Large deviations of mean-field interacting particle systems in a fast varying environment

Sarath Yasodharan and Rajesh Sundaresan

ECE Department, Indian Institute of Science, Bangalore 560 012, India

Introduction

- Goal: Study large time behaviour and metastability in networked systems.
- Metastability: A system exhibits different behaviour over different time scales.

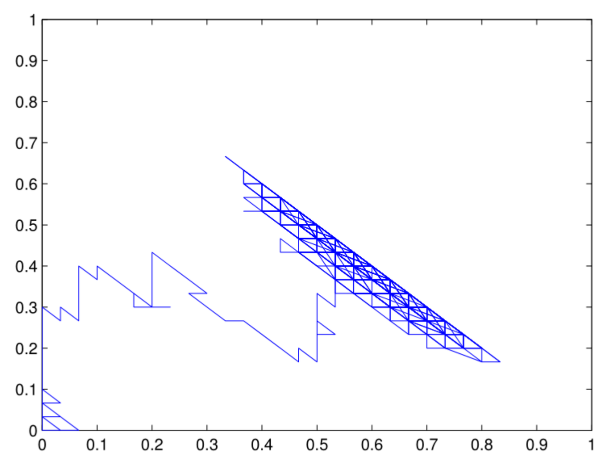

Figure 1: A sample path of the state of the system in a wireless local area network

- Model: A "fully coupled" two time scale mean-field model.
- Main result: Process-level large deviation principle for the joint law of the empirical measure of the particles and the occupation measure of the fast environment.

System Model

- N particles and an environment. Their states evolve over time.
- At time t, the state of the nth particle is denoted by $X_n^N(t) \in \mathcal{X}$; the state of the environment is denoted by $Y_N(t) \in \mathcal{Y}$.
- Certain allowed transitions.
  - Particles: a directed graph $(\mathcal{X}, \mathcal{E}_\mathcal{X})$; environment: a directed graph $(\mathcal{Y}, \mathcal{E}_\mathcal{Y})$.
- Empirical measure of the system of particles at time t:

$$\mu_N(t) = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1^N(\mathcal{X}) \subset M_1(\mathcal{X}).$$

- "Fully coupled" Markov evolution. Transition rates at time t:
  - A particle: $x \to x'$ transition at rate $\lambda_{x,x'}(\mu_N(t), Y_N(t))$;
  - Environment: $y \to y'$ transition at rate $N\gamma_{y,y'}(\mu_N(t))$.
- (μ_N, Y_N) is a Markov process with infinitesimal generator

$$f \mapsto \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} N\xi(x)\lambda_{x,x'}(\xi,y)\left[f\left(\xi + \frac{\delta_{x'}}{N} - \frac{\delta_x}{N}, y\right) - f(\xi,y)\right] + N\sum_{y':(y,y')\in\mathcal{E}_\mathcal{Y}} (f(\xi,y') - f(\xi,y))\gamma_{y,y'}(\xi), \ (\xi,y) \in M_1^N(\mathcal{X}) \times \mathcal{Y}.$$

Assumptions on the model

(A1) The graphs $(\mathcal{X}, \mathcal{E}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{E}_\mathcal{Y})$ are irreducible.

(A2) The functions $\lambda_{x,x'}(\cdot, y)$ are Lipschitz continuous and $\inf_\xi \lambda_{x,x'}(\xi,y) > 0$ for all $(x,x') \in \mathcal{E}_\mathcal{X}$ and $y \in \mathcal{Y}$.

(A3) The functions $\gamma_{y,y'}(\cdot)$ are continuous and $\inf_\xi \gamma_{y,y'}(\xi) > 0$ for all $(y,y') \in \mathcal{E}_\mathcal{Y}$.

Preliminaries

Quantities of interest

- Fix a time duration $T > 0$ and view μ_N as a random element of $D([0,T], M_1(\mathcal{X}))$.
- Consider the occupation measure of the fast environment:

$$\theta_N(t)(\cdot) = \int_0^t 1_{\{Y_N(s)\in\cdot\}} ds, \ 0 \le t \le T.$$

- θ_N is a random element of $D_\uparrow([0,T], M(\mathcal{Y}))$, the set of $\theta \in D([0,T], M(\mathcal{Y}))$ such that $\theta_t - \theta_s \in M(\mathcal{Y})$ and $\theta_t(\mathcal{Y}) = t$ for $0 \le s \le t \le T$.
- $\theta \in D_\uparrow([0,T], M(\mathcal{Y}))$ is also viewed as a measure on $[0,T]\times\mathcal{Y}$ and obeys the disintegration $\theta(dydt) = m_t(dy)dt$ where $m_t \in M_1(\mathcal{Y})$.
- We consider (μ_N, θ_N) with sample paths in $D([0,T], M_1(\mathcal{X})) \times D_\uparrow([0,T], M(\mathcal{Y}))$.

The averaging principle

- Suppose we freeze $\mu_N(t)$ to be ξ.Then for large N,
  - Y_N would quickly equilibrate to π_ξ, the unique invariant probability measure of

$$L_\xi g(y) := \sum_{y':(y,y')\in\mathcal{E}_\mathcal{Y}} (g(y') - g(y))\gamma_{y,y'}(\xi), y \in \mathcal{Y}.$$

  - An (x,x') transition occurs at rate $\sum_{y\in\mathcal{Y}} \lambda_{x,x'}(\xi,y)\pi_\xi(y) =: \bar{\lambda}_{x,x'}(\xi,\pi_\xi)$.

Theorem 1 (Bordenave et al. 2009)**.** *Suppose that $\mu_N(0) \to \nu$ in $M_1(\mathcal{X})$. Then μ_N converges in probability, in $D([0,T], M_1(\mathcal{X}))$, to the solution to the ODE*

$$\dot{\mu}_t = \bar{\Lambda}^*_{\mu_t,\pi_{\mu_t}}\mu_t, \ 0 \le t \le T, \ \mu_0 = \nu,$$

where $\bar{\Lambda}_{\mu_t,\pi_{\mu_t}}(x,x') = \bar{\lambda}_{x,x'}(\mu_t, \pi_{\mu_t})$.

- μ_N is a small random perturbation of the above ODE. We establish a large deviation principle (LDP) for (μ_N, θ_N); $P((\mu_N,\theta_N) \in A) \sim \exp\{-N \inf_{(\mu,\theta)\in A} I(\mu,\theta)\}$.

Main Result

Theorem 2. *Assume (A1), (A2), and (A3). Suppose that $\{\mu_N(0)\}_{N\ge1}$ satisfies the LDP on $M_1(\mathcal{X})$ with rate function I_0. Then the sequence $\{(\mu_N(t),\theta_N(t)), 0\le t\le T\}_{N\ge1}$ satisfies the LDP on $D([0,T], M_1(\mathcal{X})) \times D_\uparrow([0,T], M(\mathcal{Y}))$ with rate function*

$$I(\mu,\theta) = I_0(\mu(0)) + J(\mu,\theta);$$

$$J(\mu,\theta) = \int_{[0,T]} \Bigg\{ \sup_{\alpha\in\mathbb{R}^{|\mathcal{X}|}} \Bigg(\langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t)\rangle - \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} \tau(\alpha(x') - \alpha(x))\bar{\lambda}_{x,x'}(\mu_t,m_t)\mu_t(x)\Bigg) + \sup_{g\in\mathbb{R}^{|\mathcal{Y}|}} \sum_{y\in\mathcal{Y}} \Bigg(-L_{\mu_t}g(y) - \sum_{y':(y,y')\in\mathcal{E}_\mathcal{Y}} \tau(g(y') - g(y))\gamma_{y,y'}(\mu_t)\Bigg) m_t(y) \Bigg\} dt$$

whenever the mapping $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{X})$ is absolutely continuous, where $\theta(dtdy) = m_t(dy)dt$, and $J(\mu,\theta) = +\infty$ otherwise.

- $\tau(u) = e^u - u - 1, u \in \mathbb{R}$.

Proof Sketch

Exponential tightness

Theorem 3. *The sequence $\{(\mu_N(t),\theta_N(t)), t\in[0,T]\}_{N\ge1}$ is exponentially tight in $D([0,T], M_1(\mathcal{X})) \times D_\uparrow([0,T], M(\mathcal{Y}))$, i.e., given any $M > 0$, there exists a compact set K_M such that*

$$\limsup_{N\to\infty} \frac{1}{N} \log P\left(\{(\mu_N(t),\theta_N(t)), 0\le t\le T\} \notin K_M\right) \le -M.$$

An equation for a subsequential rate function

- Let $\tilde{I}$ be a subsequential rate function for $\{(\mu_N,\theta_N), N\ge1\}$.
- Let $\alpha : [0,T]\times M_1(\mathcal{X}) \to \mathbb{R}^{|\mathcal{X}|}$ and $g : [0,T]\times M_1(\mathcal{X})\times\mathcal{Y} \to \mathbb{R}$ be bounded measurable, and continuous on $M_1(\mathcal{X})$.
- Define $U_t^{\alpha,g}(\mu,\theta)$ by

$$\int_{[0,t]} \Bigg\{ \langle \alpha_s(\mu_s), \dot{\mu}_s - \bar{\Lambda}^*_{\mu_s,m_s}\mu_s\rangle - \sum_{(x,x')} \tau(\alpha_s(\mu_s)(x') - \alpha_s(\mu_s)(x))\bar{\lambda}_{x,x'}(\mu_s,m_s)\mu_s(x) + \sum_y \Bigg(-L_{\mu_s}g_s(\mu_s,\cdot)(y) - \sum_{y:(y,y')\in\mathcal{E}_\mathcal{Y}} \tau(g_s(\mu_s,y') - g_s(\mu_s,y))\gamma_{y,y'}(\mu_s)\Bigg) m_s(y)\Bigg\} ds.$$

- Using Varadhan's lemma, it can be shown that, for each α and g,

$$\sup_{(\mu,\theta)\in\Gamma} (U_T^{\alpha,g}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0,$$

where Γ is the set of (μ,θ) such that $t \mapsto \mu_t$ absolutely continuous.

A candidate rate function

- Define $I^*(\mu,\theta) = \sup_{\alpha,g} U_T^{\alpha,g}(\mu,\theta)$.
- It can be shown that $I^* = J$, and that $\tilde{I} \ge I^*$ on Γ. Outside Γ, I^* can be shown to be $+\infty$. We next show that $\tilde{I} \le I^*$ whenever $I^* < +\infty$.

Identification of I^* on "nice" elements

- Suppose $(\hat{\mu},\hat{\theta})$ is such that $I^*(\hat{\mu},\hat{\theta}) < +\infty$; $\inf_{t\in[0,T]}\min_{x\in\mathcal{X}} \hat{\mu}_t(x) > 0$; the mapping $[0,T] \ni t \mapsto \hat{\mu}_t \in M_1(\mathcal{X})$ is Lipschitz continuous; $\inf_{t\in[0,T]}\min_{y\in\mathcal{Y}} \hat{m}_t(y) > 0$ where $\hat{\theta}(dydt) = \hat{m}_t(dy)dt$.
- Then, there exists $(\hat{\alpha},\hat{g})$ that attains $\sup_{\alpha,g} U_T^{\alpha,g}(\hat{\mu},\hat{\theta})$. Then get $(\tilde{\mu},\tilde{\theta})$ that attains the supremum in $\sup_{(\mu,\theta)\in\Gamma}(U_T^{\hat{\alpha},\hat{g}}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0$.
- Hence, $I^*(\tilde{\mu},\tilde{\theta}) \ge U_T^{\hat{\alpha},\hat{g}}(\tilde{\mu},\tilde{\theta}) = \tilde{I}(\tilde{\mu},\tilde{\theta})$. Thus $I^*(\tilde{\mu},\tilde{\theta}) = \tilde{I}(\tilde{\mu},\tilde{\theta})$.
- Show that $(\tilde{\mu},\tilde{\theta}) = (\hat{\mu},\hat{\theta})$. It follows that $\tilde{I}(\hat{\mu},\hat{\theta}) = I^*(\hat{\mu},\hat{\theta})$.

Identification of I^* using approximations

- For a general (μ,θ), construct a suitable sequence of "nice" elements $\{(\mu_i,\theta_i), i\ge1\}$ and show that $(\mu_i,\theta_i) \to (\mu,\theta)$ and $I^*(\mu_i,\theta_i) \to I^*(\mu,\theta)$ as $i\to\infty$.

Acknowledgements

SY is supported by a research fellowship grant from the Centre for Networked Intelligence (a Cisco CSR Initiative), Indian Institute of Science.