Lecture 6: Binary Search Trees (BST) and Red-Black Trees (RBTs)

Instructor: Saravanan Thirumuruganathan

Outline

1. Data Structures for representing Dynamic Sets
    - Binary Search Trees (BSTs)
    - Balanced Search Trees
    - Balanced Binary Trees - Red Black Trees (RBTs)

In-Class Quizzes

- **URL:** `http://m.socrative.com/`
- **Room Name: 4f2bb99e**

Key Things to Know for Data Structures

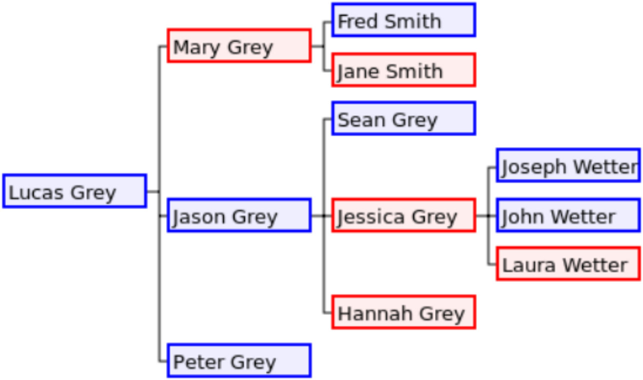
- Motivation
- Distinctive Property
- Major operations
- Key Helper Routines
- Representation
- Algorithms for major operations
- Applications

Non-Linear Data Structures:

- Very common and useful category of data structures
- Most popular one is **hierarchical**

Trees - Applications[1]

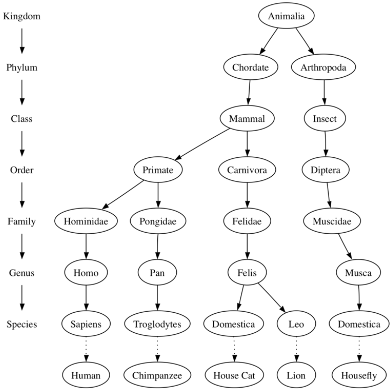
Family Tree:



[1] http://interactivepython.org/runestone/static/pythonds/Trees/trees.html

Trees - Applications[2]

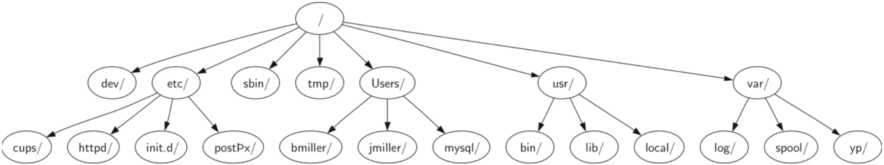
Taxonomy Tree:



[2] http://interactivepython.org/runestone/static/pythonds/Trees/trees.html

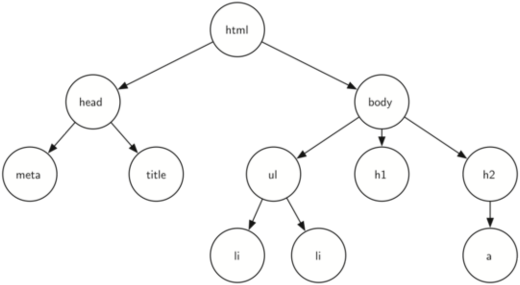
Trees - Applications[3]

Directory Tree:



[3] http://interactivepython.org/runestone/static/pythonds/Trees/trees.html

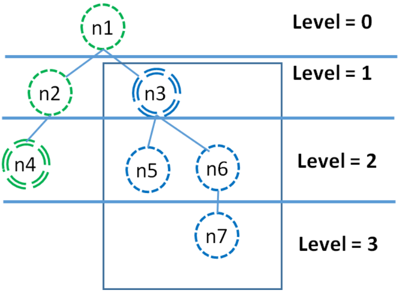
HTML DOM (Parse) Tree:



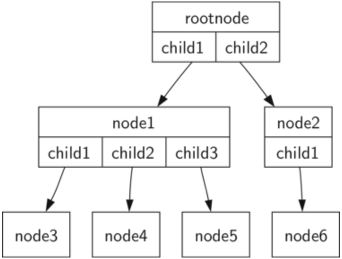
[4] http://interactivepython.org/runestone/static/pythonds/Trees/trees.html

Tree - Terminology

- Node
- Edge
- Root
- Children
- Parent
- Sibling
- Subtree
- Leaf/External node
- Internal node
- Level (node)
- Height (tree)
- Arity



Tree - Abstract Representation[5]



[5] http://interactivepython.org/runestone/static/pythonds/Trees/trees.html

Motivation

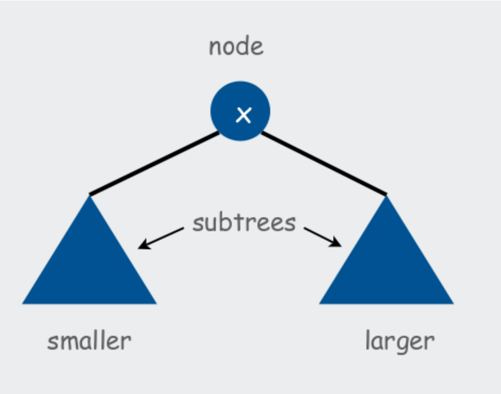
- Store dynamic set efficiently
- Use good ideas from ordered list (OL) and ordered doubly linked list (ODLL)
- Use hierarchical storage to avoid pitfalls of OL and ODLL
- First attempt at hierarchical data structure that **tries** to implement all 7 operations efficiently

Binary Trees

- Each node has at most 2 children
- Commonly referred to as *left* and *right* child
- The descendants of *left* child constitute *left* subtree
- The descendants of *right* child constitute *right* subtree

BST Property

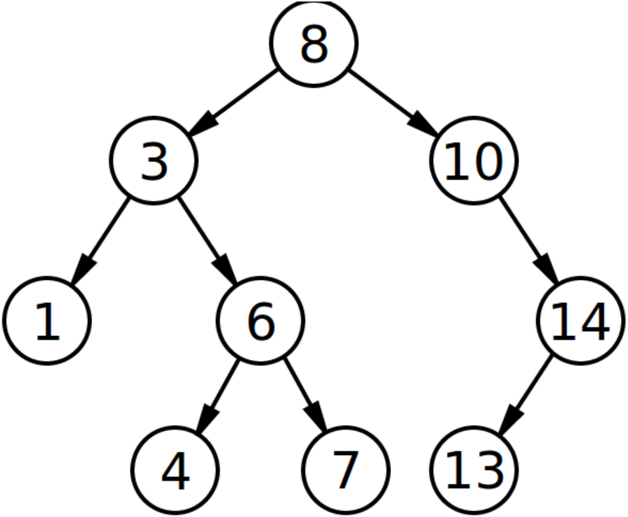
- For every node x, the keys of left subtree $\leq key(x)$
- For every node x, the keys of right subtree $\geq key(x)$



[6]

[6] http://www.cs.princeton.edu/~rs/AlgsDS07/08BinarySearchTrees.pdf

BST Examples[7]



[7] http://en.wikipedia.org/wiki/Binary_search_tree

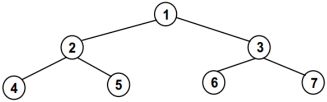
BST Height[8]

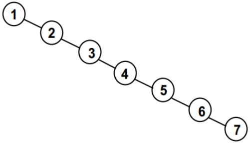
- There exists multiple possible BSTs to store same set of elements
- Minimum and Maximum Height:

[8] https://engineering.purdue.edu/~ee608/handouts/lec10.pdf

BST Height[8]

- There exists multiple possible BSTs to store same set of elements
- Minimum and Maximum Height: $\lg n$ and n
- Best and worst case analysis (or specify analysis wrt height)



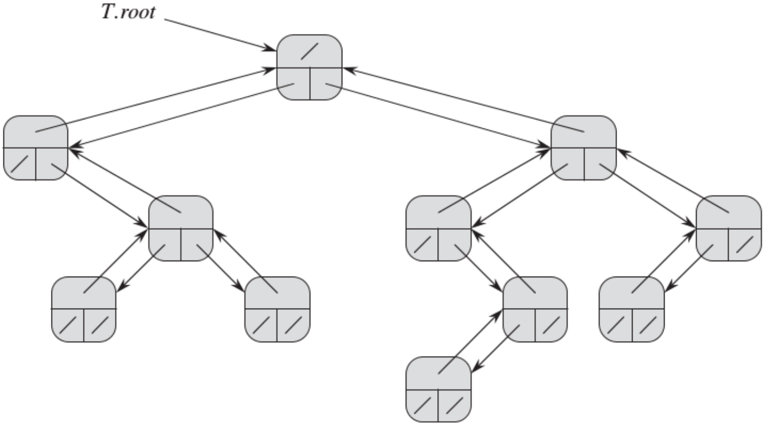


[8] https://engineering.purdue.edu/~ee608/handouts/lec10.pdf

Representation - I

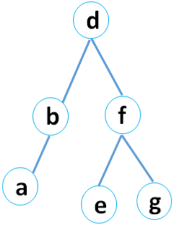
- **key:** Stores key information that is used to compare two nodes
- **value:** Stores satellite/auxillary information
- **parent:** Pointer to parent node. parent(root) $=$ NULL
- **left:** Pointer to left child if it exists. Else NULL
- **right:** Pointer to right child if it exists. Else NULL

Representation - I[9]



[9] CLRSFig10.9

Representation - II



Index	Key	Left	Right
1	d	2	3
2	b	4	NULL
3	f	5	6
4	a	NULL	NULL
5	e	NULL	NULL
6	g	NULL	NULL

Major Operations

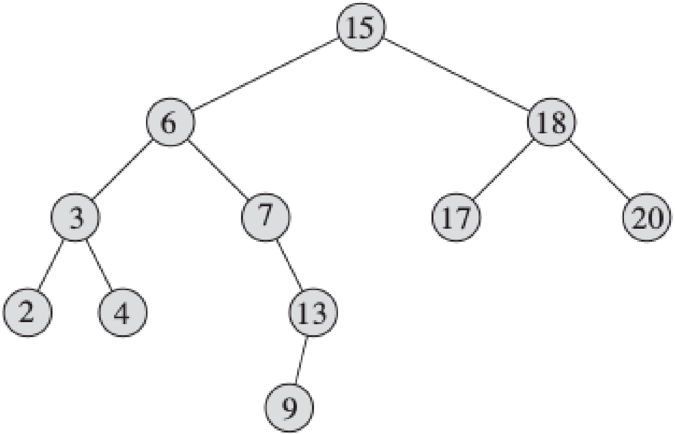
- Search
- Insert
- Minimum/Maximum
- Successor/Predecessor
- Deletion
- Traversals

Key Helper Routines

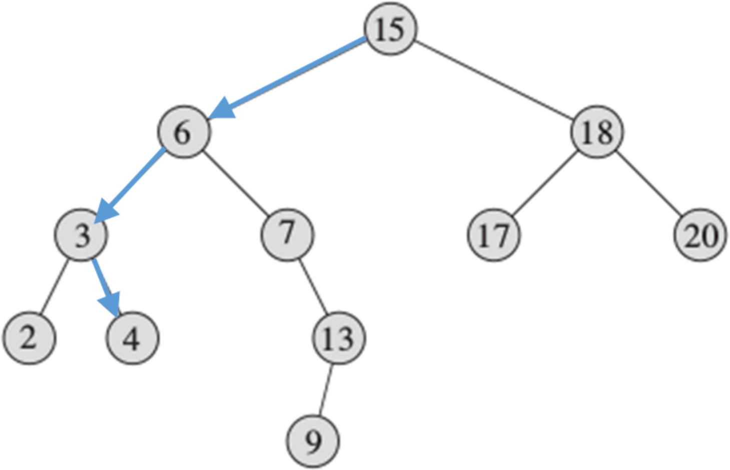
- Successor/Predecessor
- Traversals
- Case by Case Analysis:
  - Analysis by number of children : $0, 1, 2$
  - Analysis by type of children: *left*, *right*

BST: Search

Search: 4



BST: Search



BST: Search

```
Tree-Search(x, k):
    if x == NULL or k == x.key
        return x
    if k < x.key
        return Tree-Search(x.left, k)
    else
        return Tree-Search(x.right, k)
```

- **Analysis:**

BST: Search

```
Tree-Search(x, k):
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        return Tree-Search(x.left, k)
    else
        return Tree-Search(x.right, k)
```
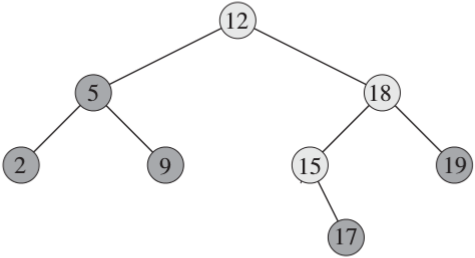

- **Analysis:** $O(h)$
- **Best Case:** $\lg n$ and **Worst Case:** $O(n)$

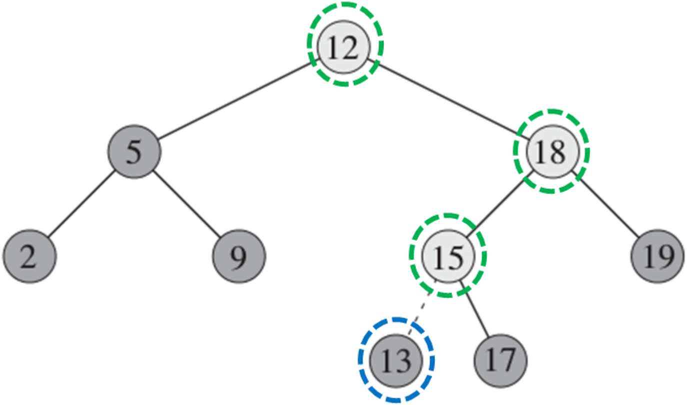
BST: Insert

Insert: 13



BST: Insert

Insert: 13



BST:

- **Analysis:**

BST:

- **Analysis:** $O(h)$
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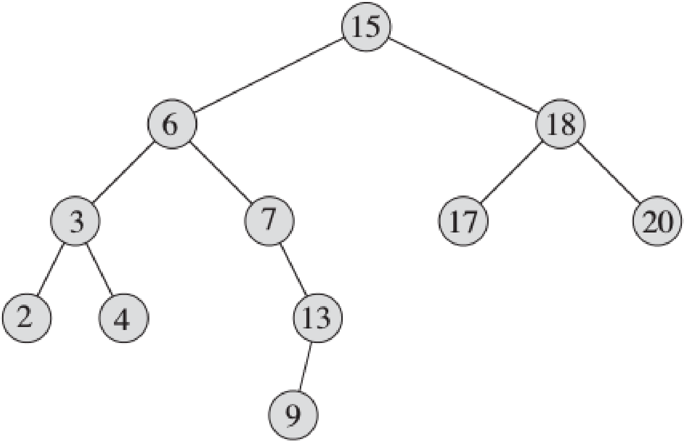
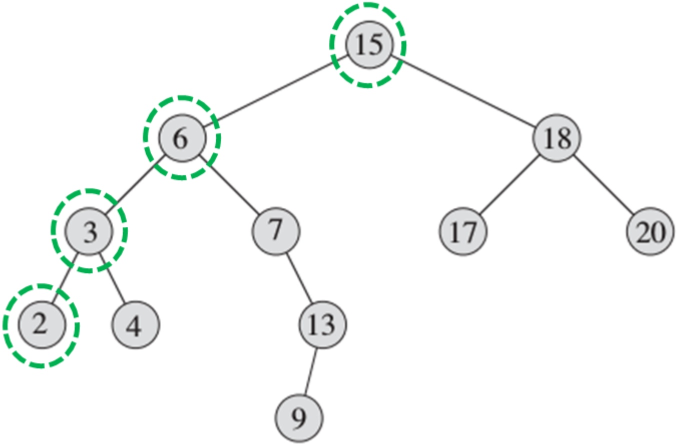
BST: Minimum



BST: Minimum

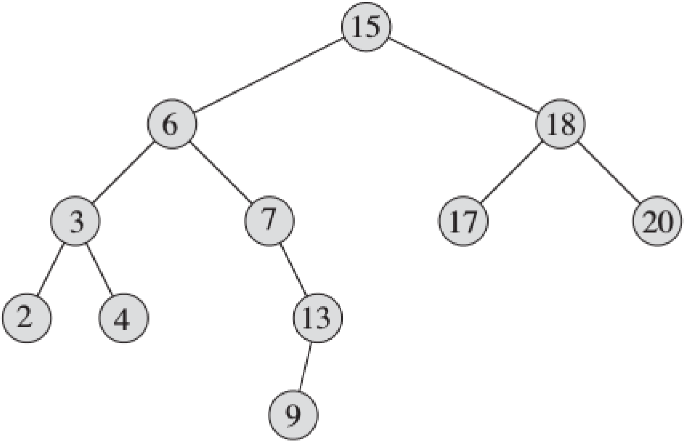


BST: Minimum

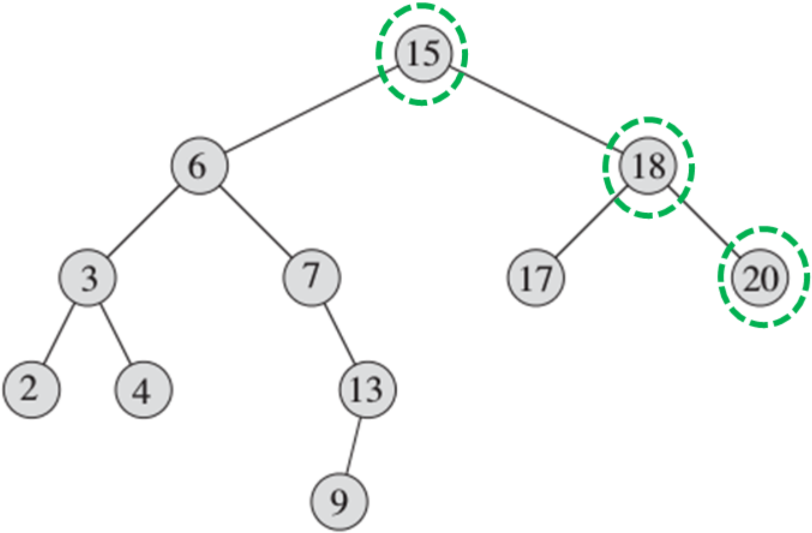
```
Tree-Minimum(x)
    while x.left is not NULL
        x = x.left
    return x
```

- **Analysis:** $O(h)$
- **Best Case:** $\lg n$ and **Worst Case:** $O(n)$

BST: Maximum



BST: Maximum



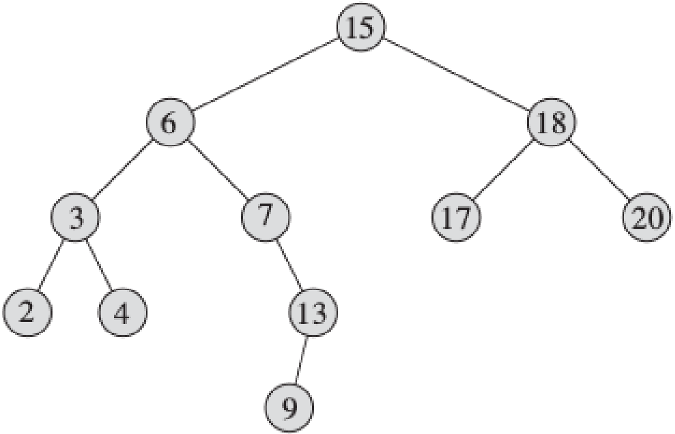
BST: Maximum

```
Tree-Maximum(x)
    while x.right is not NULL
        x = x.right
    return x
```

- **Analysis:** $O(h)$
- **Best Case:** $\lg n$ and **Worst Case:** $O(n)$

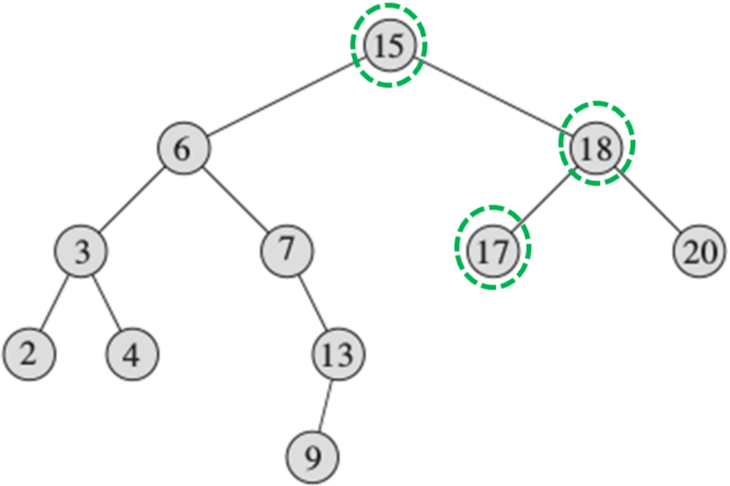
BST: Successor

Successor: 15



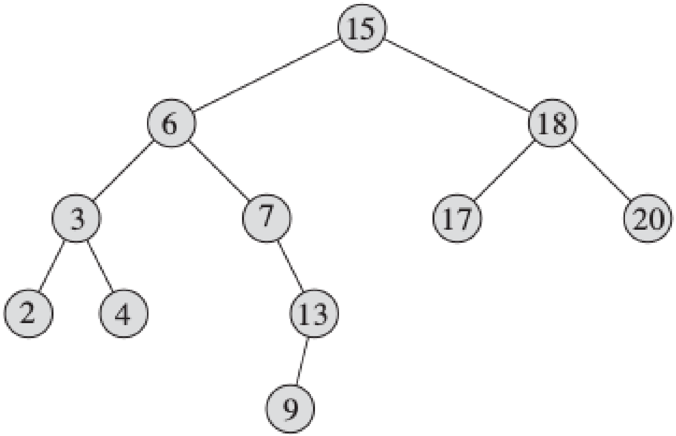
BST: Successor

Successor: 15



BST: Successor

Successor: 13



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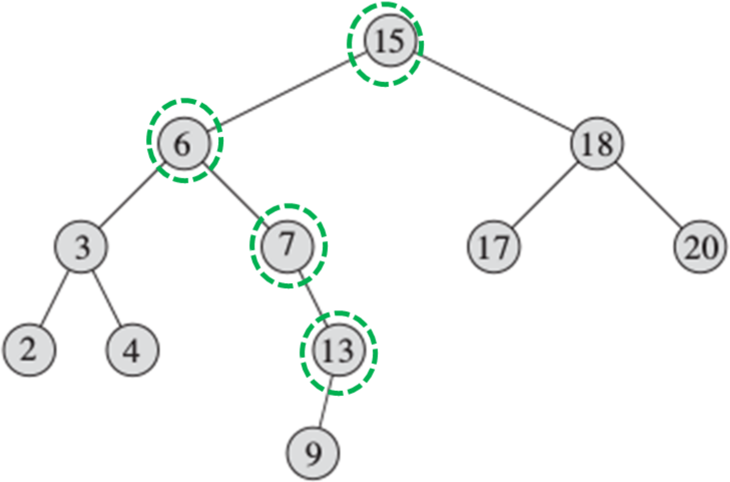
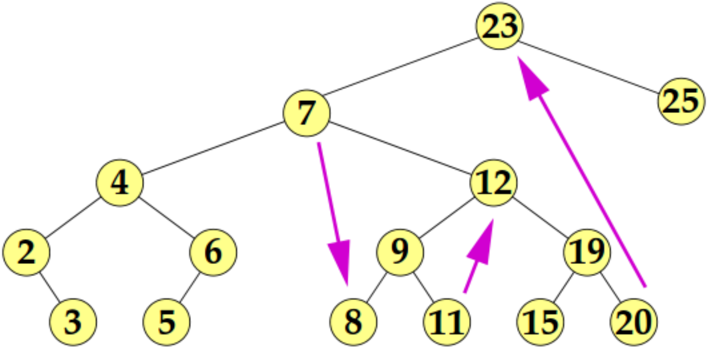


BST: Successor



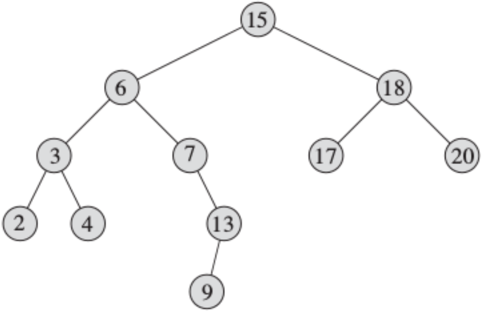
BST:

```
Tree-Successor(x):
    if x.right is not NULL
        return Tree-Minimum(x.right)
    y = x.parent
    while y is not NULL and x == y.right
        x = y
        y = y.parent
    return y
```

- BST Property allowed us to find successor **without** comparing keys
- **Analysis:** $O(h)$
- **Best Case:** $\lg n$ and **Worst Case:** $O(n)$

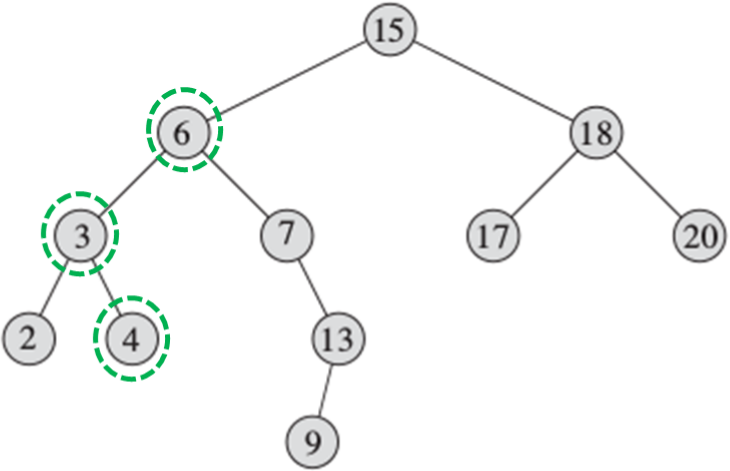
BST: Predecessor

Predecessor: 6



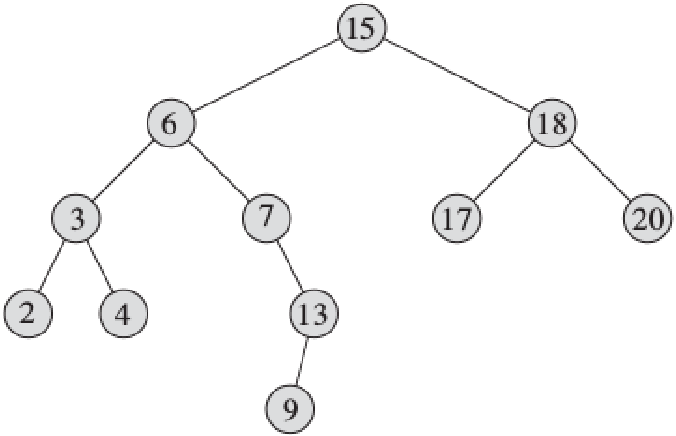
BST: Predecessor

Predecessor: 6

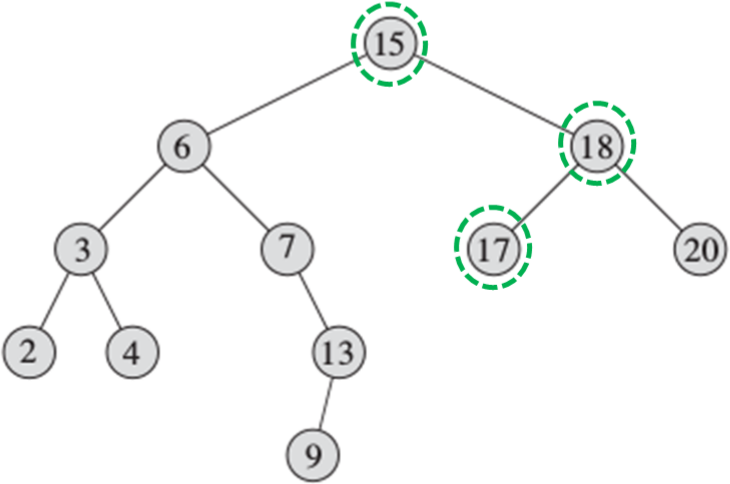


BST: Predecessor

Predecessor: 17



Predecessor: 17



BST:

```
Tree-Predecessor(x):
    if x.left is not NULL
        return Tree-Maximum(x.left)
    y = x.parent
    while y is not NULL and x == y.left
        x = y
        y = y.parent
    return y
```

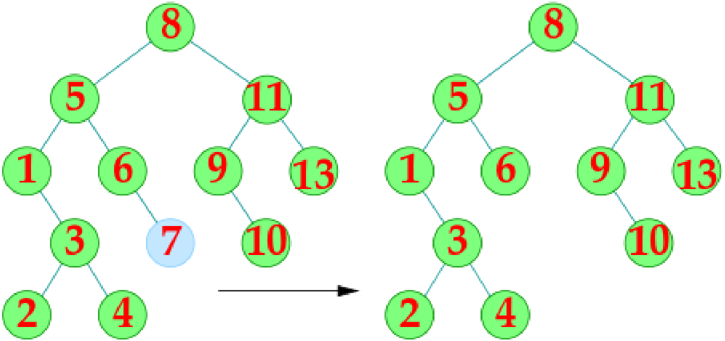
- BST Property allowed us to find predecessor **without** comparing keys
- **Analysis:** $O(h)$
- **Best Case:** $\lg n$ and **Worst Case:** $O(n)$

BST: Deletion

Trickiest Operation! Suppose we want to delete node z

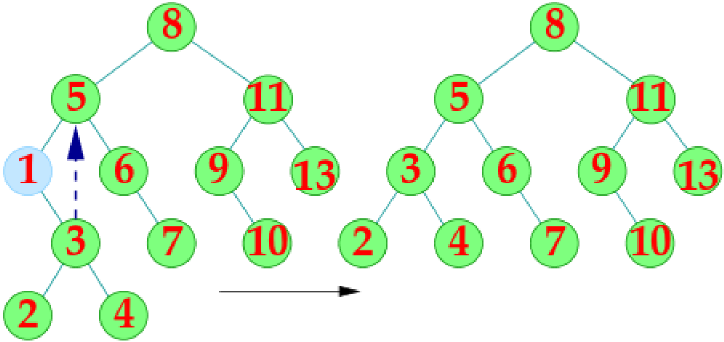
1. z has no children: Replace z with NULL
2. z has one children c: Promote c to z's place
3. z has two children:
   (a) Let z's successor be y
   (b) y is either a leaf or has **only** right child
   (c) Promote y to z's place
   (d) Fix y's loss via Cases 1 or 2

BST: Deletion Case I[10]



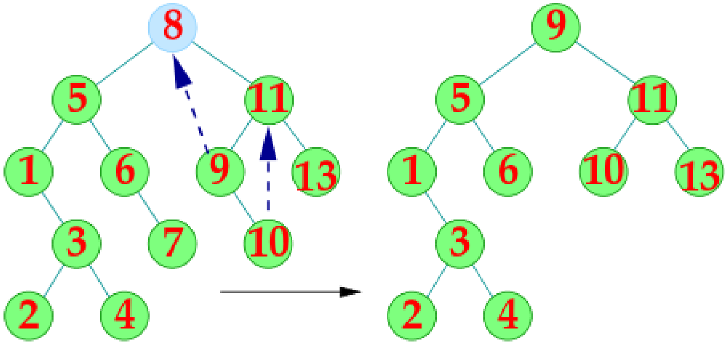
[10] https://www.cs.rochester.edu/u/gildea/csc282/slides/C12-bst.pdf

BST: Deletion Case II[11]



[11] https://www.cs.rochester.edu/u/gildea/csc282/slides/C12-bst.pdf

BST: Deletion Case III[12]



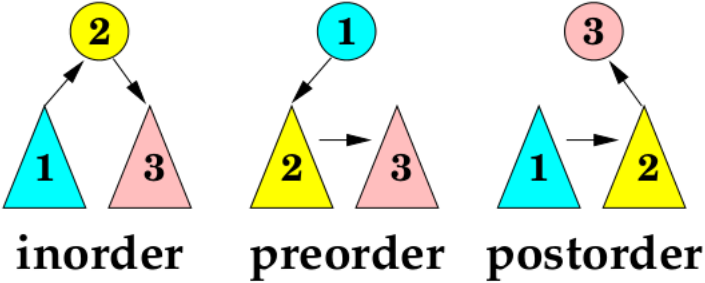
[12] https://www.cs.rochester.edu/u/gildea/csc282/slides/C12-bst.pdf

Digression

- Perfectly fine if you cannot do deletion by memory
- Things will become hairier in RBT
- As long as you remember the key ideas and operations, you will be fine

BST: Traversal

- **Traversal:** Visit all nodes in a tree
- Many possible traversal strategies
- Three are most popular: Pre-Order, In-Order, Post-Order



BST: Traversals

```
In-Order-Walk(x):
    if x == NULL
        return
    In-Order-Walk(x.left)
    Print x.key
    In-Order-Walk(x.right)
```

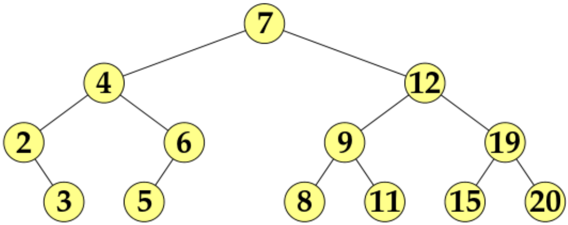
- **Analysis:**

BST: Traversals

```
In-Order-Walk(x):
    if x == NULL
        return
    In-Order-Walk(x.left)
    Print x.key
    In-Order-Walk(x.right)
```

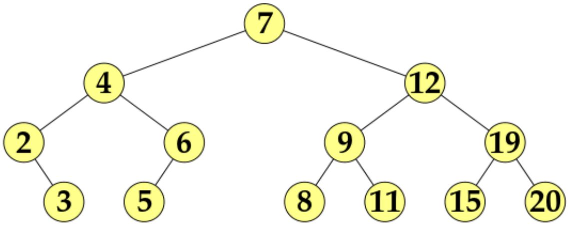
- **Analysis:** $O(n)$
- Holds true for all three traversals

BST: Traversal



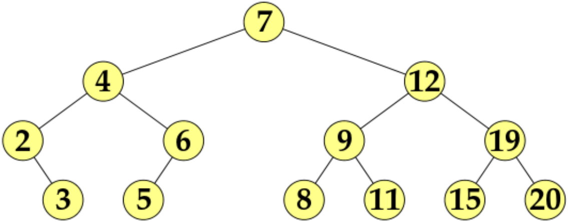
- **In-Order:**

BST: Traversal



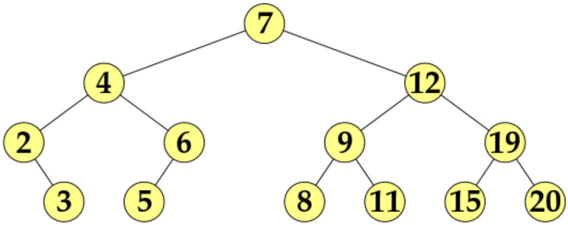
- **In-Order:** 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15, 19, 20.
- **Pre-Order:**

BST: Traversal



- **In-Order:** 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15, 19, 20.
- **Pre-Order:** 7, 4, 2, 3, 6, 5, 12, 9, 8, 11, 19, 15, 20.
- **Pre-Order:**

BST: Traversal



- **In-Order:** 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 15, 19, 20.
- **Pre-Order:** 7, 4, 2, 3, 6, 5, 12, 9, 8, 11, 19, 15, 20.
- **Pre-Order:** 3, 2, 5, 6, 4, 8, 11, 9, 15, 20, 19, 12, 7.

BST: Traversals

- Notice anything special about In-Order traversal?

BST: Traversals

- Notice anything special about In-Order traversal?
  - Returns items in sorted order!
- Successor/Predecessor can be expressed in terms on In-Order traversal

Application: Tree-Sort

Tree-Sort:

- Construct a BST out of elements
- Do In-Order traversal

Analysis:

- **Time Complexity:** Time Complexity for Constructing Tree + Time Complexity for In-Order traversal
- **Best Case:**

Application: Tree-Sort

Tree-Sort:

- Construct a BST out of elements
- Do In-Order traversal

Analysis:

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- **Best Case:** $n \times \lg n + n = O(n \lg n)$
- **Worst Case:**

Application: Tree-Sort

Tree-Sort:

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Analysis:

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- **Best Case:** $n \times \lg n + n = O(n \lg n)$
- **Worst Case:** $n \times n + n = O(n^2)$

Balanced Binary Trees

- Time complexity of BST operations depends on height
- Can vary between $O(\lg n)$ to $O(n)$
- BST operations do not take any special care to keep tree balanced
- If we can do the balancing efficiently, then all operations become faster
- Self-balancing - Do not run balancing algorithms periodically

Notion of Balance

- Maintaining perfectly balanced trees is very hard and expensive
- So we resort to BSTs that are **approximately** balanced
- Need to define notion of balance
- Ideas?

Notion of Balance

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- So we resort to BSTs that are **approximately** balanced
- Need to define notion of balance
- Ideas?
  - Informally, ensure the longest path in tree is not "too" long
  - Many ways of formally specifying it
  - Eg: $|\text{height(left subtree)} - \text{height(right subtree)}| \leq 1$
- When can balance be broken?

Notion of Balance

- Maintaining perfectly balanced trees is very hard and expensive
- So we resort to BSTs that are **approximately** balanced
- Need to define notion of balance
- Ideas?
  - Informally, ensure the longest path in tree is not "too" long
  - Many ways of formally specifying it
  - Eg: $|\text{height(left subtree)} - \text{height(right subtree)}| \leq 1$
- When can balance be broken? Insertion, Deletion

Balanced Search Trees

- Red-Black trees
- AVL trees
- $2-3$ and $2-3-4$ trees
- B-trees and other variants
- Treaps
- Skip trees
- Splay trees
- and many many more

Red-Black Trees

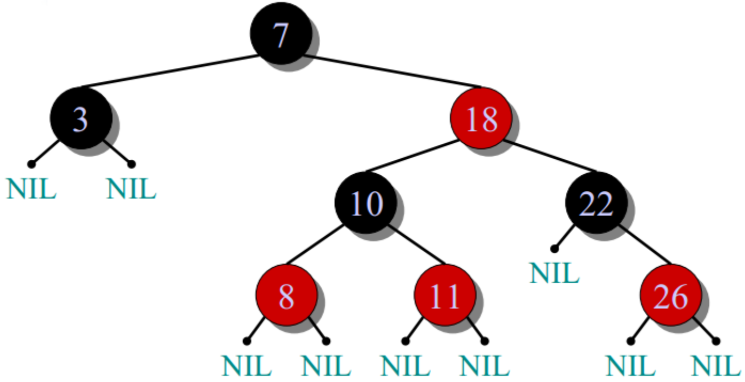
RBT: Motivation

- Most important self-balancing BST
- Invented by Guibas-Sedgewick
- Simplifies/Unifies various balanced tree algorithms
- Became popular due to its simplicity in implementation
- Stores additional information about color of node (1 bit)
- All operations are logarithmic!

RBT Property

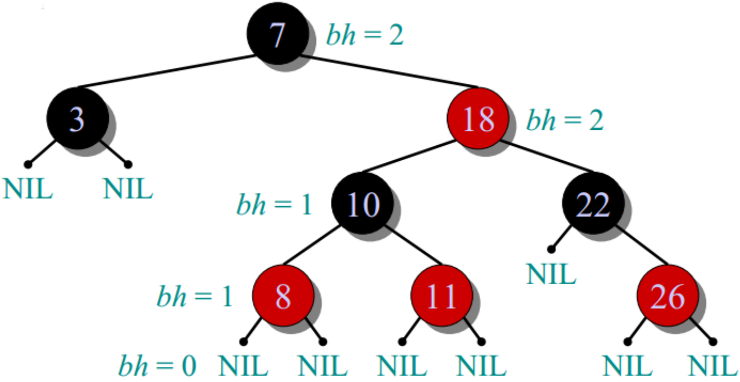
- Every node is either red or black
- Root and leaves are black
- If node is red, then its parent is black
- All *simple* paths from any node v to a descendent leaf have same number of black nodes. Aka *black-height(v)*

RBT Example[13]



[13] MIT OCW 6-046j

RBT Example[14]



[14] MIT OCW 6-046j

RBT Property: Implications

- If a **red** node has any children, then it must have **two** children and both must be **black**
- If a **black** node has **only one** child, it has to be **red**
- No root to leaf path has two consecutive red nodes
- No root to leaf path is more than twice as long as any other
- The rules **bound** the imbalance in the tree

Major Operations

- Search
- Insert
- Minimum/Maximum
- Successor/Predecessor
- Deletion

Key Helper Routines

- Rotations - Right and Left
- Case by Case Analysis:
  - Analysis by type of children, siblings and uncle (sibling of parent)
  - Analysis by color of children

RBT Theorem

Theorem (RBT Theorem)

A red-black tree with n keys has height

$$h \leq 2\lg(n+1)$$

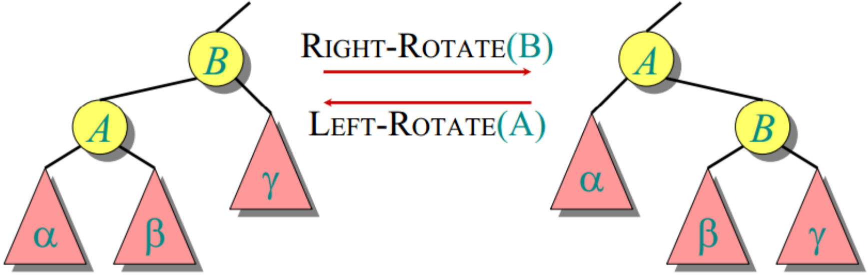
Corollary

Operations Search, Min, Max, Successor, Predecessor all run in $O(\lg n)$ time on a Red-Black tree with n nodes.

RBT - Modifying Operations

- Insert and Delete need to be more complex so as to balance the tree
- They cause "changes" to tree
  - Change of color (recoloring)
  - Change of tree structure via "rotations"

Rotations[15]



- Rotation can be done in $O(1)$ time.
- Rotations maintain in-order ordering of keys: $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \leq A \leq b \leq B \leq c$

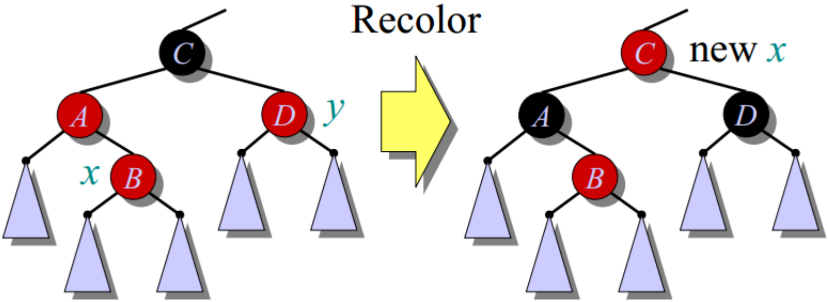
[15] MIT OCW 6-046j

RBT Insertion

- Insert x in tree (based on BST property)
- Color x red
- Only red-black property 3 can be violated
- Fix violations by rotations and recoloring

RBT : Insertion[16]

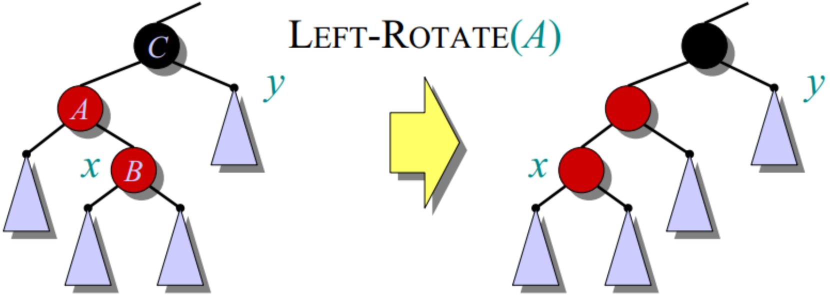
- Case I: The parent and "uncle" of x are both red
  - Color parent and uncle of x as black
  - Color grandparent of x as red
  - Recurse of grandparent of x



[16] MIT OCW 6.046j

RBT : Insertion[17]

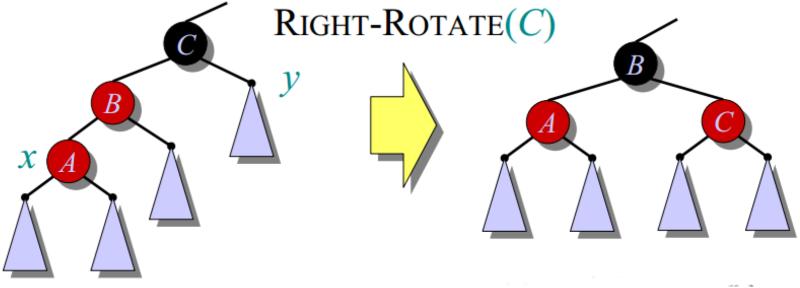
- Case II: The parent of x is red, the uncle of x is black, x's parent is a left child , x is a right child
  - Left rotate on x's parent
  - Make x's left child the new x
  - Solve this scenario by using by Case III



Transform to Case 3.

RBT : Insertion[18]

- Case III: The parent of x is Red and the uncle is black, x is a left child, and its parent is a left child
  - Right rotate on grandparent of x
  - Switch colors of x's parent and x's sibling
  - Done!



[18] MIT OCW 6.046j

RBT : Insertion Sorted Elements

- Refer `https://www.cs.utexas.edu/~scottm/cs314/handouts/slides/Topic23RedBlackTrees.pdf` for a step-by-step example of how RBT handles insertion in sorted order

RBT: Deletion

- Similarly complicated
- Refer to CLRS book for full details

Summary

Major Concepts:

- Binary Search Trees
- Concept of Self-Balancing
- Red-Black Trees