Large Time Behaviour and Eigenvalue Problems for Finite State Mean-Field Particle Systems

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System model and motivation

- N particles, each evolving on a finite connected graph $(\mathcal{Z}, \mathcal{E})$. $X_n^N(t) \in \mathcal{Z}$: state of the nth particle at time t.
- Empirical measure

$$\mu_N(t) = \frac{1}{N}\sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1(\mathcal{Z}).$$

- Mean-field interaction: when $(z, z') \in \mathcal{E}$, a particle at state z transits to state z' at rate $\lambda_{z,z'}(\mu_N(t))$.
- $\{\mu_N(t), t \geq 0\}$ is a Markov process on $M_1(\mathcal{Z})$, the space of probability measures on $\mathcal{Z}$.

Objectives:

- Study of metastability: large time behaviour of the process μ_N.
- Mixing and convergence of μ_N to its invariant measure.

Example: Interaction in WLAN

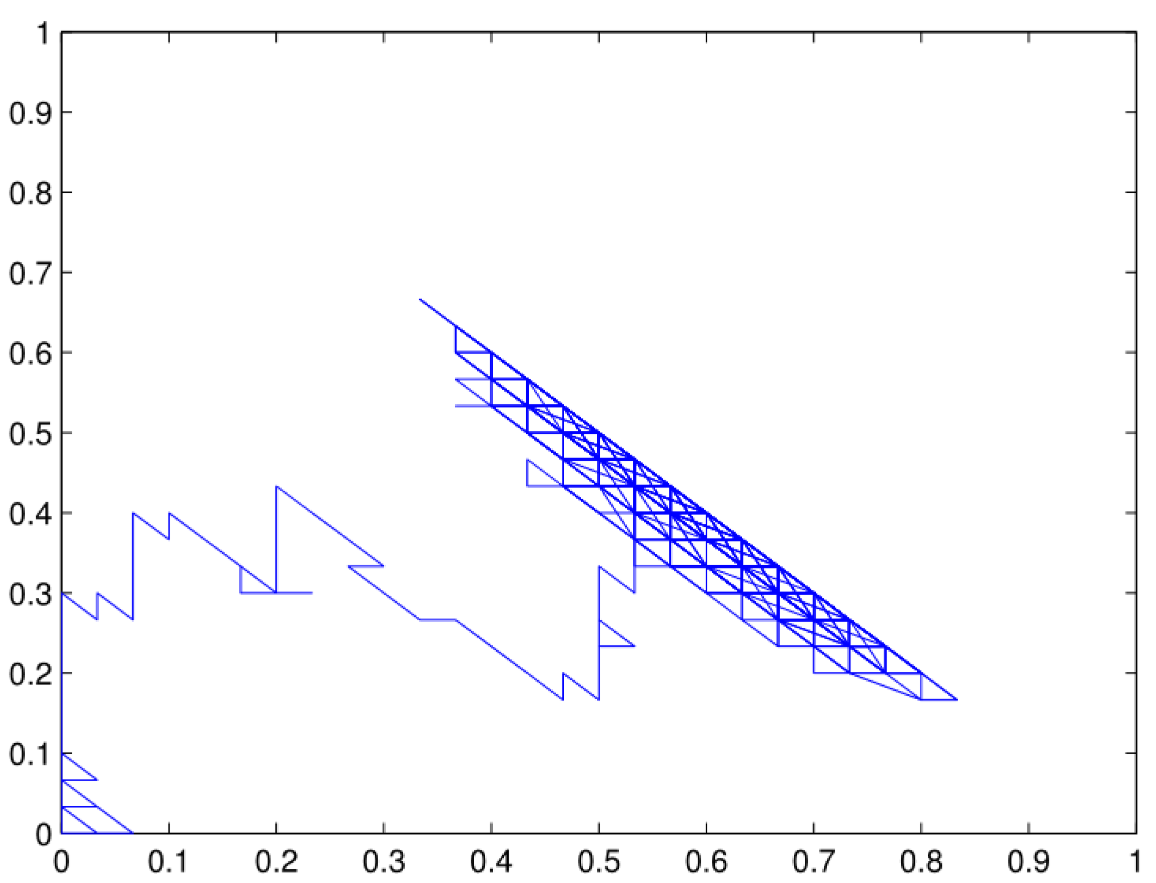

Figure 1: A sample path of μ_N in a WLAN with 30 nodes

- Another example: Randomised job assignment algorithms in the cloud.

The McKean-Vlasov equation

- Assume $\lambda_{z,z'}(\cdot)$, $(z, z') \in \mathcal{E}$, are Lipschitz.
- Let $\{\mu_N(0)\}_{N\geq 1}$ converge weakly to a deterministic measure $\nu \in M_1(\mathcal{Z})$. Then for any fixed $T > 0$, the empirical measure process $(\mu_N(t), 0 \leq t \leq T)$ converges in $D([0,T], M_1(\mathcal{Z}))$ to the solution to the ODE

$$\dot{\mu}(t) = \Lambda^*_{\mu(t)}\mu(t),\ 0 \leq t \leq T,\ \mu(0) = \nu. \tag{1}$$

- Convergence under stationarity: Let $\wp_N$ denote the unique invariant probability measure of μ_N, and let ξ^* denote the unique global attractor of (1). Then, $\wp_N \to \delta_{\xi^*}$.

Large deviations: stationary regime

- Let $p_{\nu_N}^{(N)}$ denote the law of μ_N on $D([0,T], M_1(\mathcal{Z}))$ starting at ν_N. Large deviations of $p_{\nu_N}^{(N)}$ from the McKean-Vlasov limit:

Theorem 1 ([1, Theorem 3.1])**.** *Suppose that the initial conditions $\nu_N \to \nu$ in $M_1(\mathcal{Z})$. Then the sequence of probability measures $\{p_{\nu_N}^{(N)}, N \geq 1\}$ on the space $D([0,T], M_1(\mathcal{Z}))$ satisfies the LDP with a good rate function $S_{[0,T]}(\mu|\nu)$. Moreover, if $S_{[0,T]}(\mu|\nu) < \infty$, then $\mu \in \mathcal{AC}[0,T]$, $\mu(0) = \nu$ and there exists a family of rate matrices $L(t), 0 \leq t \leq T$, such that μ is the unique solution to*

$$\dot{\mu}(t) = L(t)^*\mu(t),\ 0 \leq t \leq T,\ \mu(0) = \nu,$$

and

$$S_{[0,T]}(\mu|\nu) = \int_{[0,T]} \sum_{(i,j)\in\mathcal{E}} \mu(t)(i)\lambda_{i,j}(\mu(t))\tau^*\left(\frac{l_{i,j}(t)}{\lambda_{i,j}(\mu(t))} - 1\right) dt.$$

- Freidlin-Wentzell quasipotential $V : M_1(\mathcal{Z}) \times M_1(\mathcal{Z}) \to [0, \infty)$ defined by

$$V(\nu, \xi) = \inf\{S_{[0,T]}(\mu|\nu) : \mu(T) = \xi, T > 0\},$$

  i.e., $V(\nu, \xi)$ denotes the minimum cost of transport from ν to ξ in an arbitrary but finite time.
- $\tilde{V}(K_i, K_j) = \inf\{S_{[0,T]}(\mu|\nu) : \nu \in K_i, \mu(t) \notin \cup_{k\neq i,j}K_k \text{ for all } 0 \leq t \leq T, \mu(T) \in K_j, T > 0\}$.
- We say $\nu \sim \xi$ if $V(\nu, \xi) = 0$ and $V(\xi, \nu) = 0$. Assumption on the McKean-Vlasov equation:

(A1) There exists a finite number of compact sets $K_1, K_2, \ldots, K_l$ such that
  - For each $i = 1, 2, \ldots l$, $\nu_1, \nu_2 \in K_i$ implies $\nu_1 \sim q\nu_2$.
  - For each $i \neq j$, $\nu_1 \in K_i$ and $\nu_2 \in K_j$ implies $\nu_1 \nsim \nu_2$.
  - Every ω-limit set of the dynamical system (1) lies completely in one of the compact sets K_i.

Approximation of μ_N in a neighbourhood of the attractors

- Given $0 < \rho_1 < \rho_0$, let γ_i (resp. Γ_i) denote the ρ_1-open neighbourhood (resp. ρ_0-open neighbourhood) of K_i. Let $\gamma = \cup_{i=1}^{l}\gamma_i$, $\Gamma = \cup_{i=1}^{l}\Gamma_i$, and $C = M_1(\mathcal{Z}) \setminus \bar{\Gamma}$.
- Define hitting times: $\tau_0 = 0$, $\sigma_n = \inf\{t > \tau_{n-1} : \mu_N(t) \in C\}$, $\tau_n = \inf\{t > \sigma_n : \mu_N(t) \in \gamma\}$.
- Approximate the process μ_N by a discrete time Markov chain: $Z_n^N = \mu_N(\tau_n)$.

Lemma 1 ([1, Lemma A.6])**.** *Given $\varepsilon > 0$, there exist $\rho_0 > 0$ and $N_0 \geq 1$ such that, for any $\rho_2 < \rho_0$, there exists $\rho_1 < \rho_2$ such that for any $\nu \in [K_i]_{\rho_2} \cap M_1^N(\mathcal{Z})$ and $N \geq N_0$, the one-step transition probability of the chain Z^N satisfies*

$$\exp\{-N(\tilde{V}(K_i, K_j) + \varepsilon)\} \leq P(\nu, \gamma_j) \leq \exp\{-N(\tilde{V}(K_i, K_j) - \varepsilon)\}. \tag{2}$$

- Proof using Theorem 1 along with the strong Markov property of μ_N and continuity of V.
- For $W \subset L$, a W-graph is a directed graph on L such that (i) each element of $L \setminus W$ has exactly one outgoing arrow and (ii) there are no closed cycles in the graph. $G(W)$: set of W-graphs.

Theorem 2 ([1, Theorem 2.2])**.** *Assume (A1). Then, the sequence of invariant measures $\{\wp_N\}_{N\geq 1}$ satisfies the large deviation principle on $M_1(\mathcal{Z})$ with good rate function s given by*

$$s(\xi) = \min_{1\leq i\leq l}\{W(i) + V(K_i, \xi)\} - \min_{1\leq j\leq l} W(j), \tag{3}$$

where

$$W(i) = \min_{g\in G(i)} \sum_{(m,n)\in G} \tilde{V}(m, n).$$

Cycles

- Let $L = \{1, 2, \ldots, l\}$. If $\tilde{V}(K_i) = \min_{j\neq i}\tilde{V}(K_i, K_j)$, we say there is an arrow from i to j.
- A cycle π is a subgraph of L satisfying
  1. $i \in \pi$ and there is a sequence of arrows leading from i to j, then $j \in \pi$.
  2. For any $i \neq j$ in π, we have a sequence of arrows leading from i to j and vice-versa.
- Can define hierarchy of cycles, starting with cycle of cycles.
- Cycles are very stable subsets of L. Mean exit from a cycle π is of the order $\exp\{N\tilde{V}(\pi)\}$, uniformly in the initial condition. Another estimate:

Lemma 2. *Let π_1^k, π_2^k be k-cycles and let $\pi_1^k \to \pi_2^k$. Then, given $\varepsilon > 0$, there exist $\delta > 0$, $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma_{\pi_1^k} \cap M_1^N(\mathcal{Z})$ and $N \geq N_0$, we have*

$$P_\nu\left(\bar{\tau}_{\pi_1^k} \leq \exp\{N(\tilde{V}(\pi_1^k) - \delta)\}, \mu_N(\bar{\tau}_{\pi_1^k}) \in \gamma_{\pi_2^k}\right) \geq \exp\{-N\varepsilon\}.$$

Main results

- Define

$$\Lambda = \min\{\tilde{V}(g) : g \in G(i), i \in L\} - \min\{\tilde{V}(g) : g \in G(i,j), i, j \in L, i \neq j\}.$$

- Let i_0 be such that $\min\{\tilde{V}(g) : g \in G(i), i \in L\} = \min\{\tilde{V}(g) : g \in G(i_0)\}$.
  First result on mixing. The process μ_N mixes well if time is of the order $\exp\{N(\Lambda - O(1))\}$:

Theorem 3. *Given $\varepsilon > 0$, there exist $\delta_0 > 0$, $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $N \geq N_0$, $\nu \in M_1^N(\mathcal{Z})$, we have*

$$P_{T_0}(\nu, \gamma_{i_0}) \geq \exp\{-N\varepsilon\}, \tag{4}$$

where $T_0 = \exp\{N(\Lambda - \delta_0)\}$. Furthermore, there exist $\nu_0 \in M_1(\mathcal{Z})$ and $\beta > 0$ such that for all $N \geq N_0$ and $\nu \in [\nu_0]_{\rho_1} \cap M_1^N(\mathcal{Z})$

$$P_{T_0}(\nu, \gamma_{i_0}) \leq \exp\{-N\beta\}. \tag{5}$$

- Proof idea: follow a hierarchy of cycles, and use the estimate in Lemma 2.
- μ_N becomes very close to its invariant measure if time is of the order $\exp\{N(\Lambda + O(1))\}$:

Theorem 4. *Given $\delta > 0$, there exist $\varepsilon > 0$ and $N_0 \geq 1$ such that for all $\nu \in M_1^N(\mathcal{Z})$ and $N \geq N_0$*

$$|E_\nu(f(\mu_N(T))) - \langle f, \wp_N\rangle| \leq \|f\|_\infty \exp\{-\exp(N\varepsilon)\},$$

where $T = \exp\{N(\Lambda + \delta)\}$ and $f \in B(M_1(\mathcal{Z}))$.

- Proof idea: Use Theorem 3 and Theorem 2.
- Let μ_N be reversible with respect to $\wp_N$. For a fixed N, convergence to the invariant measure is governed by λ_2^N, the second eigenvalue of the generator.

Theorem 5.

$$\lim_{N\to\infty} \frac{1}{N}\log\lambda_2^N = -\Lambda.$$

- $\Lambda > 0$ when there are metastable states. This slows down convergence.

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