Lecture 1: Asymptotics, Recurrences, Elementary Sorting

Instructor: Saravanan Thirumuruganathan

Outline

1. Introduction to Asymptotic Analysis
    - Rate of growth of functions
    - Comparing and bounding functions: O, Θ, Ω
    - Specifying running time through recurrences
    - Solving recurrences
2. Elementary Sorting Algorithms
    - Bubble, Insertion and Selection sort
    - Stability of sorting algorithms

In-Class Quizzes

- **URL:** `http://m.socrative.com/`
- **Room Name: 4f2bb99e**

Analyzing Algorithms

Time Complexity:

- Quantifies amount of **time** an algorithm needs to complete as a *function* of input size

Space Complexity:

- Quantifies amount of **space** an algorithm needs to complete as a *function* of input size

Function: Input size Vs {Time, Space}

Analyzing Algorithms

Best Case Complexity:

- of an algorithm is the **function** that determines the **minimum** number of steps taken on any problem instance of size n

Worst Case Complexity:

- . . . **maximum** . . .

Average Case Complexity:

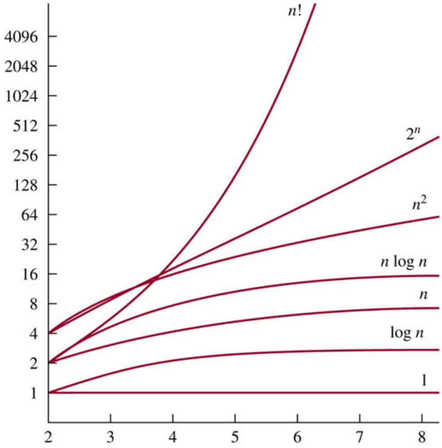
- . . . **average** . . .

Function: Input size Vs {Time, Space}

Growth Function $T(n)$

- Input is positive integers $n = 1, 2, 3, \dots$
- Asymptotically positive (returns positive numbers for large n)
- How does $T(n)$ grow when n grows?
- n is size of input
- $T(n)$ is the amount of time it takes for an algorithm to solve some problem

Rate of Growth of Functions



Quiz!

Question:

- You have a machine that can do million operations per second.
- Your algorithm requires n^2 steps
- Suppose size of input is 1 million
- How long does the algorithm takes for this input?

Quiz!

Answer:

- Algorithm will take $(1M)^2$ operations
- Machine can do $1M$ operations per second
- Running time $= \frac{(1M)^2}{1M} = 1M$ seconds
- $1M$ seconds $= \frac{1M}{60*60*24} =$ Approximately 12 days

Why does it matter?

Running time of different algorithms for various input sizes

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long

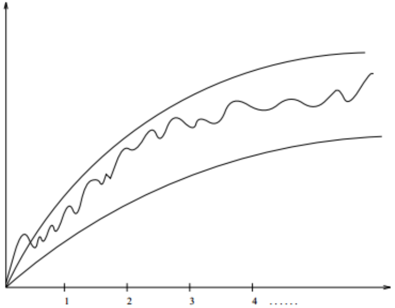
[1] Table 2.1 from K&T Algorithm Design. Very long means it takes more than 10^{25} years.

Why does it matter?

- The "Big Data" era
- Can Google/Facebook/... use it?

Functions in Real World

This is how functions look in the real world![2]



[2]Skiena Lecture notes

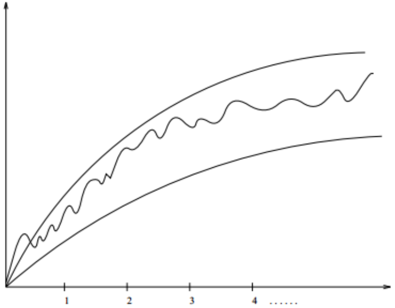
Solution: Analyze Asymptotic Behavior

- Analyze the asymptotic behavior of algorithms
- What happens to $f(n)$ when $n \to \infty$?

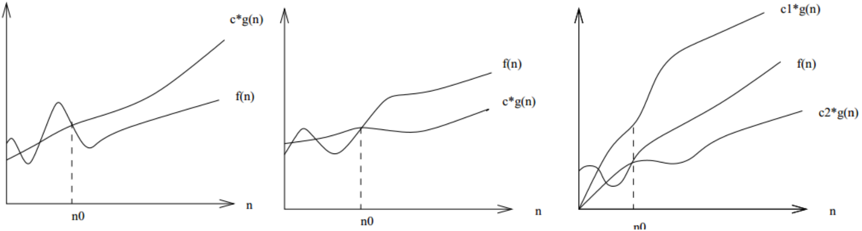
	$T(n) = 1000n$	$T(n) = n^2$
$n = 10$	$10K$	100
$n = 100$	$100K$	$10K$
$n = 1000$	$1M$	$1M$
$n = 10K$	$10M$	$100M$
$n = 100K$	$100M$	$10B$

Solution: Bound the functions

- Identify known functions (such as $n, n^2, n^3, 2^n, \ldots$) that can "bound" $T(n)$
- How to bound? - asymptotic **upper, lower** and **tight** bounds
- Find a function $f(n)$ such that $T(n)$ is **proportional** to $f(n)$
- Why proportional (as against **equal**)?
- Ignore aspects such as programming language, programmer capability, compiler optimization, machine specification etc



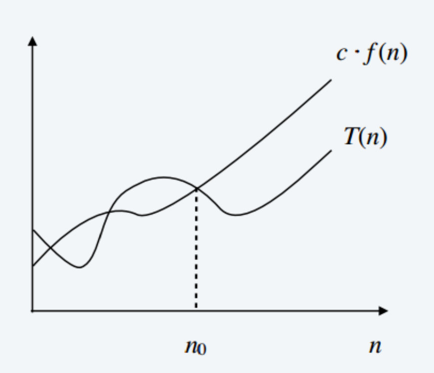
O, Ω, Θ[3]



[3] CLRS book

Big-O Notation

Upper bounds: $T(n)$ is $O(f(n))$ if there exists **constants** $c > 0$ and $n_0 \geq 0$ such that $T(n) \leq c \cdot f(n)$ for all $n \geq n_0$



[4]

[4] From K&T: Algorithm Design

Big-O Notation

Upper bounds: $T(n)$ is $O(f(n))$ if there exists **constants** $c > 0$ and $n_0 \geq 0$ such that $T(n) \leq c \cdot f(n)$ for all $n \geq n_0$

Example: $T(n) = 32n^2 + 17n + 1$. Is $T(n)$ in $O(n^2)$?

- Yes! Use $c = 50$, $n_0 = 1$
- Simple Proof:

$$\begin{aligned} T(n) &\leq 32n^2 + 17n + 1 \\ &\leq 32n^2 + 17n^2 + 1n^2 \\ &\leq 50n^2 \\ &\leq cn^2 \\ c = 50 &\text{ and } n_0 = 1 \end{aligned}$$

Note: Not necessary to find the smallest c or n_0

Quiz!

Example: $T(n) = 32n^2 - 17n + 1$. Is $T(n)$ in $O(n^2)$?

Big-O Notation

Example: $T(n) = 32n^2 - 17n + 1$. Is $T(n)$ in $O(n^2)$?

- Yes! Use $c = 50$, $n_0 = 1$
- Simple Proof:

$$
\begin{aligned}
T(n) &\leq 32n^2 - 17n + 1 \\
&\leq 32n^2 + 17n + 1 \\
&\leq 32n^2 + 17n^2 + 1n^2 \\
&\leq 50n^2 \\
&\leq cn^2
\end{aligned}
$$

$c = 50$ and $n_0 = 1$

Quiz!

Example: $T(n) = 32n^2 - 17n + 1$. Is $T(n)$ in $O(n^3)$?

Big-O Notation

Example: $T(n) = 32n^2 - 17n + 1$. Is $T(n)$ in $O(n^3)$?

- Yes! Use $c = 50$, $n_0 = 1$
- Simple Proof:

$$
\begin{aligned}
T(n) &\leq 32n^2 - 17n + 1 \\
&\leq 32n^2 + 17n + 1 \\
&\leq 32n^2 + 17n^2 + 1n^2 \\
&\leq 50n^2 \\
&\leq 50n^3 \\
&\leq cn^3
\end{aligned}
$$

$$c = 50 \text{ and } n_0 = 1$$

Quiz!

Example: $T(n) = 32n^2 + 17n + 1$. Is $T(n)$ in $O(n)$?

Example: $T(n) = 32n^2 + 17n + 1$. Is $T(n)$ in $O(n)$?

- No!
- Proof by contradiction

$$
\begin{aligned}
32n^2 + 17n + 1 &\leq c \cdot n \\
32n + 17 + \frac{1}{n} &\leq c \\
32n &\leq c \text{ (ignore constants for now)} \\
n &\leq c \text{ (ignore constants for now)}
\end{aligned}
$$

This inequality does not hold for $n = c + 1$!

Set Theoretic Perspective

- $O(f(n))$ is set of all functions $T(n)$ where there exist positive constants c, n_0 such that $0 \leq T(n) \leq c \cdot f(n)$ for all $n \geq n_0$
- Example: $O(n^2) = \{\ n^2, \ldots, 32n^2 + 17n + 1,\ 32n^2 - 17n + 1, \ldots, n, 2n, \ldots\}$
- Notation: $T(n) = O(f(n))$ or $T(n) \in O(f(n))$

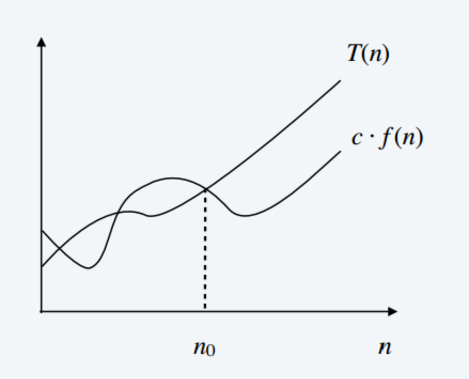
Limit based Perspective

- $T(n)$ is $O(f(n))$ if $\limsup_{n\to\infty} \frac{T(n)}{f(n)} < \infty$
- Example: $32n^2 + 17n + 1$ is $O(n^2)$

$$\begin{aligned}\limsup_{n\to\infty} \frac{T(n)}{f(n)} &= \frac{32n^2 + 17n + 1}{n^2} \\ &= 32 + \frac{17}{n} + \frac{1}{n^2} \\ &= 32 < \infty\end{aligned}$$

Big-Omega Notation

Lower bounds: $T(n)$ is $\Omega(f(n))$ if there exists **constants** $c > 0$ and $n_0 \geq 0$ such that $T(n) \geq c \cdot f(n)$ for all $n \geq n_0$



[5]

[5] From K&T: Algorithm Design

Big-Omega Notation

Lower bounds: $T(n)$ is $\Omega(f(n))$ if there exists **constants** $c > 0$ and $n_0 \geq 0$ such that $T(n) \geq c \cdot f(n)$ for all $n \geq n_0$

Example: $T(n) = 32n^2 + 17n + 1$. Is $T(n)$ in $\Omega(n^2)$?

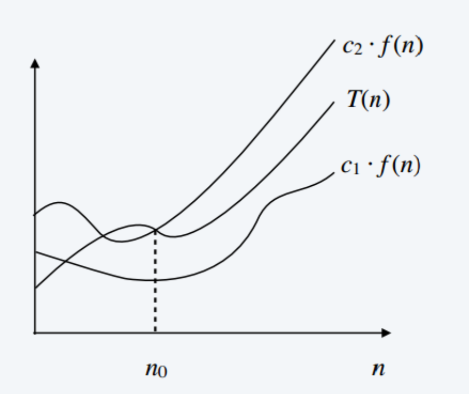
- Yes! Use $c = 32$, $n_0 = 1$
- Simple Proof:

$$\begin{aligned} T(n) &\geq 32n^2 + 17n + 1 \\ &\geq 32n^2 \\ &\geq cn^2 \end{aligned}$$

$$c = 32 \text{ and } n_0 = 1$$

Big-Theta Notation

Tight bounds: $T(n)$ is $\Theta(f(n))$ if there exists **constants** $c_1 > 0, c_2 > 0$ and $n_0 \geq 0$ such that $c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$ for all $n \geq n_0$



[6]

[6] From K&T: Algorithm Design

Big-Theta Notation

Tight bounds: $T(n)$ is $\Theta(f(n))$ if there exists **constants** $c_1 > 0, c_2 > 0$ and $n_0 \geq 0$ such that $c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$ for all $n \geq n_0$

Example: $T(n) = 32n^2 + 17n + 1$. Is $T(n)$ in $\Omega(n^2)$?

- Yes! Use $c_1 = 32$, $c_2 = 50$ and $n_0 = 1$
- Combine proofs from before

Theorem: For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Limit based Definitions

Let $\limsup_{n\to\infty} \frac{T(n)}{f(n)} = c$

- If $c < \infty$ then $T(n)$ is $O(f(n))$ (typically c is zero)
- If $c > 0$ then $T(n)$ is $\Theta(f(n))$ (also $O(f(n))$ and $\Omega(f(n))$)
- If $c = \infty$ then $T(n)$ is $\Omega(f(n))$

Some Big-O tips

- Big-O is one of the most useful things you will learn in this class!
- Big-O ignores *constant* factors through c
  - Algorithm implemented in Python might need a larger c than one implemented in C++
- Big-O ignores small inputs through n_0
  - Simply set a large value of n_0
- Suppose $T(n)$ is $O(f(n))$. Typically, $T(n)$ is messy while $f(n)$ is simple
  - $T(n) = 32n^2 + 17n + 1$, $f(n) = n^2$
- Big-O hides constant factors. Some times using an algorithm with worser Big-O might still be a good idea (e.g. sorting, finding medians)

Survey of Running Times

Complexity	**Name**	**Example**
$O(1)$	Constant time	Function that returns a constant (say 42)
$O(\log n)$	Logarithmic	Binary Search
$O(n)$	Linear	Finding Max of an array
$O(n \log n)$	Linearithmic	Sorting (for e.g. Mergesort)
$O(n^2)$	Quadratic	Selection sort
$O(n^3)$	Cubic	Floyd-Warshall
$O(n^k)$	Polynomial	Subset-sum with k elements
$O(2^n)$	Exponential	Subset-sum with no cardinality constraints

Dominance Rankings[7]

- $n! \gg 2^n \gg n^3 \gg n^2 \gg n \log n \gg n \gg \log n \gg 1$
- Exponential algorithms are useless even at $n >= 50$
- Quadratic algorithms at around $n \geq 1M$
- $O(n \log n)$ at around $n \geq 1B$

[7] Skiena lecture notes

Closer Look at $T(n)$

- So far we assumed someone gave us $T(n)$
- What is n? (Program Analysis)
- How do we get $T(n)$? (Recurrences)

Program Analysis

```
for i=1 to n
{
    constant time
    operations
}
```

```
for i=1 to n
{
    for j=1 to n
    {
        constant time
        operations
    }
}
```

Recurrences

- Typically programs are lot more complex than that
- Recurrences occur in recursion and divide and conquer paradigms
- Specify running time as a function of n and running time over inputs of smaller sizes
- Examples:
  - $\text{fibonacci}(n) = \text{fibonacci}(n-1) + \text{fibonacci}(n-2)$
  - $T(n) = 2T(\frac{n}{2}) + n$
  - $T(n) = T(n-1) + n$

Solving Recurrences

- Unrolling
- Guess and prove by induction (aka Substitution)
- Recursion tree
- Master method

Unrolling

Let $T(n) = T(n-1) + n$. Base case: $T(1) = 1$

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= n + T(n-1) \\
&= n + n - 1 + T(n-2) \\
&= n + n - 1 + n - 2 + T(n-3) \\
&= n + n - 1 + n - 2 + n - 3 + \ldots + 4 + 3 + 2 + 1 \\
&= \frac{n(n-1)}{2} \\
&= 0.5n^2 - 0.5n \\
&= O(n^2)
\end{aligned}$$

Quiz!

Solve $T(n) = 2T(n-1)$. Base case: $T(1) = 1$

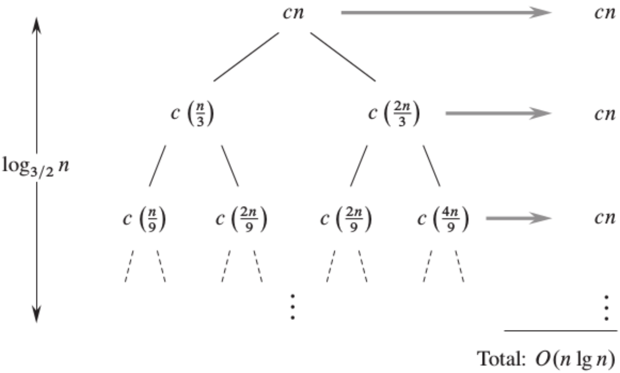
Quiz!

Solve $T(n) = 2T(n-1)$. Base case: $T(1) = 1$

$$\begin{aligned}
T(n) &= 2T(n-1) \\
&= 2(2T(n-2)) \\
&= 2(2(2T(n-3))) \\
&= 2^3 T(n-3) \\
&= 2^i \ldots 2T(n-i) \\
&= 2^n \\
&= O(2^n)
\end{aligned}$$

Recursion Tree

$$T(n) = T(\frac{2n}{3}) + T(\frac{n}{3}) + cn$$



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8 From CLRS

Logarithms[9]

- Logarithm is an *inverse* exponential function
- $b^x = n$ implies $x = \log_b n$
- If $b = 2$, logarithms reflect how many times we can double something until we get n or halve something till we get 1
- Example: $2^4 = 16$, $\log_2 16 = \lg 16 = 4$
- Example: You need $\lg 256 = 8$ bits to represent $[0, 255]$
- Identities:
  - $\log_b(xy) = \log_b(x) + \log_b(y)$
  - $\log_b a = \frac{\log_c a}{\log_c b}$
  - $\log_b b = 1$ and $\log_b 1 = 0$

[9] Skiena Lectures