Module - 2 Lecture Notes – 5

Kuhn-Tucker Conditions

Introduction

In the previous lecture the optimization of functions of multiple variables subjected to equality constraints using the method of constrained variation and the method of Lagrange multipliers was dealt. In this lecture the Kuhn-Tucker conditions will be discussed with examples for a point to be a local optimum in case of a function subject to inequality constraints.

Kuhn-Tucker Conditions

It was previously established that for both an unconstrained optimization problem and an optimization problem with an equality constraint the first-order conditions are sufficient for a global optimum when the objective and constraint functions satisfy appropriate concavity/convexity conditions. The same is true for an optimization problem with inequality constraints.

The Kuhn-Tucker conditions are both necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is concave, i.e. the problems belong to a class called the *convex programming problems.*

Consider the following optimization problem:

Minimize $f(\mathbf{X})$ subject to $g_j(\mathbf{X}) \leq 0$ for $j = 1,2,\ldots,p$; where $\mathbf{X} = [x_1 \;\; x_2 \; . \; . \; . \; x_n]$

Then the Kuhn-Tucker conditions for $\mathbf{X}^* = [x_1^* \;\; x_2^* \;\; . \; . \; . \; x_n^*]$ to be a local minimum are

$$
\begin{aligned}
\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial x_i} &= 0 \qquad i = 1, 2, \ldots, n \\
\lambda_j g_j &= 0 \qquad j = 1, 2, \ldots, m \\
g_j &\leq 0 \qquad j = 1, 2, \ldots, m \\
\lambda_j &\geq 0 \qquad j = 1, 2, \ldots, m
\end{aligned}
\qquad (1)
$$

In case of minimization problems, if the constraints are of the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be nonpositive in (1). On the other hand, if the problem is one of maximization with the constraints in the form $g_j(\mathbf{X}) \geq 0$, then λ_j have to be nonnegative.

It may be noted that sign convention has to be strictly followed for the Kuhn-Tucker conditions to be applicable.

Example 1

Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$ subject to the constraints

$$g_1 = x_1 - x_2 - 2x_3 \leq 12$$
$$g_2 = x_1 + 2x_2 - 3x_3 \leq 8$$

using Kuhn-Tucker conditions.

Solution:

The Kuhn-Tucker conditions are given by

a) $\dfrac{\partial f}{\partial x_i} + \lambda_1 \dfrac{\partial g_1}{\partial x_i} + \lambda_2 \dfrac{\partial g_2}{\partial x_i} = 0$

i.e.

$$2x_1 + \lambda_1 + \lambda_2 = 0 \quad (2)$$
$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \quad (3)$$
$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \quad (4)$$

b) $\lambda_j g_j = 0$

i.e.

$$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \quad (5)$$
$$\lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \quad (6)$$

c) $g_j \leq 0$

i.e.,

$$x_1 - x_2 - 2x_3 - 12 \le 0 \tag{7}$$
$$x_1 + 2x_2 - 3x_3 - 8 \le 0 \tag{8}$$

d) $\lambda_j \ge 0$

i.e.,

$$\lambda_1 \ge 0 \tag{9}$$
$$\lambda_2 \ge 0 \tag{10}$$

From (5) either $\lambda_1 = 0$ or, $x_1 - x_2 - 2x_3 - 12 = 0$

<u>*Case 1*:</u> $\lambda_1 = 0$

From *(2), (3) and (4)* we have $x_1 = x_2 = -\lambda_2 / 2$ and $x_3 = \lambda_2 / 2$.

Using these in *(6)* we get $\lambda_2^2 + 8\lambda_2 = 0, \therefore \lambda_2 = 0 \; or \; -8$

From *(10)*, $\lambda_2 \ge 0$, therefore, $\lambda_2 = 0$, $\mathbf{X}^* = [\ 0, 0, 0\ ]$, this solution set satisfies all of *(6)* to *(9)*

<u>*Case 2*:</u> $x_1 - x_2 - 2x_3 - 12 = 0$

Using *(2), (3) and (4)*, we have $\frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \frac{2\lambda_1 + 3\lambda_2}{3} - 12 = 0$ or,

$17\lambda_1 + 12\lambda_2 = -144$. But conditions *(9)* and *(10)* give us $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ simultaneously, which cannot be possible with $17\lambda_1 + 12\lambda_2 = -144$.

Hence the solution set for this optimization problem is $\mathbf{X}^* = [\ 0\ 0\ 0\ ]$

<u>Example 2</u>

Minimize $f = x_1^2 + x_2^2 + 60x_1$ subject to the constraints

$$g_1 = x_1 - 80 \geq 0$$
$$g_2 = x_1 + x_2 - 120 \geq 0$$

using Kuhn-Tucker conditions.

<u>Solution</u>

The Kuhn-Tucker conditions are given by

a) $\dfrac{\partial f}{\partial x_i} + \lambda_1 \dfrac{\partial g_1}{\partial x_i} + \lambda_2 \dfrac{\partial g_2}{\partial x_i} + \lambda_3 \dfrac{\partial g_3}{\partial x_i} = 0$

i.e.

$$2x_1 + 60 + \lambda_1 + \lambda_2 = 0 \qquad (11)$$
$$2x_2 + \lambda_2 = 0 \qquad (12)$$

b) $\lambda_j g_j = 0$

i.e.

$$\lambda_1 (x_1 - 80) = 0 \qquad (13)$$
$$\lambda_2 (x_1 + x_2 - 120) = 0 \qquad (14)$$

c) $g_j \leq 0$

i.e.,

$$x_1 - 80 \geq 0 \qquad (15)$$
$$x_1 + x_2 + 120 \geq 0 \qquad (16)$$

d) $\lambda_j \leq 0$

i.e.,

$$\lambda_1 \le 0 \tag{17}$$
$$\lambda_2 \le 0 \tag{18}$$

From (13) either $\lambda_1 = 0$ or, $(x_1 - 80) = 0$

<u>*Case 1*:</u> $\lambda_1 = 0$

From *(11) and (12)* we have $x_1 = -\lambda_2/2 - 30$ and $x_2 = -\lambda_2/2$

Using these in *(14)* we get $\lambda_2\left(\lambda_2 - 150\right) = 0$; $\therefore \lambda_2 = 0 \; or \; -150$

Considering $\lambda_2 = 0$, $\mathbf{X}^*$ = [30, 0].

But this solution set violates *(15)* and *(16)*

For $\lambda_2 = -150$, $\mathbf{X}^*$ = [45, 75].

But this solution set violates *(15)* .

<u>*Case 2*:</u> $(x_1 - 80) = 0$

Using $x_1 = 80$ in *(11)* and *(12)*, we have

$$\begin{aligned} \lambda_2 &= -2x_2 \\ \lambda_1 &= 2x_2 - 220 \end{aligned} \tag{19}$$

Substitute (19) in (14), we have

$$-2x_2\left(x_2 - 40\right) = 0 .$$

For this to be true, either $x_2 = 0$ *or* $x_2 - 40 = 0$

For $x_2 = 0$, $\lambda_1 = -220$. This solution set violates *(15)* and *(16)*

For $x_2 - 40 = 0$, $\lambda_1 = -140$ *and* $\lambda_2 = -80$. This solution set is satisfying all equations from (15) to (19) and hence the desired. Therefore, the solution set for this optimization problem is $\mathbf{X}^*$ = [80 40].

References / Further Reading:

1. Rao S.S., *Engineering Optimization – Theory and Practice*, Third Edition, New Age International Limited, New Delhi, 2000.
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