Lecture 7: Binary Heaps, Heapsort, Union-Find

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Outline

1. Data Structures to speed up algorithms
    - Binary Heap
        - Heapsort
    - Union Find

In-Class Quizzes

- **URL:** `http://m.socrative.com/`
- **Room Name: 4f2bb99e**

Key Things to Know for Data Structures

- Motivation
- Distinctive Property
- Major operations
- Key Helper Routines
- Representation
- Algorithms for major operations
- Applications

Data Structures for Algorithmic Speedup

- BST and RBT are two examples of data structures to represent dynamic set
- Today's topic, Heap and Union-Find, can also be used to represent dynamic set
- However, these are used more often to speed up algorithms

Binary Heap

Motivation

- Heap Sort (CLRS is organized that way!)
- Priority Queue
- Most space efficient data structure

Priority Queue

- "Queue" data structure has a FIFO property
- Some times it is useful to consider **priority**
- Output element with highest priority first

Priority Queue - Major Operations

- Insert
- FindMin (resp. FindMax)
- DeleteMin (resp. DeleteMax)
- DecreaseKey (resp. IncreaseKey)

Priority Queue - Applications[1]

- Dijkstra's shortest path algorithm
- Prim's MST algorithm
- Heapsort
- Online median
- Huffman Encoding
- A* Search (or any Best first search)
- Discrete event simulation
- CPU Scheduling
- ...
- See Wikipedia entry for priority for details

[1] Kleinberg-Tardos Book and Wikipedia

Priority Queue - Candidate Implementations

- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
    - Insert:

Priority Queue - Candidate Implementations

- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
    - Insert: $O(1)$
    - FindMin:

Priority Queue - Candidate Implementations

- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
  - Insert: $O(1)$
  - FindMin: $O(n)$
  - DeleteMin:

Priority Queue - Candidate Implementations

- Assume: for DeleteMin and DecreaseKey, pointer to element is given
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  - Insert: $O(1)$
  - FindMin: $O(n)$
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  - DecreaseKey:

Priority Queue - Candidate Implementations

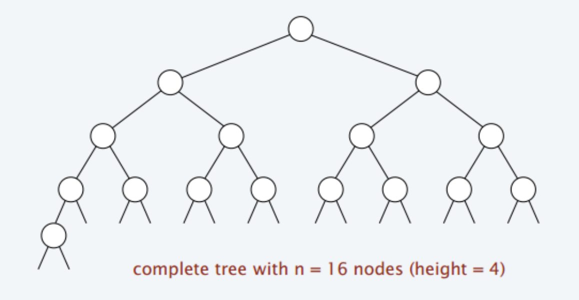
- Assume: for DeleteMin and DecreaseKey, pointer to element is given
- LinkedList
  - Insert: $O(1)$
  - FindMin: $O(n)$
  - DeleteMin: $O(1)$
  - DecreaseKey: $O(1)$
- Binary Heap
  - Insert: $O(\lg n)$
  - FindMin: $O(1)$
  - DeleteMin: $O(\lg n)$
  - DecreaseKey: $O(\lg n)$
- Binomial Heaps, Fibonacci Heaps etc.

Binary Heaps

- Perfect data structure for implementing Priority Queue
- MaxHeap and MinHeap
- We will focus on MaxHeaps in this lecture

Complete Tree[2]

- Perfectly balanced, except for bottom level
- Elements were inserted top-to-bottom and left-to-right

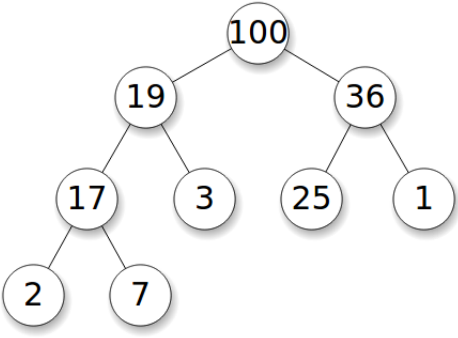


[2] http://www.cs.princeton.edu/courses/archive/spring13/cos423/lectures/BinomialHeaps.pdf

Heap Property

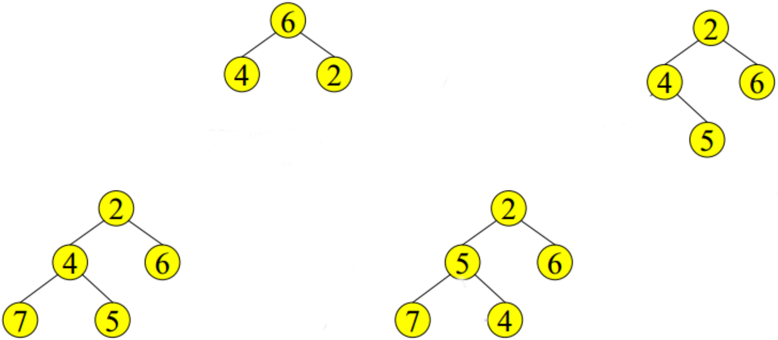
- Heap is a binary tree (**NOT BST**)
- Heap:
    - **Completeness** Property: Heap has restricted structure. It must be a complete binary tree .
    - **Ordering** Property: Relates parent value with that of its children
- MaxHeap property: Value of parent must be greater than **both** its children
- MinHeap property: Value of parent must be less than **both** its children
- Heap with n elements has height $O(\lg n)$

Max Heap Example[3]



[3] Wikipedia page for Heap

Heap Property[4]



[4] http://courses.cs.washington.edu/courses/cse373/06sp/handouts/lecture10.pdf

Major Operations

- Insert
- FindMax
- DeleteMax (aka ExtractMax)
- IncreaseKey

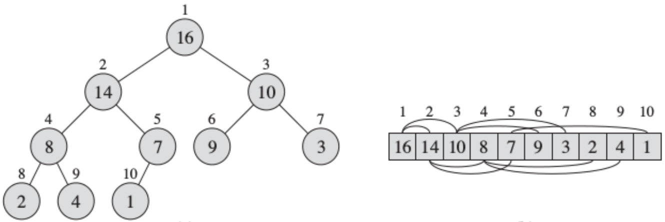
Key Helper Routines

- Max-Heapify (or Min-Heapify)
- Bubble-Up
- Bubble-Down
- Heapify

Representation: Arrays

- Very efficient implementation using arrays
- Possible due to completeness property
- Parent(i): return $\lfloor i/2 \rfloor$
- LeftChild(i): return $2i$
- RightChild(i): return $2i + 1$

Representation: Arrays[5]



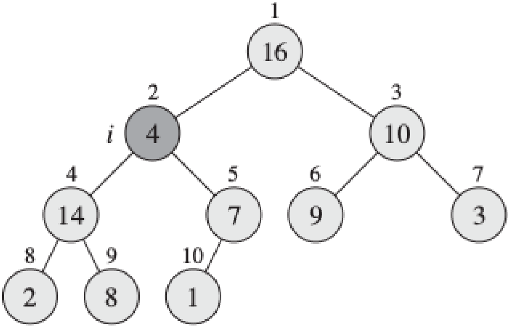
(a) (b)

[5]CLRS Fig 6.1

Max-Heapify

- Objective: Maintain heap property
- Invocation: Max-Heapify(A, i)
- Assume: Left(i) and Right(i) are valid max-heaps
- $A[i]$ might violate max-heap property
- Bubble-Down the violation
- **Analysis:** $O(\lg n)$

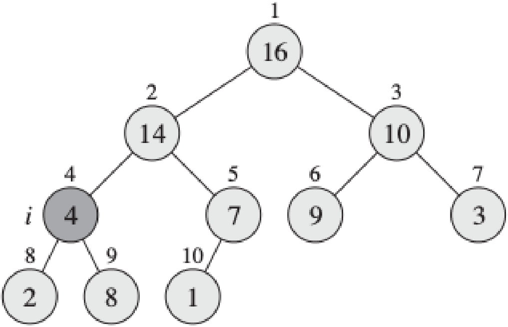
Max-Heapify: Example[6]



(a)

[6]CLRS Fig 6.2

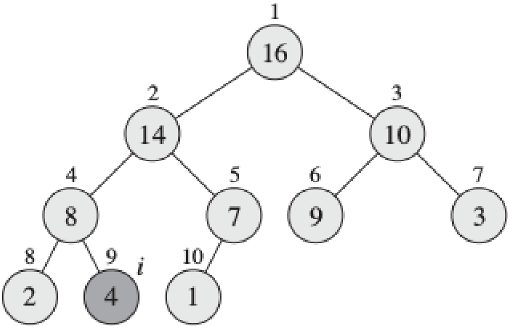
Max-Heapify: Example[7]



(b)

[7] CLRS Fig 6.2

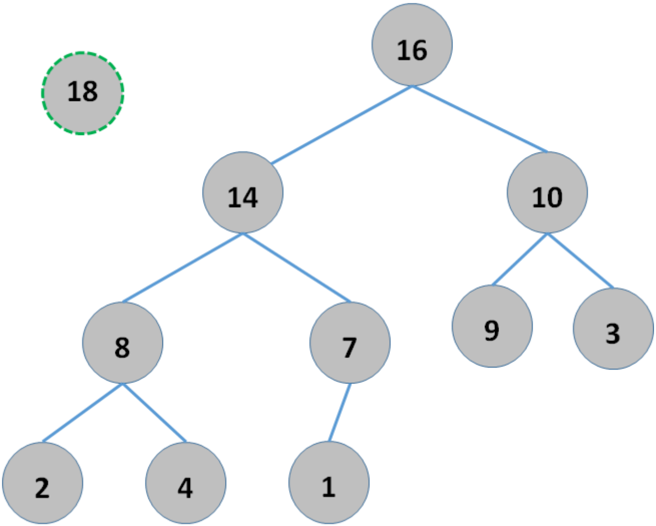
Max-Heapify: Example[8]



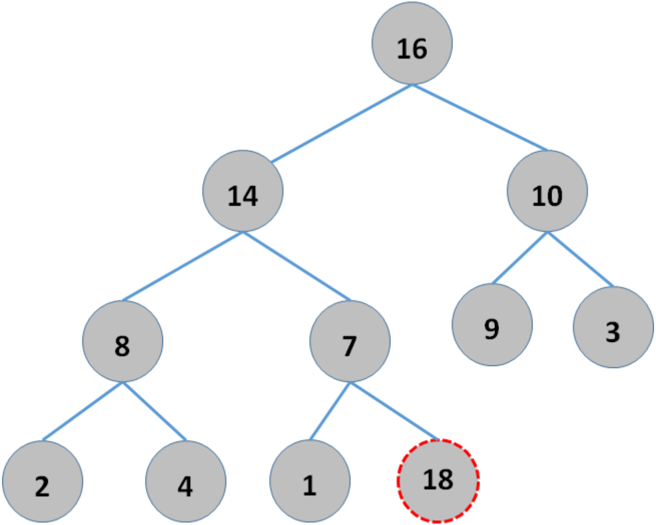
(c)

[8] CLRS Fig 6.2

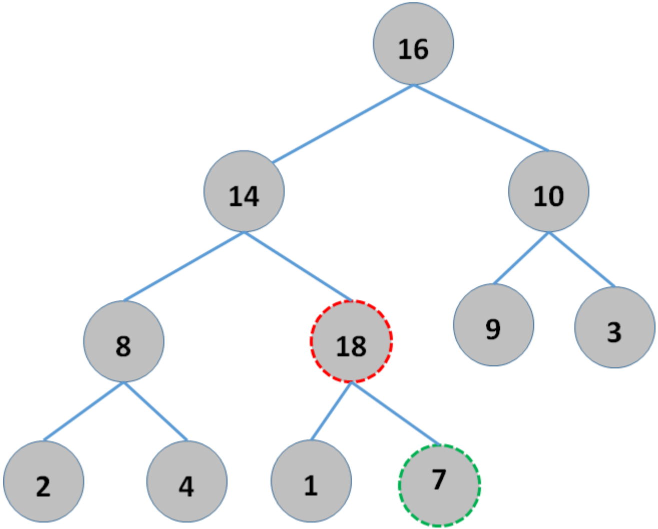
Heap : Insert



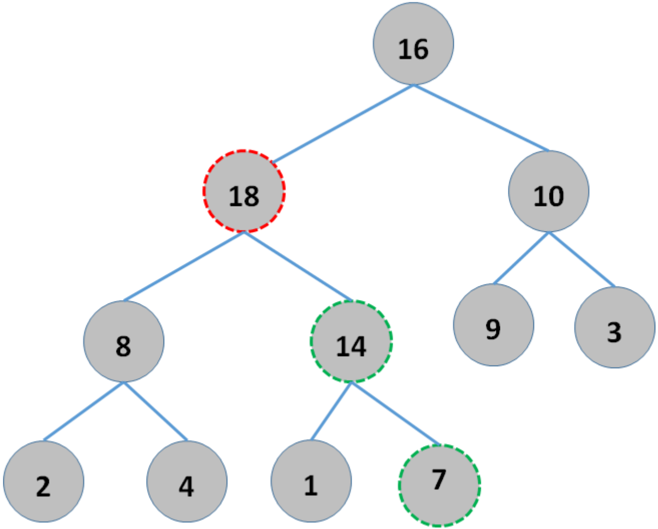
Heap : Insert



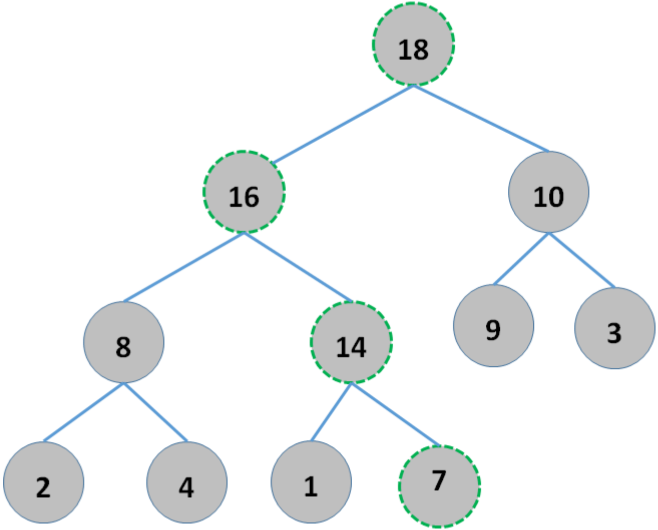
Heap : Insert



Heap : Insert



Heap : Insert



Heap : Insert

- Insert element at first available slot (no completeness property violation!)
- Fix heap property violations by bubbling up the vilolation till it is fixed
- Complexity:

Heap : Insert

- Insert element at first available slot (no completeness property violation!)
- Fix heap property violations by bubbling up the vilolation till it is fixed
- Complexity: $O(\lg n)$

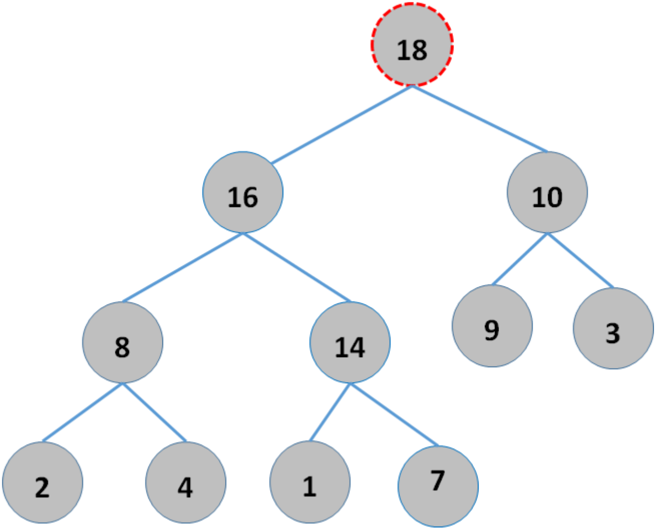
Heap : FindMax

- Look at the root element
- Time complexity: $O(1)$

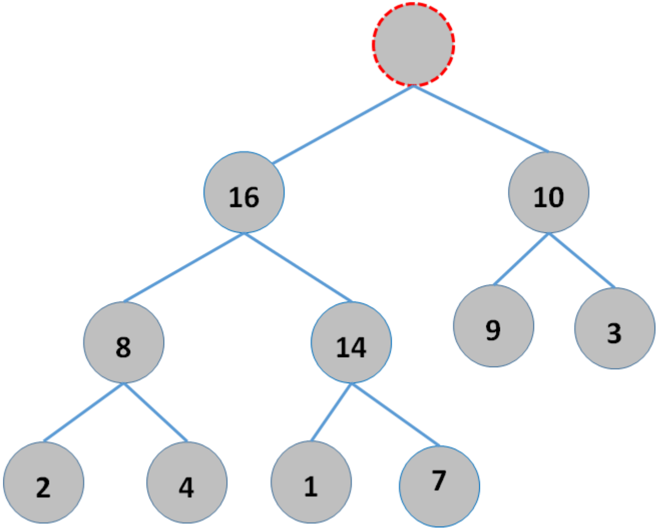
Heap : DeleteMax

- Delete the maximum element (root)
- Fix the heap

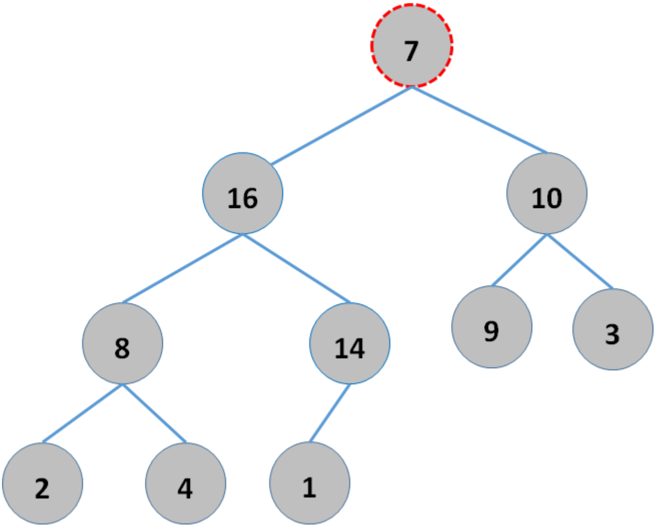
Heap : DeleteMax



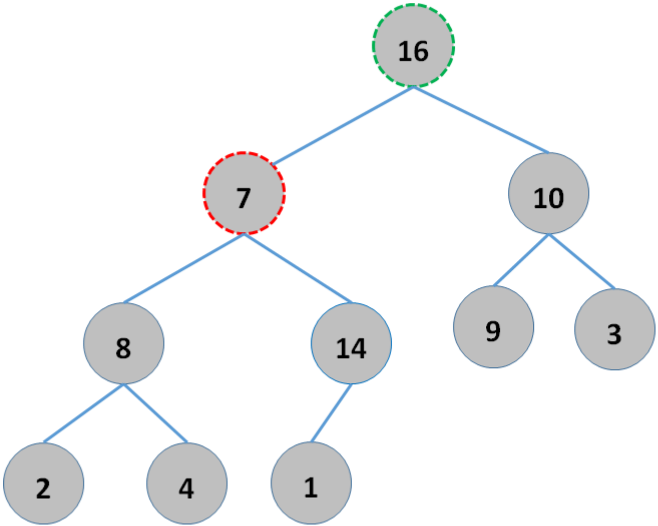
Heap : DeleteMax



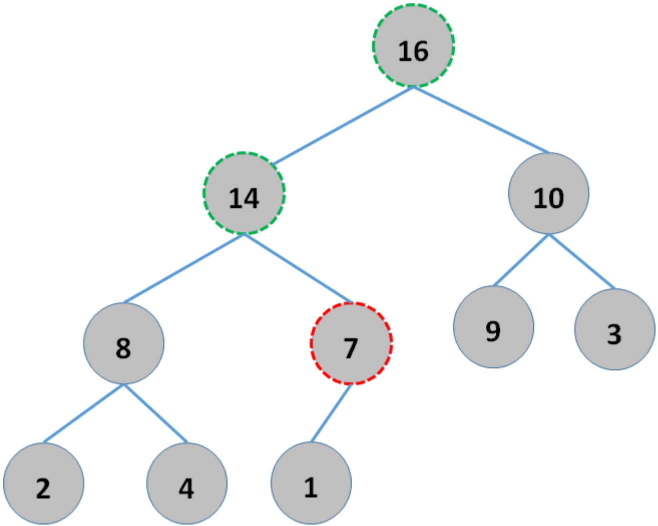
Heap : DeleteMax



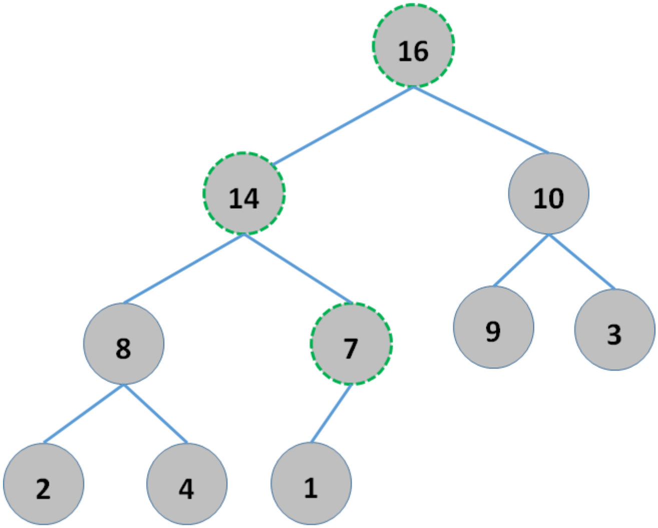
Heap : DeleteMax



Heap : DeleteMax



Heap : DeleteMax



Heap : DeleteMax

- Remove root
- Replace root with last element (does not affect Completeness property)
- Fix heap violations by bubbling it down till it is fixed
- Complexity:

Heap : DeleteMax

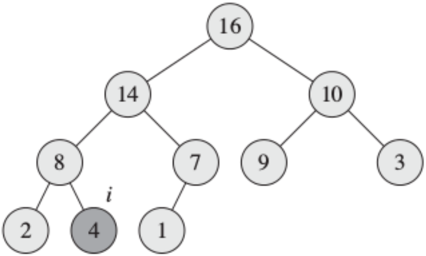
- Remove root
- Replace root with last element (does not affect Completeness property)
- Fix heap violations by bubbling it down till it is fixed
- Complexity: $O(\lg n)$

Heap : IncreaseKey

- Given a node, increase its priority to a new, higher value
- Fix heap property violations

Heap : IncreaseKey[9]

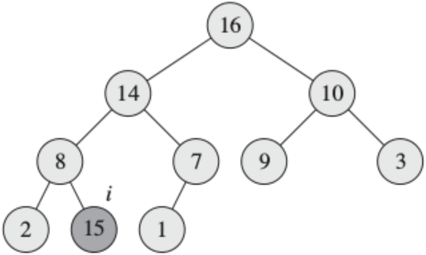
IncreaseKey: Increase value of 4 to 15



(a)

[9] CLRS Fig 6.5

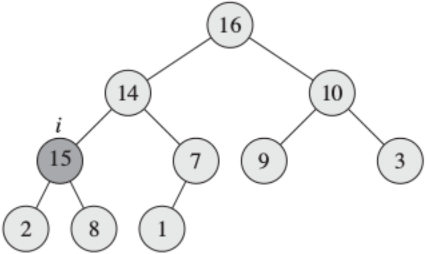
Heap : IncreaseKey[10]



(b)

[10] CLRS Fig 6.5

Heap : IncreaseKey[11]



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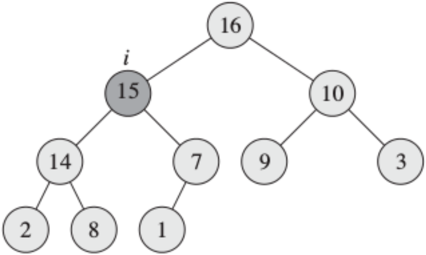
[11] CLRS Fig 6.5

Heap : IncreaseKey[12]



(d)

[12] CLRS Fig 6.5

Heap : IncreaseKey

- Update element
- Fix heap violations by bubbling it up till it is fixed
- Complexity:

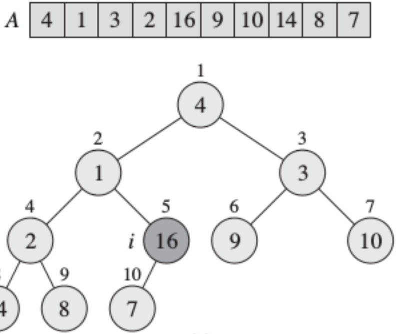
Heap : IncreaseKey

- Update element
- Fix heap violations by bubbling it up till it is fixed
- Complexity: $O(\lg n)$

Build-Max-Heap

- Given an array A, convert it to a max-heap
- $A.length$: Length of the array
- $A.heapSize$: Elements from 1 ... A.heapSize form a heap
- Build-Max-Heap(A):
  - A.heapSize = A.length
  - for $i = \lfloor A.length/2 \rfloor$ down to 1
    Max-Heapify(A, i)
- **Analysis:** $O(n)$ (See book for details)

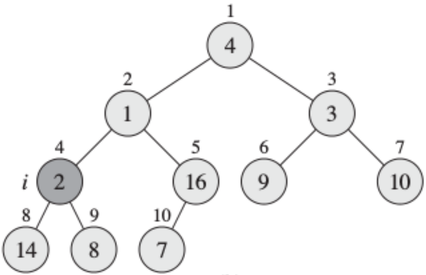
Build-Max-Heap : Example [13]



(a)

[13] CLRS Fig 6.3

Build-Max-Heap : Example [14]



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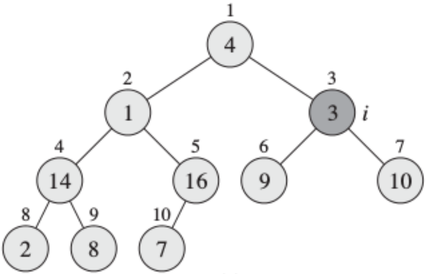
[14] CLRS Fig 6.3

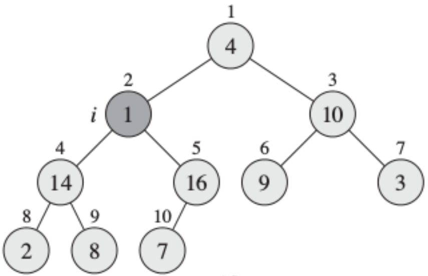
Build-Max-Heap : Example [15]



(c)

[15] CLRS Fig 6.3

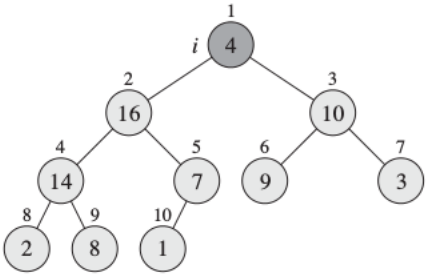
Build-Max-Heap : Example [16]



(d)

[16]CLRS Fig 6.3

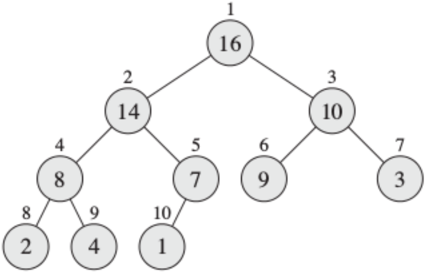
Build-Max-Heap : Example [17]



(e)

[17] CLRS Fig 6.3

Build-Max-Heap : Example [18]



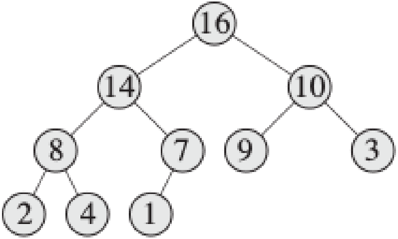
(f)

[18] CLRS Fig 6.3

HeapSort

```
HeapSort(A):
    Build-Max-Heap(A)
    for i = A.length down to 2
        Exchange A[1] with A[i]
        A.heapSize = A.heapSize - 1
        Max-Heapify(A, 1)
```

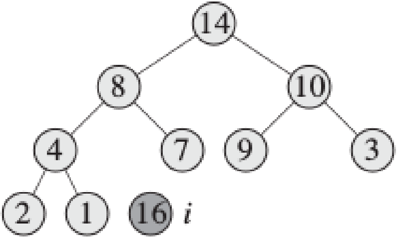
Heap Sort: Example[19]



(a)

[19] CLRS Fig 6.4

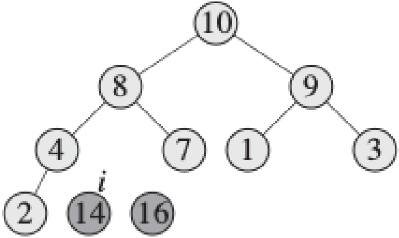
Heap Sort: Example[20]



(b)

[20] CLRS Fig 6.4

Heap Sort: Example[21]



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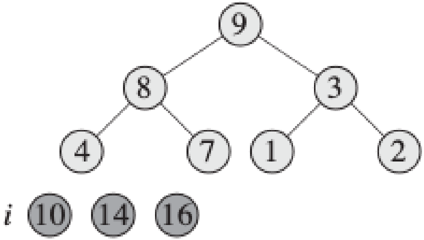
[21] CLRS Fig 6.4

Heap Sort: Example[22]



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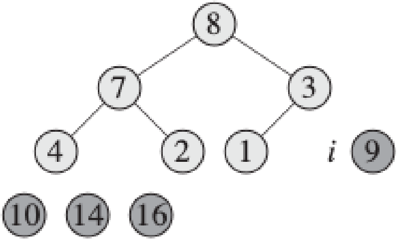
[22] CLRS Fig 6.4

Heap Sort: Example[23]



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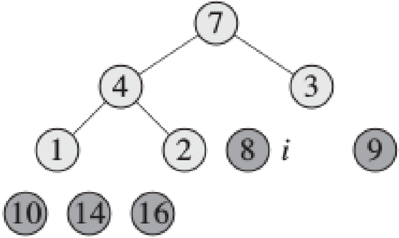
[23] CLRS Fig 6.4

Heap Sort: Example[24]



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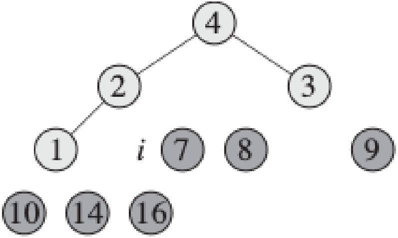
[24]CLRS Fig 6.4

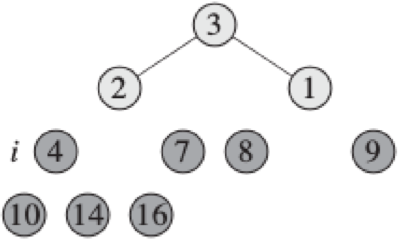
Heap Sort: Example[25]



(g)

[25] CLRS Fig 6.4

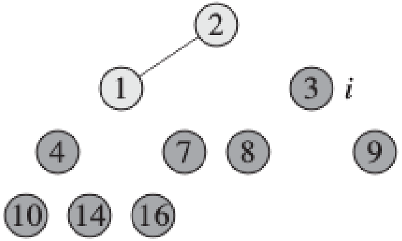
Heap Sort: Example[26]



(h)

[26] CLRS Fig 6.4

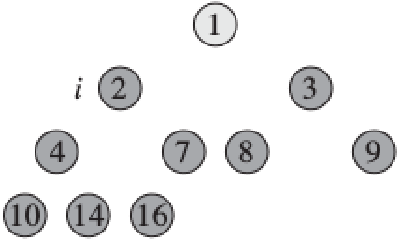
Heap Sort: Example[27]



(i)

[27]CLRS Fig 6.4

Heap Sort: Example[28]



(j)

[28] CLRS Fig 6.4

Heap Sort: Example[29]

A	1	2	3	4	7	8	9	10	14	16

[29] CLRS Fig 6.4

HeapSort: Analysis

- Operations:
  - Build-Max-Heap:

HeapSort: Analysis

- Operations:
  - Build-Max-Heap: $O(n)$
  - n Max-Heapify:

HeapSort: Analysis

- Operations:
  - Build-Max-Heap: $O(n)$
  - n Max-Heapify: $n \times \lg n = O(n \lg n)$
  - Complexity: $O(n) + O(n \lg n) = O(n \lg n)$

HeapSort

- Very efficient in practice - often competitive with QuickSort
- In-Place but not stable (why?)
- Requires constant extra space
- Best, average and worst case complexity is $O(n \lg n)$ (unlike Quicksort)

Data Structures for Disjoint Sets

Disjoint Sets ADT

- Objective: Represent and manipulate disjoint sets (sets that do not overlap)
- Required Operations
  - MakeSet(x): Create a new set $\{x\}$ with single element x
  - Find(x): Find the set containing x
  - Union(x, y): Merge sets containing x and y

Disjoint Sets: Example[30]

- Objects.

```
0 1 2 3 4 5 6 7 8 9
```

- Disjoint sets of objects.

```
0 1 { 2 3 9 } { 5 6 } 7 { 4 8 }
```

- Find query: are objects 2 and 9 in the same set?

```
0 1 { 2 3 9 } { 5 6 } 7 { 4 8 }
```

- Union command: merge sets containing 3 and 8.

```
0 1 { 2 3 4 8 9 } { 5 6 } 7
```

[30] https://www.cs.princeton.edu/~rs/AlgsDS07/01UnionFind.pdf

Disjoint Sets: Applications[31]

- Network connectivity: are two computers connected?
- Compilers: are two variables aliases?
- Image segmentation: are both pixels in same segment?
- Chip design: are two transistors connected to each other?
- Maze design
- Speeding up Kruskal's MST algorithm
- Many many more

[31] `http://courses.cs.washington.edu/courses/cse326/08sp/lectures/18-disjoint-union-find.pdf`

Disjoint Sets: Naming

- Represent each disjoint set by a unique name
- For convenience, the name is one of its elements
- This element is called the **leader** of the set
- Find(x) returns the leader of set containing x
- Typically, Union takes leaders as input. For eg, Union(a, b). If not easily fixable by Union(Find(a), Find(b))

Data Structures for Disjoint Sets

Objective:

- Design an efficient data structure to represent DS ADT
- Assume that there are N elements - represented by $1, \ldots, N$
- M operations (any mixture of union and find)

Candidate Representations:

- Array based
- Linked List based
- Tree based

Disjoint Sets Implementation: Arrays

Idea:

- Maintain an array A with N elements
- $A[i]$ stores the leader for set containing element i

Find: Find(3)=4, Find(6)=8

i	1	2	3	4	5	6	7	8	9	10
A[i]	1	2	4	4	5	8	5	8	8	5

Union: Union$(4, 8)$

i	1	2	3	4	5	6	7	8	9	10
A[i]	1	2	8	8	5	8	5	8	8	5

Analysis:

- Find:

Disjoint Sets Implementation: Arrays

Analysis:

- Find: $O(1)$
- Union:

Disjoint Sets Implementation: Arrays

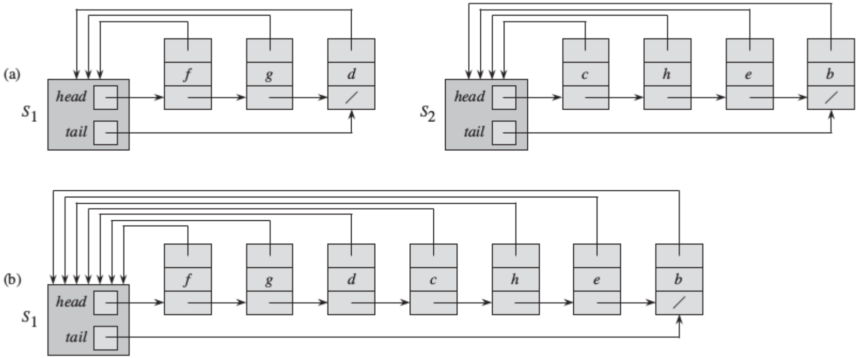
Analysis:

- Find: $O(1)$
- Union: $O(N)$
- Complexity for M operations: $O(MN)$

Disjoint Sets Implementation: Linked List[32]

Idea:

- Represent each set as a linked list
- Set first element of each linked list as the leader



[32] CLRS Fig 21.2

Disjoint Sets Implementation: Linked List

Analysis:

- Find:

Disjoint Sets Implementation: Linked List

Analysis:

- Find: $O(1)$
- Union:

Disjoint Sets Implementation: Linked List

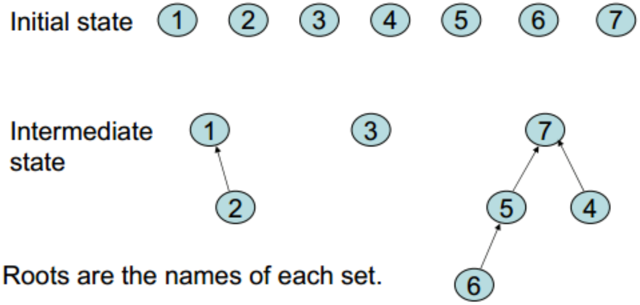
Analysis:

- Find: $O(1)$
- Union: $O(N)$
- Complexity for M operations: $O(MN)$

Disjoint Sets Implementation: Tree

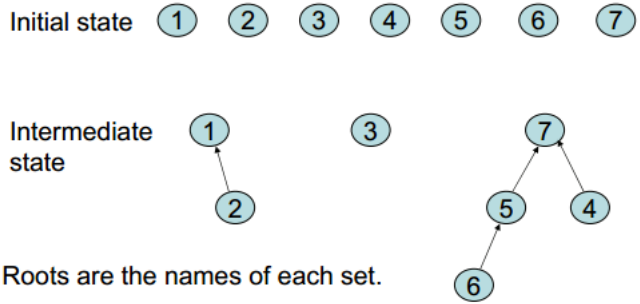
- Also called as Union-Find data structure
- Idea: Represent each set as a tree
- Store all sets as a forest (collection of disconnected trees)
- Allow each node to have arbitrary number of children
- Root of each tree is the leader

Union-Find: Up-Tree Representation[33]



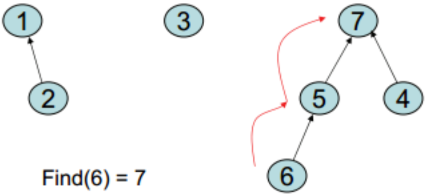
[33] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Up-Tree Representation [34]



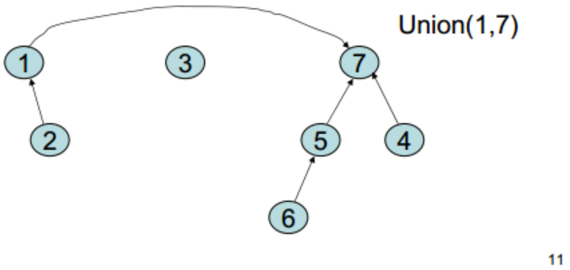
[34] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Find Operation [35]



[35] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Union Operation [36]

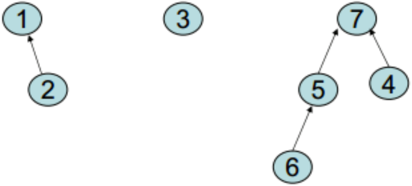


[36] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Up-Tree Implementation [37]

	1	2	3	4	5	6	7
up	-1	1	-1	7	7	5	-1

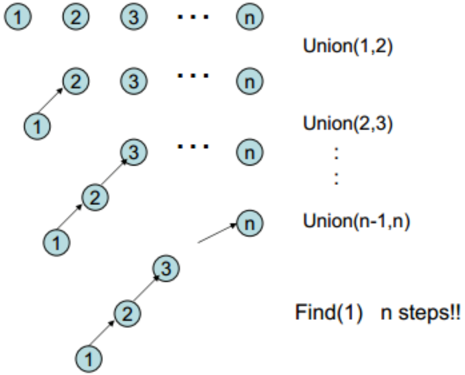
Up[x] = -1 means x is a root.



12

[37] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

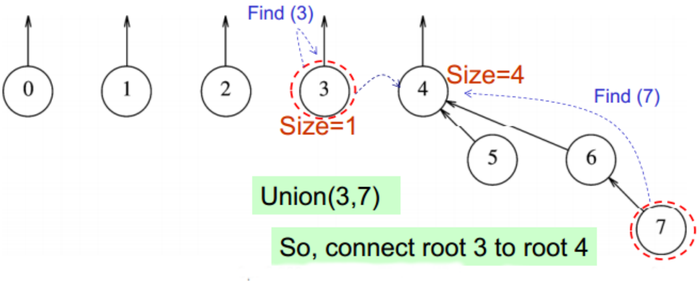
Union-Find: Worst Case[38]



[38] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Candidate Improvements[39]

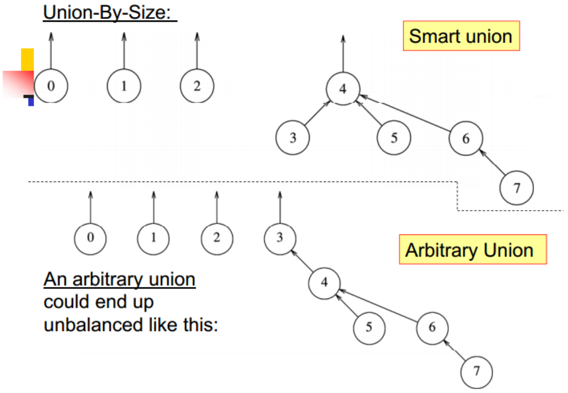
- Improve Union
  - Union by Size
  - Union by Rank (depth)
- Improve Find



[39] http://www.eecs.wsu.edu/~ananth/CptS223/Lectures/UnionFind.pdf

Union-Find: Improved Union[40]

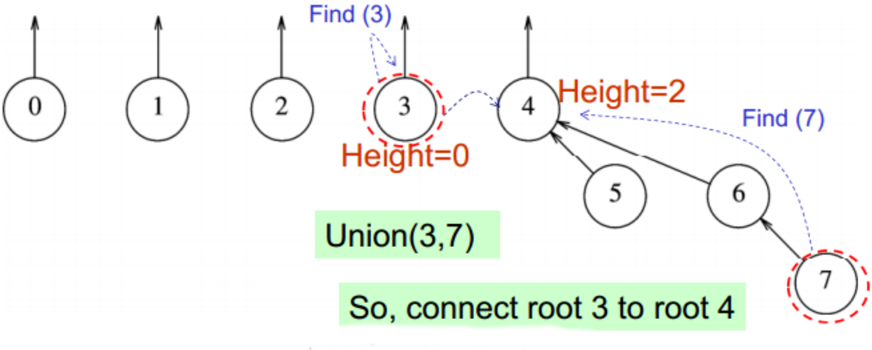
Union by Size (Weighted Union): Always point smaller tree to root of larger tree. Break ties arbitrarily.



[40] http://www.eecs.wsu.edu/~ananth/CptS223/Lectures/UnionFind.pdf

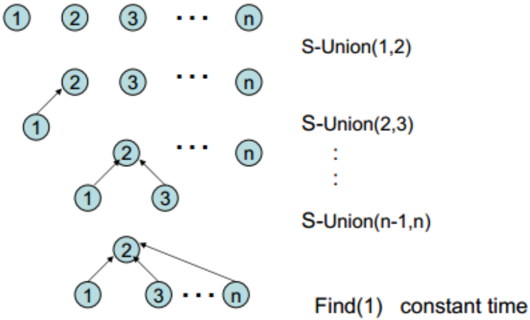
Union-Find: Improved Union[41]

Union by Rank: Always point tree with smaller rank (depth) to root of larger tree. Break ties arbitrarily.



[41] http://www.eecs.wsu.edu/~ananth/CptS223/Lectures/UnionFind.pdf

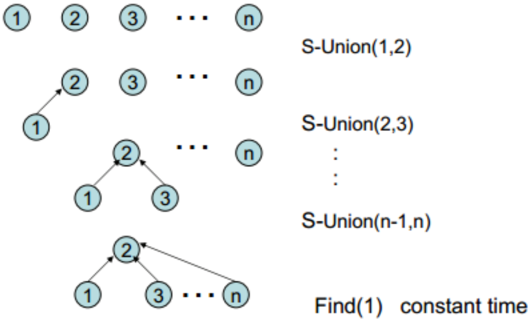
Union by Size: Best Case[42]



Complexity:

[42] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

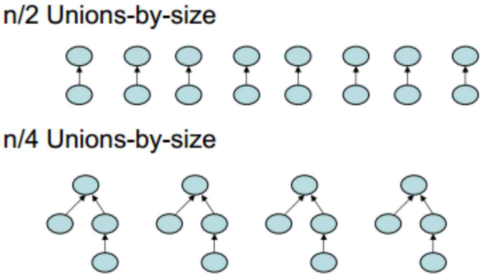
Union by Size: Best Case[42]



Complexity: Union and Find take $O(1)$ time

[42] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

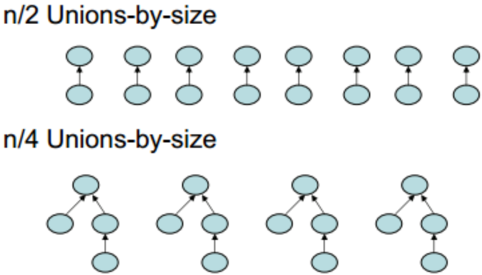
Union by Size: Worst Case[43]



Complexity:

[43] `http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf`

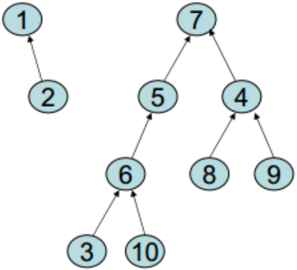
Union by Size: Worst Case[43]



Complexity: Union takes $O(1)$ and Find take $O(\lg n)$ time

[43] `http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf`

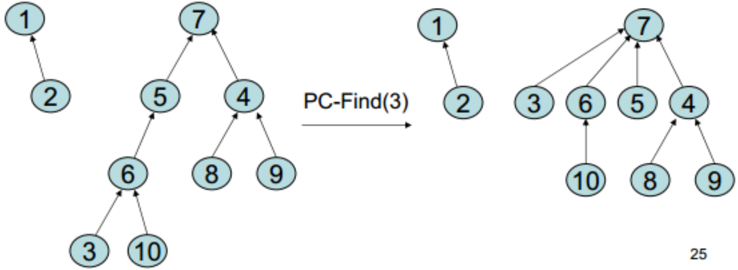
Union-Find: Improved Find[44]



[44] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Improved Find[45]

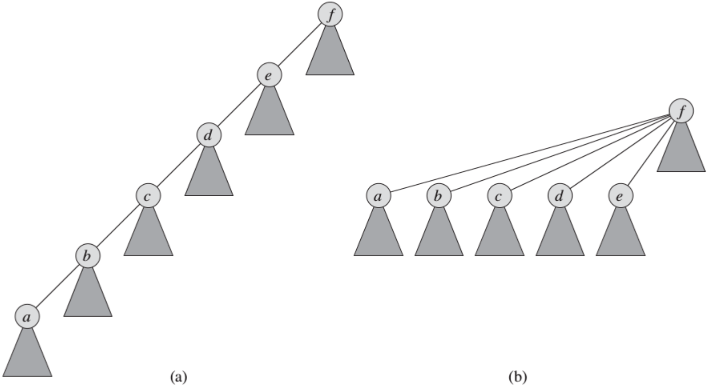
Path Compression: When doing find, point all nodes on search path to root.



[45] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Union-Find: Improved Find[46]

Path Compression:



(a) (b)

[46] CLRS Fig 21.5

Union-Find

- Amortized analysis
- Similarity with Red-Black trees
- Worst Case Analysis of Union-by Size + Path Compression
  - Single Union-by-Size: $O(1)$
  - Single Find with Path Compression: $O(\lg n)$
  - Amortized Complexity for $M \geq N$ operations: $O(m \log^* n)$

Log^* Function[47]

- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 =$

[47] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Log^* Function[47]

- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 = \log^* 2^{2^2} = 3$ (i.e. $\log \log \log 16 = 1$)
- $\log^* 65536 =$

[47] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Log^* Function[47]

- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 = \log^* 2^{2^2} = 3$ (i.e. $\log\log\log 16 = 1$)
- $\log^* 65536 = \log^* 2^{2^{2^2}} = 4$ (i.e. $\log\log\log\log 65536 = 1$)
- $\log^* 2^{65536} =$

[47] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Log^* Function[47]

- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 = \log^* 2^{2^2} = 3$ (i.e. $\log\log\log 16 = 1$)
- $\log^* 65536 = \log^* 2^{2^{2^2}} = 4$ (i.e. $\log\log\log\log 65536 = 1$)
- $\log^* 2^{65536} = \ldots = 5$ (i.e. $\log\log\log\log\log 2^{65536} = 1$)
- In summary for all reasonable n , $\log^* n \leq 5$

[47] http://courses.cs.washington.edu/courses/cse326/08sp/lectures/19-disjoint-union-find-part-2.pdf

Tighter Bound

- Tarjan's tighter bound when $M \geq N$, $\Theta(M\,\alpha(M, N))$
- $\alpha(a, b)$ is the inverse Ackerman function
- It grows even slower than $\log^* n$!

Summary

Major Concepts:

- Binary Heap
- Heapsort
- Disjoint set data structures
- Union-Find