISBN 0-534-10050-3. MR 982264. Very thorough treatment, particularly for probabilists with good notes and historical references.

- Folland, Gerald B. (1999). *Real analysis: Modern techniques and their applications*. Pure and Applied Mathematics (New York) (Second ed.). New York: John Wiley & Sons Inc. xvi+386. ISBN 0-471-31716-0. MR 1681462.

- Halmos, Paul R. (1950). *Measure Theory*. New York, N. Y.: D. Van Nostrand Company, Inc. pp. xi+304.

MR 0033869. A classic, though somewhat dated presentation.

- Hazewinkel, Michiel, ed. (2001), "Lebesgue integral", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Lebesgue, Henri (1904). "Leçons sur l'intégration et la recherche des fonctions primitives". Paris: Gauthier-Villars.
- Lebesgue, Henri (1972). *Oeuvres scientifiques (en cinq volumes)* (in French). Geneva: Institut de Mathématiques de l'Université de Genève. p. 405. MR 0389523.
- Lieb, Elliott; Loss, Michael (2001). *Analysis*. Graduate Studies in Mathematics. **14** (2nd ed.). American Mathematical Society. ISBN 978-0821827833.
- Loomis, Lynn H. (1953). *An introduction to abstract harmonic analysis*. Toronto-New York-London: D. Van Nostrand Company, Inc. pp. x+190. MR 0054173. Includes a presentation of the Daniell integral.
- Munroe, M. E. (1953). *Introduction to measure and integration*. Cambridge, Mass.: Addison-Wesley Publishing Company Inc. pp. x+310. MR 0053186. Good treatment of the theory of outer measures.
- Royden, H. L. (1988). *Real analysis* (Third ed.). New York: Macmillan Publishing Company. pp. xx+444. ISBN 0-02-404151-3. MR 1013117.
- Rudin, Walter (1976). *Principles of mathematical analysis*. International Series in Pure and Applied Mathematics (Third ed.). New York: McGraw-Hill Book Co. pp. x+342. MR 0385023. Known as *Little Rudin*, contains the basics of the Lebesgue theory, but does not treat material such as Fubini's theorem.
- Rudin, Walter (1966). *Real and complex analysis*. New York: McGraw-Hill Book Co. pp. xi+412. MR 0210528. Known as *Big Rudin*. A complete and careful presentation of the theory. Good presentation of the Riesz extension theorems. However, there is a minor flaw (in the first edition) in the proof of one of the extension theorems, the discovery of which constitutes exercise 21 of Chapter 2.
- Saks, Stanisław (1937). "Theory of the Integral". Monografie Matematyczne. **7** (2nd ed.). Warszawa-Lwów: G.E. Stechert & Co.: VI+347. JFM 63.0183.05. Zbl 0017.30004.. English translation by Laurence Chisholm Young, with two additional notes by Stefan Banach.
- Shilov, G. E.; Gurevich, B. L. (1977). *Integral, measure and derivative: a unified approach. Translated from the Russian and edited by Richard A. Silverman*. Dover Books on Advanced Mathematics. New York: Dover Publications Inc. xiv+233. ISBN 0-486-63519-8. MR 0466463. Emphasizes the Daniell integral.
- Siegmund-Schultze, Reinhard (2008), "Henri Lebesgue", in Timothy Gowers, June Barrow-Green, Imre Leader, *Princeton Companion to Mathematics*, Princeton University Press .
- Teschl, Gerald. *Topics in Real and Functional Analysis*. (lecture notes).
- Yeh, James (2006). *Real Analysis: Theory of Measure and Integral 2nd. Edition Paperback*. Singapore: World Scientific Publishing Company Pte. Ltd. p. 760. ISBN 978-981-256-6.

Categories: Definitions of mathematical integration | Measure theory