



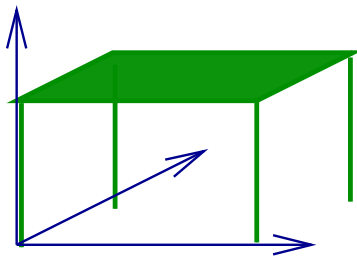
CS7.302
Basics of Computer Graphics
Module: Graphics Pipeline

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Spring 2021

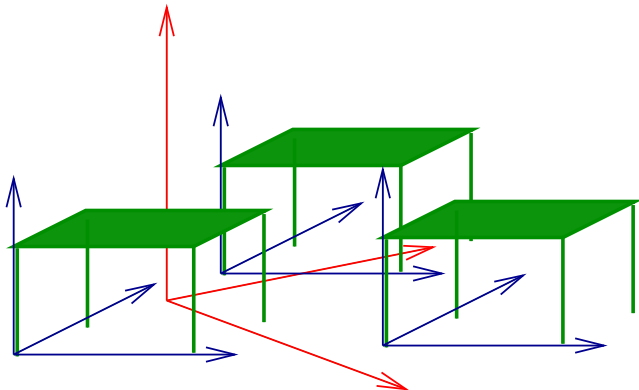
A Single Table

A table defined in its own coordinate system.

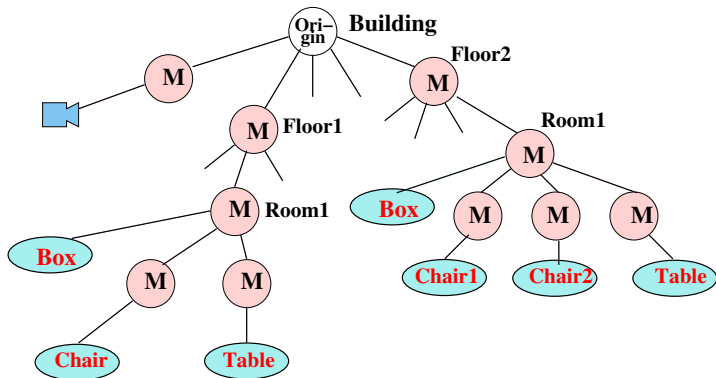


Many Tables in a Room

Place many tables in a **common world coord system**!

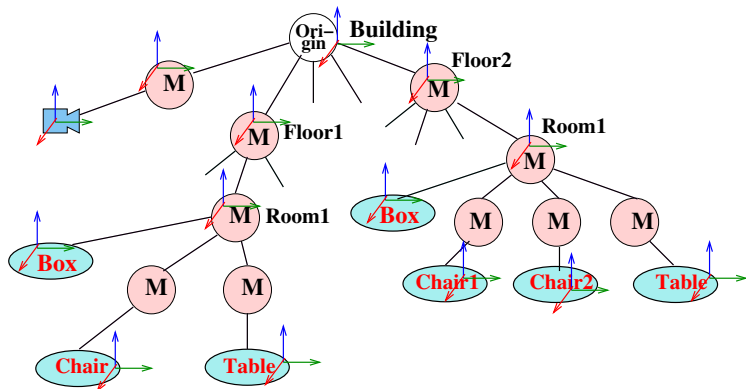


A Building Model



Hierarchical model with root representing whole scene.

A Building Model



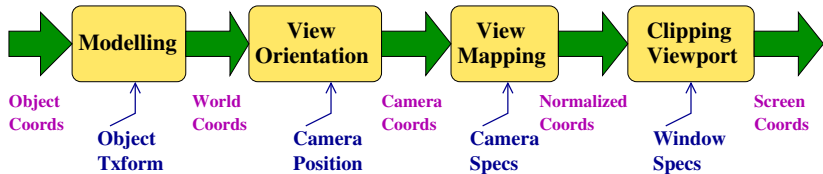
Each matrix **M** aligns parent frame to child frame

Different Coordinates

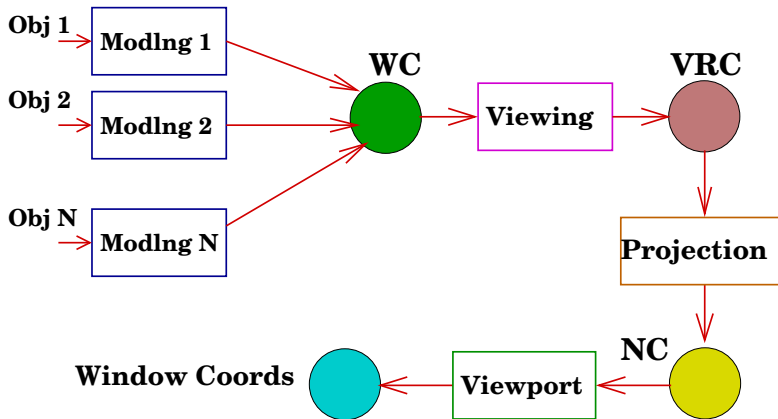
- ▶ **Object Reference:** Object is described in an internal coordinate frame called **ORC**.
- ▶ **World:** Common reference frame to describe different objects, called **WC**.
- ▶ **Camera/View Reference:** Describe with respect to the current camera position/orientation, called **VRC**
- ▶ Ultimately, **how the scene appears to the camera** determines the image produced
- ▶ Goal of Computer Graphics: describe the scene in the camera coordinate frame

3D Graphics Pipeline

- ▶ Objects are specified in their own coordinate system and placed in the world coordinate frame.
- ▶ Camera is also placed in the world coordinate frame.
- ▶ Camera-to-world geometry is first projected to normalized coordinates and then to screen.



3D Graphics: Block Diagram



Different Coordinates

- ▶ **Object Reference:** Object is described in an internal coordinate frame called **ORC**.
- ▶ **World:** Common reference frame to describe different objects, called **WC**.
- ▶ **Camera/View Reference:** Describe with respect to the current camera position/orientation, called **VRC**.
- ▶ **Normalized Projection:** A standard space from which projection is easy, called **NPC**.
- ▶ **Screen:** Coordinates in the output device space.

Transformations

- ▶ **Modelling:** Convert from object coordinates to world coordinates (ORC to WC).
- ▶ **View Orientation or Viewing:** From world coordinates to camera coordinates (WC to VRC).
- ▶ Simple coordinate transformations.
- ▶ **View Mapping or Projection:** From VRC to Normalized Coordinates (NC).
- ▶ **Viewport:** From NC to window coordinates.

Modelling and Viewing

- ▶ Transform points from object coordinates (ORC) to world coordinates (WC) to camera coordinates (VRC)
- ▶ A series of transformations for each object or point

$$\mathbf{P}_{\text{VRC}} = \mathbf{V}_{\text{VRC}} \mathbf{M}_{\text{WC}} \mathbf{P}_{\text{ORC}}$$

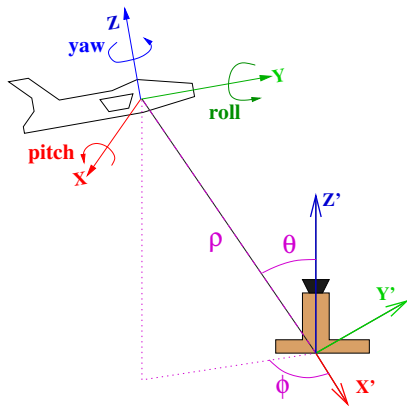
- ▶ Goal: Evaluate the coordinates of each point/line/triangle with respect to the camera

Modelling

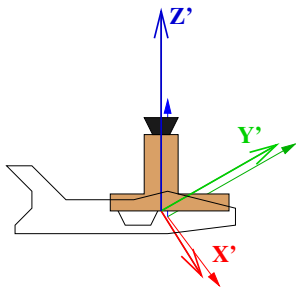
- ▶ Goal: Transform object coordinates to world coordinates.
- ▶ Method: Place ORC frame in the world coordinate frame.
- ▶ A single transformation matrix or **modelling matrix** with translation, rotation, scaling.
- ▶ A unit cube at origin can generate any cuboid using translation/rotation/scaling.
- ▶ Different objects have different modelling matrices.

Example: Aircraft in a Polar World

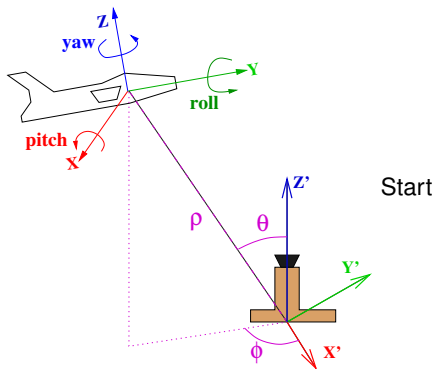
- ▶ WC frame on ground, ORC frame on the aircraft.
- ▶ Controllers think in polar coordinates for position and roll-pitch-yaw for orientation.
- ▶ What are the modelling steps?



Example: Aircraft in a Polar World (cont.)



at origin and move to new location



Aircraft in a Polar World

- ▶ Start with both axes aligned
- ▶ Translate to the location given by (ρ, θ, ϕ)
- ▶ Apply yaw, pitch, and roll: In which order ??

Aircraft in a Polar World

- ▶ Start with both axes aligned
- ▶ Translate to the location given by (ρ, θ, ϕ)
- ▶ Apply yaw, pitch, and roll in that order. (Why?)
- ▶ Coordinate axes undergoing transformation!
- ▶ Net effect: $\mathbf{T}(\rho, \theta, \phi) \mathbf{R}_z(\mathbf{y}) \mathbf{R}_x(\mathbf{p}) \mathbf{R}_y(\mathbf{r})$
- ▶ What is $\mathbf{T}(\rho, \theta, \phi)$? Compute (x, y, z) and translate
- ▶ Alternate: Rotate to align aircraft's Z-axis to translation direction, translate by ρ and unrotate
$$\mathbf{T}(\rho, \theta, \phi) = \mathbf{R}_z(-\phi) \mathbf{R}_y(\theta) \mathbf{T}(\mathbf{0}, \mathbf{0}, \rho) \mathbf{R}_y(-\theta) \mathbf{R}_z(\phi)$$

Why yaw, pitch, roll?

- ▶ Let **Y be East**, **X be South**, and **Z be Up**
- ▶ Consider a **pitch of 30 degrees** and a **yaw of 90 degrees**
- ▶ **Yaw** followed by **pitch**: what happens?
- ▶ **Pitch** followed by **yaw**: what happens?

Why yaw, pitch, roll?

- ▶ Let **Y be East**, **X be South**, and **Z be Up**
- ▶ Consider a **pitch of 30 degrees** and a **yaw of 90 degrees**
- ▶ **Yaw** followed by **pitch**: Flight going North, climbing 30°
 - ▶ Flight goes from Hyderabad to Delhi, still climbing
- ▶ **Pitch** followed by **yaw**: what happens?

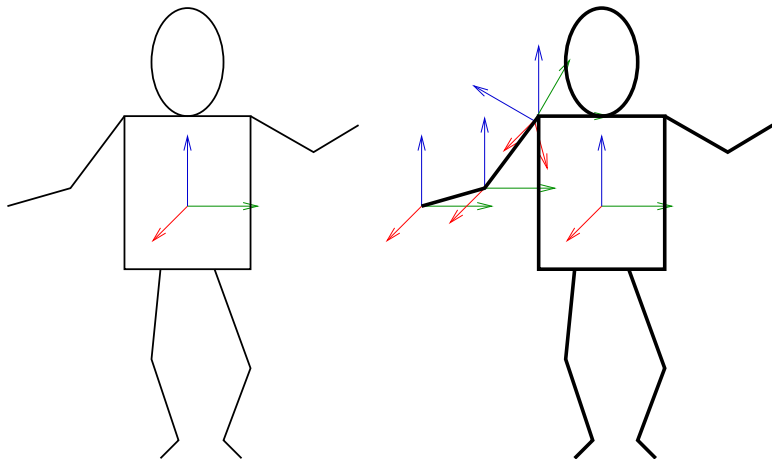
Why yaw, pitch, roll?

- ▶ Let **Y be East**, **X be South**, and **Z be Up**
- ▶ Consider a **pitch of 30 degrees** and a **yaw of 90 degrees**
- ▶ **Yaw** followed by **pitch**: Flight going North, climbing 30°
 - ▶ Flight goes from Hyderabad to Delhi, still climbing.
 - ▶ (In reality, aircraft will also roll while turning left).
- ▶ **Pitch** followed by **yaw**.
 - ▶ Yaw happens in a different plane
 - ▶ Flight won't be climbing, but will have a roll!
 - ▶ Not what one wants!

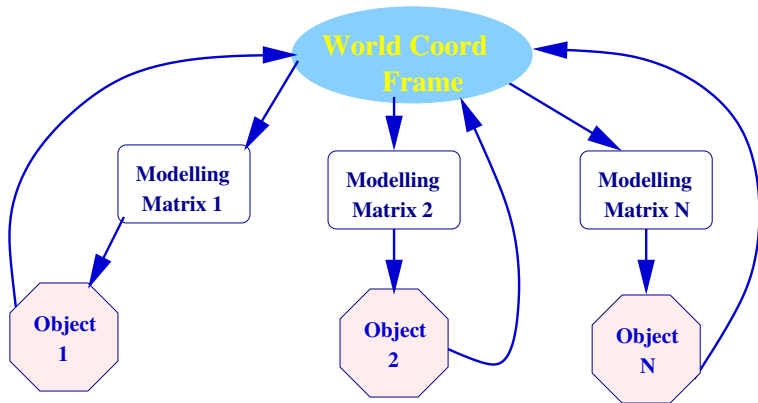
Hierarchy of Transformations

- ▶ A hierarchy of transformations needed to setup the world and the camera.
- ▶ A humanoid robot could have a coordinate frame on its body, another one on the shoulder, a third on the shoulder that moves with the upper arm, a fourth on the elbow, a fifth on the elbow that moves with the forearm, etc.
- ▶ Remember the wheel with an ant moving on its spoke!
- ▶ $\mathbf{M} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3 \cdots$ captures the composite transform as a shift in coordinate frames.

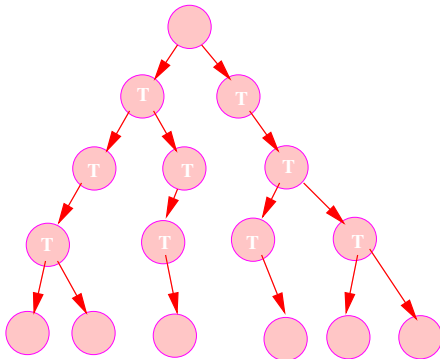
Humanoid Robot



Modelling Different Objects



Scene Graph



- Objects organized hierarchically with transforms.

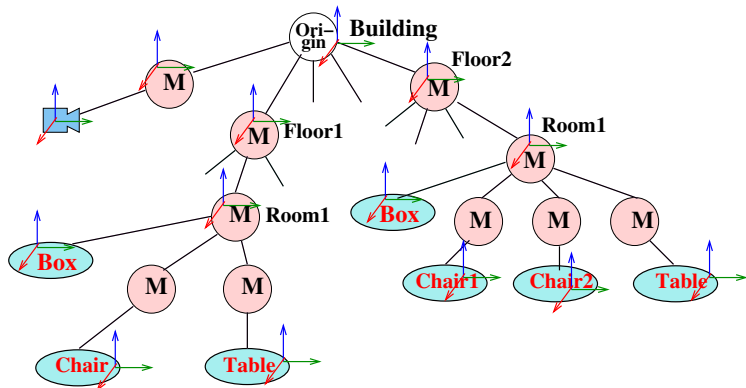
Modelling in OpenGL

- ▶ OpenGL 3.0 takes a single matrix that transforms object coordinates to normalized projection coordinates **directly**.
- ▶ You can devise separate **Projection**, **Viewing**, and **Modelling** matrices for ease of understanding
- ▶ Multiply them into **P V M** and send to the shader
- ▶ Shader transforms coordinates in the vertex array to projection/screen coordinates using this matrix
- ▶ Modelling matrix for the aircraft in polar coordinates:
$$\mathbf{M} = \mathbf{T} \mathbf{R}_Z(\mathbf{y}) \mathbf{R}_X(\mathbf{p}) \mathbf{R}_Y(\mathbf{r})$$

View Orientation or Viewing

- ▶ Placing the camera in the world and orienting it right.
- ▶ Has 6 degrees of freedom: 3 for position and 3 for orientation.

Building: Scene Graph



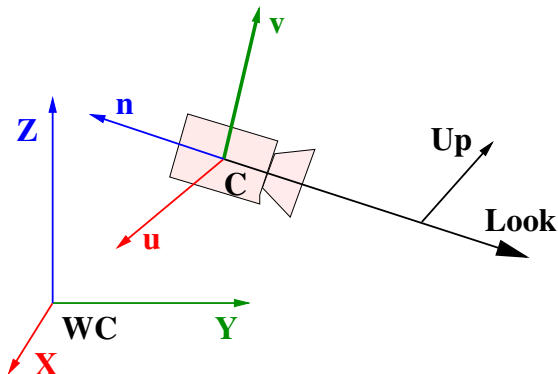
Each matrix **M** aligns parent frame to child frame

View Orientation or Viewing

- ▶ Placing the camera in the world and orienting it right.
- ▶ Has 6 degrees of freedom: 3 for position and 3 for orientation.
- ▶ Goal: Transform points expressed in WC to VRC.
- ▶ Let $\mathbf{u}, \mathbf{v}, \mathbf{n}$ be the VRC or camera coordinate axes
- ▶ Viewing Transformation can be specified in many ways.
- ▶ Commonly using: Camera location, Look point, and Up direction.

Viewing Specification

- ▶ Camera-center, Look-point and Up-vector specified in the world coordinates.



Transformation Steps

How do we align WC to VRC?

- ▶ Translate to $\mathbf{C} = (x, y, z)$.
- ▶ Rotate to align Z-axis to $-(\text{Look Vector})$ or $-\tilde{\mathbf{L}}$
- ▶ Rotate to align Y-axis to Up.
- ▶ Translation is easy. How do we get the rotation matrix?
- ▶ Remember columns of the matrix give directions **to** which the axes rotate!!

Rotation

- ▶ Let $\bar{\mathbf{l}} = \bar{\mathbf{L}}/|\bar{\mathbf{L}}|$ and $\bar{\mathbf{t}} = \bar{\mathbf{U}}/|\bar{\mathbf{U}}|$ be the unit vectors in those directions.
- ▶ Third column of the matrix: $\bar{\mathbf{n}} = -\bar{\mathbf{l}}$.
- ▶ Up vector needn't be orthogonal to the look vector. The $\bar{\mathbf{L}}$ and $\bar{\mathbf{U}}$ vectors define the "vertical" plane. A **plane in the world** that projects to a **vertical line in the image**. Or the camera's **vn** plane.
- ▶ First column: $\bar{\mathbf{u}} = \bar{\mathbf{t}} \times \bar{\mathbf{n}}/|\bar{\mathbf{t}} \times \bar{\mathbf{n}}|$
- ▶ Second column: $\bar{\mathbf{v}} = \bar{\mathbf{n}} \times \bar{\mathbf{u}}$.

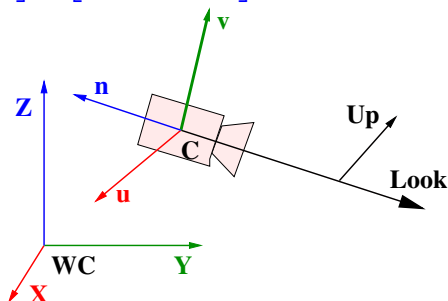
View Orientation Transformation

$$\mathbf{A} = \begin{bmatrix} & x \\ \mathbf{I} & y \\ & z \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{n}} & 0 \\ & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & x \\ \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{n}} & y \\ & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What have we achieved?

$$\mathbf{P}_{WC} = \mathbf{A} \mathbf{P}_{VRC} \text{ or}$$

$$\mathbf{P}_{VRC} = \mathbf{A} \mathbf{P}_{WC} \quad ?$$



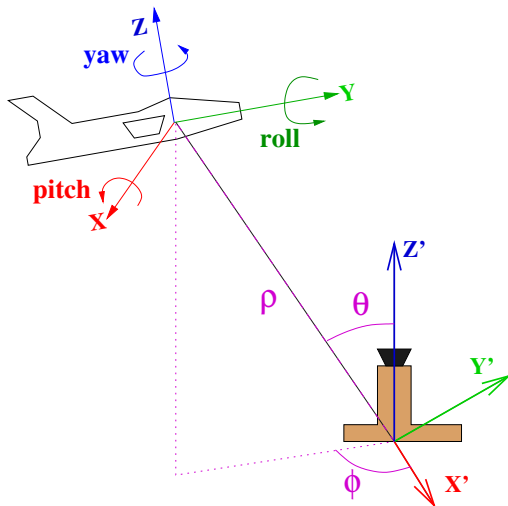
View Orientation Transformation

$$\mathbf{A} = \begin{bmatrix} & x \\ \mathbf{I} & y \\ & z \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{n}} & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & x \\ \bar{\mathbf{u}} & \bar{\mathbf{v}} & \bar{\mathbf{n}} & y \\ & & & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ We have achieved: $\mathbf{P}_{WC} = \mathbf{A} \mathbf{P}_{VRC}$.
- ▶ We need the reverse, everything to be in VRC
- ▶ Viewing transform: $\mathbf{V} = \mathbf{A}^{-1} = \mathbf{R}^T \mathbf{T}(-\mathbf{C})$.

Viewing from the Aircraft

- ▶ Need to give the pilot's view from aircraft.
- ▶ What are the viewing steps?



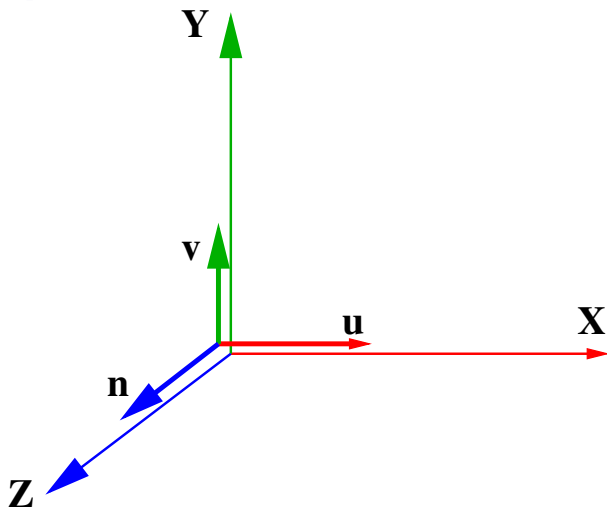
Aircraft in Polar World: Viewing

- ▶ Start with both axes aligned.
- ▶ Inverse of modelling or placing aircraft in WC
- ▶ Viewing transform: $\mathbf{R}_y(-\mathbf{r}) \mathbf{R}_x(-\mathbf{p}) \mathbf{R}_z(-\mathbf{y}) \mathbf{T}^{-1}(\rho, \theta, \phi)$

Modelling and Viewing in OpenGL

- ▶ Modelling and Viewing are not truly independent.
- ▶ What ultimately matters is only the **relative geometry** between the camera and the object(s).
- ▶ What we want is the description of each point in VRC, with respect to the camera.
- ▶ It is convenient to think of each object being placed in a WC and then the WC being transformed to VRC.
- ▶ Thus, **each object has its** modelling matrix.
The scene **has one** viewing matrix

When OpenGL Starts

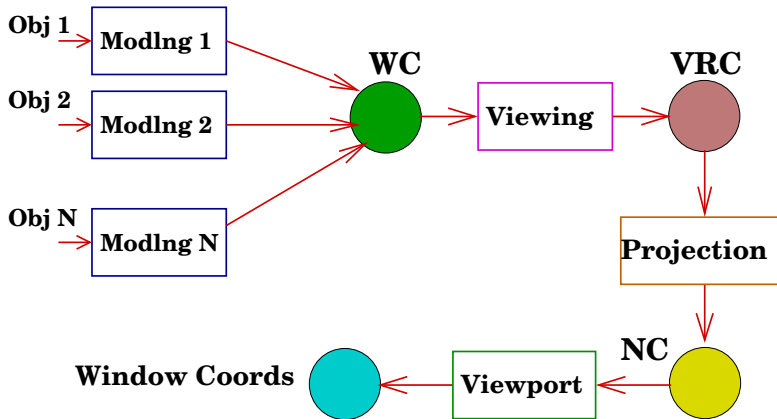


Modelling and Viewing matrices are Identity.

Setting up Objects and Camera

- ▶ WC is at VRC at start. First push it away to where WC should be. This is the Viewing Transformation matrix \mathbf{V}
- ▶ **Stay here** and draw objects in the scene one by one
 - ▶ Move to ORC of each object and draw its own model
 - ▶ Each object i has its Modelling Matrix \mathbf{M}_i
- ▶ Create matrix $\mathbf{P} \mathbf{V} \mathbf{M}_i$ and send to shader
 - ▶ Draw the object using description in its own frame

Block Diagram



Structure of an OpenGL Program

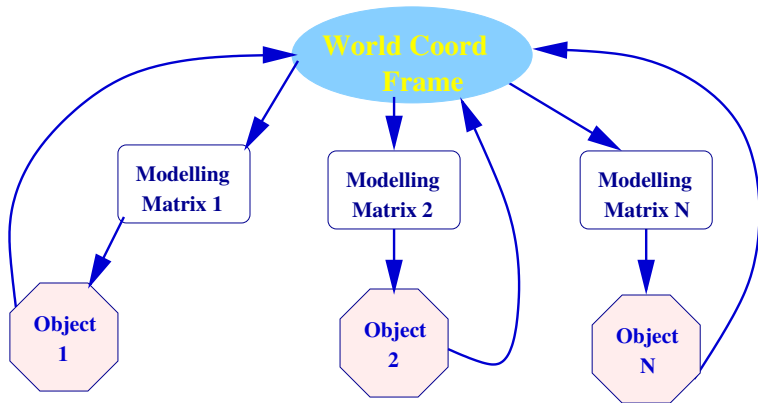
```
// Set projection matrix P (covered later)
// WC is aligned to VRC on start
// Camera is given by Pos & Orientation in WC

 $V = R(-\text{Orient}) T(-\text{Pos})$     // WC moved away from VRC

// WC is set. Model each object with it as reference

// Draw object i with respect to WC
 $M = T(i)R(i)$                 // Modelling matrix for object i
 $Mat = P V M$                 // from MVP matrix
send Mat to Shader            // send to shader
drawObject(i)                // Draw object polygon
// Start next object with respect to WC
```

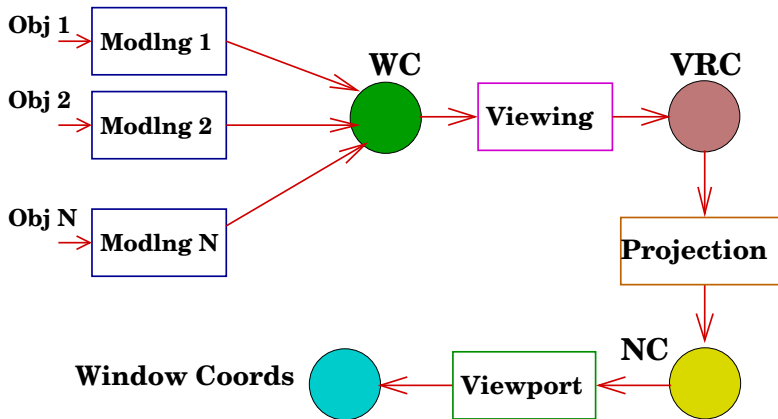
Modelling Different Objects



Modelling & Viewing: Summary

- ▶ Place objects in the world coordinate frame
- ▶ Place camera in the world coordinate frame
- ▶ Can compute object points in camera coordinate frame
- ▶ $P_{VRC} = V \cdot M \cdot P_{ORC}$

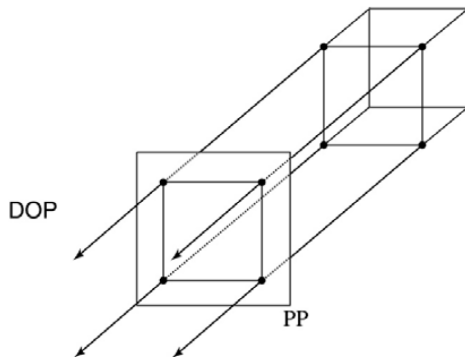
3D Graphics: Block Diagram



Projections

- ▶ Projection involves *projectors* starting from 3D points and hitting the 2D *projection plane*, forming the *image* of the point.
- ▶ Two types of projections.
- ▶ **Parallel projection**: Projectors are parallel to each other, all have the same *direction of projection* (**DOP**).

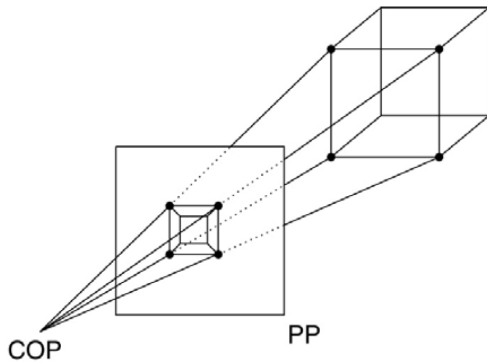
Projections (cont.)



Projections

- ▶ Projection involves *projectors* starting from 3D points and hitting the 2D *projection plane*, forming the *image* of the point.
- ▶ Two types of projections.
- ▶ **Parallel projection**: Projectors are parallel to each other, all have the same *direction of projection* (**DOP**).
- ▶ **Perspective projection**: All projects pass through a point in space called the *centre of projection* (**COP**).

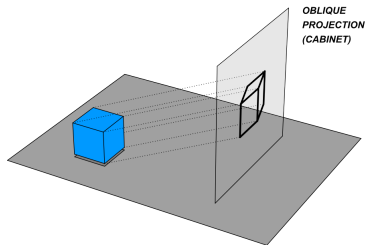
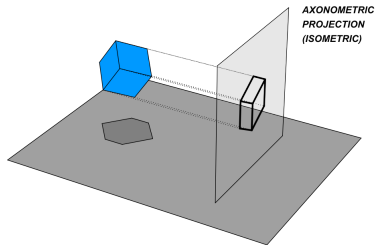
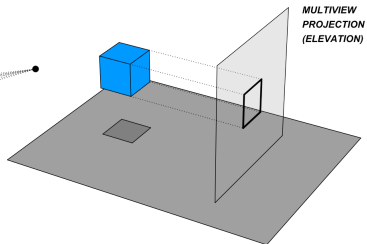
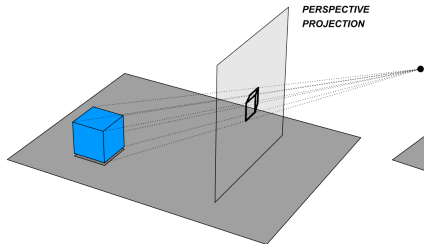
Projections (cont.)



Parallel Projections

- ▶ **Orthographic:** Projection plane is perpendicular to the direction of projection.
 - ▶ If direction of projection parallel to the axes:
plan, elevation, side elevation.
 - ▶ If PP intersect all axels at equal distance : *isometric* projection.
- ▶ **Oblique:** Otherwise.
 - ▶ *Cavalier* when projectors make 45 degrees with the projection plane.
 - ▶ *Cabinet* when they make $\arctan(2)$ degrees with the projection plane.

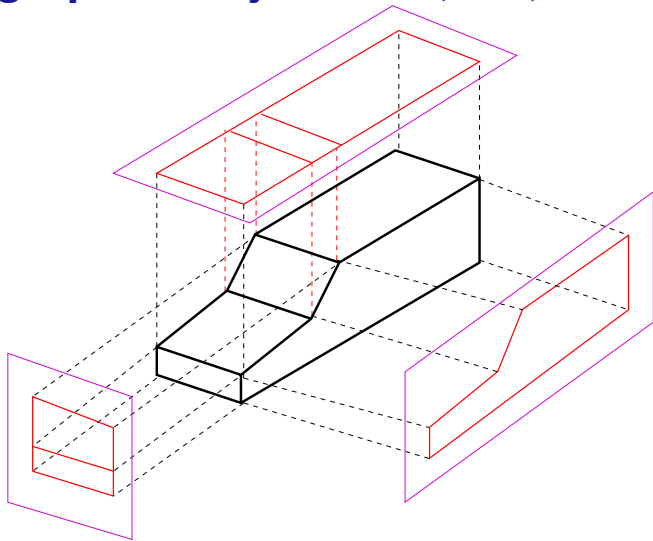
Parallel Projections (cont.)



Orthographic Projections

- ▶ Lengths parallel to the projection plane are preserved.
- ▶ Only direction of projection matters; distance from the point to the projection plane doesn't.
- ▶ Good approximation for a camera with a long focal length. (Orthographic with uniform scaling).
- ▶ Plan, elevation, side views etc.

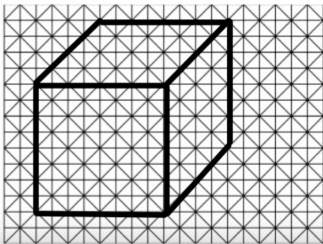
Orthographic Projections (cont.)



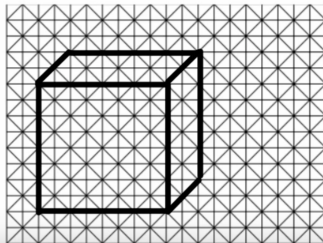
Oblique Projections

- ▶ Cavalier: Length along the depth axis preserved.
- ▶ Cabinet: Length along depth axis halved. More realistic.

Cavalier



Cabinet



Orthographic Projection Equation

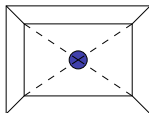
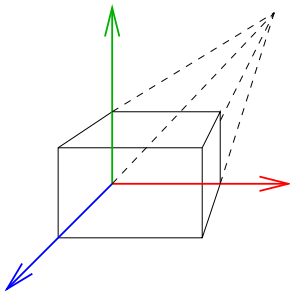
- ▶ Can be expressed as a matrix equation:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- ▶ If uniform scaling is involved, the top two 1's should be the scale factor.

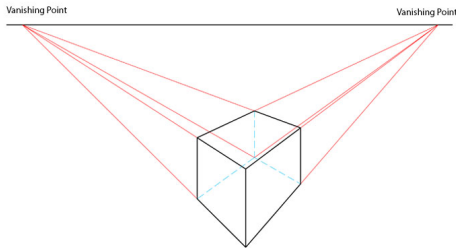
Perspective Projections

- ▶ Can be characterized by the number of **vanishing points**. (projections of points at infinity).
- ▶ Depends on the number of axes the projection plane intersects.
- ▶ 1-point, 2-point, and 3-point perspective projections.



Perspective Projections

- ▶ Can be characterized by the number of **vanishing points**.
(projections of points at infinity)
- ▶ Depends on the number of axes the projection plane intersects
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Perspective Projections 1-Point Example



Perspective Projections 2-Point Example



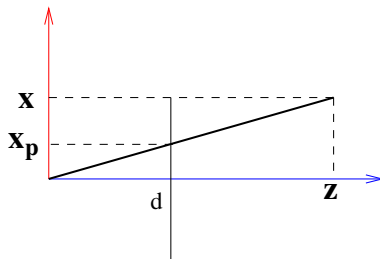
Perspective Projections 3-Point Example



Geometry of Perspective Projection

- ▶ What is x_p, y_p, z_p ?
- ▶ We know X, Z , and d .
- ▶ Remember similar triangles?

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



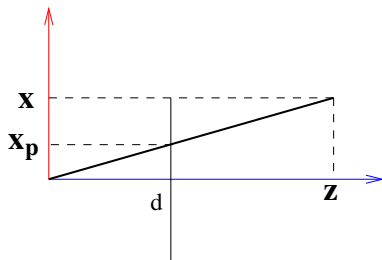
Geometry of Perspective Projection

► $\frac{x_p}{d} = \frac{X}{Z}, \quad \frac{y_p}{d} = \frac{Y}{Z}, \quad z_p = d.$

► In matrix form,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

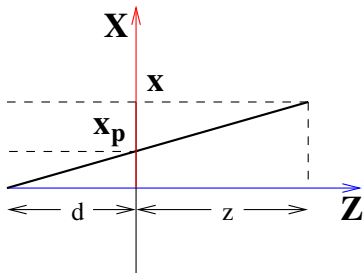
► Coordinates scaled down proportional to the depth or z values.



Another View

- Shift origin to lie on the projection plane, CoP at $(0, 0, -d)$, we get the matrix:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

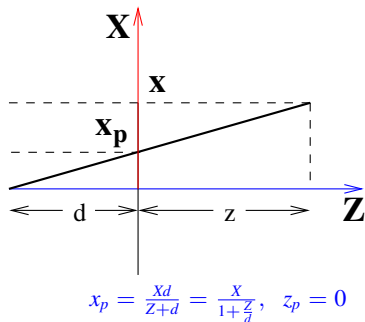


Another View

- ▶ Shift origin to lie on the projection plane, CoP at $(0, 0, -d)$, we get the matrix:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- ▶ Orthographic Projection matrix is a special case when $d \rightarrow \infty$.



Projections: Summary

Perspective

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$d < \infty$$

Orthographic

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$d = \infty$$

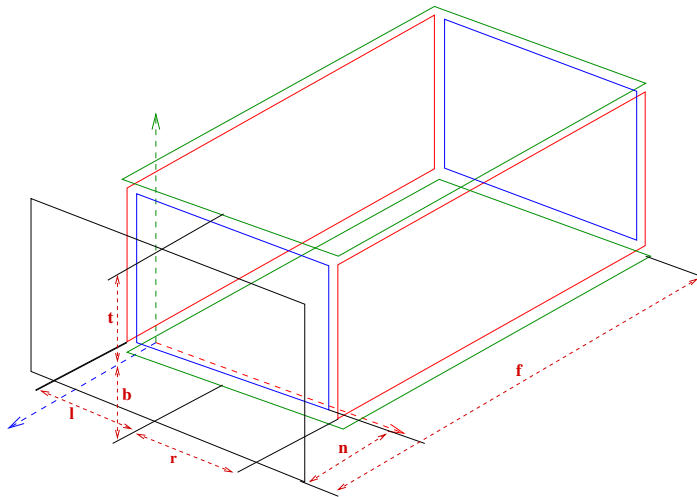
Volume of Visibility

- ▶ Cameras have finite fields of view in horizontal and vertical directions.
- ▶ What is the shape of its visible space?
- ▶ A cylinder for orthographic projections and a cone starting from the CoP for perspective may seem natural.
- ▶ Mathematics is difficult for cones; rectangular structures are easier!
- ▶ **View Volume:** The volume of potentially visible space.

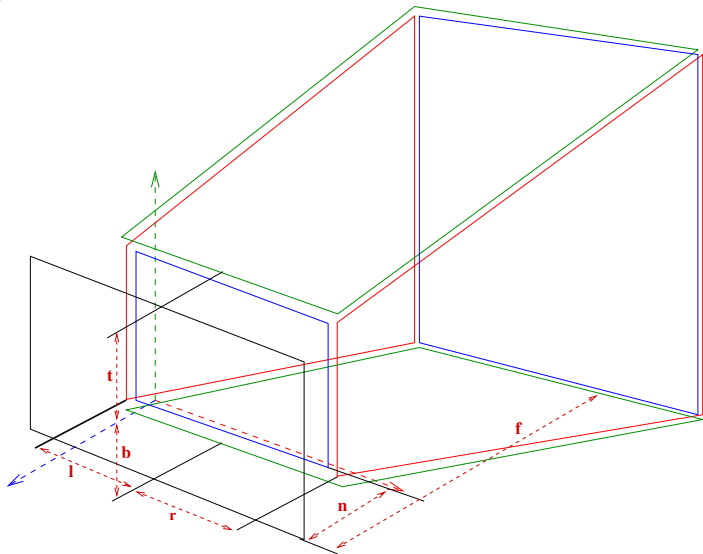
View Volume

- ▶ View volume is a cube for orthographic cameras and a (truncated) pyramid for perspective projections.
- ▶ 4 planes (**left, right, top, bottom**) define the view volume.
- ▶ Graphics cameras use 2 additional planes to limit visibility: **near & far!**
- ▶ Planes are specified in VRC; they move with the camera

Orthographic View Volume

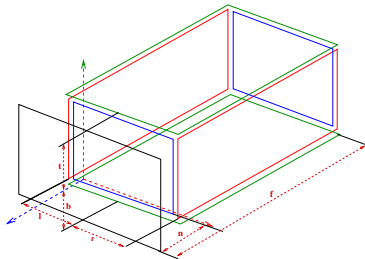


Perspective View Volume



Orthographic

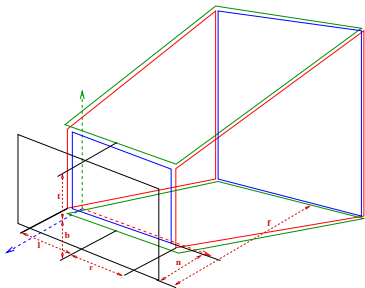
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix}$$



&

Perspective

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}$$



- View volume: Top, Bottom, Left, Right, Near, Far!

What About the Focal Length?

- ▶ An ideal pin-hole camera has the whole world in focus.
- ▶ Finite focal-length lenses introduce the effect of focus in real cameras.
- ▶ Even for them, the **depth of field** (region in focus) increases as the f-stop increases or the aperture gets smaller.
- ▶ Computer Graphics simulates ideal pin-hole cameras.
- ▶ Depth of field can be simulated by intentional blurring.

View Volume Specification

- ▶ View volume is specified by 6 planes:
`left, right, top, bottom, near, far`. All values in VRC

`glm::frustum()` or `glm::ortho()`

- ▶ `left, right, top, bottom`: signed distances.
`near, far`: positive distances to planes.
- ▶ Needn't be symmetric!

Alternate Specification

- ▶ Symmetric view volumes: horizontal and vertical fields of view θ_h , θ_v
- ▶ For symmetric perspective view volumes:

$$\tan \frac{\theta_h}{2} = \dots?$$

$$\tan \frac{\theta_v}{2} = \dots?$$

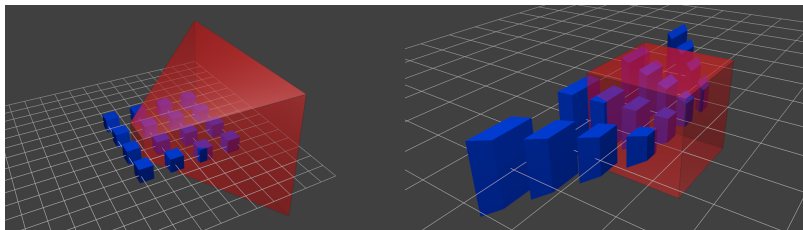
- ▶ Aspect ratio: `top / right`.

Using `glm::perspective()`

Canonical View Volume

- ▶ Projection is not performed right away; instead, map the view volume to a cube of fixed dimensions, called the **canonical view volume** or a standard view volume
- ▶ A **normalizing matrix** performs this transformation.
- ▶ Why?
 - ▶ Easier to eliminate objects outside the view volume.
 - ▶ Orthographic & perspective aren't different.
 - ▶ The z -coordinates not thrown away. (Used later!)

Canonical View Volume: Visualization



<http://www.opengl-tutorial.org/beginners-tutorials/tutorial-3-matrices/>

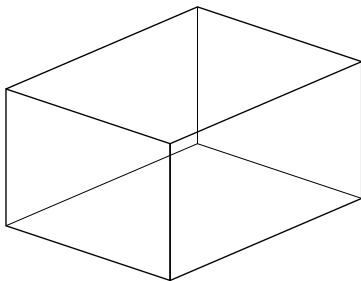
Canonical View Volume: Dimensions

- ▶ OpenGL:

$$-1 \leq x \leq 1$$

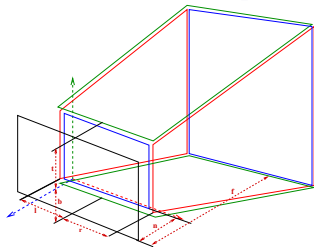
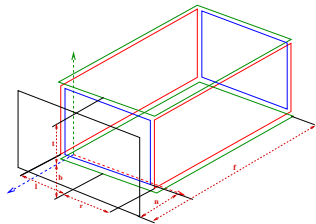
$$-1 \leq y \leq 1$$

$$-1 \leq z \leq 1$$

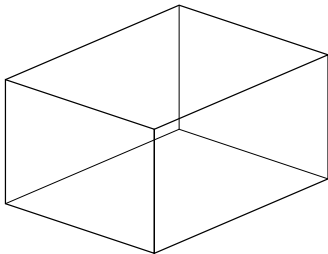


- ▶ Canonical view volume is the Orthographic View Volume, with appropriate scaling.

Orthographic & Perspective



To target view volume:



Orthographic Normalizing Matrix

- ▶ What are the side lengths on start?
- ▶ What are the side lengths at end?
- ▶ Whats the scale factor?
- ▶ Where is the origin at start? At end?
- ▶ How do we achieve that?

Orthographic Normalizing Matrix

- ▶ Lengths (right - left), (top - bottom) and (far - near) scaled to 2.
- ▶ Shift origin so as to range from -1 to +1.
- ▶ Matrix??

Orthographic Normalizing Matrix (cont.)

- ▶ Matrix:

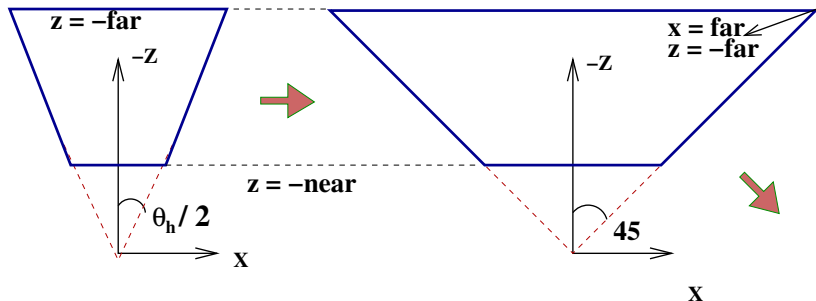
$$\begin{bmatrix} \frac{2}{\text{right}-\text{left}} & 0 & 0 & -\frac{\text{right}+\text{left}}{\text{right}-\text{left}} \\ 0 & \frac{2}{\text{top}-\text{bottom}} & 0 & -\frac{\text{top}+\text{bottom}}{\text{top}-\text{bottom}} \\ 0 & 0 & \frac{2}{\text{far}-\text{near}} & \frac{\text{far}+\text{near}}{\text{far}-\text{near}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ $(0, 0, -\text{near})$ maps to $+1$ and $(0, 0, -\text{far})$ maps to -1 .
- ▶ Drop z and use the (x, y) coordinates as the (normalized) window coordinates.

Symmetric Perspective Proj Matrix

- ▶ More complicated than orthographic case, as a frustum has to be mapped to a cube. Do it in steps.
- ▶ First, scale the horizontal and vertical extents so that the vertical and horizontal fields of view are 90 degrees.
- ▶ A scaling transformation with $s_x = ??, s_y = ??, s_z = 1$
- ▶ View volume is almost right except for a uniform scale.
- ▶ Next, scale uniformly so that the far plane is at -1. We will also have $-1 \leq x, y \leq 1$ at the far plane after this.
- ▶ $s_x = s_y = s_z = ??$

Symmetric Perspective Proj Matrix (cont.)



Symmetric Perspective Proj Matrix (cont.)

► M_1 with $s_x = \cot \frac{\theta_h}{2}$, $s_y = \cot \frac{\theta_v}{2}$, $s_z = 1$

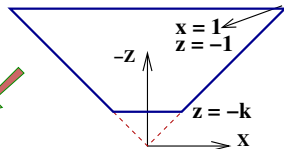
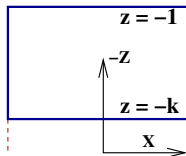
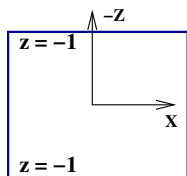
► M_2 with $s_x = s_y = s_z = \frac{1}{\text{far}}$

► $M_2 M_1 = \begin{bmatrix} \frac{\cot \theta_h / 2}{\text{far}} & 0 & 0 & 0 \\ 0 & \frac{\cot \theta_v / 2}{\text{far}} & 0 & 0 \\ 0 & 0 & \frac{1}{\text{far}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

► The near plane is now at $k = \frac{\text{near}}{\text{far}}$.

► View volume fits into the canonical view volume, but is still a frustum!

Symmetric Perspective Proj Matrix (cont.)



$k = \text{near/far}$

Symmetric Perspective Proj Matrix (cont.)

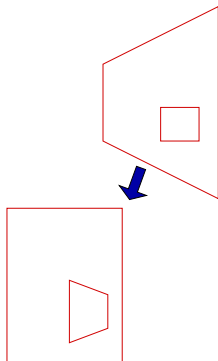
- ▶ Scale by to $-z$ to convert to a cube using a matrix with last row $\begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$.
- ▶ Simultaneously send third component to range $[-1, 1]$.
 $z = -k$ maps to 1 and $z = -1$ maps to -1 .
- ▶ Scale z by $\frac{1+k}{1-k}$ and translate by $\frac{2k}{1-k}$.
- ▶ Perspective division: convert from homogeneous to cartesian by dividing by the last component.
- ▶ Keep third component for later use. Relative ordering needs to be preserved. The values are not important.

Perspective Normalizing Txform

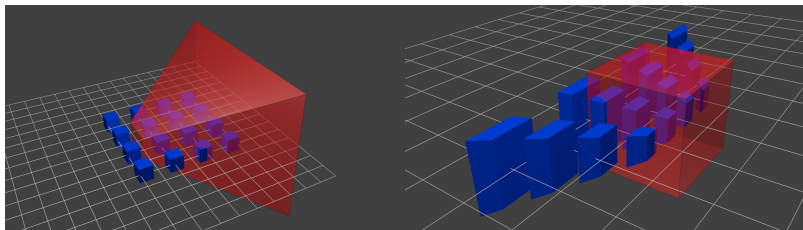
- ▶ Matrix M_3 for this step:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+k}{1-k} & \frac{2k}{1-k} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- ▶ $(x, y, -k)$ & $(x, y, -1)$ go to?
- ▶ Final matrix: $M = M_3 M_2 M_1$
- ▶ Frustum becomes a cube



Canonical View Volume: Visualization



<http://www.opengl-tutorial.org/beginners-tutorials/tutorial-3-matrices/>

Final 2D Coordinates

$$(u, v, d) \equiv \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{\text{VRC}}$$

- ▶ **Perspective division:** Divide x', y' coordinates by the w to get the normalized coordinates (u, v) . (z' maintains ordering and can be used without division.)
- ▶ The normalized d component has non-linear precision. Higher around the *near* plane and lower around the *far* plane due to the division by z .

OpenGL Normalizing/Perspective Matrix

- ▶ The projection matrix in OpenGL is given by

$$P = \begin{bmatrix} \frac{2n}{r-l} & 0 & A & 0 \\ 0 & \frac{2n}{t-b} & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

where,

$$A = \frac{r+l}{r-l}, B = \frac{t+b}{t-b}, C = \frac{f+n}{f-n}, D = \frac{2nf}{f-n}$$

*Please refer to the support material for derivation.

Actual Projection

- ▶ We have already performed the perspective division.
- ▶ Projection involves simply dropping the z coordinate and scaling x - y to the viewport.
- ▶ Why go through with the z -coordinates?
- ▶ The ordering is preserved along the depth dimension.
 z values can be used for visibility determination.

Where is the Film?

- ▶ Turns out: **It does not matter.**
- ▶ A final scaling is in the viewport transformations.
- ▶ As long as the film is in front of the camera, we will see an upright image.
- ▶ Can consider the near plane as the film plane.

Viewport Txformation: To Window

- ▶ Image of size -1 to +1 in X and Y is ready. The **viewport** transformation maps it to the actual window on screen.
- ▶ From $[-1, 1]$, map x and y to $[0, W]$ and $[0, H]$.
- ▶ First step: set sizes by scaling: $S(\frac{W}{2}, \frac{H}{2})$.
Next: Translate origin to South-West corner: $T(\frac{W}{2}, \frac{H}{2})$

▶ Overall: $M = T(\frac{W}{2}, \frac{H}{2}) S(\frac{W}{2}, \frac{H}{2}) = \begin{bmatrix} \frac{W}{2} & 0 & \frac{W}{2} \\ 0 & \frac{H}{2} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix}$

General Viewport Txform

- ▶ General command: `glViewport(l, b, r, t)`.
- ▶ Translate so the range of x, y is $0 \cdots 2$.
- ▶ Scale so x varies from 0 to $(r - l)$ and y varies from 0 to $(t - b)$.
- ▶ Translate so x range is l to r and y range is b to t .

▶ Matrix for this? $\mathbf{T}(l, b) \mathbf{S}(\frac{r-l}{2}, \frac{t-b}{2}) \mathbf{T}(1, 1) = \begin{bmatrix} \frac{r-l}{2} & 0 & \frac{r+l}{2} \\ 0 & \frac{t-b}{2} & \frac{t+b}{2} \\ 0 & 0 & 1 \end{bmatrix}$

- ▶ $[-1 \ -1 \ 1]^T$ maps to $[l \ b \ 1]^T$.
- ▶ $[1 \ 1 \ 1]^T$ maps to $[r \ t \ 1]^T$.

(Point) Pipeline in Action



- ▶ Points are transformed from Object to World to Canonical to Window coordinates.
- ▶ Each 3D point maps to a pixel (i,j) in the window space.
- ▶ Lines are made out of two points. Triangles and polygons are made out of 3 or more points.

Recapitulation

- ▶ 3D Graphics additionally involves projecting the 3D world to the 2D image plane of the camera.
- ▶ Compute the 3D world with respect to the camera. Or compute the relative geometry first.
- ▶ This involves a series of rigid transformations. For complex objects/environments, each object or its part is described in its own coordinate system.
- ▶ **Modelling** places these different objects in the world coordinate system. This could involve a hierarchy of transforms for objects made up of complex parts.
- ▶ **View Orientation** computes the world in the VRC.

Recapitulation (cont.)

- ▶ Camera can be perspective or parallel (orthographic, oblique). 6 planes give the view volume and defines the camera.
- ▶ **View Mapping** involves mapping the world to a canonical view volume, which is an orthographic view volume. This **Normalizing Transformation** has different forms for parallel and perspective cameras.
- ▶ **Projection** and **Clipping** are easy to perform in the canonical view volume. An image with dimensions from -1 to +1 results.
- ▶ **Viewport** transformation is the final step, involving a 2D scaling and translation to map to window coordinates that can be used to address the frame buffer.

Recapitulation (cont.)

- ▶ Given a description of the 3D world primitives, project each point to 2D to get 2D primitives. These can be scan-converted using standard algorithms.