Lecture 7: Binary Heaps, Heapsort, Union-Find

Instructor: Saravanan Thirumuruganathan

Outline

1. Data Structures for representing Dynamic Sets
   - Binary Heap
     - Heapsort
   - Union Find

In-Class Quizzes

- **URL:** `http://m.socrative.com/`
- **Room Name: 4f2bb99e**

Key Things to Know for Data Structures

- Motivation
- Distinctive Property
- Major operations
- Key Helper Routines
- Representation
- Algorithms for major operations
- Applications

Binary Heap

Motivation

- Heap Sort (CLRS is organized that way!)
- Priority Queue
- Most space efficient data structure

Priority Queue

- "Queue" data structure has a FIFO property
- Some times it is useful to consider **priority**
- Output element with highest priority first

Priority Queue - Major Operations

- Insert
- FindMin (resp. FindMax)
- DeleteMin (resp. DeleteMax)
- DecreaseKey (resp. IncreaseKey)

Priority Queue - Applications[1]

- Dijkstra's shortest path algorithm
- Prim's MST algorithm
- Heapsort
- Online median
- Huffman Encoding
- A* Search (or any Best first search)
- Discrete event simulation
- CPU Scheduling
- ...
- See Wikipedia entry for priority for details

[1] Kleinberg-Tardos Book and Wikipedia

Priority Queue - Candidate Implementations

- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
  - Insert:

Priority Queue - Candidate Implementations

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- LinkedList
  - Insert: $O(1)$
  - FindMin:

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Priority Queue - Candidate Implementations

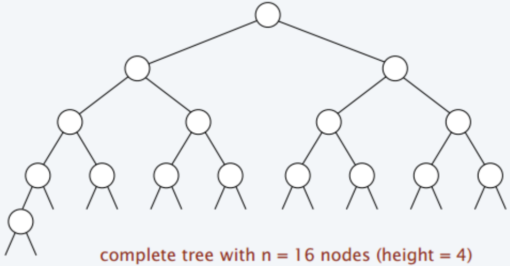
- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
  - Insert: $O(1)$
  - FindMin: $O(n)$
  - DeleteMin: $O(1)$
  - DecreaseKey: $O(1)$
- Binary Heap
  - Insert: $O(\lg n)$
  - FindMin: $O(1)$
  - DeleteMin: $O(\lg n)$
  - DecreaseKey: $O(\lg n)$
- Binomial Heaps, Fibonacci Heaps etc.

Binary Heaps

- Perfect data structure for implementing Priority Queue
- MaxHeap and MinHeap
- We will focus on MaxHeaps in this lecture

Complete Tree[2]

- Perfectly balanced, except for bottom level
- Elements were inserted top-to-bottom and left-to-right



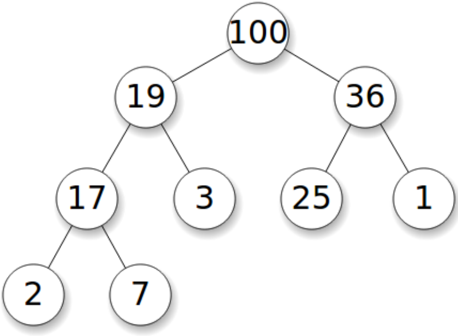
complete tree with n = 16 nodes (height = 4)

[2] http://www.cs.princeton.edu/courses/archive/spring13/cos423/lectures/BinomialHeaps.pdf

Heap Property

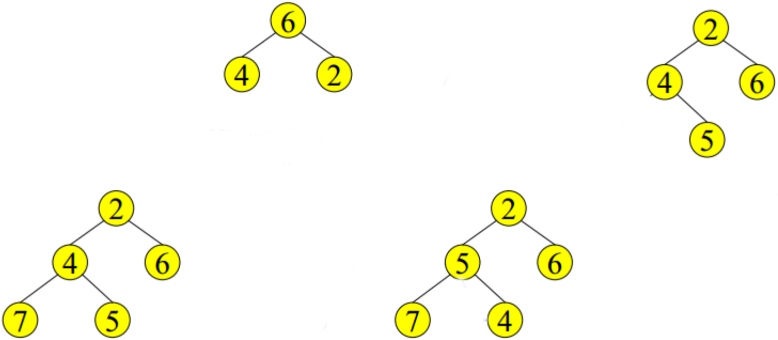
- Heap is a binary tree (**NOT BST**)
- Heap:
  - **Completeness** Property: Heap has restricted structure. It must be a complete binary tree .
  - **Ordering** Property: Relates parent value with that of its children
- MaxHeap property: Value of parent must be greater than **both** its children
- MinHeap property: Value of parent must be less than **both** its children
- Heap with n elements has height $O(\lg n)$

Max Heap Example[3]



[3]Wikipedia page for Heap

Heap Property[4]



[4] http://courses.cs.washington.edu/courses/cse373/06sp/handouts/lecture10.pdf

Major Operations

- Insert
- FindMax
- DeleteMax (aka ExtractMax)
- IncreaseKey

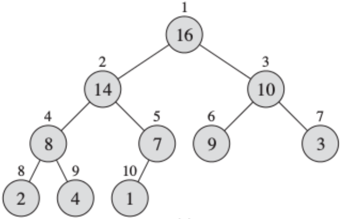
Key Helper Routines

- Max-Heapify (or Min-Heapify)
- Bubble-Up
- Bubble-Down
- Heapify

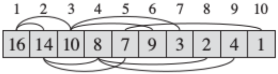
Representation: Arrays

- Very efficient implementation using arrays
- Possible due to completeness property
- Parent(i): return $\lfloor i/2 \rfloor$
- LeftChild(i): return $2i$
- RightChild(i): return $2i + 1$

Representation: Arrays[5]



(a)



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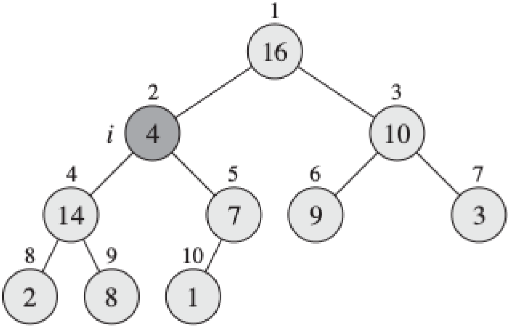
[5] CLRS Fig 6.1

Max-Heapify

- Objective: Maintain heap property
- Invocation: Max-Heapify(A, i)
- Assume: Left(i) and Right(i) are valid max-heaps
- $A[i]$ might violate max-heap property
- **Analysis:** $O(\lg n)$

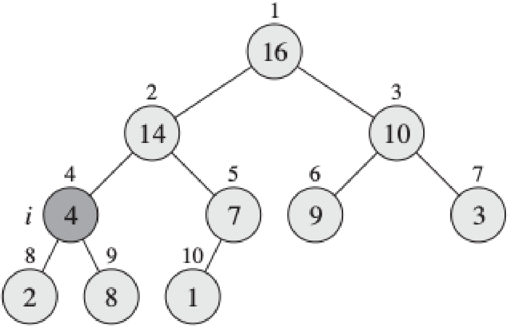
Max-Heapify: Example[6]



(a)

[6] CLRS Fig 6.2

Max-Heapify: Example[7]



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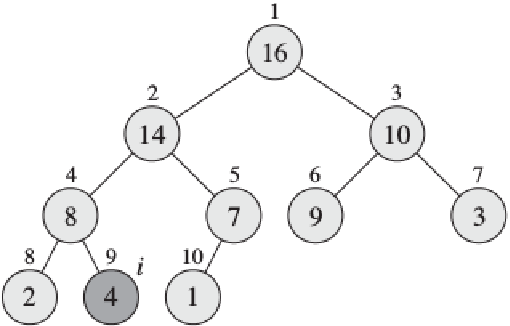
[7] CLRS Fig 6.2

Max-Heapify: Example[8]



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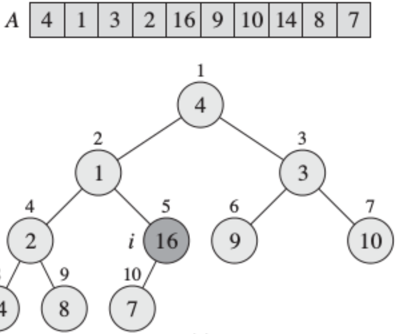
[8] CLRS Fig 6.2

Build-Max-Heap

- Given an array A, convert it to a max-heap
- *A.length*: Length of the array
- *A.heapSize*: Elements from 1 ... A.heapSize form a heap
- Build-Max-Heap(A):
  - A.heapSize = A.length
  - for $i = \lfloor A.length/2 \rfloor$ down to 1
    Max-Heapify(A, i)
- **Analysis:** $O(n)$

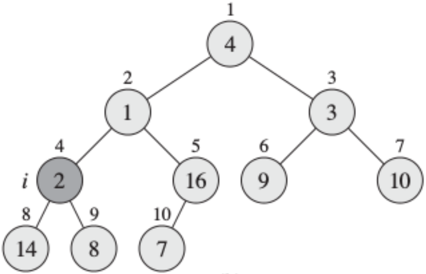
Build-Max-Heap : Example [9]



(a)

[9] CLRS Fig 6.3

Build-Max-Heap : Example [10]



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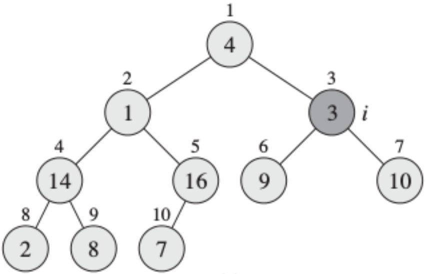
[10] CLRS Fig 6.3

Build-Max-Heap : Example [11]



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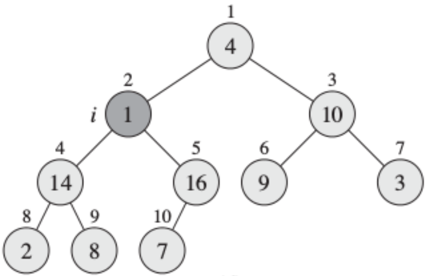
[11] CLRS Fig 6.3

Build-Max-Heap : Example [12]



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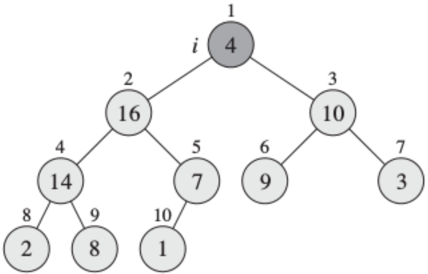
[12] CLRS Fig 6.3

Build-Max-Heap : Example [13]



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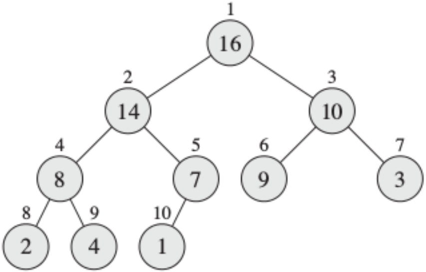
[13] CLRS Fig 6.3

Build-Max-Heap : Example [14]



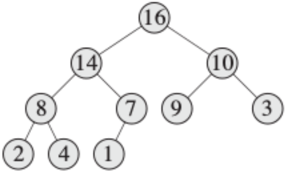
(f)

[14]CLRS Fig 6.3

HeapSort

```
HeapSort(A):
    Build-Max-Heap(A)
    for i = A.length down to 2
        Exchange A[1] with A[i]
        A.heapSize = A.heapSize - 1
        Max-Heapify(A, 1)
```

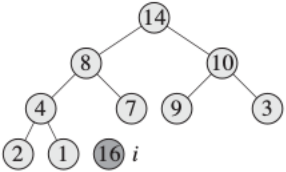
Heap Sort: Example[15]



(a)

[15] CLRS Fig 6.4

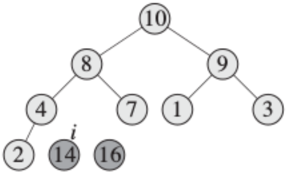
Heap Sort: Example[16]



(b)

[16]CLRS Fig 6.4

Heap Sort: Example[17]



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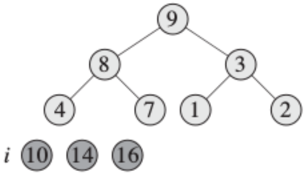
[17] CLRS Fig 6.4

Heap Sort: Example[18]



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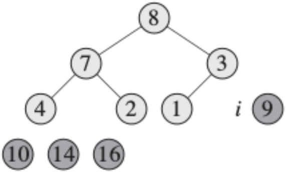
[18] CLRS Fig 6.4

Heap Sort: Example[19]



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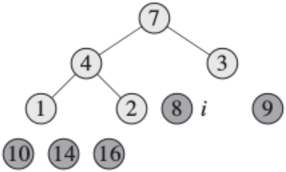
[19] CLRS Fig 6.4

Heap Sort: Example[20]



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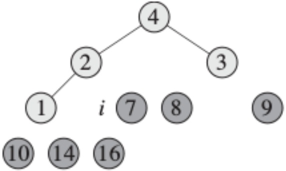
[20]CLRS Fig 6.4

Heap Sort: Example[21]



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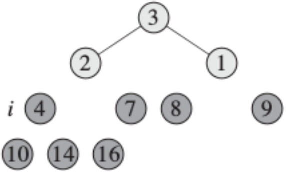
[21] CLRS Fig 6.4

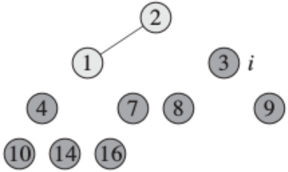
Heap Sort: Example[22]



(h)

[22] CLRS Fig 6.4

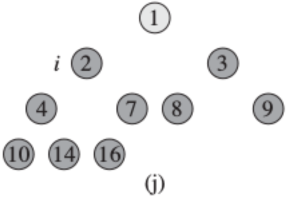
Heap Sort: Example[23]



(i)

[23]CLRS Fig 6.4

Heap Sort: Example[24]



[24] CLRS Fig 6.4

Heap Sort: Example[25]

A	1	2	3	4	7	8	9	10	14	16

[25] CLRS Fig 6.4

- GiG

Summary

Major Concepts:

- Binary Heap
- Heapsort
- Union-Find