Large deviations of mean-field interacting particle systems in a fast varying environment

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$$\mu_N(t) := \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)} \in M_1^N(\mathcal{X}) \subset M_1(\mathcal{X}).$$

- We are given functions $\lambda_{x,x'}(\cdot, y)$, $(x, x') \in \mathcal{E}_\mathcal{X}$, $y \in \mathcal{Y}$ and $\gamma_{y,y'}(\cdot)$, $(y, y') \in \mathcal{E}_\mathcal{Y}$ on $M_1(\mathcal{X})$.

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- Markov evolution. Transition rates at time t:
  - Particles: $x \to x'$ at rate $\lambda_{x,x'}(\mu_N(t), Y_N(t))$;
  - Environment: $y \to y'$ at rate $N\gamma_{y,y'}(\mu_N(t))$.

System model

- (μ_N, Y_N) is a Markov process with infinitesimal generator

$$f \mapsto \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} N\xi(x)\lambda_{x,x'}(\xi,y)\left[f\left(\xi+\frac{\delta_{x'}}{N}-\frac{\delta_x}{N},y\right)-f(\xi,y)\right]$$
$$+N\sum_{y':(y,y')\in\mathcal{E}_{\mathcal{Y}}}(f(\xi,y')-f(\xi,y))\gamma_{y,y'}(\xi),$$

$(\xi, y) \in M_1^N(\mathcal{X}) \times \mathcal{Y}$.

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- A "fully coupled" two time scale mean-field model.
- Assumptions:
    - The graphs $(\mathcal{X},\mathcal{E}_{\mathcal{X}})$ and $(\mathcal{Y},\mathcal{E}_{\mathcal{Y}})$ are connected.
    - The functions $\lambda_{x,x'}(\cdot,y)$ are Lipschitz continuous and $\inf_\xi \lambda_{x,x'}(\xi,y)>0$ for all $(x,x')\in\mathcal{E}_{\mathcal{X}}$ and $y\in\mathcal{Y}$.
    - The functions $\gamma_{y,y'}(\cdot)$ are continuous and $\inf_\xi \gamma_{y,y'}(\xi)>0$ for all $(y,y')\in\mathcal{E}_{\mathcal{Y}}$.

Quantities of interest

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- View μ_N as a random element of $D([0,T], M_1(\mathcal{X}))$.
- Consider the occupation measure of the fast environment:

$$\theta_N(t)(\cdot) := \int_0^t 1_{\{Y_N(s)\in\cdot\}} ds,\ 0 \le t \le T.$$

- θ_N is a random element of $D_\uparrow([0,T], M(\mathcal{Y}))$, the set of θ such that $\theta_t - \theta_s \in M(\mathcal{Y})$ and $\theta_t(\mathcal{Y}) = t$ for $0 \le s \le t \le T$.
- $\theta \in D_\uparrow([0,T], M(\mathcal{Y}))$ is also viewed as a measure on $[0,T] \times \mathcal{Y}$ and obeys the disintegration $\theta(dydt) = m_t(dy)dt$ where $m_t \in M_1(\mathcal{Y})$.

Main result

Theorem

Suppose that $\{\mu_N(0)\}_{N\geq 1}$ *satisfies the large deviation principle (LDP) on* $M_1(\mathcal{X})$ *with rate function* I_0*. Then the sequence* $\{(\mu_N(t), \theta_N(t)), 0 \leq t \leq T\}_{N\geq 1}$ *satisfies the LDP on* $D([0,T], M_1(\mathcal{X})) \times D_{\uparrow}([0,T], M(\mathcal{Y}))$ *with rate function*

$$I(\mu, \theta) := I_0(\mu(0)) + J(\mu, \theta).$$

The rate function J

- Let $\tau(u) = e^u - u - 1, u \in \mathbb{R}$.
- Consider (μ, θ). If the mapping $[0, T] \ni t \mapsto \mu_t \in M_1(\mathcal{X})$ is absolutely continuous, then

$$J(\mu, \theta) := \int_{[0,T]} \Bigg\{ \sup_{\alpha \in \mathbb{R}^{|\mathcal{X}|}} \Bigg(\langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t, m_t} \mu_t) \rangle - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} \tau(\alpha(x') - \alpha(x)) \bar{\lambda}_{x,x'}(\mu_t, m_t) \mu_t(x) \Bigg) + \sup_{g \in \mathbb{R}^{|\mathcal{Y}|}} \sum_{y \in \mathcal{Y}} \Bigg(-L_{\mu_t} g(y) - \sum_{y' : (y,y') \in \mathcal{E}_{\mathcal{Y}}} \tau(g(y') - g(y)) \gamma_{y,y'}(\mu_t) \Bigg) m_t(y) \Bigg\} dt$$

where $\theta(dtdy) = m_t(dy)dt$. $J(\mu, \theta) = +\infty$ otherwise.

Outline of the proof

- Show exponential tightness. This gives a subsequential LDP.
- Get a condition for any subsequential rate function in terms of an exponential martingale.
- Identify the subsequential rate function on "nice" elements of the space.
  - Study of a certain variational problem, and parametric continuity of optimisation problems.
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Reference: arXiv:2008.06855

Thank you