Thompson Sampling for Bandit Convex Optimisation

Alireza Bakhtiari[1]   Tor Lattimore[2]   Csaba Szepesvári[1,2]

University of Alberta[1] & Google DeepMind[2]

COLT 2025

Paper in One Slide

- Goal: Understand how **Thompson Sampling (TS)** behaves in **bandit convex optimization (BCO)**.
- **Positive**:
  - $\mathrm{BReg}_n(\mathrm{TS}) = \widetilde{O}(\sqrt{n})$ in *1-D* convex bandits.
  - $\mathrm{BReg}_n(\mathrm{TS}) = \widetilde{O}\big(d^{2.5}\sqrt{n}\big)$ for *d-D monotone convex ridge* losses.
- **Negative**:
  - TS can suffer $\Omega(\exp(d))$ regret for general d-D convex losses.
  - Classical info-ratio tricks $\Rightarrow$ no better than $\widetilde{O}(d^{1.5}\sqrt{n})$ for general adversarial BCO, through duality.

Problem Setup

- Convex action set $\mathcal{K} \subset \mathbb{R}^d$.
- $\mathcal{F}$ be a set of convex functions from $\mathcal{K}$ to $[0, 1]$.
- ξ is a known prior on $\mathcal{F}$.

[1] The Bernoulli noise assumption is just for convenience.

Problem Setup

- Convex action set $\mathcal{K} \subset \mathbb{R}^d$.
- $\mathcal{F}$ be a set of convex functions from $\mathcal{K}$ to $[0, 1]$.
- ξ is a known prior on $\mathcal{F}$.
- At round t:
  1. Learner plays $X_t \in \mathcal{K}$.
  2. Observes scalar loss $Y_t \in \{0, 1\}$[1].

[1] The Bernoulli noise assumption is just for convenience.

Problem Setup

- Convex action set $\mathcal{K} \subset \mathbb{R}^d$.
- $\mathcal{F}$ be a set of convex functions from $\mathcal{K}$ to $[0, 1]$.
- ξ is a known prior on $\mathcal{F}$.
- At round t:
  1. Learner plays $X_t \in \mathcal{K}$.
  2. Observes scalar loss $Y_t \in \{0, 1\}$[1].
- $\mathbb{E}\left[Y_t | X_1, Y_1, \cdots, X_t, f\right] = f(X_t)$, where $f \in \mathcal{F}$ is convex.

[1] The Bernoulli noise assumption is just for convenience.

Problem Setup

- Convex action set $\mathcal{K} \subset \mathbb{R}^d$.
- $\mathcal{F}$ be a set of convex functions from $\mathcal{K}$ to $[0,1]$.
- ξ is a known prior on $\mathcal{F}$.
- At round t:
  1. Learner plays $X_t \in \mathcal{K}$.
  2. Observes scalar loss $Y_t \in \{0,1\}$[1].
- $\mathbb{E}[Y_t | X_1, Y_1, \cdots, X_t, f] = f(X_t)$, where $f \in \mathcal{F}$ is convex.
- $\mathcal{A}$ (learner) is a mapping from histories & prior to $\mathcal{K}$.

[1] The Bernoulli noise assumption is just for convenience.

Problem Setup

- Convex action set $\mathcal{K} \subset \mathbb{R}^d$.
- $\mathcal{F}$ be a set of convex functions from $\mathcal{K}$ to $[0, 1]$.
- ξ is a known prior on $\mathcal{F}$.
- At round t:
  1. Learner plays $X_t \in \mathcal{K}$.
  2. Observes scalar loss $Y_t \in \{0, 1\}$[1].
- $\mathbb{E}[Y_t | X_1, Y_1, \cdots, X_t, f] = f(X_t)$, where $f \in \mathcal{F}$ is convex.
- $\mathcal{A}$ (learner) is a mapping from histories & prior to $\mathcal{K}$.
- **Bayesian Regret**

$$\mathrm{BReg}_n(\mathcal{A}, \xi) = \mathbb{E}\left[\sup_{x \in \mathcal{K}} \sum_{t=1}^{n} (f(X_t) - f(x))\right],$$

where $f \sim \xi$.

[1] The Bernoulli noise assumption is just for convenience.

Thompson Sampling for BCO

Algorithm 1 Thompson Sampling

Prior ξ over convex losses.
for $t = 1, \ldots, n$ **do**
  Sample $f_t \sim \mathbb{P}(f = \cdot | X_1, Y_1, \cdots, X_{t-1}, Y_{t-1})$
  Play $x_t \in \arg\min_{x \in \mathcal{K}} f_t(x)$ and observe Y_t
end for

- Our analysis studies how

$$\mathrm{BReg}^{\mathrm{TS}}(\mathcal{F}) = \sup_{\xi \in \mathcal{P}(\mathcal{F})} \mathrm{BReg}_n(\mathrm{TS}, \xi)\,,$$

behaves for various natural classes of convex functions $\mathcal{F}$.

Upper bounds - main results

- Positive results for two sets of functions $f : \mathcal{K} \to [0, 1]$:

$\mathcal{F}_{\texttt{bl}}$: $\mathcal{K} \subset \mathbb{R}$ and f is a 1-Lip and bounded convex function,

$\mathcal{F}_{\texttt{blrm}}$: $\mathcal{K} \subset \mathbb{R}^d$ and $f(x) = \ell(\langle x, \theta \rangle)$ for some $\theta \in \mathbb{R}^d$ and a convex 1-Lip bounded *monotone* $\ell : \mathbb{R} \to \mathbb{R}$.

Upper bounds – 1-D convex functions

TS achieves

$$\mathrm{BReg}^{\mathrm{TS}}(\mathcal{F}_{\texttt{bl}}) \;=\; \widetilde{O}(\sqrt{n}) \,,$$

And

$$\mathrm{BReg}^{\mathrm{TS}}(\mathcal{F}_{\texttt{blrm}}) \;=\; \widetilde{O}(d^{2.5}\sqrt{n}) \,.$$

Upper bounds – analysis

(Generalized) **Information Ratio (IR)**

- Let $x_g = \text{argmin}_{x \in \mathcal{K}}\, g(x)$.
- Let (X, f) have law $\pi \otimes \xi$, and $\bar{f} = \mathbb{E}[f]$.
- Define

$$\Delta(\pi, \xi) = \mathbb{E}[\bar{f}(X) - f_\star] \quad \text{and} \quad \mathcal{I}(\pi, \xi) = \mathbb{E}[(f(X) - \bar{f}(X))^2]\,.$$

Generalized IR

Define generalized information ratio associated with TS as

$$\text{IR}(\mathcal{F}) = \left\{(\alpha, \beta) \in \mathbb{R}_+^2 : \Delta(\pi_{\text{TS}}, \xi) \le \alpha + \sqrt{\beta \mathcal{I}(\pi_{\text{TS}}, \xi)}\,, \forall \xi\right\}\,.$$

- $(0, \beta) \in \text{IR}(\mathcal{F}) \Rightarrow \Delta(\pi_{\text{TS}}, \xi)^2 / \mathcal{I}(\pi_{\text{TS}}, \xi) \le \beta$.

Upper bounds – regret & IR

Proposition 1 – regret bound through IR

Suppose that $\mathcal{F} \in \{\mathcal{F}_{\texttt{bl}}, \mathcal{F}_{\texttt{blrm}}\}$ and $(\alpha, \beta) \in \mathrm{IR}(\mathcal{F})$. Then, the regret of TS is at most

$$\mathrm{BReg}_n(\mathrm{TS}, \xi) \leq n\alpha + O\left(\sqrt{\beta n d \log(n \operatorname{diam}(\mathcal{K}))}\right) .$$

Upper bounds – regret & IR

Proposition 1 – regret bound through IR

Suppose that $\mathcal{F} \in \{\mathcal{F}_{\texttt{bl}}, \mathcal{F}_{\texttt{blrm}}\}$ and $(\alpha, \beta) \in \mathrm{IR}(\mathcal{F})$. Then, the regret of TS is at most

$$\mathrm{BReg}_n(\mathrm{TS}, \xi) \leq n\alpha + O\left(\sqrt{\beta n d \log(n \,\mathrm{diam}(\mathcal{K}))}\right) .$$

It remains to bound the IR of TS.

Upper bounds – regret & IR

Proposition 1 – regret bound through IR

Suppose that $\mathcal{F} \in \{\mathcal{F}_{\texttt{bl}}, \mathcal{F}_{\texttt{blrm}}\}$ and $(\alpha, \beta) \in \text{IR}(\mathcal{F})$. Then, the regret of TS is at most

$$\mathsf{BReg}_n(\text{TS}, \xi) \leq n\alpha + O\left(\sqrt{\beta n d \log(n \operatorname{diam}(\mathcal{K}))}\right) .$$

Proposition 2 – TS-IR Decomposition

If $\exists k, m \in \mathbb{N}$ such that for all $\bar{f} \in \operatorname{conv}(\mathcal{F})$ there is $\mathcal{F} = \cup_{i=1}^{m} \mathcal{F}_i$ for which

$$\sup_{f \in \mathcal{F}_i} (\bar{f}(x_f) - f(x_f)) \leq \alpha + \sqrt{\beta \underbrace{\inf_{f_1, \dots, f_k \in \mathcal{F}_i} \sum_{j \neq l \in [k]} (f_j(x_{f_l}) - \bar{f}(x_{f_l}))^2}_{\text{DIS}(\mathcal{F}_i, k)}},$$

for all i, then $(\alpha, k(k-1)m\beta) \in \text{IR}(\mathcal{F})$.

Upper bounds – regret & IR

Proposition 1 – regret bound through IR

Suppose that $\mathcal{F} \in \{\mathcal{F}_{\texttt{bl}}, \mathcal{F}_{\texttt{blrm}}\}$ and $(\alpha, \beta) \in \mathrm{IR}(\mathcal{F})$. Then, the regret of TS is at most

$$\mathrm{BReg}_n(\mathrm{TS}, \xi) \leq n\alpha + O\left(\sqrt{\beta n d \log(n \operatorname{diam}(\mathcal{K}))}\right).$$

Proposition 2 – TS-IR Decomposition

If $\exists k, m \in \mathbb{N}$ such that for all $\bar{f} \in \operatorname{conv}(\mathcal{F})$ there is $\mathcal{F} = \cup_{i=1}^{m} \mathcal{F}_i$ for which

$$\boxed{\sup_{f \in \mathcal{F}_i} (\bar{f}(x_f) - f(x_f))} \leq \alpha + \sqrt{\beta \underbrace{\inf_{f_1, \ldots, f_k \in \mathcal{F}_i} \sum_{j \neq l \in [k]} (f_j(x_{f_l}) - \bar{f}(x_{f_l}))^2}_{\mathrm{DIS}(\mathcal{F}_i, k)}},$$

worst case *average* regret within $\mathcal{F}_i$

for all i, then $(\alpha, k(k-1)m\beta) \in \mathrm{IR}(\mathcal{F})$.

Upper bounds – regret & IR

Proposition 1 – regret bound through IR

Suppose that $\mathcal{F} \in \{\mathcal{F}_{\mathtt{bl}}, \mathcal{F}_{\mathtt{blrm}}\}$ and $(\alpha, \beta) \in \textsc{ir}(\mathcal{F})$. Then, the regret of $\textsc{ts}$ is at most

$$\mathrm{BReg}_n(\textsc{ts}, \xi) \leq n\alpha + O\left(\sqrt{\beta n d \log(n \operatorname{diam}(\mathcal{K}))}\right) .$$

Proposition 2 – $\textsc{ts}$-IR Decomposition

If $\exists k, m \in \mathbb{N}$ such that for all $\bar{f} \in \operatorname{conv}(\mathcal{F})$ there is $\mathcal{F} = \cup_{i=1}^{m} \mathcal{F}_i$ for which

$$\boxed{\sup_{f \in \mathcal{F}_i} (\bar{f}(x_f) - f(x_f))} \leq \alpha + \sqrt{\beta \boxed{\underbrace{\inf_{f_1,\ldots,f_k \in \mathcal{F}_i} \sum_{j \neq l \in [k]} (f_j(x_{f_l}) - \bar{f}(x_{f_l}))^2}_{\textsc{dis}(\mathcal{F}_i, k)}}},$$

worst case *average* regret within $\mathcal{F}_i$

$\textsc{ts}$ minimum inf. gain (discrepancy)

for all i, then $(\alpha, k(k-1)m\beta) \in \textsc{ir}(\mathcal{F})$.

Upper bounds – IR bound in 1-D

Theorem 3 – IR bound for 1-D

Suppose that $d = 1$ and $\alpha \in (0, 1)$. Then $(\alpha, 10^4 \lceil \log(1/\alpha) \rceil) \in \mathrm{IR}(\mathcal{F}_{\mathrm{bl}})$.

Proof sketch:

- Use the decomposition lemma $\Rightarrow$ partition based on x_f and $f(x_f)$.
- Analyze functions with similar gaps (average regret) in $\mathcal{F}_i$:

  Similar gaps: $\bar{f}(x_f) - f(x_f) \in [\epsilon, 2\epsilon]$,
  $\bar{f}$ monotone: $x_{\bar{f}} \le x_f \Rightarrow \bar{f}$ (begin 1-D is crucial here).

- Prove that $\mathrm{DIS}(\mathcal{F}_i, k) \ge c^2 \epsilon_i^2$

$$\inf_{f_1, \ldots, f_k \in \mathcal{F}_i} \sum_{j \ne l \in [k]} (f_j(x_{f_l}) - \bar{f}(x_{f_l}))^2 \ge c^2 \epsilon_i^2 \, .$$

Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\inf_{f_1, f_2, f_3, f_4 \in \mathcal{F}_i} \sum_{j \neq l \in [4]} (f_j(x_{f_l}) - \bar{f}(x_{f_l}))^2 \leq c^2 \epsilon_i^2$$

$$\Rightarrow \forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \;\; f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \,.$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$

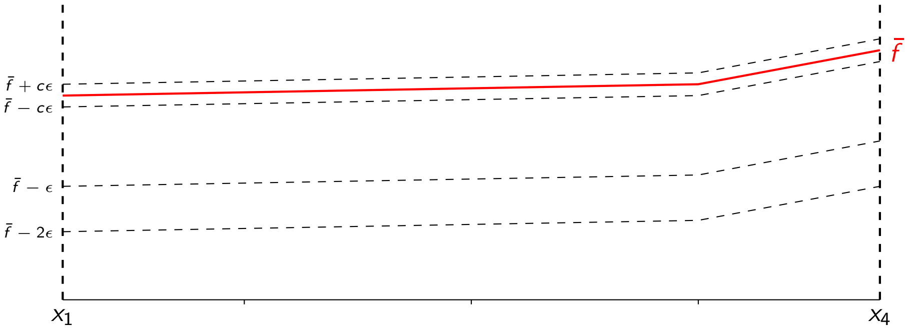
Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \;\; f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \,.$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$



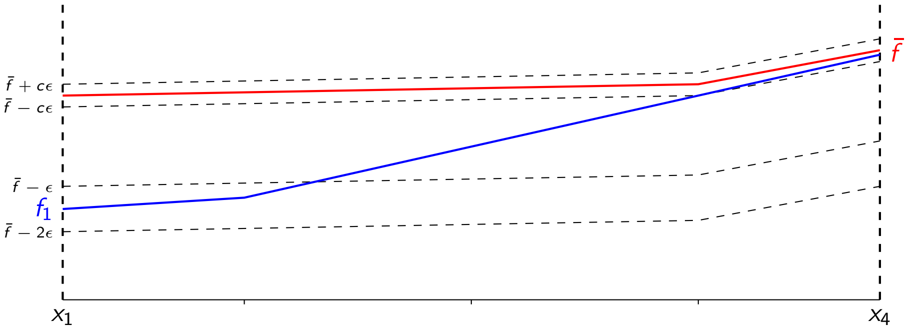
Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \ \ f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \, .$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$



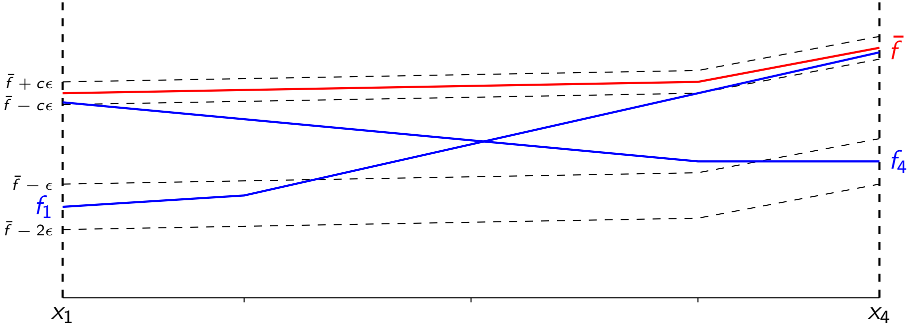
Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \;\; f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \, .$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$



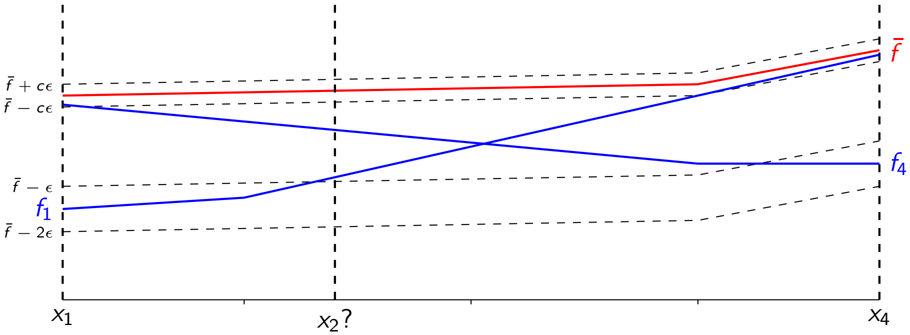
Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \;\; f_j(x_{f_l}) - \bar{f}(x_{f_l}) \le c\epsilon_i \,.$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \le x_2 \le x_3 \le x_4$



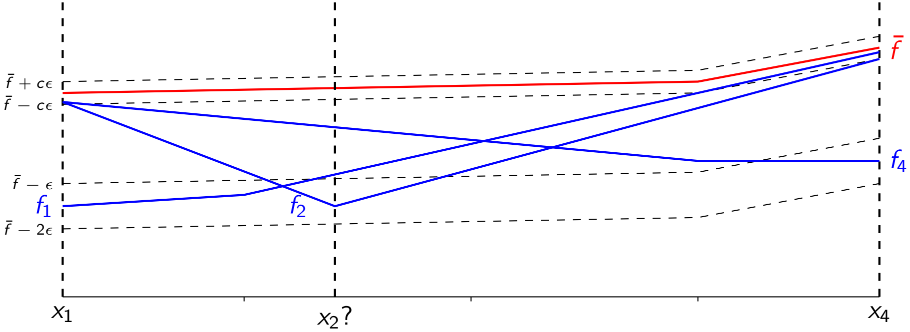
Upper bounds – IR bound in 1-D

Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \quad f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \,.$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$

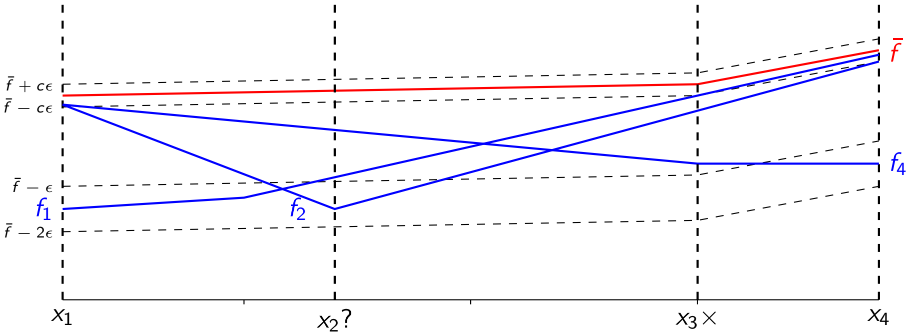


Proof sketch:(proof by contradiction)

- Fix $k = 4, c \approx 1/16$, suppose that

$$\forall f_1, f_2, f_3, f_4 \in \mathcal{F}_i, \;\; f_j(x_{f_l}) - \bar{f}(x_{f_l}) \leq c\epsilon_i \,.$$

- WLOG, suppose that $x_{\bar{f}}, x_1 \leq x_2 \leq x_3 \leq x_4$

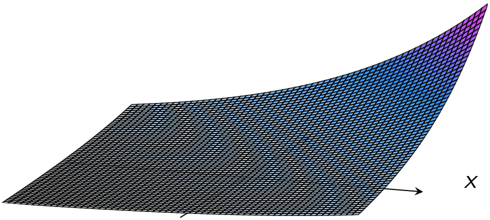


Upper bounds – IR bound in d-D

Theorem 7 – IR bound in d-D

$(\alpha, \beta \lceil \log(1/\alpha) \rceil) \in \mathrm{IR}(\mathcal{F}_{\mathtt{blrm}})$ whenever $\alpha \in (0, 1)$ and

$$\mathrm{BReg}^{\mathrm{TS}}(\mathcal{F}_{\mathtt{blrm}}) = \widetilde{O}\big(d^{2.5}\sqrt{n}\big)\,.$$



$f(x, y) = e^{x+y}$ — a convex ridge function with $\theta = (1, 1)$ and $\ell(t) = e^t$.

Lower bounds – main results

The following lower bounds share the same construction.

Lower Bound 4

When $\mathcal{K} = \mathbb{B}_1 \subset \mathbb{R}^d$, there exists a prior ξ on $\mathcal{F}_{\mathtt{bl}}$ such that

$$\mathrm{BReg}_n(\mathrm{TS}) \;\geq\; \tfrac{1}{2}\min\Big\{n,\; \frac{1}{4}e^{\Omega(d/32)}\Big\}.$$

Lower Bound 2

Suppose that $\mathcal{K} = \mathbb{B}_1 \subset \mathbb{R}^d$ and $d > 256$. Then there exists a prior ξ on $\mathcal{F}_{\mathtt{bl}}$ such that

$$\Delta(\pi, \xi) \geq 2^{-19} \frac{d}{\log(d)} \sqrt{\mathcal{I}(\pi, \xi)}\,,$$

for all policies π on K.

Lower bounds – construction

- $f(x) = \epsilon + \frac{1}{2}\left\|x\right\|^2$, a simple bowl shape function in $\mathbb{B}_1$,
- f_θ which is the largest convex function g that is smaller than f, and $g(\theta) = 0$.
- ξ is the *uniform* prior over f_θ. ($\theta \sim \text{unif}(\mathbb{S}_1)$)
- TS lower bound:
    1. The "perimeter" of these valleys gets smaller exponentially with d.
    2. TS only plays on the surface where the minimizers are.
    3. TS will suffer $O(\exp(d))$ regret.
- IR lower bound:
    1. Both δ and $\mathcal{I}$ of a point $x \in \mathbb{B}_1$ are a function of its radius r.
    2. Optimize for r.

Paper in One Slide

- Goal: Understand how **Thompson Sampling (TS)** behaves in **bandit convex optimization (BCO)**.
- **Positive**:
  - $\mathrm{BReg}_n(\mathrm{TS}) = \widetilde{O}(\sqrt{n})$ in *1-D* convex bandits.
  - $\mathrm{BReg}_n(\mathrm{TS}) = \widetilde{O}\big(d^{2.5}\sqrt{n}\big)$ for *d-D monotone convex ridge* losses.
- **Negative**:
  - TS can suffer $\Omega(\exp(d))$ regret for general d-D convex losses.
  - Classical info-ratio tricks $\Rightarrow$ no better than $\widetilde{O}(d^{1.5}\sqrt{n})$ for general adversarial BCO.

Thanks!