Metastability phenomenon: Large deviations in the stationary regime

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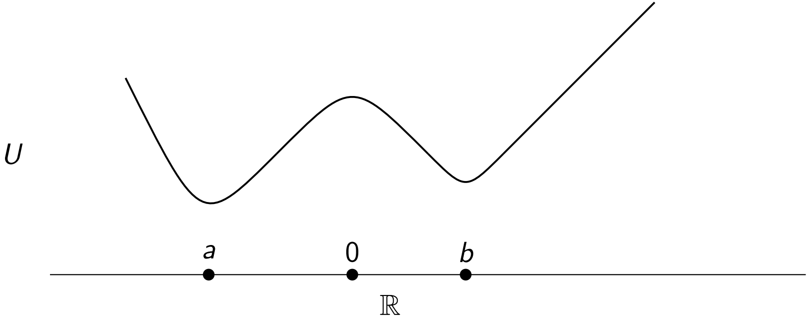
28 July 2020

Part 1: Metastability

Example 1 – Double well potential

- Consider the ODE

$$dX_t = -U'(X_t)dt.$$



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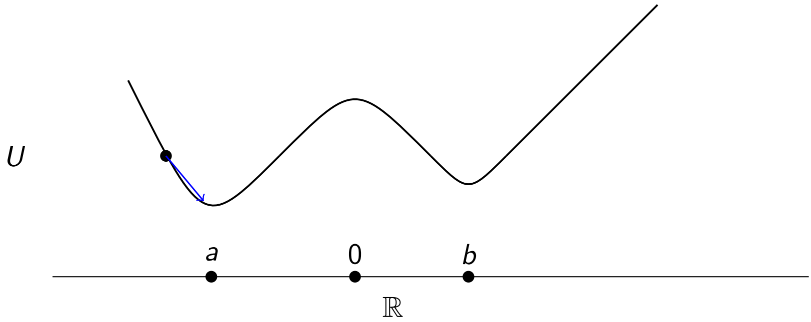

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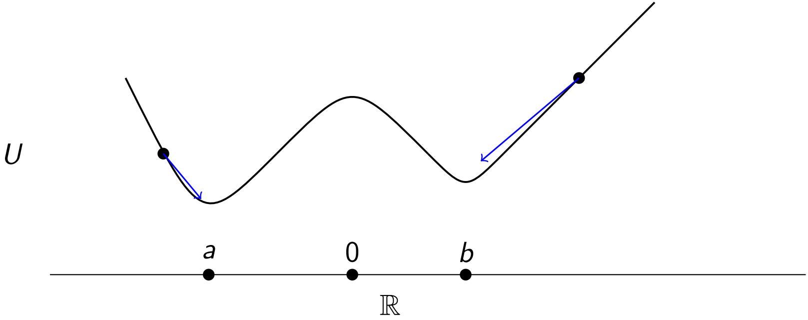

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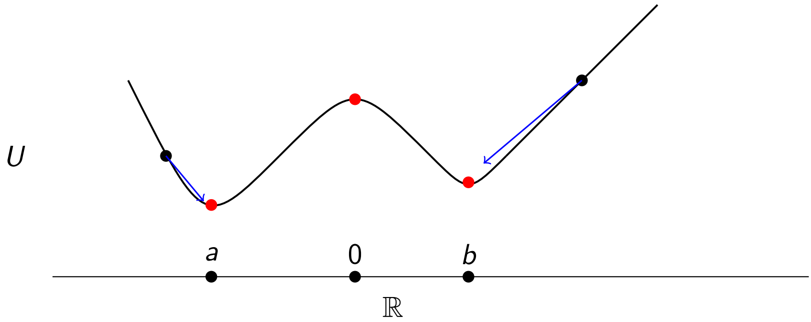

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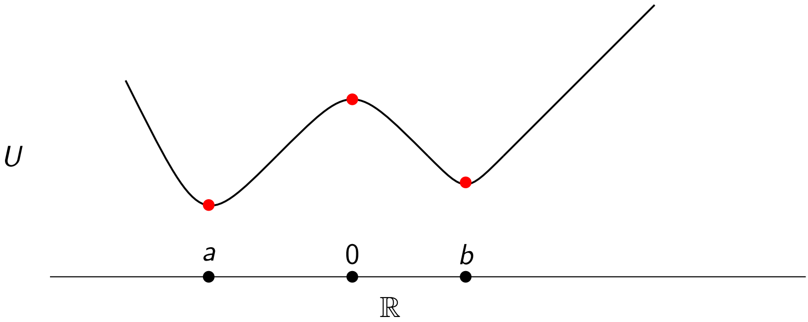

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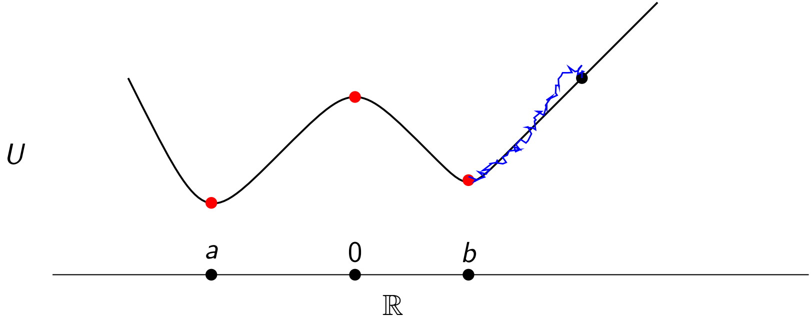

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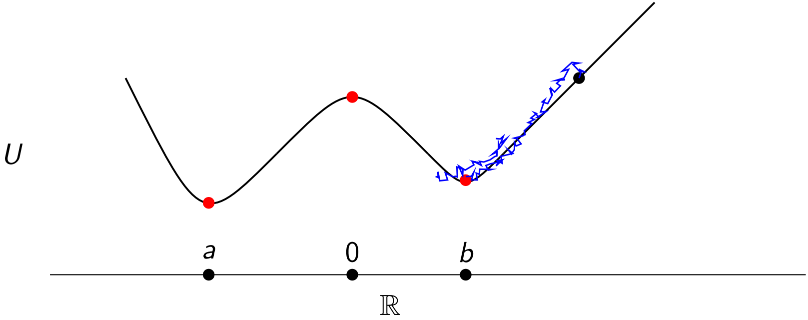

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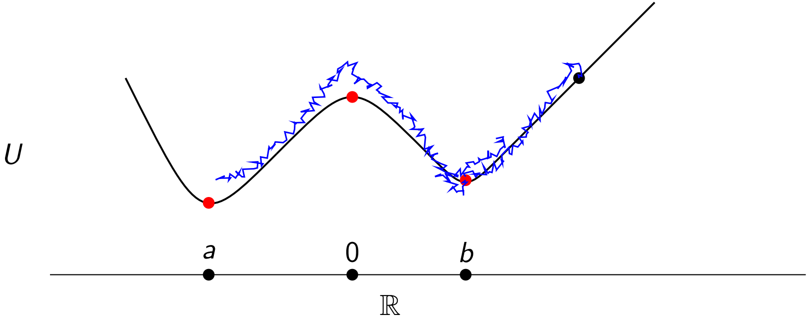

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- For each $(z, z') \in \mathcal{E}$, we have a function $\lambda_{z,z'} : M_1(\mathcal{Z}) \to [0, +\infty)$.
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- Particle transitions: at time t, a $z \to z'$ transition occurs at rate $\lambda_{z,z'}(\mu_N(t))$. Mean-field interaction.
- $\{(X_n^N(t), 1 \le n \le N), t \ge 0\}$ is a Markov process on $\mathcal{Z}^N$.

The empirical measure process μ_N

- $\{\mu_N(t), t \geq 0\}$ is a Markov process on $M_1(\mathcal{Z})$ with infinitesimal generator

$$L^N f(\xi) = \sum_{(z,z')\in\mathcal{E}} N\xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right].$$

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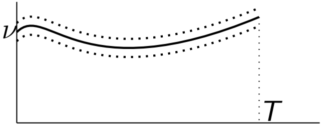
- Typical behaviour of μ_N (mean-field limit):
  Let $\mu_N(0) \to \nu$ weakly as $N \to \infty$. Then $\{\mu_N(t), 0 \leq t \leq T\}$, w.h.p., is "close to" to the solution to the McKean-Vlasov equation:

$$\dot{\mu}_t = \Lambda^*_{\mu_t}\mu_t, \; \mu_0 = \nu.$$



- Thus, μ_N is an ODE+"noise". Metastability occurs here as well.

Interaction in WiFi networks

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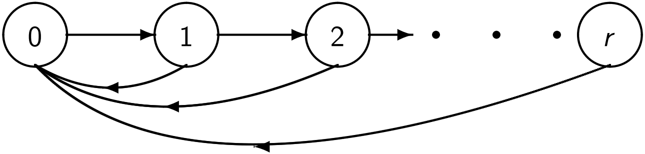

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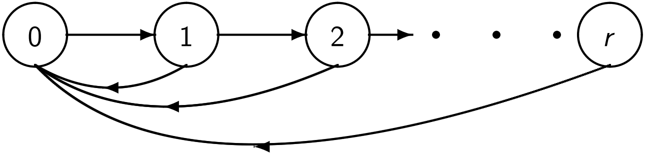

Figure: Set of allowed transitions in WiFi example

- State evolution:
  - Becomes less aggressive after a collision.
  - Moves to the most aggressive state after a successful packet transmission.

A sample path of μ_N in WiFi example

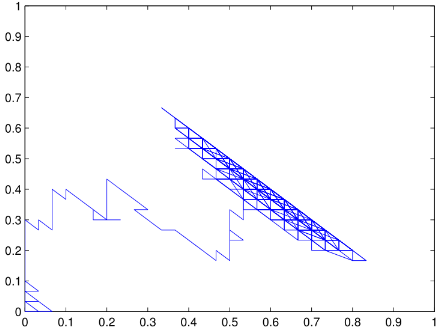

Figure: Evolution of states in a WiFi network under the MAC protocol

- Metastability phenomenon: Multiple stable regions in the system. Transition between two stable regions occur over large time durations.

Random perturbations of dynamical systems

- Example 1 - Double well potential
  - Perturbation of an ODE, “Gaussian” noise.
  - Markov process, a.s.-continuous sample paths
  - State space: $\mathbb{R}$
- Example 2 - Mean-field model
  - Perturbation of the McKean-Vlasov equation, “Poissonian noise”.
  - Markov process, with jumps
  - State space: $M_1(\mathcal{Z})$
- Abstractly,
  - A dynamical system, taking values in some space. $\mathbb{R}, M_1(\mathcal{Z}), C([0,1])$, etc.
  - A small random perturbation of this dynamical system. Mostly, Gaussian or Poissonian noise. This generates a Markov process.
  - As noise becomes small, the process converges to its typical behaviour.
  - Goal of this talk: Explain the metastability phenomenon of randomly perturbed dynamical system.

Back to Example 1 – A more concrete goal

- The potential U explains metastability phenomenon.
- It turns out that X^ε is reversible and possesses a unique invariant probability measure

$$\wp^\varepsilon(dx) = \text{const} \times \exp\left\{-\frac{1}{\varepsilon^2} U(x) dx\right\}.$$

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- But, there is no closed for expression for $\wp^\varepsilon$ in most cases!
  - The WiFi example has no closed form expression, even when the nodes do not interact.
- We want to identify a suitable "U" and obtain asymptotics of the above form.
  - Other questions: Asymptotics of exit times, mixing time etc. (not in this talk).

Part 2: Large deviations

Large deviations

- S: a metric space. $\{X_N\}_{N\geq 1}$ is a sequence of S-valued random variables.
- Roughly, $P(X_N \in A) \sim \exp\{-NI_A\}$ where $I_A = \inf_{x\in A} I(x)$.

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  $$P(d(X_N, \Phi(s)) \geq \delta) \leq \exp\{-N(s - \gamma)\}$$

  for any $s \leq s_0$ and $N \geq N_0$.

More suited to study the stationary regime. We'll come back later.

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Theorem (Sanov)

$\{M_N\}$ satisfies the LDP on $M_1(S)$ with rate function $D(\cdot \| \nu)$.

- In particular, $P\left(M_N \in A\right) \sim \exp\{-N \inf_{\mu \in A} D(\mu \| \nu)\}$.

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  - Compactness of level sets: $\{y \in T : J(y) \leq M\} = f(\{x \in S : I(x) \leq M\})$.
  - Upper and lower bounds: $P(Y_N \in A) = P(X_N \in f^{-1}(A))$

Back to our goal

- Recall that we have to find a suitable "U" that governs asymptotics of the form

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- We only know that $\wp^\varepsilon$ is invariant for X^ε.
- Strategy is to study large deviations of the trajectories and then move to stationarity.

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in $C([0, T], \mathbb{R})$, in probability, as $\varepsilon \to 0$ where X^0 solves the ODE

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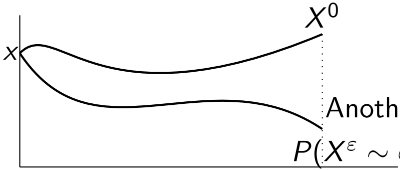
- LDP for X^ε. A functional $S_{[0,T]}$ on $C([0,T],\mathbb{R})$ so that



Another path φ.

$P(X^\varepsilon \sim \varphi) \sim e^{-\frac{1}{\varepsilon^2} S_{[0,T]}(\varphi)}$.

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Theorem

Let $X_0^\varepsilon \to x$.

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Theorem

Let $X_0^\varepsilon \to x$*. Then* $(X_t^\varepsilon, t \in [0, T])$ *satisfies the LDP on* $C([0, T], \mathbb{R})$ *with rate function* $S_{[0,T]}$ *defined as follows.*

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Theorem

Let $X_0^\varepsilon \to x$. Then $(X_t^\varepsilon, t \in [0, T])$ satisfies the LDP on $C([0, T], \mathbb{R})$ with rate function $S_{[0,T]}$ defined as follows.

$$S_{[0,T]}(\varphi) = \frac{1}{2} \int_{[0,T]} |\dot{\varphi}_t - (-U'(\varphi_t))|^2 dt$$

if $\varphi_0 = x$ and $t \mapsto \varphi_t$ is absolutely continuous; $S_{[0,T]}(\varphi) = +\infty$ otherwise.

Mean-field example – Large deviations in path-space

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$$S_{[0,T]}(\mu) = \sup_{\alpha \in C([0,T], \mathbb{R}^{|\mathcal{Z}|})} \int_{[0,T]} \Bigg\{ \langle \alpha_t, \dot{\mu}_t - \Lambda^*_{\mu_t}\mu_t \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha_t(z') - \alpha_t(z)) \lambda_{z,z'}(\mu_t)\mu_t(z) \Bigg\} dt,$$

$S_{[0,T]}(\mu) = +\infty$ otherwise.

- $\tau(u) = e^u - u - 1,\ u \in \mathbb{R}$.

Some remarks

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- The variational form in the mean-field rate function is due to the non-uniqueness of h.
- Many proof techniques in the mean-field case
  - Sanov's theorem + Girsanov transformation (Dawson and Gärtner, Léonard).
  - Characterise the rate function using exponential martingales (Puhalskii, Liptser).
  - Weak convergence approach using variational formulae for Poisson random measure (Boué and Dupuis, Budhiraja).
  - Using semigroup convergence (Feng and Kurtz).

Part 3: Large deviations in the stationary regime

The unique equilibrium case

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- Turns out that

$$\wp_N \to \delta_{\xi^*}$$

  weakly as $N \to \infty$.

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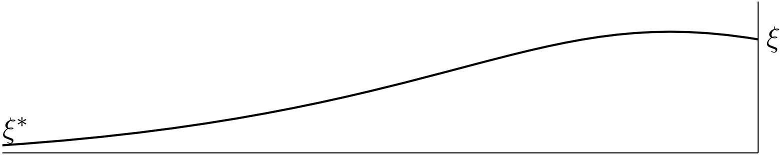

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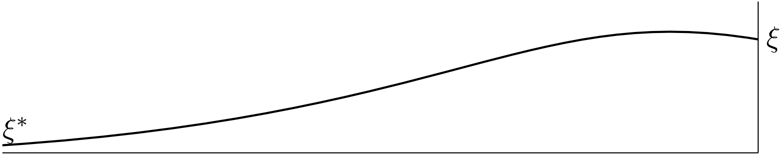

- Guess the cost:
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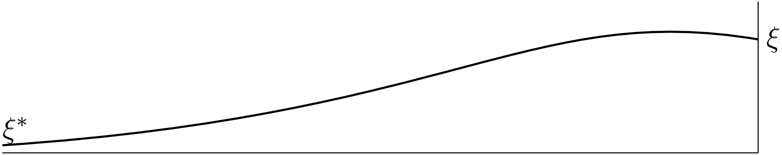

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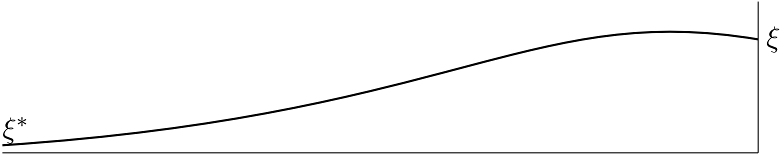

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  - Natural to ask if
    $\inf\{S_{[-T,0]}(\mu), \mu_{-T} = \xi^*, \mu_0 = \xi, T > 0\} \stackrel{?}{=}$
    $\inf\{S_{(-\infty,0]}(\mu), \mu_t \to \xi^*, \mu_0 = \xi, T > 0\}$
    $\geq$ is clear. $\leq$ needs a "small-cost connection" from ξ^* to a nearby point.

Large deviations of $\mu_N(T)$

- The mapping $\pi_T : D([0, T], M_1(\mathcal{Z})) \to M_1(\mathcal{Z})$ is continuous. Use the contraction principle.

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"Cost" to go from ν to ξ in arbitrary time.

- If $\mathcal{Z}$ is a finite set, we can show that $S_T(\cdot|\cdot)$ is continuous (hence uniformly continuous). "Small-cost connection" follows.

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  for all $\nu \in A$, $s \leq s_0$ and $N \geq N_0$.

- In many cases, one can essentially "transfer" the sample-path LDP to this uniform version (usually, $\mathcal{A}$ is the power set).

Uniform LDP – Examples

- Freidlin and Wentzell (1984): Finite-dimensional diffusions.
- Sowers (1992): A reaction diffusion equation (RDE).
- Cerrai and Röckner (2004): A multidimensional RDE.
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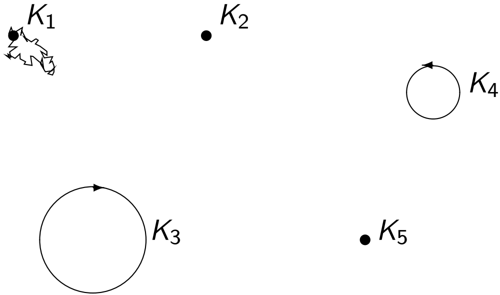
- Unique equilibrium case: Uniform LDP on compact sets + "small-cost" connection.

Multiple equilibria



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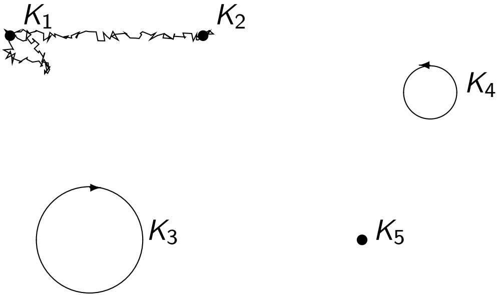

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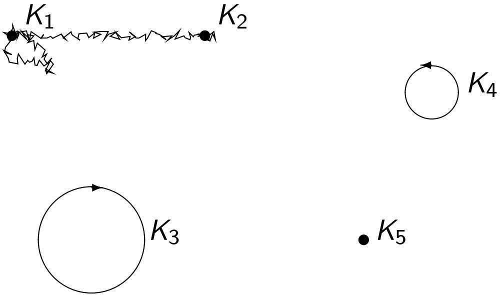

- τ_n: hitting time of μ_N in a given neighbourhood of K_i's.
- Hitting time chain: $Z_n^N = \mu_N(\tau_n)$, $n \geq 1$.

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- Key properties used:
  - Strong Markov property of μ_N.
  - Uniform LDP for μ_N on open sets of $M_1(\mathcal{Z})$.
  - Continuity of the quasipotential V.

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  - Khasminskii reconstruction needs to be extended.

Some recent progress

- Martirosyan (2017): Considers a situation where an invariant measure for Z^N may not exist. Works with "generalised stationary distributions".
- Puhalskii (2020): Provides some general conditions under which LDP for the invariant measure can be shown from the process-level LDP.
- Budhiraja et al (2018): Derives uniform LDP for a Banach space valued SDE using the weak convergence approach.
- Salins and Spiliopoulos (2018): Shows some continuity properties of the quasipotential for an infinite dimensional SDE.

Some open questions

- Uniform LDP in infinite dimensions for the mean-field setting
  - Not aware of such a result even in the non-interacting case.
  - However, uniform LDP on compact sets can be established (this is sufficient for the problem of invariant measure).
- A general result from which LDP for the invariant measure can be derived from process-level LDP
  - Puhalskii (2020) answers this partially, but this is not applicable in infinite dimensional settings.
  - Borkar and Sundaresan (2012) does this for the mean-field example using a control theoretic approach.

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Thank you