Large Time Behaviour and the Second Eigenvalue Problem for Finite State Mean-Field Interacting Particle Systems

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- Goal: understand large time behaviour of μ_N, and convergence to stationarity.

A sample path of μ_N in WiFi example

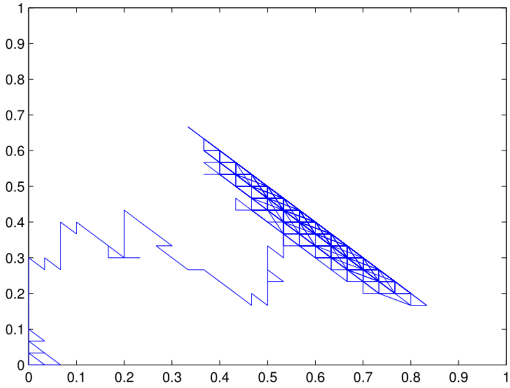

Figure: Evolution of states in a WiFi network under the MAC protocol

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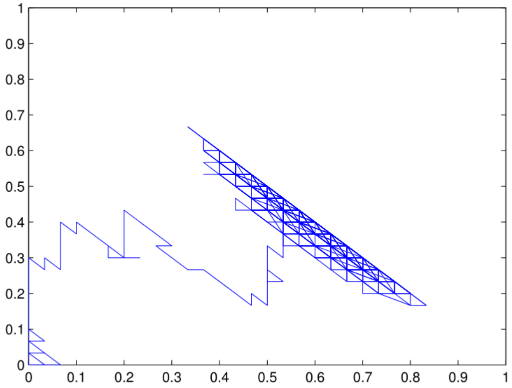

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- Multiple stable regions in the system. Transitions between two stable regions occur over large time durations.

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  - Estimates on probability of reaching a given ω-limit set.
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- Scaling of the second largest eigenvalue (assuming reversibility): $\lambda_2^N \simeq \exp\{-N\Lambda\}$.
  - Consequence: μ_N mixes slowly if there are metastable states ($\Lambda > 0$).