Sequences and Series

Real life applications of sequences and series link:

https://www.quora.com/How-can-you-apply-series-and-sequences-in-real-life

Sequences and Series, arithmetic progression and geometric progression basic concepts, theory and practice questions link:

http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-apgp-2009-1.pdf

1. Sequences and Summation Notation

Question: Find a general term for the sequence:
$\frac{1}{2}, -\frac{5}{2}, \frac{25}{2}, -\frac{125}{2}, \ldots$

Also, find the 10th term of the sequence.

Solution: General term $a_n = \frac{1}{2}(-5)^{n-1}$, $n = 1, 2, 3, \ldots$

10th term = $a_{10} = \frac{1}{2}(-5)^{10-1} = \frac{1}{2}(-5)^9$

Question: Find a general term for the sequence: $2, 4, 10, 30,$ ...

Also, find the 9th term of the sequence.

Solution: General term:

$a_1 = 2, \; a_n = \frac{1}{2}(n+2)a_{n-1}, \; n = 2, 3, \ldots$

$a_5 = 105, a_6 = 420, a_7 = 1890, a_8 = 9450$

$a_9 = \frac{1}{2}(9+2)a_{9-1} = 5.5a_8 = 5.5 \times 9450 =$

Question: Find the first four terms of the recursive relation $a_1 = 1028, a_n = 1/2a_{n-1}$

Solution: Put n = 2, 3, 4 in $a_n = 1/2a_{n-1}$

$$a_2 = \frac{1(1028)}{2} \quad , \; a_3 = \frac{1(1028)}{4}, a_4 = \frac{1(1028)}{8}$$

Question Simplify the expressions, which involve factorials**:**

a) $\frac{49!-51!}{48!}$ b) $\frac{28!}{23!5!}$

Solution: a) $\frac{(49)48!-(51)(50)(49)(48!)}{48!} = \frac{48!(49-51\times 50\times 49)}{48!} = 49 - 51 \times 50 \times 49 = -124901$

b) $\frac{28\times 27\times 26\times 25\times 24\times 23!}{23!(5\times 4\times 3\times 2\times 1)} = \frac{28\times 27\times 26\times 25\times 24}{(5\times 4\times 3\times 2\times 1)} =$

$$= \frac{\overset{7}{\cancel{28}} \times \overset{9}{\cancel{27}} \times 26 \times \overset{5}{\cancel{25}} \times \overset{12}{\cancel{24}}}{(\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1)} = 98280$$

Question: Calculate the sum $\sum_{n=-1}^{4}(4 + 3n^2)$

Solution: (4+3)+(4+0)+(4+3)+(4+12)+(4+27)+(4+48) = 117

Question: Suppose that John borrows \$12000, interest free, from his father-in-law. He agrees to pay back the loan in monthly installments of \$550 . Find a summation that gives the balance owed by John after n months and use the summation to find his balance owed after 12 months.

Solution: $12000 - \sum_{n=1}^{n} 550$ = 12000 – 550n

After 12 months money left = $12000 - 12 \times 550$ =5400

2. Arithmetic Sequences:

Question: Determine the sequence 1, 4, 7, 10, … is arithmetic or not. If yes, find the common difference, first term and general term.

Solution: The differences are: 4-1 = 3, 7-4= 3, 10-7 = 3 and so on.

First term = a_1= 1 , and common difference = d = 3

General term = $a_n = a_1 + (n-1)d$ = 1+ (n-1)3 = 3n-2

a_n = 3n-2

Question: Determine the sequence -1, 5, 9, 15, … is arithmetic or not. If yes, find the common difference, first term and general term.

Solution: No because common differences are different.

5-(-1) = 6, 9-5 = 4, 15-9 = 6

Question: For the arithmetic sequence: -4, -10, -16, -22, …

Find the a) general term, b) 200th term and c) sum of first 50 terms

Solution: a) First term = $a_1 = -4$, Common difference = $d = -6$

General term = $a_n = a_1 + (n-1)d = -4 + (n-1)(-6) = -6n + 2$

$a_n = -6n + 2$

b) For the 200th term substitute n = 200 in the general term
$a_n = -6n + 2$

$a_{200} = -6 \times 200 + 2 = -1200 + 2 = -1198$

200th term =
$a_{200} = -1198$

c) Sum of first k terms = $S_k = \frac{k}{2}(a_1 + a_k)$

Sum of first 50 terms = $S_{50} = \frac{50}{2}(a_1 + a_{50})$ (1)

To find 50th term substitute n = 50 in the general term:
$a_n = -6n + 2$

$a_{50} = -6 \times 50 + 2 = -300 + 2 = -298$

Sum of first 50 terms = $S_{50} = \frac{50}{2}(a_1 + a_{50}) = \frac{50}{2}(-4 - 298) = 25(-302) = -7550$

Question: For the arithmetic sequence: 10, 17, 24, 31, …

Find a) the general term, b) 100th term and c) sum of first 500 terms

Solution: a) First term = $a_1 = 10$, Common difference = $d = 7$

General term = $a_n = a_1 + (n-1)d = 10 + (n-1)(7) = 7n + 3$

$a_n = 7n + 3$

b) For the 100th term substitute n = 100 in the general term $a_n = 7n + 3$

$a_{100} = 7 \times 100 + 3 = 700 + 3 = 703$

100th term = $a_{100} = 703$

c) Sum of first k terms = $S_k = \frac{k}{2}(a_1 + a_k)$

Sum of first 500 terms = $S_{500} = \frac{500}{2}(a_1 + a_{500})$ (1)

To find 500th term substitute n = 500 in the general term: $a_n = 7n + 3$

$a_{500} = 7 \times 500 + 3 = 3500 + 3 = 3503$

Sum of first 500 terms = $S_{500} = \frac{500}{2}(a_1 + a_{500}) = \frac{500}{2}(10 + 3503) = 878250$

Question: For the sequence where $a_n = 5/4 + (n-1)7/4$, find the sum of first 30 terms.

Solution: Sum of first k terms **:** $S_k = \frac{k}{2}(a_1 + a_k)$

First term = $a_1 = \frac{5}{4} + \frac{(1-1)7}{4} = \frac{5}{4}$

30th term: $a_{30} = \frac{5}{4} + \frac{(30-1)7}{4} = \frac{5}{4} + \frac{203}{4} = \frac{208}{4} = 52$

Sum of first 30 terms **:** $S_{30} = \frac{30}{2}(a_1 + a_{30}) = 15\left(\frac{5}{4} + 52\right) = 798.75$

3. Geometric Sequences:

Question: Check the sequence 3, 12, 48, 192, … is a geometric sequence or not. If yes, find the a) common ratio, b) first term, c) general term, d) 55th term and e) sum of first 10 terms of the geometric sequence.

Solution: a) Common ratios = $r = 12/3 = 48/12 = 192/48 = \ldots = 4$

Where $\left[common\ ratio = r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots\right]$

b) First term: $a_1 = 3$

c) General term = $a_n = a_1\, r^{n-1}$ (1)

Put $r = 4$ and $a_1 = 3$

$a_n = a_1 r^{n-1} = 3(4)^{n-1}$

Therefore the general term is:

$a_n = 3(4)^{n-1}$ (2)

d) $55th$ term can be obtained by putting $n = 55$ in (2):

$a_{55} = 3(4)^{55-1} = 3(4)^{54} = 9.7 \times 10\text{^}32$

e) Sum of first n terms of a geometric sequence:

$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(r^n-1)}{r-1}$, $r \neq 1$

$where\ r = common\ ratio\ and\ a_1 = first\ term$

Sum of first 10 terms of a geometric sequence = $S_{10} = \frac{3(1-4^{10})}{1-4} = 1048575$

Question: Check the sequence $9, -3, 1, -1/3, \ldots$ is a geometric sequence or not. If yes, find the **a)** common ratio, **b)** first term, **c)** general term, **d)** $10th$ term and **e)** sum of first 30 terms of the geometric sequence.

Solution: a) Common ratios $= r = -\frac{3}{9} = \frac{1}{-3} = -\frac{\frac{1}{3}}{1} = \ldots = -1/3$

Where $[common\ ratio = r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots]$

b) First term: $a_1 = 9$

c) General term = $a_n = a_1\, r^{n-1}$ (1)

Put $r = -1/3$ and $a_1 = 9$

$a_n = a_1\, r^{n-1} = 9(-1/3)^{n-1}$

Therefore the general term is:

$a_n = 9(-1/3)^{n-1}$ (2)

d) $10th$ term can be obtained by putting $n = 10$ in (2):

$a_{10} = 9(-1/3)^{10-1} = 9(-1/3)^9 =$

e) Sum of first n terms of a geometric sequence = $S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(r^n-1)}{r-1}$, $r \neq 1$

$where\ r = common\ ratio\ and\ a_1 = first\ term$

Sum of first 30 terms of a geometric sequence = $S_{30} = \frac{9\left(1-(-1/3)^{30}\right)}{1-(-1/3)} = -6.751687922$

Question: Find the sum of the given finite geometric sequence: $\sum_{n=1}^{n=12} 3(2/3)^n$.

Solution: Expanding the sum:

$$\sum_{n=1}^{n=12} 3(2/3)^n = 3\left(\frac{2}{3}\right) + 3(2/3)^2 + 3(2/3)^3 + \cdots + 3(2/3)^{12}$$

First term = $a_1 = 3(2/3) = 2$

Common ratio = r = $\frac{3(2/3)^2}{3\left(\frac{2}{3}\right)} = \frac{3(2/3)^3}{3(2/3)^2} = \cdots = 2/3$

Sum of first n terms of a geometric sequence = $S_n = \frac{a_1(1-r^n)}{1-r}$

$$\sum_{n=1}^{n=12} 3(2/3)^n = 3\left(\frac{2}{3}\right) + 3(2/3)^2 + 3(2/3)^3 + \cdots + 3\left(\frac{2}{3}\right)^{12}$$

$$= \frac{2\left(1-\left(\frac{2}{3}\right)^{12}\right)}{1-\frac{2}{3}} = 5.953755$$

Question: Check the type of sequence 1, 3, 9, 36, …

Solution: Not a geometric sequence because all ratios are not equal. There is no one common ratio.

Common ratio r = $= \mathbf{\frac{3}{1} = \frac{9}{3} \neq \frac{36}{9}}$

Applications of arithmetic and geometric sequences and series links:

https://www.youtube.com/watch?v=_ultmutIDu4

https://www.youtube.com/watch?v=8qBZ-oPRKF0

Question: Anthony wants to open an account by depositing \$4000 on the first day of the year. Suppose he intends to make subsequent deposits on the first day of each subsequent year for the next 20 years, increasing his deposit by 10% each year. Ignoring interest, how much will Anthony's deposit be in years 5 and 10, and how much will he deposit in total over the course of the 20 years ?

Solution: Formula $A = P(1+i)^{n-1}$

A = Amount paid into an annuity in nth year

i= decimal form of increase of a set percentage

n= years

P = Initial payment

5th year's deposit = $4000\left(1+\frac{10}{100}\right)^{5-1}$

$= 4000(1.1)^4 = \$5856.4$

10th year's deposit = $4000\left(1+\frac{10}{100}\right)^{10-1}$ = \$9431.8

Total deposited over 20 years:
4000+4000(1.1)+4000$(1.1)^2$+…

Formula: Sum of first n =20 terms=

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$S_{20} = \frac{4000\left(1-(1.1)^{20}\right)}{1-1.1}$ =$ 229099.998

Find more practice questions for the arithmetic and geometric sequences by clicking on the links below:

http://www.purplemath.com/modules/series3.htm

http://downloads.cambridge.edu.au/education/extra/209/PageProofs/Further%20Maths%20TINCP/Core%209.pdf (Study from page number 259 to 291)

http://www.emathzone.com/tutorials/algebra/application-of-arithmetic-sequence-and-series.html

http://www.emathzone.com/tutorials/algebra/application-of-geometric-sequence-and-series.html

http://www.tutor-homework.com/Math_Help/college_algebra/m6l4notes1.pdf

4. The Principles of Mathematical Induction:

Question: Prove the following statement using mathematical induction.

$1 + 3 + 5 + \cdots\ 2n - 1\ =\ n^2$ (1)

Solution: Part I:

Show that it's true for n=1 If n = 1 then the LHS of (1) = 1 and RHS of (1) = 1^2= 1.

So (1) is true for n=1.

Part II: Assume that it's true for n=k; i.e., assume that for a fixed k,

$1\ +\ 3\ +\ 5 + \ldots + 2k - 1\ =\ \ k^2$ (2)

(the induction hypothesis)

PART III: Show that it's true for n=k+1; i.e., show that

1 + 3 + 5 + ... + 2k-1 + 2k+1 = $(k+1)^2$ (3)

The proof: L.H.S. of (3)= 1 + 3 + 5 + ... + 2k+1 = (1 + 3 + 5 + ... + 2k-1) + 2k+1 =

= $k^2 + 2k + 1 = (k+1)^2$ = R.H.S of (3)

$[using\ \ equation\ (2)\ 1 + 3 + 5 + \ldots + 2\text{k} - 1 = k^2]$

Result is true for $n = k + 1$

Equation (1) is true for all positive integers n.

Principle of Mathematical induction Links:

1) http://home.cc.umanitoba.ca/~thomas/Courses/textS1-21.pdf (Most of the questions from 6.4 section of the book are solved)

2) http://www.purplemath.com/modules/inductn.htm

Pascal Triangle and Binomial Theorem Link:

http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-pascal-2009-1.pdf

5. The Binomial Theorem:

The binomial theorem:

When n is a positive whole number

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \ldots + b^n$$

Or

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3(2)}a^{n-3}b^3 + \cdots + b^n$$

Or

$$(a+b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \cdots + {}_nC_n b^n$$

Or

$$(a+b)^n = \sum_{k=0}^{k=n} C_k^n a^{n-k} b^k$$

where $\binom{n}{k} = C_k^n = {}_nC_k = \frac{n!}{k!(n-k)!}$

The binomial expansion of $(a+b)^n$ has $n+1$ terms. In general, the rth term in the expansion is calculated by

$${}_nC_{r-1}a^{n-r+1}b^{r-1}$$

Question: Calculate the binomial coefficient ${}_7C_4$.

Solution: $\frac{7!}{4!(7-4)!} = \frac{7(6)(5)(4!)}{4!(3!)} = \frac{7(6)(5)}{3(2)(1)} = 35$

Question: Use the binomial theorem to expand the following expression:

$(3a - 5b)^5$

Solution:
$(3a-5b)^5 = {}_5C_0\,(3a)^5 + {}_5C_1\,(3a)^4(-5b)^1 + {}_5C_2\,(3a)^3(-5b)^2 + {}_5C_3\,(3a)^2(-5b)^3 + \ldots$

$\ldots + {}_5C_4\,(3a)^1(-5b)^4 + {}_5C_5\,(3a)^{5-5}\,(-5b)^5$

$= \frac{5!}{0!(5-0)!}(3a)^5 - \frac{5!}{1!(5-1)!}(\,(3a)^4(5b) + + \frac{5!}{2!(5-2)!}(3a)^3(5b)^2 - \frac{5!}{3!(5-3)!}(3a)^2(5b)^3 + \frac{5!}{4!(5-4)!}(3a)(5b)^4 - \frac{5!}{5!(5-5)!}(5b)^5$

$=243a^5 - 2025a^4b + 6750a^3b^2 - 11{,}250\ a^2b^3 + 9375ab^4 - 3125b^5$

Question: Find the coefficient of x^5 in $(x-3)^8$

Solution: The rth term of $(a+b)^n$ is given by

$${}_nC_{r-1}a^{n-r+1}b^{r-1}$$

Since the power of x should be 5 therefore

n-r+1 = 5

8-r+1 = 5

-r = 5-9 = -4

r= 4

Put r= 4, n= 8, a = x and b = -3 in the rth term:

$= \quad {}_nC_{r-1}a^{n-r+1}b^{r-1} \quad =$

${}_8C_{4-1}\, x^{8-4+1}(-3)^{4-1} = C_3^8 x^{8-4+1}(-3)^{4-1} = \frac{8!}{3!(5!)} x^5(-3)^3 =$

$= 8.7.6.5!/(3\times 2)(5!)x^5(-3)^3 = -1512x^5$

NOTE: ${}_8C_3 = C_3^8$

Question: Find the 4th term of the expansion: $(5x+3)^9$

Solution: The rth term of $(a+b)^n$ is given by:

$$ {}_nC_{r-1}a^{n-r+1}b^{r-1} \qquad \textbf{(1)} $$

4th term is: Substitute r= 4 in equation (1)

$= {}_9C_3(5x)^{9-4+1}(3^{4-1})$

$= 9!/3!\,(9-3)!\,(5x)^6(3^3) = 9!/3!\,(6)!(5^6)(3^3)x^6 =$ $[(9.8.7.6!)/((3.2.1)6!\,)][(5^6)(3^3)x^6]$

$= 35437500x^6$

Question: Find first three terms in the expression: $\frac{1}{(x+2y)^3}$.

Solution: Using formula:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3(2)}a^{n-3}b^3 + \cdots + b^n \qquad (1)$$

Rewriting the given expression as: $\frac{1}{(x+2y)^3} = (x+2y)^{-3}$

For finding first three terms put n= -3 in the formula (1)

$$(x+2y)^{-3} = x^{-3} - 3x^{-3-1}(2y) + \frac{(-3)(-3-1)}{2}x^{-3-2}(2y)^2) + \cdots$$

$=x^{-3} - 6x^{-4}y + 24x^{-5}y^2$

Binomial theorem Links: *Students can practice more questions from the links listed below:*

http://www.purplemath.com/modules/binomial.htm

Homework/Practice Problems:

Odd numbered questions from the Chapter 6 (6.1-6.5) exercises from the book "*Mathematics for information technology by Alfred Basta, Stephan Delong and Nadine Basta*"

References:

1) Mathematical induction http://home.cc.umanitoba.ca/~thomas/Courses/textS1-21.pdf

2) Principle of Mathematical induction

http://www.purplemath.com/modules/inductn.htm

3) Binomial theorem and pascal triangle problems

http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-pascal-2009-1.pdf

4) Mathematics for Information Technology by Alfred Basta, Stephan Delog and Nadine Basta Cengage Learning

5) Binomial theorem problems http://www.purplemath.com/modules/binomial.htm

6) All the links provided on this file