Large Time Behaviour and Eigenvalue Problems for Finite State Mean-Field Particle Systems

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- Goal: understand large time behaviour of μ_N, and convergence to stationarity.

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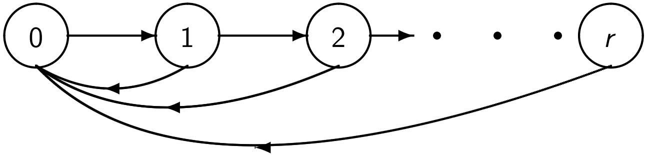
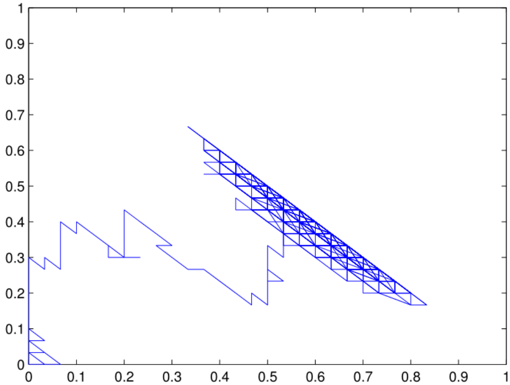


Figure: Set of allowed transitions in WiFi example

- State evolution:
  - Becomes less aggressive after a collision.
  - Moves to the most aggressive state after a successful packet transmission.

A sample path of μ_N in WiFi example



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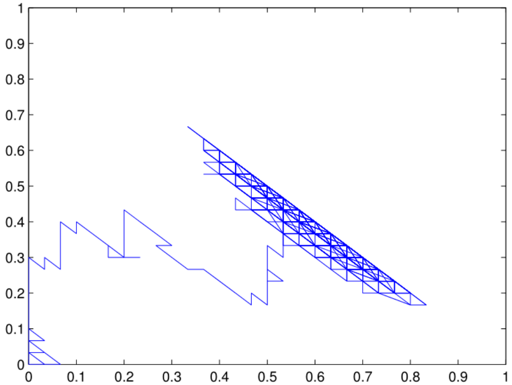

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- Multiple stable regions in the system. Transition between two stable region occur over large time durations.

Background on large deviations of μ_N

- Let $\mu_N(0) \to \nu$. Then, over $[0, T]$, the process μ_N is close to the McKean-Vlasov equation:

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- Freidlin-Wentzell quasipotential $\tilde{V}(K_i, K_j)$: minimum cost of moving from K_i to K_j.
- Obtain one-step transition probability of μ_N near the ω-limits sets:

$$P(K_i, K_j) \simeq \exp\{-N\tilde{V}(K_i, K_j)\}.$$

Main results

- Large time behaviour of μ_N (in terms of $\tilde{V}$):
  - Mean exit time from ω-limit sets.
  - Estimates on probability of reaching a given ω-limit set.
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- Scaling of the second largest eigenvalue (assuming reversibility): $\lambda_2^N \simeq \exp\{-N\Lambda\}$.
  - Consequence: μ_N mixes slowly if there are metastable states ($\Lambda > 0$).