Introduction and Basic Concepts

(ii) Optimization Problem and Model Formulation

Objectives

- To study the basic components of an optimization problem.
- Formulation of design problems as mathematical programming problems.

Introduction - Preliminaries

- Basic components of an optimization problem :
  - An **objective function** expresses the main aim of the model which is either to be minimized or maximized.
  - A set of **unknowns** or **variables** which control the value of the objective function.
  - A set of **constraints** that allow the unknowns to take on certain values but exclude others.

Introduction (contd.)

- The optimization problem is then to:
  - **find values of the *variables* that minimize or maximize the *objective function* while satisfying the *constraints*.**

Objective Function

- As already defined the objective function is the mathematical function one wants to maximize or minimize, subject to certain constraints. Many optimization problems have a single objective function (When they don't they can often be reformulated so that they do). The two interesting exceptions are:
  - ***No objective function****.* The user does not particularly want to optimize anything so there is no reason to define an objective function. Usually called *a feasibility problem.*
  - ***Multiple objective functions****.* In practice, problems with multiple objectives are reformulated as single-objective problems by either forming a weighted combination of the different objectives or by treating some of the objectives by constraints.

Statement of an optimization problem

$$\textit{To find } \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \textit{ which maximizes } \mathbf{f(X)}$$

Subject to the constraints

$$g_i(\mathbf{X}) <= 0 , \qquad i = 1, 2, \ldots, m$$

$$l_j(\mathbf{X}) = 0 , \qquad j = 1, 2, \ldots, p$$

Statement of an optimization problem

where

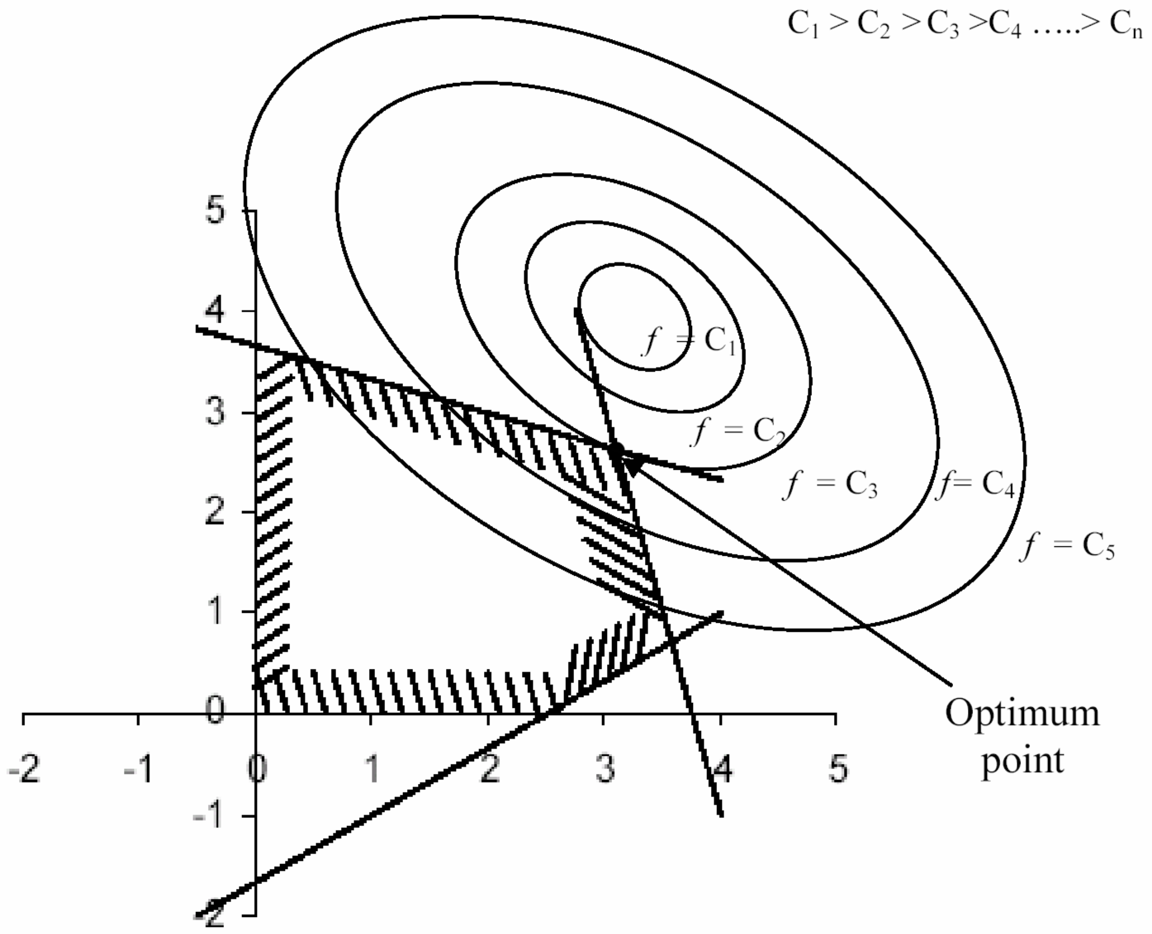
- $\mathbf{X}$ is an *n*-dimensional vector called the design vector
- $f(\mathbf{X})$ is called the *objective function*, and
- $g_i(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as inequality and equality constraints, respectively.

- This type of problem is called *a constrained optimization problem*.
- Optimization problems can be defined without any constraints as well. Such problems are called *unconstrained optimization problems.*

Objective Function Surface

- If the locus of all points satisfying ***f***(**X**) = a constant **c** is considered, it can form a family of surfaces in the design space called the ***objective function surfaces***.
- When drawn with the constraint surfaces as shown in the figure we can identify the optimum point (maxima).
- This is possible graphically only when the number of design variable is two.
- When we have three or more design variables because of complexity in the objective function surface we have to solve the problem as a mathematical problem and this visualization is not possible.

Objective function surfaces to find the optimum point (maxima)



Variables and Constraints

- Variables
  - These are essential. If there are no variables, we cannot define the objective function and the problem constraints.
- Constraints
  - Even though Constraints are not essential, it has been argued that almost all problems really do have constraints.
  - In many practical problems, one cannot choose the design variable arbitrarily. *Design constraints* are restrictions that must be satisfied to produce an acceptable design.

Constraints (contd.)

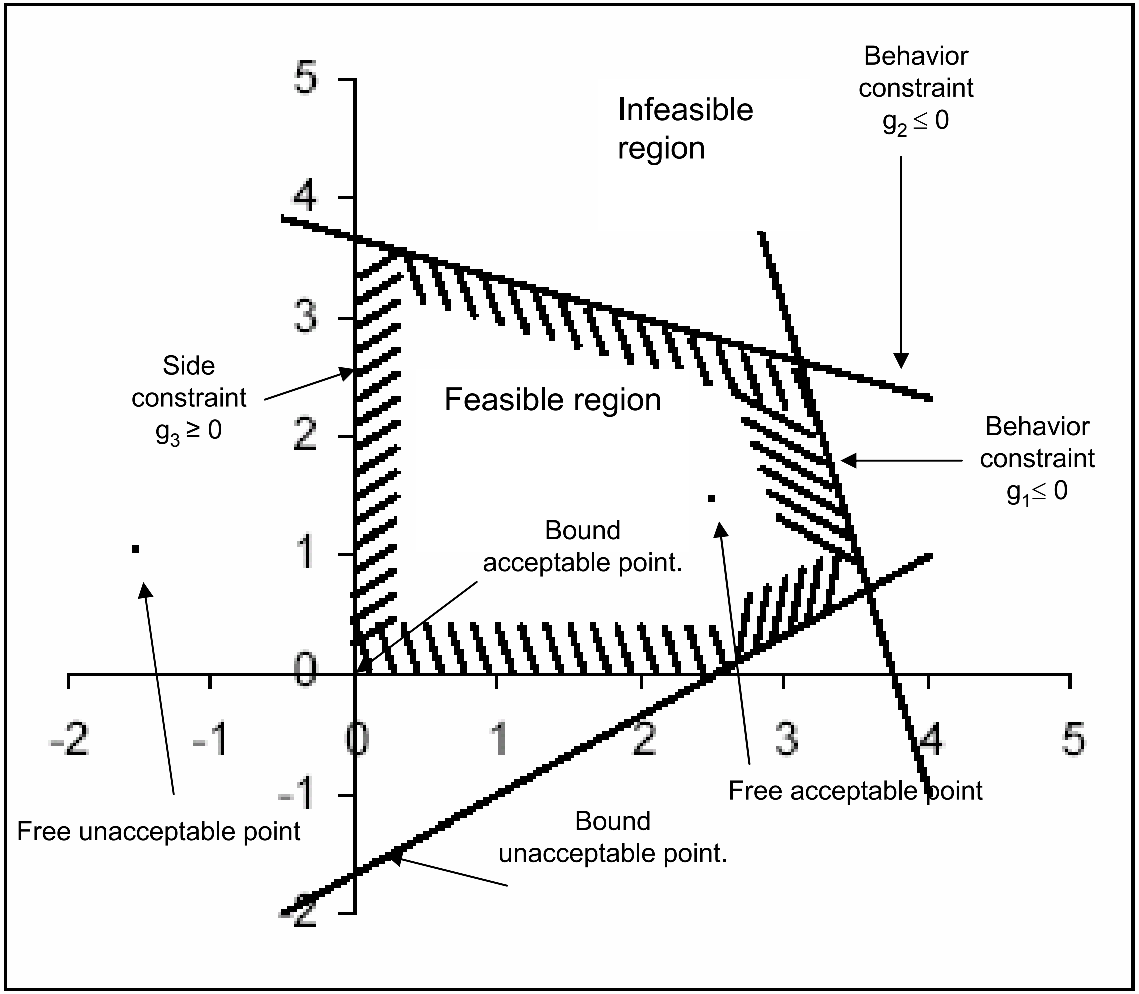
- Constraints can be broadly classified as :
  - Behavioral or Functional constraints : These represent limitations on the behavior and performance of the system.
  - Geometric or Side constraints : These represent physical limitations on design variables such as availability, fabricability, and transportability.

Constraint Surfaces

- Consider the optimization problem presented earlier with only inequality constraints $g_i(\mathbf{X})$. The set of values of **X** that satisfy the equation $g_i(\mathbf{X})$ forms a boundary surface in the design space called a *constraint surface.*

- The constraint surface divides the design space into two regions: one with $g_i(\mathbf{X}) < 0$ (feasible region) and the other in which $g_i(\mathbf{X}) > 0$ (infeasible region). The points lying on the hyper surface will satisfy $g_i(\mathbf{X}) = 0$.

The figure shows a hypothetical two-dimensional design space where the feasible region is denoted by hatched lines.



Formulation of design problems as mathematical programming problems

- **The following steps summarize the procedure used to formulate and solve mathematical programming problems.**

1. Analyze the process to identify the process variables and specific characteristics of interest i.e. make a list of all variables.
2. Determine the criterion for optimization and specify the objective function in terms of the above variables together with coefficients.

3. Develop via mathematical expressions a valid process model that relates the input-output variables of the process and associated coefficients.
   a) Include both equality and inequality constraints
   b) Use well known physical principles
   c) Identify the independent and dependent variables to get the number of degrees of freedom

4. If the problem formulation is too large in scope:
   a) break it up into manageable parts/ or
   b) simplify the objective function and the model
5. Apply a suitable optimization technique for mathematical statement of the problem.
6. Examine the sensitivity of the result to changes in the coefficients in the problem and the assumptions.

Thank You