Probabilistic Concurrent Constraint Programming

Vineet Gupta * Radha Jagadeesan † Vijay Saraswat ‡

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Abstract

We extend cc to allow the specification of a discrete probability distribution for random variables. We demonstrate the expressiveness of Probabilistic cc by synthesizing combinators for default reasoning. We extend Probabilistic cc uniformly over time, to get a synchronous reactive probabilistic programming language, Timed Probabilistic cc.

We describe operational and denotational models for Probabilistic cc (and Timed Probabilistic cc). The key feature of the denotational model(s) is that parallel composition is essentially set intersection. We show that the denotational model of Probabilistic cc (resp. Timed Probabilistic cc) is conservative over cc (resp. tcc). We also show that the denotational models are fully abstract for an operational semantics that records probability information.

*Xerox PARC, 3333 Coyote Hill Road, Palo Alto Ca 94304; vgupta@parc.xerox.com

†Dept. of Mathematical Sciences, Loyola University-Lake Shore Campus, Chicago, Il 60626; radha@math.luc.edu

‡AT & T Research, 180 Park Avenue, Florham Park, NJ 07932; vj@research.att.com

1 Introduction

Concurrent constraint programming (CCP, [Sar93]) is an approach to computation grounded in the use of constraints for the compositional specification of concurrent systems. It replaces the traditional notion of a store as a valuation of variables with the notion of a store as a constraint on the possible values of variables. Computation progresses by accumulating constraints in the store, and by checking whether the store entails constraints. A salient aspect of the cc computation model is that programs may be thought of as imposing constraints on the evolution of the system. cc provides four basic constructs: (tell) a (for a a primitive constraint), parallel composition (A, B), positive ask (**if** a **then** A) and hiding (**new** X **in** A). The program a imposes the constraint a. The program (A, B) imposes the constraints of both A and B — logically, this is the conjunction of A and B. **new** X **in** A imposes the constraints of A, but hides the variable X from the other programs — logically, this can be thought of as a form of existential quantification. The program **if** a **then** A reduces to A as and when the store entails a — logically, this is an (intuitionistic) implication.

A primary domain of applicability for CCP ideas is the compositional modeling of physical systems [FBB+94, FS95, GSS95]. To handle reactive systems, such as controllers for reprographics system components such as paper-feed mechanisms, CCP was extended in [SJG94] to handle discrete time. The basic idea was to adapt to CCP the Synchrony Hypothesis of Berry et al: Time is measured by the sequence of instantaneous interactions that the program has with its environment. The need to handle interrupts instantaneously (e.g., a paper jam causing power to be interrupted) rather than at the next occasion when the environment interacts with the system was handled by introducing a new idea [SJG]: a combinator (**if** a **else** A) which triggers A on the *absence* of e, that is under the assumption that e has not been produced in the store, and will not be produced throughout system execution at the present time instant. This causes the input-output relation of processes to be possibly non-monotonic but [SJG] presents a simple denotational model. The resulting theory is considerably smoother and enables the extension to hybrid systems in [GJS], which is necessary to handle continuously varying components such as variable-speed motors.

This entire conceptual framework is based, however, on the assumption that enough information will be available to model a given physical system in as accurate detail as is needed so that appropriate causal, determinate constraint-based models can be constructed. However, this assumption is violated in many physical situations — it becomes necessary to work with approximate, incomplete or uncertain information. We consider three paradigmatic examples. Consider first a telephony system that has to respond to alarms being generated from another complicated system that may only be available as a black-box. A natural model to consider for the black-box is a stochastic one, which represents the timing and duration of the alarm by random variables with some given probability distribution. Consider next situations in which it becomes necessary to model the system stochastically: for instance, not enough may be known about the mechanisms controlling wear-and-tear of rollers so that the actual departure time of a sheet of paper from a worn roller may have to be approximated by a suitably chosen distribution of a random variable. It becomes necessary then to compute system response in a setting in which system models and inputs may behave stochastically. Third, consider model-based diagnosis settings. Often information is available about *failure models* and their associated probabilities, for instance from field studies and studies of manufacturing practices. Failure models can be incorporated by assigning a variable, typically

called the *mode* of the component to represent the physical state of the component, and associating a failure model with each value of the mode variable. Probabilistic information can be incorporated by letting the mode vary according to the given probability distribution [dKW89]. The computational task at hand becomes the calculation of the most probable diagnostic hypothesis, given observations about the current state of the system.

In this paper we develop the underlying theory for (timed) *probabilistic* cc — (timed) cc augmented with the ability to describe stochastic inputs and system components – thereby making it possible to address the phenomena above within CCP. Our basic move is to allow the introduction of (discrete) random variables (RVs) with a given probability distribution. A run of the system will choose a value for an RV, with the given probability; these probability values accumulate as more and more choices are made in the course of the run. Alternate choices lead to alternate runs, with their own accumulated probability values. Inconsistencies between chosen values of RVs and constraints in the store lead to some runs being dropped. Now the possibly multiple consistent (normalized) outcomes of the various runs can be declared as the probabilistic outputs of the program. The extension to timed Probabilistic cc is done in exactly the same way as the extension of cc to `tcc` [SJG94], using the Synchrony Hypothesis.

Probabilistic cc In more detail we proceed as follows. We add a single combinator **new** (r, f) **in** A where r is a variable, and f is a probability measure for the variable r, and A is an agent. Intuitively, such an expression is executed by making a choice for the r according to the distribution f.

Example 1.1 *Consider the program:*

$$\textbf{new}\ (r, f : f(0) = f(1) = f(2) = f(3) = 0.25)\ \textbf{in}\ [\textbf{if}\ r = 0\ \textbf{then}\ a, \textbf{if}\ r = 1\ \textbf{then}\ b]$$

On input `true`*, this program can produce outputs* a*,* b *or* `true`*. The probability of* a *being the output is* .25*; similarly for* b*. The probability of the output* `true` *is* .5*.*

Random variables, like any other variables, may be constrained. These constraints may cause portions of the space of the joint probability distribution to be eliminated due to inconsistency.

Example 1.2 *Consider the program below, with the same* f *as before, and with the constraint* $r \in \{0, 1\}$ *interpreted as " r is 0 or 1" [HSD92].*

$$\textbf{new}\ (r, f)\ \textbf{in}\ [\textbf{if}\ r = 0\ \textbf{then}\ a, \textbf{if}\ r = 1\ \textbf{then}\ b, r \in \{0, 1\}]$$

On input `true`*, this program will produce outputs* a *or* b*;* `true` *is not a valid output, because of the constraint* $r \in \{0, 1\}$*. Each of the two execution paths are associated with the number* 0.25 *because of* f*; however to compute the probability of each path, we must normalize these numbers with the sum of the numbers associated with all successful execution paths. This yields the probability* $0.5 = (0.25/(0.25 + 0.25))$ *for each path.*

We make the following further assumptions on RVs (such as r in the above program)

1. RVs are uniquely associated with probability distributions — this is achieved syntactically by requiring that the probability distribution is imposed when the variable is declared.

2. Distinct RVs are assumed to be independent — correlations between RVs, *i.e.* joint probability distributions, are achieved by constraints.

Example 1.3 *Consider the program (with f as before):*

$$[\textbf{new } (r, f) \textbf{ in } x = r], [\textbf{new } (r', f) \textbf{ in } [\textbf{if } r' \in \{0, 1\} \textbf{ then } x = r']]$$

The first agent causes the generation of four execution paths, each with associated number 0.25. *For each of these, the second generates four more; however paths corresponding to six choices for $r \times r'$ are ruled out. Consequently the following results (with associated probabilities) are obtained: $x = 0(0.3), x = 1(0.3), x = 2(0.2), x = 3(0.2)$.*

The presence of RVs such as above allows us to construct program combinators reminiscent of combinators that detect "negative information" in synchronous programming languages [Hal93, BB91, Har87, SJG].

Example 1.4 *Consider the program*

$$\textbf{new } (r, g : g(0) = \epsilon, g(1) = (1 - \epsilon)) \textbf{ in } [\textbf{if } a \textbf{ then } r = 0, \textbf{if } r = 1 \textbf{ then } b]$$

This program on input `true` *produces constraint b with probability $(1 - \epsilon)$ and* `true` *with probability ϵ. On input a the program results in a with probability* 1. *Thus, this program can be thought of as*

$$\textbf{if } a \textbf{ else }_{(1-\epsilon)} b$$

i.e. *if a is not present, produce b with probability $(1 - \epsilon)$. Note however that if a is present, b is* not *produced.*

In essence, we have used the RV to set up a very high expectation that b will be produced; however, this expectation can be categorically denied on the production of a since the entire probability mass is shifted to the hitherto low end possibility.[1] The construct can be generalized to arbitrary agents A by: [2]

$$\textbf{if } a \textbf{ else }_{(1-\epsilon)} A \stackrel{d}{=} \textbf{new } X \textbf{ in } [\textbf{if } a \textbf{ else }_{(1-\epsilon)} X, \textbf{if } X \textbf{ then } A]$$

As another example of the expressiveness of Probabilistic cc, we show:

Example 1.5 (Probabilistic Or) *The probabilistic choice operator of [JP89], $P +_r Q$, can be defined as follows:*

$$\textbf{new } (X, f : (f(0) = r, f(1) = (1 - r))) \textbf{ in } (\textbf{if } X = 0 \textbf{ then } P, \textbf{if } X = 1 \textbf{ then } Q)$$

$P +_r Q$ reduces to P with probability r and to Q with probability $(1 - r)$.

[JP89] also require that this operator satisfies the laws of commutativity, associativity and absorption ($P +_r P = P$); this will be the case in the model we now develop.

[1] Note however, that the program **if** a **else** $_{1-\epsilon} b$, **if** b **else** $_{1-\epsilon} a$ is *probabilistically determinate* rather than *indeterminate*: assuming ϵ is very small, it produces on every execution the distribution $(a, 0.5), (b, 0.5)$, rather than producing either a or b indeterminately.

[2] Note that the same result can be obtained if only uniform probability distributions are available, by running n-fold parallel composition of $\epsilon = 0.5$ components, for arbitrarily large n.

What is a model for Probabilistic cc? We briefly review the model for cc programs, referring the reader to [SRP91] for details. An observation of a cc program A is a store u in which it is quiescent, i.e. running A in the store u adds no further information to the store. Formally we define the relation $A \downarrow^u$, read as A *converges on* u or A *is quiescent on* u, with the evident axioms:

$$\frac{a \in u}{a \downarrow^u} \quad \frac{A_1 \downarrow^u \quad A_2 \downarrow^u}{(A_1, A_2) \downarrow^u} \quad \frac{c \notin u}{(\textbf{if } c \textbf{ then } A) \downarrow^u} \quad \frac{A \downarrow^u}{(\textbf{if } c \textbf{ then } A) \downarrow^u} \quad \frac{A \downarrow^v \quad \exists_X u = \exists_X v}{(\textbf{new } X \textbf{ in } A) \downarrow^u}$$

The denotation of a program A can be taken to be the set of all u such that $A \downarrow^u$. The semantics is compositional since the axioms above are compositional, i.e. the convergence of A on u depends only upon the convergence of the sub-programs of A on u. The output of A on any given input a is now the least u containing a on which A converges. This immediately tells us that the set denoting A must be closed under $\sqcap$. Such a set is called a *closure operator*. It is easy to adjust this view in the presence of divergence, where divergence may arise because of infinite execution sequences or inconsistency of generated constraints [SRP91]. In essence, the denotation becomes a *partial* closure operator, characterized by sets of constraints closed under *non-empty* glbs.

We turn now to Probabilistic cc. We make two crucial observations. First, note that an RV-valuation (i.e., an assignment of values to hidden RVs) reduces a Probabilistic cc program to a cc program, which can be represented as a set of its quiescent points. Consider Example 1.4. The choice of r as 0 yields the fixed-point set $\{\texttt{true}, a, b, a \wedge b\}$. Similarly the choice of r as 1 yields $\{b\}$ — this is the fixed-point set of a partial closure operator undefined on a.

One might consider, then, the denotation of a Probabilistic cc process to be a set of pairs (c, p) where c is a fixed-point, and p is the sum of the probabilities associated with the RV-valuations which realize c, that is, for which c is a fixed-point. However, from just this information it is not possible to recover the fixed-point *set* generated by an RV-valuation; hence it is necessary to record correlations explicitly. That is, for an arbitrary set of fixed-points X, record (the sum of the probabilities associated with) those valuations on RVs for which all elements of X are fixed-points. Technically, this is best done by associating with the set S of fixed-points a probability lattice, defined formally in Section 2.2, which associates a probability with each element in the freely-generated complete and completely distributive lattice (= free profinite lattice, [Joh82]) on S. A process P can then be taken to be a pair (P_A, P_C) where P_A is a set of constraints, and P_C a probability lattice on P_A; P must satisfy conditions that ensure that it is "generable" from a program, intuitively, that it is possible to recover from it a (finite) set of (partial) closure operators, namely those generated by the reduction of the given program under different RV-valuations.

Thus for instance the program in Example 1.4 has the fixed-point set $\{\texttt{true}, a, b, a \wedge b\}$. The probability lattice associated with this process yields 0.5 for $\texttt{true}, a, a \wedge b$ and 1 for b; 0.5 for the collection $\{\texttt{true}, a, a \wedge b, b\}$ (since each element in this set is a fixed-point for the RV-valuation $X = 0$, which has probability 0.5); 0.5 for the collection $\{a, b\}$ etc.

The resulting model has the following key properties: (1) parallel composition is essentially set intersection (Section 2.3); (2) there is a natural embedding of the space of cc processes into the space of Probabilistic cc processes (Section 2.4); (3) it is fully abstract with respect to a notion of observation that includes probabilities (Section 2.6).

Timed Probabilistic cc arises from Probabilistic cc by the integration of a notion of time. This allows the representation of (possibly probabilistic) timed reactive systems. We use the same mechanism to extend Probabilistic cc that was used to extend Default cc to Default tcc in [SJG]. At each time step the computation executed is a Probabilistic cc program. Computation progresses in cycles: input a constraint from the environment, compute to quiescence and compute the program to be executed at subsequent time instants.[3]

As for Default tcc we add to the untimed Probabilistic cc a single temporal control construct: **hence** A. Declaratively, **hence** A imposes the "constraints" of A at every time instant after the current one. Operationally, if **hence** A is invoked at time t, a new copy of A is invoked at each instant in $t' > t$. As shown in [SJG] **hence** can combine with ask operations to yield rich patterns of temporal evolution. In particular, Timed Probabilistic cc satisfies probabilistic variants of following key features of synchronous programming languages: (1) The notion of time is *multiform* — any signal can serve as the notion of time. (2) All the ESTEREL-style combinators, including (the probabilistic versions of) the strong preemption combinators such as "`do A watching a`", are expressible.

The construction of the denotational model of Timed Probabilistic cc from Probabilistic cc follows the idea that "processes are relations extended over time" ([Abr93, Abr94]). Formally, the construction follows the definition of the Default tcc model from the Default cc model described in [SJG]. Consequently, the model for Timed Probabilistic cc "inherits" the good formal properties of Probabilistic cc (Section 3): (1) Parallel composition is essentially set intersection in this model; (2) The Timed Probabilistic cc model is conservative over the tcc model of [SJG94]; (3) The Timed Probabilistic cc model is fully abstract.

Related work The role of probability has been extensively studied in the context of several models of concurrency. Typically, these studies have involved a marriage of a concurrrent computation model with a model of probability. For example, stochastic Petri nets [Mar89] add Markov chains to the underlying Petri net model — the success of this approach is vouched for by the existence of several software tools to compute performance statistics [VN92]. Similarly probabilistic process algebras add a notion of randomness to the underlying process algebra model. This theory is well developed and is primarily about the interaction between probability and non-determinism, see for example [HJ90, vGSST90, JY95, LS91, HS86, CSZ92]. These studies have been carried out in the traditional framework of semantic theories of process algebras, *e.g.* theories of (probabilistic) testing, relationship with (probabilistic) temporal logics etc.

We start with the underlying concurrent model being cc. However, inspired by [BLFG95], we build a more integrated treatment of probability and the underlying concurrent programming paradigm. This is revealed in the dual roles played by the underlying cc paradigm. In addition to being utilized in the specification of system (this is similar to the use of the Petri nets/process algebras in the above approaches), the cc paradigm is exploited to build and specify joint probability distributions of several variables (as illustrated in earlier examples). This integration is not merely motivated by aesthetics; indeed, the earlier examples show how we exploit it to *synthesize* instantaneous detection of negative information, the key in-

[3]Note that in order to implement the Synchrony Hypothesis it is necessary that computation at each step be bounded. It is evident that recursion-free Probabilistic cc programs have bounded execution, under standard assumptions about the underlying constraint system [SJG].

gredient of synchronous programming languages. Consequently, we inherit the expressive power of synchronous programming, e.g. preemption constructs, dynamic priorities. In contrast, the above approaches force the addition of specialized concurrency constucts to achieve preemption/priorities. Also, our models remains determinately probabilistic [MMS96]. The key feature of our technical development is that that we maintain the simple view of parallel composition as set intersection.

The development of probabilistic frameworks in knowledge representation has been extensive [Pea88]. It is easy to see how to express the joint probability distributions of Bayesian networks within Probabilistic cc, e.g. by associating a random variable for each row in the joint conditional dependence matrix for each node in the network. In this sense, Probabilistic cc provides a simple but powerful notation for Bayesian networks. It also seems feasible to represent probabilistic Dempster-Shafer rules of the form "If it is Sunday, John will go to baseball game" with a given strength (say 0.8) as the agent: **if** $sunday(today)$ **then new** $(x, f : f(1) = 0.8, f(0) = 0.2)$ **in if** $x = 1$ **then** $will\_go(john, b\text{-}game)$. However, Probabilistic cc does not allow the direct assertion and manipulation of conditional probability assertions e.g. `p(fly(tweety) | bird(tweety)) = 0.9` or as in the logics of [Nil86, FJM90]. We expect to address this in the future.

2 Probabilistic cc Model

2.1 Constraint systems with discrete random variables

A constraint system with discrete random variables extends the usual notion of constraint systems with special variables called random variables, that can take values from non-negative integers $\mathbb{N}$. Such constraint systems have an explicit notion of inconsistencies of random variables.

Such a constraint system $\mathcal{D}$ is a system of partial information, consisting of a set of primitive constraints (first-order formulas) or *tokens* D, closed under conjunction and existential quantification, and an inference relation (logical entailment) $\vdash$ that relates tokens to tokens. We use $a, b, \ldots$ to range over tokens. The entailment relation induces through symmetric closure the logical equivalence relation, $\approx$.

Definition 2.1 *A* constraint system with random variables *is a structure* $\langle D, \vdash, \mathbf{Var}, \mathbf{RandVar} \subseteq \mathbf{Var}, \{\exists_X \mid X \in \mathbf{Var}\}, \mathtt{false} \in D\rangle$ *such that:*

1. *D is closed under conjunction($\wedge$); $\vdash \subseteq D \times D$ satisfies:*
   - *(a)* $(\forall a)$ $[\mathtt{false} \vdash a]$
   - *(b) Identity:* $a \vdash a$, $\quad a \vdash a'$ *and* $a' \wedge a'' \vdash b$ *implies that* $a \wedge a'' \vdash b$
   - *(c) Conjunction:* $a \wedge b \vdash a$, $\quad a \wedge b \vdash b$, $\quad a \vdash b_1$ *and* $a \vdash b_2$ *implies that* $a \vdash b_1 \wedge b_2$
2. **Var** *is an infinite set of* variables*, such that for each variable* $X \in \mathbf{Var}$, $\exists_X : D \to D$ *is an operation satisfying usual laws on existentials:*

   $$a \vdash \exists_X a,\ \exists_X(a \wedge \exists_X b) \approx \exists_X a \wedge \exists_X b,\ \exists_X \exists_Y a \approx \exists_Y \exists_X a,\ a \vdash b \text{ implies } \exists_X a \vdash \exists_X b$$

3. **RandVar** *is a set of* random variables *satisfying:*

(a) $(\forall r \in \mathbf{RandVar})\ (\forall i \in \mathbb{N})\ [r = i \in D]$.

(b) $(\forall r \in \mathbf{RandVar})\ (\forall i, j \in \mathbb{N})\ [i \neq j$ *implies* $(r = i) \wedge (r = j) \vdash \mathtt{false}]$

4. $\vdash$ *is decidable.*

The last condition is necessary to have an effective operational semantics.

A *constraint* is an entailment closed subset of D. For any set of tokens S, we let $\overline{S}$ stand for the constraint $\{a \in D \mid \exists\{a_1, \ldots, a_k\} \subseteq S.\ a_1 \wedge \ldots \wedge a_k \vdash a\}$. For any token a, $\overline{a}$ is just the constraint $\overline{\{a\}}$. The set of constraints, written $|D|$, ordered by inclusion($\subseteq$), forms a complete algebraic lattice with least upper bounds induced by $\wedge$, least element $\mathtt{true} = \{a \mid \forall b \in D.\ b \vdash a\}$ and greatest element $\mathtt{false} = \overline{\mathtt{false}} = D$. Reverse inclusion is written $\supseteq$. $\exists, \vdash$ lift to operations on constraints.

An example of such a system is the system FD [HSD92]. In this system, variables are assumed to range over finite domains. In addition to tokens representing equality of variables, there are tokens that that restrict the range of a variable to some finite set.

In the rest of this paper we will assume that we are working in some constraint system $\langle D, \vdash, \mathbf{Var}, \mathbf{RandVar}, \{\exists_X \mid X \in \mathbf{Var}\}, \mathtt{false}\rangle$. We will let $a, b \ldots$ range over D. We use $u, v, w \ldots$ to range over constraints.

2.2 Modeling probability information

We model probability information using probability lattices. Given a set A let $L(A)$ be the free profinite lattice generated by it. In the following definition, one should think of A as the fixed-point set of constraints, and think of C as mapping each $x \in L(A)$ to the sum of the probabilities associated with the RV-valuations realizing x, where an RV-valuation is considered to realize $\bigvee\{x_i\}$ if it realizes some x_i and with $\bigwedge\{x_i\}$ if it realizes each x_i. For instance, with this interpretation, the Modularity condition below is seen to correspond to taking the union of two overlapping sets.[4]

Definition 2.2 *A probability lattice C on A is a function from $L(A) \to [0,1]$ satisfying:*

Monotonicity: $x \leq y \Rightarrow C(x) \leq C(y)$.

Join Continuity: *If $\{x_i\}$ is a directed set then* $C(\vee x_i) = \mathbf{sup}\ C(x_i)$.

Meet Continuity: *If $\{x_i\}$ is a filter then* $C(\wedge_i x_i) = \mathbf{inf}\ C(x_i)$.

Normality: $C(\vee A) = 1$.

Modularity: $C(x \vee y) = C(x) + C(y) - C(x \wedge y)$.

Extensionality: *If* $C(x) = C(x') = C(x \vee x')$, *then* $(\forall y)\ [C(x \vee y) = C(x' \vee y)]$

[4]These conditions are motivated by Choquet capacities [Cho53]; given a relation $R \subseteq M \times A$ where M is a space equipped with a measure, the "projection" of R on A, *i.e.* hiding random variables, yields a structure that is not a measure since it does not satisfy additivity on disjoint sets. However, this structure is a Choquet capacity.

In what follows we will find useful the following definitions: For a probability lattice C on A, define $u \preceq_C v$ if $C(u) = C(u \vee v)$; intuitively, the set of RV-valuations realizing u is contained in the set of RV-valuations realizing v. Let $\equiv_C$ be the associated symmetric closure. Note that by extensionality, $\equiv_C$ is an equivalence relation. Say that $S \subseteq A$ is *C-consistent* if $C(\bigwedge S) > 0$; intuitively there is at least one RV-valuation that jointly realizes every element of a consistent set. Note that to specify a probability lattice f on A, it is enough to specify the values of f on all finite meets (or joins) of A; such an f can be uniquely extended (via monotonicity, continuity, modularity and extensionality) to a map on $L(A)$.
We define the following operations on probability lattice.

Quotient. Suppose C is a probability lattice on A, and $h : A \to B$ is a function. If $C(\vee(h^{-1}(B))$ is greater than 0, define $\frac{C}{h}$, the *quotient* of C by h, as follows. For b in the image of h, let $h^{-1}(y) = \bigvee\{x \in A \mid h(x) = y\}$ and freely extend it to the lattice $L(h(A))$ by $h^{-1}(y \wedge y') = h^{-1}(y) \wedge h^{-1}(y')$ and so on. (Alternatively we could have extended h to a homomorphism between $L(A)$ and $L(h(A))$, and then defined h^{-1}.) We can *quotient* C to a probability lattice C' on $h(A)$, by defining $C'(b) = C(h^{-1}(b))$. C' inherits monotonicity, modularity and extensionality from h^{-1}. Multiplying all values of $C'(b)$ by $\frac{1}{C(\vee h^{-1}(B))}$ normalizes the resulting probability lattice.

Expansion. Suppose C is a probability lattice on B, and $h : A \to B$ is a partial function. If $C(\vee h(A)) > 0$, define $C' = \texttt{Exp}(C, h)$, the *expansion* of C by h, as follows. C' is a probability lattice on $\texttt{dom}(h)$ (the domain of h) given by $C'(x) = C(h(x)), C'(\vee\{x_i\}) = C(\vee\{h(x_i)\}), C'(\wedge\{x_i\}) = C(\wedge\{h(x_i)\})$ etc. Multiplying all values of $C'(a)$ by $\frac{1}{C(\vee h(A))}$ normalizes the resulting probability lattice.

Product. The *product* of two probability lattices C_1 on A_1 and C_2 on A_2 is a probability lattice C on $A_1 \times A_2$. $C(\langle x, y\rangle) = C_1(x) \times C_2(y)$ for $\langle x, y\rangle \in A_1 \times A_2$. Similarly $C(\langle x, y\rangle \wedge \langle x', y'\rangle) = C_1(x \wedge x') \times C_2(y \wedge y')$; use modularity, and continuity to define C on all elements of $L(A_1 \times A_2)$.

We can now define a process:

Definition 2.3 (Process) *A* Probabilistic cc process P *is a pair* (P_A, P_C) *where* $P_A \subseteq |D| - \{\texttt{false}\}$ *and* P_C *is a probability lattice on* P_A *that satisfies the following conditions:*

Consistency *For every* $u \in P_A$, $\{u\}$ *is* P_C*-consistent, i.e.* $P_C(u) > 0$.

Glb-closure *For every* P_C*-consistent subset* S *of* P_A, $\sqcap S \in P_A$ *and* $\sqcap S \preceq_{P_C} \bigwedge S$.

Finiteness *The number of* $\equiv_{P_C}$*-equivalence classes are finite.*

The first condition ensures that every purported fixed-point is consistent (generated by at least one RV-valuation). The second forces every P_C-consistent set S of fixed-points to have a glb that is realized by at least every RV-valuations which realizes S; this ensures that every P_C-consistent set of fixed-points can be consistently extended to the range of a closure operator. The third condition forces finiteness of the total number of possible closure operators that can thus be generated.

2.3 Semantics of combinators

Tell. $\mathcal{P}[\![c]\!]_A \stackrel{d}{=} \{u \in |D| \mid c \in u\}$. $\mathcal{P}[\![c]\!]_C = 1$ (the constant function 1).

Ask. $\mathcal{P}[\![\textbf{if } c \textbf{ then } A]\!]_A \stackrel{d}{=} \{u \in |D| \mid c \in u \Rightarrow u \in \mathcal{P}[\![A]\!]_A\}$. If $c \notin u$, then $\mathcal{P}[\![\textbf{if } c \textbf{ then } A]\!]_C(u) = 1$, otherwise $\mathcal{P}[\![\textbf{if } c \textbf{ then } A]\!]_C(u) = \mathcal{P}[\![A]\!]_C(u)$. This is extended to the rest of the lattice in the usual way.

Parallel Composition. $\mathcal{P}[\![A_1, A_2]\!]_A \stackrel{d}{=} \mathcal{P}[\![A_1]\!]_A \cap \mathcal{P}[\![A_2]\!]_A$. The probability lattice is defined as $\texttt{Exp}(\mathcal{P}[\![A_1]\!]_C \times \mathcal{P}[\![A_2]\!]_C, \Delta)$ where $\Delta : |D| \to |D| \times |D|$ is the diagonal function.

Hiding. $\mathcal{P}[\![\textbf{new } X \textbf{ in } A]\!]_A \stackrel{d}{=} \{u \in |D| \mid \exists v \in \mathcal{P}[\![A]\!], \exists_X u = \exists_X v\}$. The probability lattice is defined as $\texttt{Exp}(\frac{\mathcal{P}[\![A]\!]_C}{\exists_X}, \exists_X)$, where $\exists_X : |D| \to |D|$ is the existential function on constraints from the given constraint system.

Distributions. Let f be a probability lattice of the domain of X[5]. The denotation of **new** (X, f) **in** A can be best understood as follows. Define the process $\texttt{distribution}(X, f)$ as $\texttt{distribution}(X, f)_A = \{u \in |D| \mid \exists r.(X = r) \in u, f(r) > 0\}$ and $\texttt{distribution}(X, f)_C = \texttt{Exp}(f, h)$, where $h(u) = r$ if $(X = r) \in u$.

Now $\mathcal{P}[\![\textbf{new } (X, f) \textbf{ in } A]\!] \stackrel{d}{=} \mathcal{P}[\![\textbf{new } X \textbf{ in } (\texttt{distribution}(X, f), P)]\!]$.

2.4 Conservativity results

cc programs are embedded easily into the Probabilistic cc model. Let A be a cc program. Then, it is immediate from the above definitions that $\mathcal{P}[\![A]\!] = (Q, C)$, where Q is the set of quiescent points of A and C is the constant function 1. Conservativity of Probabilistic cc over cc follows immediately.

2.5 Operational semantics

We define a transition relation to give the operational semantics of Probabilistic cc programs, and then prove that this operational semantics is equivalent to the denotational semantics. The transition relation is similar to the transition relation for cc. A configuration is a multiset of agents Γ. The transition relation is:

$$\frac{\sigma(\Gamma) \vdash a}{\Gamma, \textbf{if } a \textbf{ then } B \longrightarrow \Gamma, B} \qquad \Gamma, (A, B) \longrightarrow \Gamma, A, B$$

$$\Gamma, \textbf{new } X \textbf{ in } A \longrightarrow \Gamma, A[Y/X] \quad (Y \text{ not free in } \Gamma)$$

$$\frac{f(r) > 0}{\Gamma, \textbf{new } (X, f) \textbf{ in } A \longrightarrow \Gamma, Y = r, A[Y/X] \quad (Y \text{ not free in } \Gamma)}$$

Consider $\{\Gamma_i \mid A, a \longrightarrow^* \Gamma_i \not\longrightarrow, \sigma(\Gamma_i) \neq \texttt{false}\}$. Define a probability lattice C on Γ_i by setting $C(\Gamma_i) = \Pi_{Y \in \Gamma_i} f_Y(\{r_Y\})$, where $Y = r_Y \in \Gamma_i$, and f_Y is the probability distribution on Y. $C(\Gamma_i \wedge \Gamma_j) = 0$, so we get a probability lattice on by modularity and normalization.

[5] A probability measure on a finite set extends to a probability lattice structure.

The output of a process A on an input a, denoted $\texttt{OpsemIO}(A, a)$ is

$$\{\exists_{\vec{Y}}\sigma(\Gamma) \mid P, a \longrightarrow^* \Gamma \not\longrightarrow, \sigma(\Gamma) \neq \texttt{false}, \vec{Y} \text{ new vars in derivation}\}$$

with the probability lattice structure given by $\frac{C}{h}$, where $h(\Gamma) = \exists_{\vec{Y}}\sigma(\Gamma)$.

2.6 Correspondence Theorems

The key ingredient of the correspondence theorems relating the operational and denotational semantics is a representation theorem on Probabilistic cc processes, sketched below. We show that each process corresponds to a set of closure operators with associated probability, and conversely, given such a set, we can recover a process.

Let P be a process. A *consistent closure operator* (cco) of P is any P_C-consistent subset of P closed under glbs of non-empty subsets. Note that to every cco S there corresponds at least one RV-valuation jointly realizing S. However, it could be that the valuation also realizes other constraints, that is, it realizes a cco $T \supset S$. Intuitively, the closure operators exhibited by P are going to be those cco's S for which there is an RV-valuation realizing S and not any cco $T \supset S$. Such cco's can be determined as follows. Let $p(S) = P_C(\bigwedge S) - P_C(\bigvee\{\bigwedge T \mid T \supset S\})$. If $p(S) > 0$ then S is an exhibited closure operator. The probability of S is $p(S)$.

Conversely, given a set Z of closure operators, each associated with a probability $p : Z \to [0, 1]$, we can recover a process P as follows. $P_A = \bigcup\{S \mid S \in Z\}$. P_C is defined as follows. For any finite subset V of P_A, let $P_C(\wedge V) = \Sigma\{p(S) \mid S \in Z, S \supseteq V\}$. P_C can be extended to be a probability lattice in the usual way.

2.6.1 The input-output relation

The representation theorem permits the *semantic* recovery of the input output relation from a process P. Let c be an input. P will associate with c an output o iff there is closure operator f exhibited by P which maps c to o. The probability associated with o is the sum of the probabilities associated with each closure operator that maps c to o, divided by the sum of the probabilities associated with each closure operator defined for c.

2.6.2 Full abstraction

The operational and denotational semantics are equivalent. The proof exploits the representation theorem sketched above, and is omitted for lack of space.

Theorem 2.1 Computational Adequacy: *For any* Probabilistic cc *program* A, $\texttt{IO}(\mathcal{P}[\![A]\!])(u) = \texttt{OpsemIO}(A, u)$

Full Abstraction: *For any two* Probabilistic cc *programs* A_1 *and* A_2, *if* $\mathcal{P}[\![A_1]\!] \neq \mathcal{P}[\![A_2]\!]$ *then there is a context* C *such that* $\texttt{OpsemIO}(C[A_1], \texttt{true})$ *and* $\texttt{OpsemIO}(C[A_2], \texttt{true})$ *are different.*

3 Adding time — Timed Probabilistic cc

Timed Probabilistic cc arises from Probabilistic cc by adding a notion of discrete time. In adding time, we would also like to keep the characteristic properties of synchronous programming languages alluded to earlier. We ensure this by extending Probabilistic cc to Timed Probabilistic cc using the same method used to extend Default cc to Default tcc in [SJG, SJG94]. Concretely, we add a single temporal construct **hence** A — when this is invoked at time t, then a new copy of A is started at each time instant $t' > t$.

Notation. We will be working with sequences, *i.e.* partial functions on the natural numbers — their domains will be initial segments of the natural numbers of the form $0..n$. We let s, t and their variations, $s', s'', \ldots$ denote sequences. We use "ϵ" to denote the empty sequence. The concatenation of sequences is denoted by "$\cdot$"; for this purpose a singleton u is regarded as the one-element sequence $\langle u \rangle$. Given a subset of sequences S, and a sequence s, we will write S **after** s for the set $\{t \in \mathbf{Obs} \mid s \cdot t \in S\}$. $s(n)$ denotes its n'th element of s. We also define $S(0) = \{u \mid \exists s.\ u \cdot s \in S\}$.

We regard the set of all sequences S as a *sequence algebra* with the signature $\langle S, Pref_i, \text{length}() \rangle$, where $\text{length}(s) : S \to \mathbb{N}$ and $Pref_i : S \to S$, $Pref_i(s) = s$ if $\text{length}(s) \leq i$, otherwise $Pref_i(s) = \langle s(0), s(1), \ldots, s(i-1) \rangle$, the sequence consisting of the first i elements of s. Homomorphisms on sequence algebras will preserve prefixes and lengths.

In the rest of this paper we will assume that we are working in some constraint system with random variables $\langle D, \vdash, \mathbf{Var}, \mathbf{RandVar}, \{\exists_X \mid X \in \mathbf{Var}\}, \texttt{false} \rangle$.

3.1 Denotational Model

Observations. An observation is a *quiescent sequence* of a program. Thus an observation of the system will be a sequence of consistent constraints, such that if this sequence were the input, the output will be the same sequence. **Obs** is the the set of all such observations. A process is a collection of observations that satisfies the following condition — Instantaneous execution at any time instant is modeled by a Probabilistic cc process. The probability lattice for the Probabilistic cc process is the conditional probability, given the history up till this instant.

Formally, we proceed as follows.

Definition 3.1 *P is a* Timed Probabilistic cc *process if $P_A \subseteq \mathbf{Obs}$, and for each $s \in P_A$ we are given a probability lattice P_s, satisfying the following conditions:*

Non-emptiness $\epsilon \in P_A$,

Prefix-closure $s \in P_A$ *whenever* $s \cdot t \in P_A$,

Extension *If* $s \in P_A$ *then* $\exists t \neq \epsilon . s \cdot t \in P_A$,

Point execution *Whenever* $s \in P_A$, $(P_A$ **after** $s)(0)$ *with the probability lattice P_s is a* Probabilistic cc *process.*

We can combine the probability lattices into a single indexed set of probability lattices by defining $P_C(s \cdot z) \stackrel{d}{=} P_s(z)$, and similarly for joins and meets in P_s.

Probability Lattice operations on P_C. We generalize the definitions of operations on probability lattices to indexed sets of probability lattices.

Quotienting. Let $h : A \to B$ be a sequence algebra homomorphism. Let P be an indexed set of probability lattices on A, we want to define an indexed set Q on the image of h. We will do this in an inductive fashion on the tree of sequences. The basic idea is that we will at each stage collapse the sequences identified by h, and assume that for any two sequences $t, t' \in A$, if $t \neq t'$ then the probability lattices P_t and $P_{t'}$ are independent.

For each sequence $s \in B$ we will also define inductively a probability lattice R^s on the set $\{t \cdot v \in A \mid h(t) = s\}$. R^s will be quotiented to produce Q_s.

Define $R^\epsilon = P_\epsilon$, as $h(u) = u$. Then $Q_\epsilon = \frac{R^\epsilon}{h}$.

Assume we have defined Q_s and R^s for $s \in B$. Let $s \cdot u \in B$. If $h(t \cdot v) = s \cdot u$, define $R^{s \cdot u}(t \cdot v \cdot w) = R^s(t \cdot v) \times P_{t \cdot v}(w)$. Also, $R^{s \cdot u}(t \cdot v \cdot w \wedge t \cdot v \cdot w') = P^{t \cdot v}(w \wedge w')$. If $h(t' \cdot v') = s \cdot u$, and $t' \cdot v' \neq t \cdot v$ then $R^{s \cdot u}(t \cdot v \cdot w \wedge t' \cdot v' \cdot w') = R^s(t \cdot v \wedge t' \cdot v') \times P_{t \cdot v}(w) \times P_{t' \cdot v'}(w')$. This follows from the independence assumption. Now $Q_{s \cdot u} = \frac{R^{s \cdot u}}{h'}$, where $h'(t \cdot v \cdot w) = u'$ if $h(t \cdot v \cdot w) = s \cdot u \cdot u'$.

Expansion. Let $h : A \to B$ be a partial sequence algebra homomorphism, i.e. the domain of h is a subalgebra of A. Let Q be an indexed set of probability lattices on B. Then we define an indexed set of probability lattices P on the domain of h by $P_s = \texttt{Exp}(Q_s, h')$, where $h'(u) = h(s \cdot u)$.

Product. Let P_1 and P_2 be two indexed sets of probability lattices on A_1 and A_2. Define the fibered product of A_1 and A_2 as the set $\{s \cdot \langle u, v \rangle \mid s \cdot u \in A_1, s \cdot v \in A_2\}$. Define Q on this fibered product as $Q_s = P_{1_s} \times P_{2_s}$.

We will overload the symbols for quotient, expansion and product to stand for the corresponding operations on indexed sets of probability lattices also.

Combinators of Timed Probabilistic cc. c, **if** c **then** $A, (A, B)$ are inherited from Probabilistic cc and their denotations are induced by their Probabilistic cc definitions.

Tell. $\mathcal{D}[\![a]\!]_A \stackrel{d}{=} \{\epsilon\} \cup \{u \cdot s \in \textbf{Obs} \mid a \in u\}$. $\mathcal{D}[\![a]\!]_C = 1$.

Ask. $\mathcal{D}[\![\textbf{if } a \textbf{ then } A]\!]_A \stackrel{d}{=} \{\epsilon\} \cup \{u \cdot s \in \textbf{Obs} \mid a \in u \Rightarrow u \cdot s \in \mathcal{D}[\![A]\!]_A\}$. For any sequence $u \cdot s$, if $a \in u$ then $\mathcal{D}[\![\textbf{if } a \textbf{ then } A]\!]_C(u \cdot s) = \mathcal{D}[\![A]\!]_C(u \cdot s)$, otherwise $\mathcal{D}[\![\textbf{if } a \textbf{ then } A]\!]_C(u \cdot s) = 1$. The rest of $\mathcal{D}[\![\textbf{if } a \textbf{ then } A]\!]_C$ is defined by monotonicity, continuity and modularity.

Parallel Composition. $\mathcal{D}[\![A, B]\!]_A \stackrel{d}{=} \mathcal{D}[\![A]\!]_A \cap \mathcal{D}[\![B]\!]_A$. $\mathcal{D}[\![A, B]\!]_C$ is given as before by $\texttt{Exp}(\mathcal{P}[\![A_1]\!]_C \times \mathcal{P}[\![A_2]\!]_C, \Delta)$ where $\Delta(\epsilon) = \epsilon$ and $\Delta(s \cdot u) = s \cdot \langle u, u \rangle$.

Hiding. Every observation $s \in \mathcal{D}[\![\textbf{new } X \textbf{ in } A]\!]_A$ is induced by some observation $s' \in \mathcal{D}[\![A]\!]_A$, *i.e.* at every time instant t, $s(t)$ must equal the result of hiding X in the Probabilistic cc process given by A at time t after history s^{t-1}.

Formally, let $\exists_X s = \exists_X s'$ denote $|s| = |s'|$, and $\forall i < |s|, \exists_X s(i) = \exists_X s'(i)$. Then

$$\mathcal{D}[\![\textbf{new } X \textbf{ in } A]\!] \stackrel{d}{=} \{s \in \textbf{Obs} \mid \exists s' \in \mathcal{D}[\![A]\!].\exists_X s = \exists_X s'\}$$

The index of probability lattices is defined as $\texttt{Exp}(\frac{\mathcal{D}[\![A]\!]_C}{\exists_X}, \exists_X)$, where $\exists_X : |D| \to |D|$ is the existential on sequences.

Distributions. $\texttt{distribution}(X, f)$ ensures that X follows distribution f for all time, *i.e.* after every sequence, we must get the Probabilistic cc process $\texttt{distribution}(X, f)$.

Thus $\texttt{distribution}(X, f)_A = \{s \in \textbf{Obs} \mid \forall i < \text{length}(s).\ \exists r_i.\ (X = r_i) \in s(i), f(r_i) > 0\}$. The indexed set of probability lattices is given by the f — $\texttt{distribution}(X, f)_s = \texttt{Exp}(f, h)$, where $h(u) = r$ if $(X = r) \in u$.

Now we use the equation $\mathcal{D}[\![\textbf{new } (X, f) \textbf{ in } A]\!] \stackrel{d}{=} \mathcal{D}[\![\textbf{new } X \textbf{ in } (\texttt{distribution}(X, f), P)]\!]$ to define $\mathcal{D}[\![\textbf{new } (X, f) \textbf{ in } A]\!]$.

Hence. The definition for **hence** is as expected — observations have to "satisfy" A everywhere after the first instant; the probability lattice codes the fact that at time t there are $t-1$ copies of B running in parallel.

$$\mathcal{D}[\![\textbf{hence } B]\!]_A \stackrel{d}{=} \{u \cdot s \in \textbf{Obs} \mid (\forall s_1, s_2) s = s_1 \cdot s_2 \Rightarrow\ s_2 \in \mathcal{D}[\![B]\!]_A\}$$

$\mathcal{D}[\![\textbf{hence } B]\!]_C(u \cdot s) = 1$ if $s = \epsilon$, otherwise $\mathcal{D}[\![\textbf{hence } B]\!]_C(u \cdot s) = \Pi\{\mathcal{D}[\![B]\!]_C(s_2) \mid s = s_1 \cdot s_2\}$.

3.2 Definable combinators

Since all the basic combinators of Default tcc[SJG] are available here, all the defined combinators of Default tcc are definable in Timed Probabilistic cc. We list them below, showing that Timed Probabilistic cc is an expressive synchronous language.

next $_\epsilon A$	start A at the next instant with probability $1 - \epsilon$
first a **then** $_\epsilon A$	whenever a becomes true start A with probability $1 - \epsilon$
do A **watching** $_\epsilon c$	do A with probability $1 - \epsilon$ at each instant, until c becomes true
time $_\epsilon A$ **on** c	do A with probability $1 - \epsilon$ during the instants c holds

The other combinators described are also definable. Note that for the time and watchdog combinators the process A occurs with probability $1 - \epsilon$ at each instant, thus at any time these processes can randomly stop with probability ϵ.

3.3 Input-output behavior.

Given a process Z, we define its input-output behavior after a history s as the input output behavior of the Probabilistic cc process $(Z \textbf{ after } s)(0)$.

Formally, we extend the Probabilistic cc input-output definition to get the input-output relation for a Timed Probabilistic cc process — given finite sequences of constraints s, s':

$$\begin{aligned} \texttt{TimedIO}(Z)(\epsilon) &= \{\epsilon\} \\ \texttt{TimedIO}(Z)(s' \cdot a) &= \{s \cdot b \mid s \in \texttt{TimedIO}(Z)(s'), b \in \texttt{IO}((Z \textbf{ after } (s))(0))(a)\} \end{aligned}$$

The conditional (given s) probability lattice associated with the outputs is given at each stage by the corresponding probability lattice for the outputs of the Probabilistic cc process $(Z \textbf{ after } s)(0)$.

3.4 Operational semantics

The operational semantics for Timed Probabilistic cc is built on the operational semantics for Probabilistic cc — in this section, we focus on the aspects of the transition system that involve time. As before, we assume that the program is operating in isolation —interaction with the environment can be coded as an observation and run in parallel with the program. We use $\Gamma, \Delta, \ldots$ for multisets of programs; $\sigma(\Gamma)$ is defined as before — the tell tokens in Γ.

A configuration consists of a pair(Γ, Δ) — the agents currently active and the "continuation" — the program to be executed at subsequent times. The rules for asks, hiding and parallel composition and new random variables remain as before, in each case the Δ is unchanged after the transition. The rule for **hence** is

$$((\Gamma, \textbf{hence } A), \Delta) \longrightarrow (\Gamma, (A, \textbf{hence } A, \Delta))$$

The instantaneous outputs of Γ are $\{\exists_{\vec{Y}}\sigma(\Gamma') \mid (\Gamma, \emptyset) \longrightarrow^* (\Gamma', \Delta') \not\longrightarrow\}$. The associated probability lattice C on the set of pairs (Γ', Δ') is derived as before in Probabilistic cc.

If the output is u, we now define the timestep transition relation $\rightsquigarrow$

$$\begin{aligned}\Gamma \rightsquigarrow \textbf{new } (X, f) \textbf{ in } (\ & \textbf{if } X = 1 \textbf{ then new } \vec{Y} \textbf{ in } \Delta_1, \\ & \ldots \\ & \textbf{if } X = n \textbf{ then new } \vec{Y} \textbf{ in } \Delta_n)\end{aligned}$$

Here $\Delta_1 \ldots \Delta_n$ are those Δ's for which $\exists_{\vec{Y}}\sigma(\Gamma') = u$. $f = \frac{C}{h}$, where $h((\Gamma_i, \Delta_i)) = i$. For any random variable in $\vec{Y}$, we instead write **new** (Y, g) **in** Δ_i, where g was the distribution of Y.

The transition system gives an input-output relation — analogous to the definition for Probabilistic cc, we define $\texttt{TimedOpsemIO}(A)(s)$, the observed input output relation —

$$\begin{aligned}\texttt{TimedOpsemIO}(P)(s) = \{s' \mid\ & |s'| = |s| = n, A \stackrel{d}{=} A_0, \forall i < n.(A_i, s(i)) \rightsquigarrow A_{i+1}, \\ & \text{output at step } i \text{ is } s'(i)\}\end{aligned}$$

The probability lattice structure at each step given the history is computed from the probability lattice C defined above.

3.5 Correspondence theorems

The operational and denotational semantics are equivalent. The proof is essentially just a "lifting" of the proof for Probabilistic cc, and is omitted lack of space.

Theorem 3.1 Computational Adequacy: *For any* Timed Probabilistic cc *program* A, $\texttt{TimedIO}(\mathcal{P}[\![A]\!])(u) = \texttt{TimedOpsemIO}(A, u)$

Full Abstraction: *For any* Timed Probabilistic cc *programs* A_1 *and* A_2, *if* $\mathcal{D}[\![A_1]\!] \neq \mathcal{D}[\![A_2]\!]$, *there is a context* $C[]$ *such that* $\texttt{TimedOpsemIO}(C[A_1], \texttt{true})$ *and* $\texttt{TimedOpsemIO}(C[A_2], \texttt{true})$ *are different.*

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