Large Time Behaviour and Metastability in Mean-Field Interacting Particle Systems

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Outline

Part 1: Motivation and Background

Part 2: Large Deviations

Part 3: Main Results

Part 4: Summary and Future Directions

Part 1: Motivation and Background

A mean-field model

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- Empirical measure at time t

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- Particle transitions: at time t, a $z \to z'$ transition occurs at rate $\lambda_{z,z'}(\mu^N(t))$. Mean-field interaction.
- $\{(X_n^N(t), 1 \leq n \leq N), t \geq 0\}$ is a Markov process on $\mathcal{Z}^N$. $\{\mu^N(t), t \geq 0\}$ is a Markov process on $M_1^N(\mathcal{Z})$.

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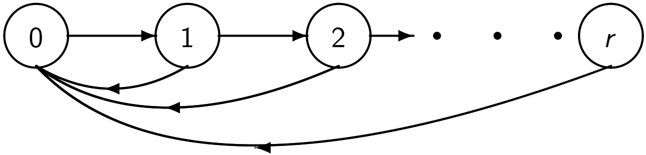

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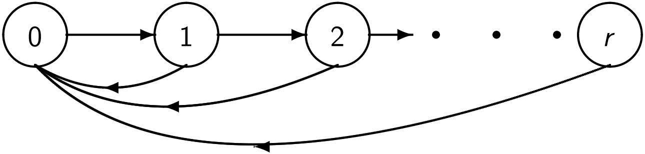

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- State evolution:
  - Becomes less aggressive after a collision.
  - Moves to the most aggressive state after a successful packet transmission.

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- Scaling each time slot by $1/N$, the transition rates of the continuous time caricature are

$$\lambda_{z,0}(\xi) = c_z \exp\{-\langle c, \xi\rangle\},$$
$$\lambda_{z,z+1}(\xi) = c_z(1-\exp\{-\langle c, \xi\rangle\}).$$

- Transition rates of a node depend on the states of the other nodes through the empirical measure.

The mean-field limit

- Recall μ^N. This is a Markov process on $M_1(\mathcal{Z})$ with infinitesimal generator

$$L^N f(\xi) = \sum_{(z,z')\in\mathcal{E}} N\xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right].$$

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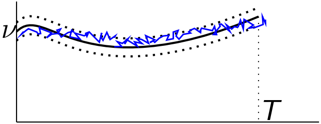
- Typical behaviour of μ^N (mean-field limit):



Let $\mu^N(0) \to \nu$ weakly as $N \to \infty$. Then $\{\mu^N(t), 0 \le t \le T\}$, w.h.p., is “close to” to the solution to the McKean-Vlasov equation:

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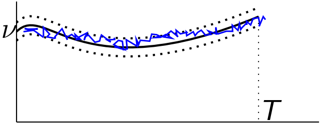

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- Thus, μ^N is a small random perturbation of the above ODE.

A sample path of μ^N in WiFi example

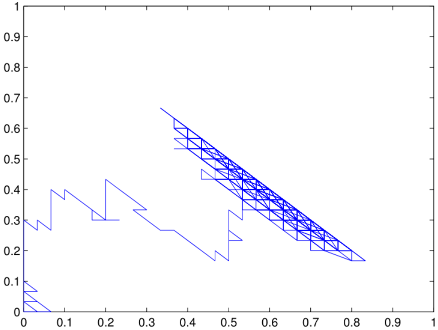
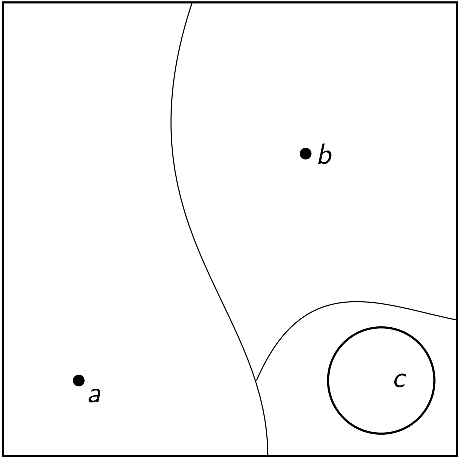


Figure: Evolution of states in a WiFi network under the MAC protocol

- Metastability phenomenon: Multiple stable regions in the system. Transition between two stable regions occur over large time durations.

μ^N is an ODE + "noise"



- Multiple stable equilibria. Transitions over large time durations.
- Goal: understand and quantify such metastable phenomena in mean-field models.

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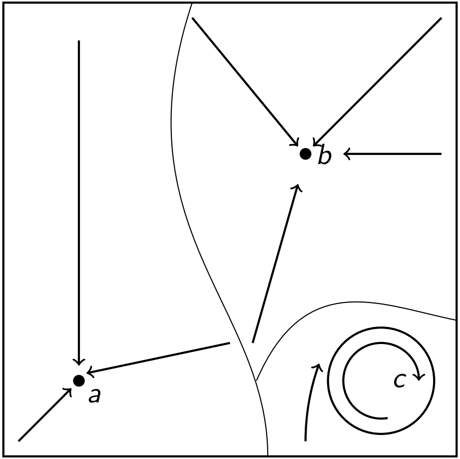

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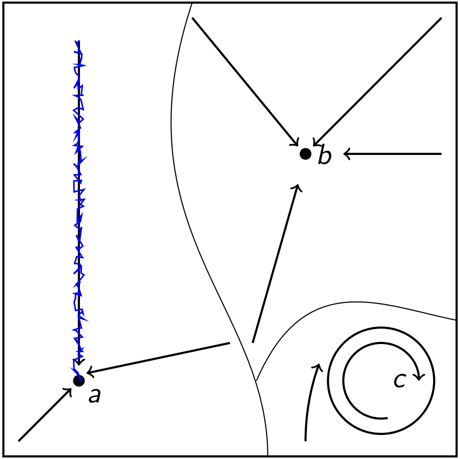

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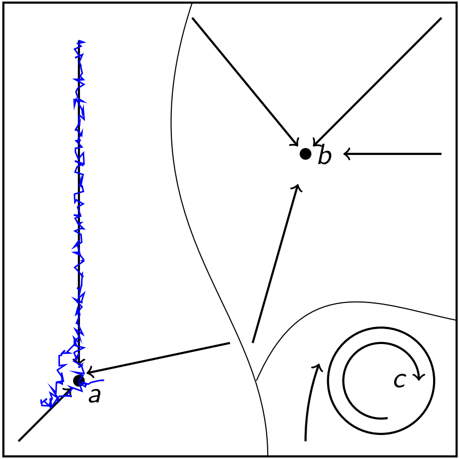

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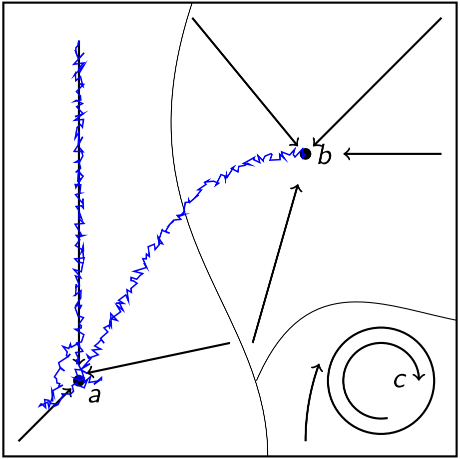

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Part 2: Large Deviations

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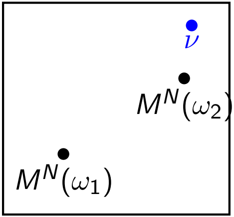

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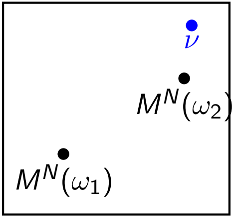

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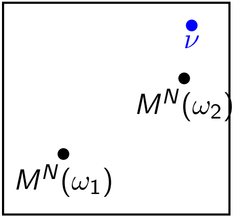

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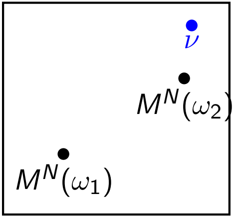

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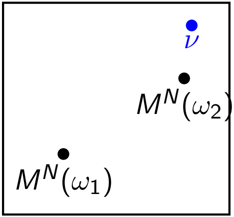

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- In particular, if $\operatorname{dist}(\nu, A) > 0$, then $P\left(M^N \in A\right) \to 0$ exponentially fast as $N \to \infty$.

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- Example: $\{M^N, N \geq 1\}$ satisfies the LDP on $M_1(S)$ with rate function $D(\cdot \| \nu)$ (Sanov's theorem).

Large deviations: An equivalent formulation

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- (Compactness of level sets). For any $s \geq 0$, $\Phi(s) := \{x \in S : I(x) \leq s\}$ is a compact subset of S;
- (LDP lower bound). For any $\gamma > 0$, $\delta > 0$, and $x \in S$, there exists $N_0 \geq 1$ such that

  $$P(d(X^N, x) < \delta) \geq \exp\{-N(I(x) + \gamma)\}$$

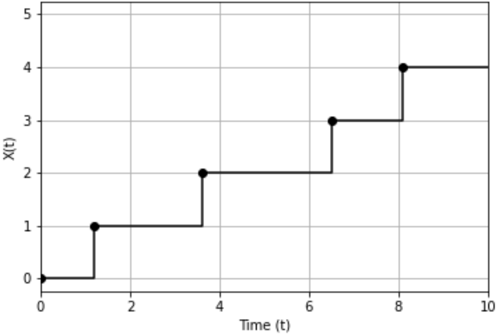
  for any $N \geq N_0$;
- (LDP upper bound). For any $\gamma > 0$, $\delta > 0$, and $s > 0$, there exists $N_0 \geq 1$ such that

  $$P(d(X^N, \Phi(s)) \geq \delta) \leq \exp\{-N(s - \gamma)\}$$

  for any $N \geq N_0$.

Example: LDP on the space of trajectories

- Consider the standard Poisson point process $X(t)$ for $t \in [0, T]$.



- Consider $D([0, T], \mathbb{R})$: space of $\mathbb{R}$-valued functions that are right continuous with left limits.
- X is a $D([0, T], \mathbb{R})$-valued random variable.

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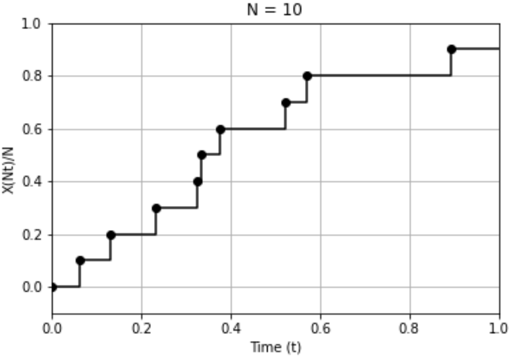
- Consider the time-scaled and amplitude-scaled process: $\frac{1}{N}X(Nt)$.



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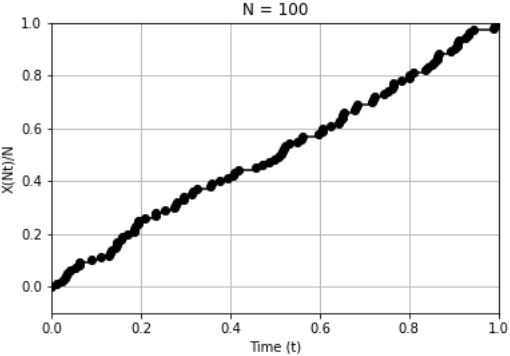

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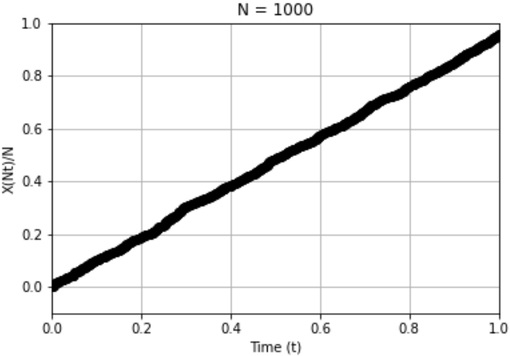

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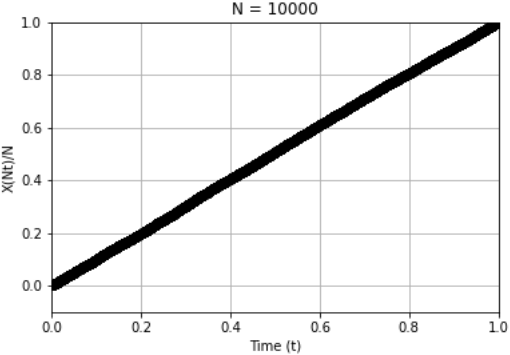

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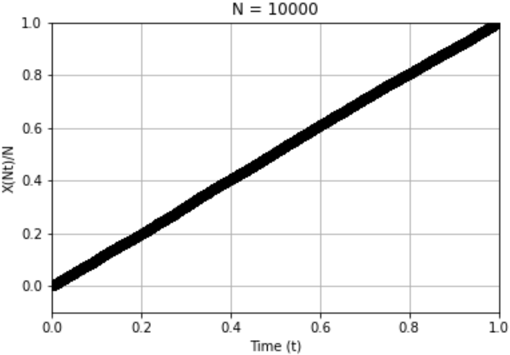
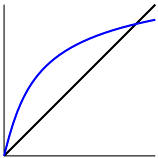


- The process $\frac{1}{N}X(Nt)$ is a small random perturbation of the ODE

$$\dot{x}(t) = 1,\ x(0) = 0,\ t \in [0, 1].$$

Example: LDP on the space of trajectories

- Question: probability that $\frac{1}{N}X(Nt)$ tracks a given function φ?



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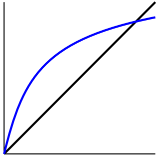

- One can show that $\{\frac{1}{N}X(Nt), N \geq 1\}$ satisfies the LDP on $D([0,T],\mathbb{R})$ with rate function

$$S(\varphi) = \int_{[0,T]} \tau^*(\dot{\varphi}(t) - 1)dt,$$

if $t \mapsto \varphi(t)$ is absolutely continuous, increasing, and $\varphi(0) = 0$; $S(\varphi) = \infty$ otherwise.

- Here,

$$\tau^*(x) = \begin{cases} (x+1)\log(x+1) - x, & \text{if } x \geq -1, \\ \infty, & \text{if } x < -1. \end{cases}$$

Part 3: Main Results

Large time behaviour of finite-state mean-field models

System model

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- The empirical measure μ^N is a Markov process with infinitesimal generator

$$L^N f(\xi) = N \sum_{(z,z') \in \mathcal{E}} \xi(z) \lambda_{z,z'}(\xi) \left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N} \right) - f(\xi) \right].$$

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- Recall the finite-state mean-field model. N particles. Allowed transitions $(\mathcal{Z}, \mathcal{E})$. $z \to z'$ transition at rate $\lambda_{z,z'}(\mu^N(t))$.
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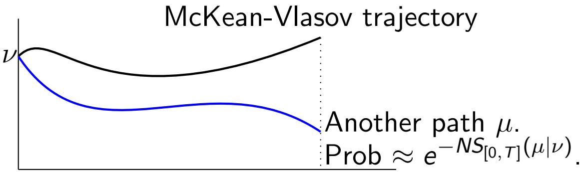
  - The graph $(\mathcal{Z}, \mathcal{E})$ is irreducible.
  - The functions $\lambda_{z,z'} : M_1(\mathcal{Z}) \to \mathbb{R}_+$ are Lipschitz continuous, and bounded away from 0.

Large deviations of μ^N



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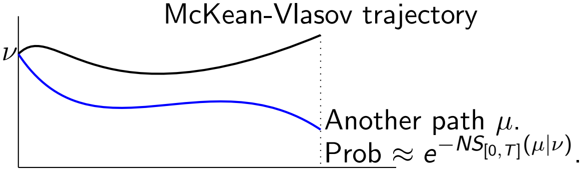


Theorem (Léonard (1995), Borkar and Sundaresan (2012))

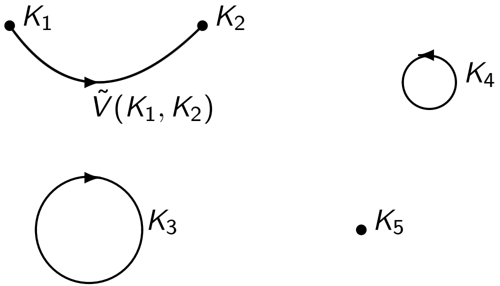
Let $\nu_N \to \nu$ weakly. Then $\mu^N_{\nu_N}$ satisfies the LDP on $D([0,T], M_1(\mathcal{Z}))$ with rate function $S_{[0,T]}(\cdot|\nu)$ defined as follows. If $\mu_0 = \nu$ and $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{Z})$ is absolutely continuous,

$$S_{[0,T]}(\mu|\nu) = \int_{[0,T]} \sup_{\alpha \in \mathbb{R}^{|\mathcal{Z}|}} \Big\{ \langle \alpha, \dot{\mu}_t - \Lambda^*_{\mu_t}\mu_t \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha(z') - \alpha(z)) \lambda_{z,z'}(\mu_t)\mu_t(z) \Big\} dt,$$

else $S_{[0,T]}(\mu|\nu) = \infty$. Here, $\tau(u) = e^u - u - 1$.

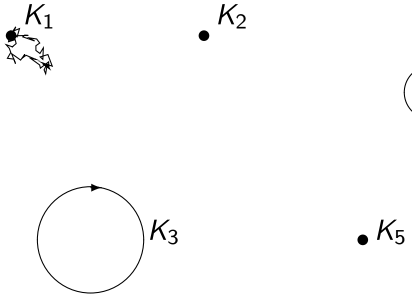
Some notation

- Assumptions on the McKean-Vlasov equation: There exists a finite number of compact sets $K_1, K_2, \ldots, K_l$ such that
  - Every ω-limit set of the McKean-Vlasov equation lies completely in one of the compact sets K_i.
  - No cost of movement within K_i. Positive cost to go out of (or come into) K_i.



- $\tilde{V}(K_i, K_j) = \inf\{S_{[0,T]}(\varphi|\varphi_0) : \varphi_0 \in K_i, \varphi_T \in K_j, \varphi_t \notin \cup_{i' \neq i,j} K_{i'}, T > 0\}$ (communication cost from K_i to K_j).

Approximation of μ^N using a discrete chain



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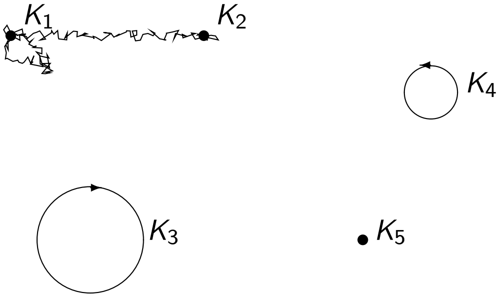

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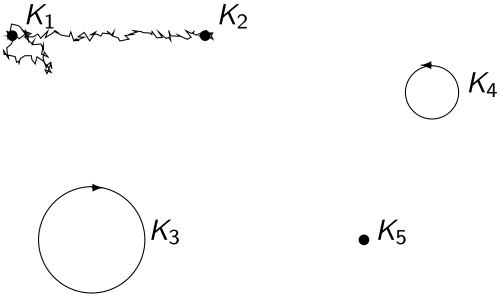

- τ_n: hitting time of μ^N in a given neighbourhood of K_i's.
- Hitting time chain: $Z_n^N = \mu^N(\tau_n)$, $n \geq 1$.
- To quantify the transitions between K_i's, we need large deviation estimates of μ^N *uniformly* with respect to the initial condition.

Uniform large deviations

- μ_ν^N: process starting from ν. Indexed by two parameters.

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Definition

$\{\mu_\nu^N\}$ is said to satisfy the uniform LDP over a class of subsets $\mathcal{A} \subset M_1(\mathcal{Z})$ if

- for each $K \subset M_1(\mathcal{Z})$ compact and $s > 0$, $\mathcal{K} = \bigcup_{\nu \in K} \Phi_\nu(s)$ is a compact subset of $D([0, T], M_1(\mathcal{Z}))$;
- for any $\gamma > 0, \delta > 0, s > 0$ and $A \in \mathcal{A}$, there exists $N_0 \geq 1$ such that

$$P_\nu(\rho(\mu_\nu^N, \varphi) < \delta) \geq \exp\{-N(S_{[0,T]}(\varphi|\nu) + \gamma)\},$$

  for all $\nu \in A$, $\varphi \in \Phi_\nu(s)$ and $N \geq N_0$;
- for any $\gamma > 0, \delta > 0, s_0 > 0$ and $A \in \mathcal{A}$, there exists $N_0 \geq 1$ such that

$$P_\nu(\rho(\mu_\nu^N, \Phi_\nu(s)) \geq \delta) \leq \exp\{-N(s - \gamma)\},$$

  for all $\nu \in A$, $s \leq s_0$ and $N \geq N_0$.

- Theorem: $\{\mu_\nu^N\}$ satisfies the uniform LDP over $M_1(\mathcal{Z})$.

Estimates on one step transition probability

Lemma (Borkar and Sundaresan (2012))

The one-step transition probability of the chain Z^N satisfies

$$\exp\{-N(\tilde{V}(K_i, K_j) + \varepsilon)\} \leq P(K_i, K_j) \leq \exp\{-N(\tilde{V}(K_i, K_j) - \varepsilon)\}.$$

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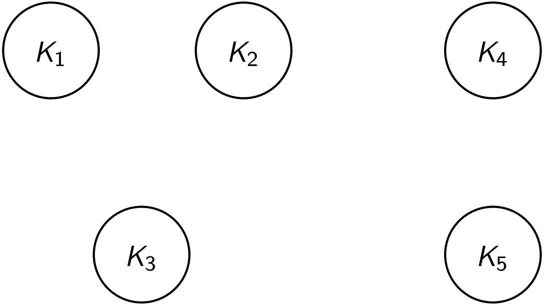
- Upon exit from K_i, μ^N is most likely to visit K_j that attains $\min_{j'} \tilde{V}(K_i, K_{j'})$ $(= \tilde{V}(K_i))$.
- The mean exit time from K_i is of the order of $\exp\{N\tilde{V}(K_i)\}$.

Decomposition into cycles - example

$$\tilde{V} \text{ values: } \begin{pmatrix} 0 & 4 & 9 & 13 & 12 \\ 7 & 0 & 5 & 10 & 11 \\ 6 & 8 & 0 & 17 & 15 \\ 3 & 6 & 8 & 0 & 2 \\ 5 & 7 & 10 & 3 & 0 \end{pmatrix}$$



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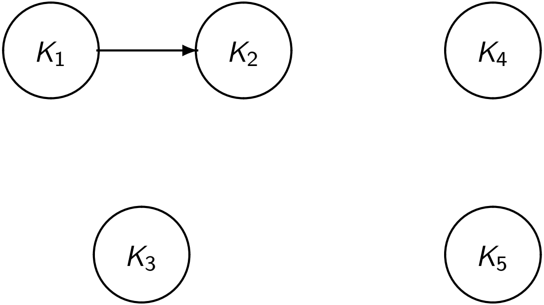

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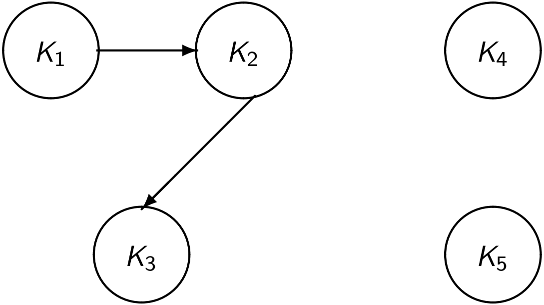

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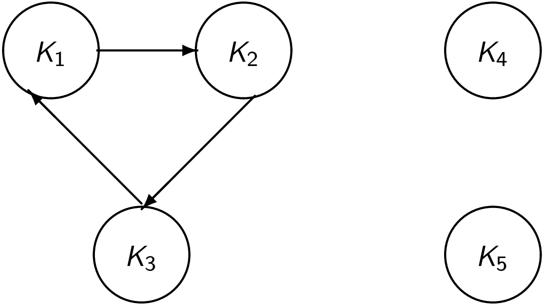

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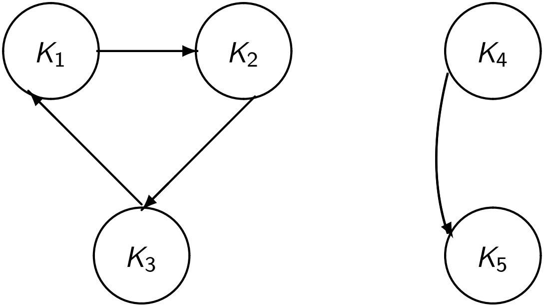

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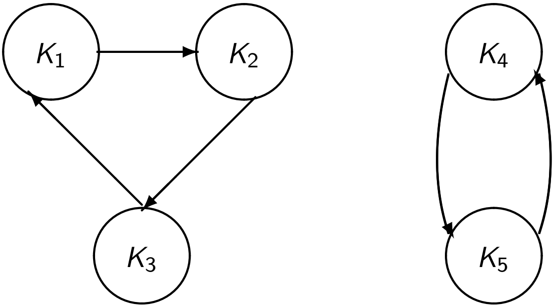

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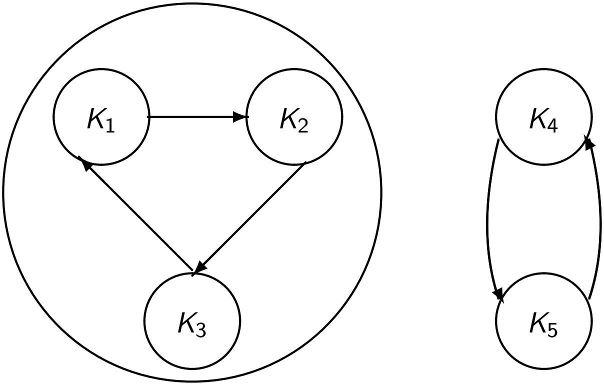

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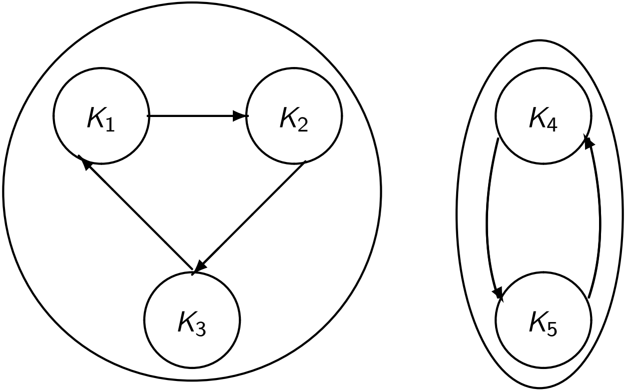

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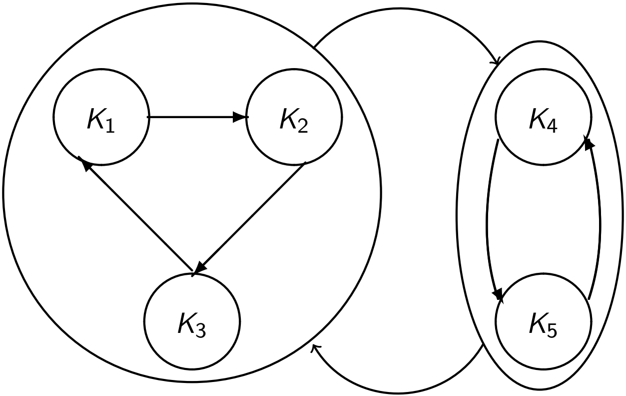

Large time behaviour

- Cycles are "very stable" subsets of $M_1(\mathcal{Z})$.
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Theorem (Informal statement)

Let $W = L \setminus K_1$. We have

$$\begin{aligned}\exp\{-N(\tilde{V}(K_i,K_j) - \tilde{V}(K_i) + \varepsilon)\} &\leq P_{K_i}(\mu^N(\hat{\tau}_W) \in K_j) \\ &\leq \exp\{-N(\tilde{V}(K_i,K_j) - \tilde{V}(K_i) - \varepsilon)\}.\end{aligned}$$

- Proof via the application of the uniform LDP for μ^N.

Mixing of μ^N

- Recall the discrete time chain $Z^N = \mu^N(\tau_n)$.
- $(i,j)th$ entry of its transition probability matrix (P^Z) is $\exp\{-N\tilde{V}(K_i, K_j)\}$.

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- $\mathrm{Re}(1 - \tilde{\lambda}_2^N)$ governs the mixing time of Z^N.
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- It can be shown that $\mathrm{Re}(1-\tilde{\lambda}_2^N) \sim \exp\{-N\Lambda\}$, where $\Lambda > 0$ is described using minimum of sums of $\tilde{V}$.
- When time is of the order of $\exp\{N\Lambda\}$, we expect Z^N to mix well.

Convergence to the invariant measure

Theorem

Given $\delta > 0$, there exist $\varepsilon > 0$ and $N_0 \geq 1$ such that for all $\nu \in M_1^N(\mathcal{Z})$ and $N \geq N_0$

$$\left| E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle \right| \leq \|f\|_\infty \exp\{-\exp(N\varepsilon)\},$$

where $T = \exp\{N(\Lambda + \delta)\}$ and $f \in B(M_1(\mathcal{Z}))$.

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- This constant Λ is sharp; over time durations $\exp\{N(\Lambda - \delta)\}$, there are points in $M_1(\mathcal{Z})$ starting from which the process μ^N would not mix well.

Asymptotics of the second eigenvalue

- If μ^N is reversible, the spectral decomposition of L^N tells us that

$$E_\nu f(\mu^N(t)) = \langle f, \wp^N \rangle + \sum_{k \geq 2} e^{-t\lambda_k^N} (f, u_k^N) u_k^N(\nu),$$

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Theorem

Assume that L^N is reversible with respect to $\wp^N$. Then,

$$\lim_{N \to \infty} \frac{1}{N} \log \lambda_2^N = -\Lambda.$$

Convergence to a global minimum

- Fix $c > 0$. Start with $N_0 = \min\{n \in \mathbb{N} : \exp\{nc\} - 2 \geq 0\}$ particles.
- Let $t_{N_0} = 0$. Add a particle at times $t_N = \exp\{Nc\} - 2$, $N > N_0$, with a certain state.
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- Small c: particles are added too frequently; $\bar{\mu}$ could get trapped in a local minimum of s depending on $\bar{\mu}(0)$.
- Large c: sufficient time for exploration, $\bar{\mu}$ converges to a desired equilibrium (K_{i_0}).

Theorem

For $c > c^*$ and any $\rho_1 > 0$,

$$P_{0,\nu}(\bar{\mu}(t) \in K_{i_0}) \to 1$$

as $t \to \infty$, uniformly for all $\nu \in M_1^{N_0}(\mathcal{Z})$.

Large deviations of two time scale mean-field models

System model

- N particles and an environment.

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- At time t,
  - The state of the nth particle is $X_n^N(t) \in \mathcal{X}$;
  - The state of the environment is $Y^N(t) \in \mathcal{Y}$.
- Certain allowed transitions.
  - Particles: a directed graph $(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$;
  - Environment: a directed graph $(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$.
- Empirical measure of the system of particles at time t:

$$\mu^N(t) := \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1^N(\mathcal{X}) \subset M_1(\mathcal{X}).$$

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- We are given functions $\lambda_{x,x'}(\cdot, y)$, $(x, x') \in \mathcal{E}_{\mathcal{X}}$, $y \in \mathcal{Y}$ and $\gamma_{y,y'}(\cdot)$, $(y, y') \in \mathcal{E}_{\mathcal{Y}}$ on $M_1(\mathcal{X})$.
- Markovian evolution at time t:
  - Particles: $x \to x'$ at rate $\lambda_{x,x'}(\mu^N(t), Y^N(t))$;

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- (μ^N, Y^N) is a Markov process with infinitesimal generator

$$f \mapsto \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} N\xi(x)\lambda_{x,x'}(\xi,y)\left[f\left(\xi+\frac{\delta_{x'}}{N}-\frac{\delta_x}{N},y\right)-f(\xi,y)\right]$$
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System model

- (μ^N, Y^N) is a Markov process with infinitesimal generator

$$f \mapsto \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} N\xi(x)\lambda_{x,x'}(\xi,y)\left[f\left(\xi+\frac{\delta_{x'}}{N}-\frac{\delta_x}{N},y\right)-f(\xi,y)\right]$$
$$+N\sum_{y':(y,y')\in\mathcal{E}_{\mathcal{Y}}}(f(\xi,y')-f(\xi,y))\gamma_{y,y'}(\xi),$$

  $(\xi,y)\in M_1^N(\mathcal{X})\times\mathcal{Y}$.
- A "fully coupled" two time scale process.
- Assumptions:
  - The graphs $(\mathcal{X},\mathcal{E}_{\mathcal{X}})$ and $(\mathcal{Y},\mathcal{E}_{\mathcal{Y}})$ are irreducible.
  - The functions $\lambda_{x,x'}(\cdot,y)$ are Lipschitz continuous and $\inf_\xi \lambda_{x,x'}(\xi,y)>0$ for all $(x,x')\in\mathcal{E}_{\mathcal{X}}$ and $y\in\mathcal{Y}$.
  - The functions $\gamma_{y,y'}(\cdot)$ are continuous and $\inf_\xi \gamma_{y,y'}(\xi)>0$ for all $(y,y')\in\mathcal{E}_{\mathcal{Y}}$.

The occupation measure process

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- θ^N is a random element of $D_\uparrow([0,T], M(\mathcal{Y}))$, the set of θ such that $\theta_t - \theta_s \in M(\mathcal{Y})$ and $\theta_t(\mathcal{Y}) = t$ for $0 \le s \le t \le T$.

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- $\theta \in D_\uparrow([0,T], M(\mathcal{Y}))$ is also viewed as a measure on $[0,T] \times \mathcal{Y}$ and obeys the disintegration $\theta(dydt) = m_t(dy)dt$ where $m_t \in M_1(\mathcal{Y})$.

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- We consider the process (μ^N, θ^N) with sample paths in $D([0, T], M_1(\mathcal{X})) \times D_\uparrow([0, T], M(\mathcal{Y}))$.

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Theorem (Bordenave et al. 2009)

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$$\dot{\mu}_t = \bar{\Lambda}^*_{\mu_t, \pi_{\mu_t}} \mu_t,\ 0 \le t \le T,\ \mu_0 = \nu.$$

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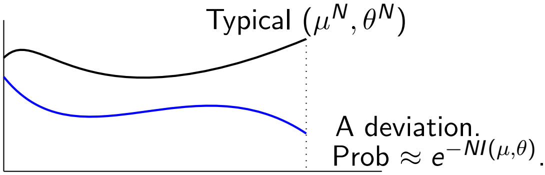
- μ^N is a small random perturbation of the above ODE. We study large deviations of (μ^N, θ^N).

Main result

Theorem

Suppose that $\{\mu^N(0), N \geq 1\}$ *satisfies the LDP on* $M_1(\mathcal{X})$ *with rate function* I_0. *Then the sequence* $\{(\mu^N(t), \theta^N(t)), t \in [0,T], N \geq 1\}$ *satisfies the LDP on* $D([0,T], M_1(\mathcal{X})) \times D_\uparrow([0,T], M(\mathcal{Y}))$ *with rate function*

$$I(\mu, \theta) := I_0(\mu(0)) + J(\mu, \theta).$$



The rate function J

$$
\begin{aligned}
J(\mu,\theta) := \int_{[0,T]} \Bigg\{ \sup_{\alpha\in\mathbb{R}^{|\mathcal{X}|}} \Bigg(& \langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t)\rangle \\
& - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} \tau(\alpha(x')-\alpha(x))\bar{\lambda}_{x,x'}(\mu_t,m_t)\mu_t(x)\Bigg) \\
+ \sup_{g\in\mathbb{R}^{|\mathcal{Y}|}} \sum_{y\in\mathcal{Y}} \Bigg(& -L_{\mu_t} g(y) \\
& - \sum_{y':(y,y')\in\mathcal{E}_{\mathcal{Y}}} \tau(g(y')-g(y))\gamma_{y,y'}(\mu_t)\Bigg) m_t(y)\Bigg\}\, dt
\end{aligned}
$$

whenever the mapping $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{X})$ is absolutely continuous, where $\theta(dtdy) = m_t(dy)dt$, and $J(\mu,\theta) = +\infty$ otherwise.

- $\tau(u) = e^u - u - 1, u \in \mathbb{R}$.

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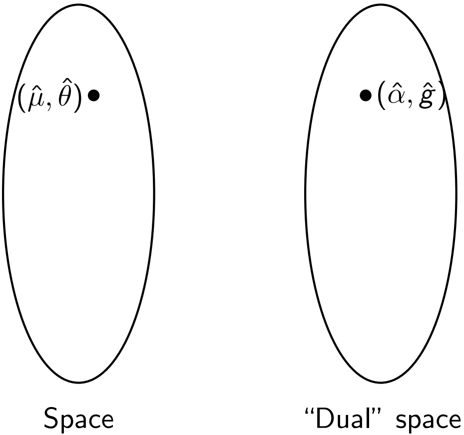
- The problem now is to uniquely identify $\tilde{I}$ (i.e., show that $\tilde{I} = I^*$).

Identification of $\tilde{I}$



- For "nice" elements of $D([0,T], M_1(\mathcal{X})) \times D_{\uparrow}([0,T], M(\mathcal{Y}))$, we show that $\tilde{I} = I^*$ (convex analysis, variational problems).
- Approximate general elements using "nice" elements and pass to the limit (parametric continuity of optimisation problems, dominated convergence).

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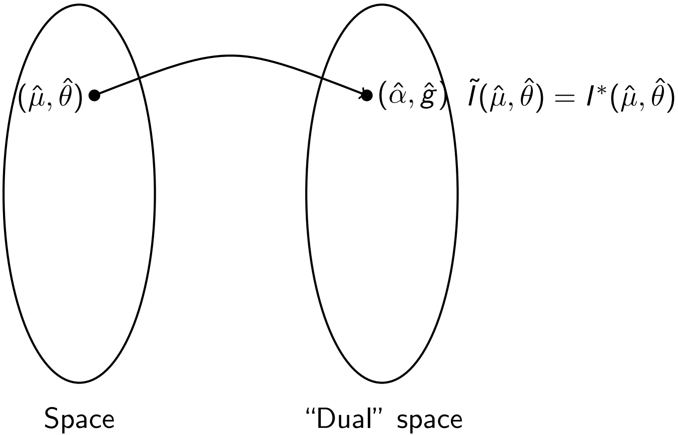

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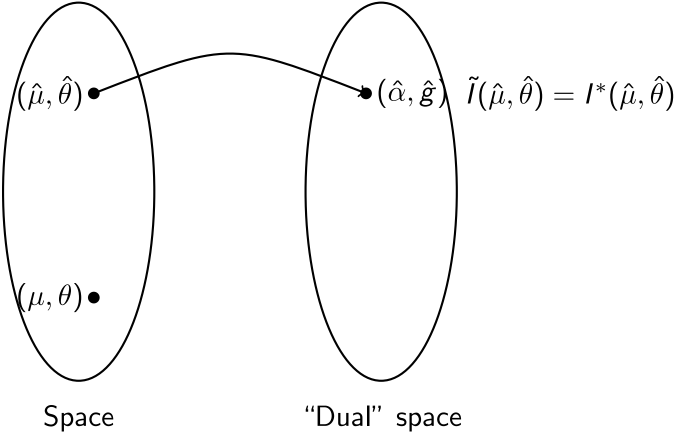

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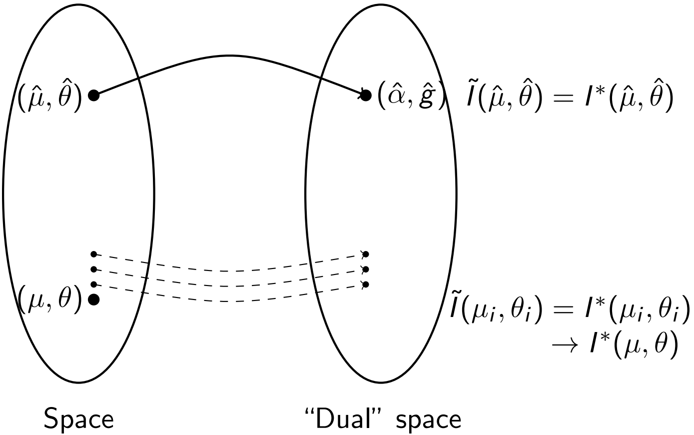

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Large deviations of the invariant measure in countable-state mean-field models

System model

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- N particles. The state of the nth particle at time t is $X_n^N(t) \in \mathcal{Z}$.

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- The empirical measure of the states of all the particles at time t:

$$\mu^N(t) = \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1^N(\mathcal{Z}).$$

- We are given the functions $\lambda_{z,z'} : M_1(\mathcal{Z}) \to \mathbb{R}_+$, $(z, z') \in \mathcal{E}$.
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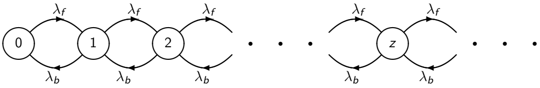
- Goal: study the large deviations of the family $\{\wp^N, N \geq 1\}$.

A counterexample

- Consider N independent, identical, positive recurrent M/M/1 queues. Let ξ^* be the stationary distribution of one queue.



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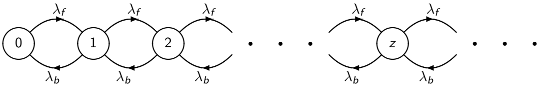

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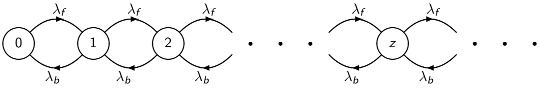

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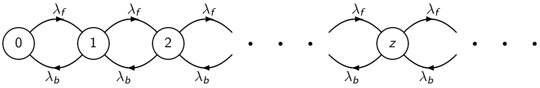

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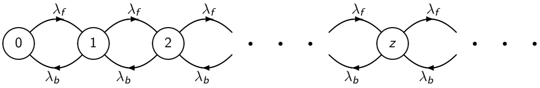

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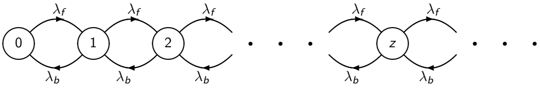

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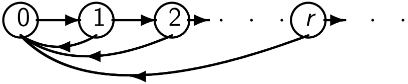
- We show that $V \neq D(\cdot\|\xi^*)$. Let $\vartheta(z) = z \log z$.
  - If $\xi \in M_1(\mathcal{Z})$ is such that $\sum z\xi(z) < \infty$ and $\sum \vartheta(z)\xi(z) = \infty$, then $V(\xi) = \infty$ but $D(\xi\|\xi^*) < \infty$.

Assumptions and main result

- We make the following assumptions on the mean-field model.
  - Transition graph:



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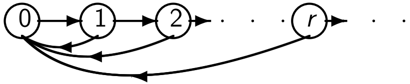

  - There exist positive constants $\overline{\lambda}$ and $\underline{\lambda}$ such that

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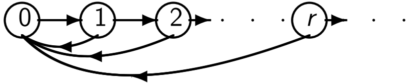

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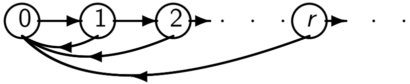

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  - There is a unique globally asymptotically stable equilibrium for the McKean-Vlasov equation (ξ^*).

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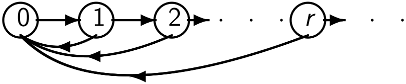
- We make the following assumptions on the mean-field model.
  - Transition graph:



  - There exist positive constants $\bar{\lambda}$ and $\underline{\lambda}$ such that

$$\frac{\underline{\lambda}}{z+1} \leq \lambda_{z,z+1}(\xi) \leq \frac{\bar{\lambda}}{z+1}, \text{ and } \underline{\lambda} \leq \lambda_{z,0}(\xi) \leq \bar{\lambda},$$

    for each $\xi \in M_1(\mathcal{Z})$.
  - The functions $(z+1)\lambda_{z,z+1}(\cdot)$, $z \in \mathcal{Z}$, and $\lambda_{z,0}(\cdot)$, $z \in \mathcal{Z} \setminus \{0\}$, are uniformly Lipschitz continuous on $M_1(\mathcal{Z})$.
  - There is a unique globally asymptotically stable equilibrium for the McKean-Vlasov equation (ξ^*).
- Under the above assumptions, we first show that, for each $N \geq 1$, there is a unique invariant measure $\wp^N$ for μ^N.

Main result

Theorem

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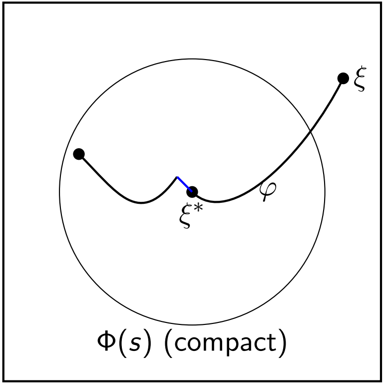
- We transfer the process-level uniform LDP for μ^N to the stationary regime (Sowers (1992)).
- Main ingredient is a continuity property of V: If $\xi_n \to \xi$ in $M_1(\mathcal{Z})$ and $\langle \xi_n, \vartheta \rangle \to \langle \xi, \vartheta \rangle$ as $n \to \infty$, then $V(\xi_n) \to V(\xi)$ as $n \to \infty$.

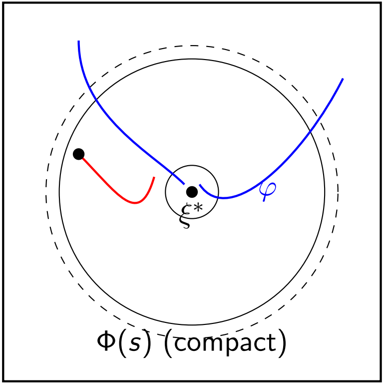
Proof sketch: Lower bound



- $\wp^N(\text{nbd}(\xi)) \geq \frac{1}{2} P(\mu^N_{\text{nbd}(\xi^*)} \in \text{nbd}(\varphi)) \geq \exp\{-N(V(\xi)+\gamma)\}$.
- The second inequality uses the uniform LDP over compact subsets of $M_1(\mathcal{Z})$.

Proof sketch: Upper bound

$$\begin{aligned}
&\wp^N(\sim \mathrm{nbd}(\Phi(s))) \\
&\le \exp\{-Ns\} + P(\mu^N_{\Phi(s)}(T) \notin \mathrm{nbd}(\Phi(s))) \\
&\le \exp\{-Ns\} \\
&\quad + P(\mu^N_{\Phi(s)} \text{ does not hit } \mathrm{nbd}(\xi^*)) \\
&\quad + P(\mu^N_{\mathrm{nbd}(\xi^*)} \in \mathrm{nbd}(\varphi)) \\
&\le \exp\{-N(s-\gamma)\}
\end{aligned}$$



- The first inequality uses exponential tightness.
- The second inequality uses the continuity of V under the convergence of $z \log z$-moments, and the strong Markov property.
- The third inequality uses the uniform LDP over compact subsets of $M_1(\mathcal{Z})$.

Part 4: Summary and Future Directions

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- We studied large time behaviour and metastability in models of weakly interacting Markov processes with jumps.
  - Finite-state mean-field models.
  - Two time scale mean-field models.
- Large deviations of the invariant measure in countable-state mean-field models.
- General strategy:
  - Study the process-level large deviations of the empirical measure process.
  - Use this to study the large time behaviour, and the large deviations of the invariant measure.

Future Directions

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Thank you!