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Generalized synchronization on the onset of auxiliary system approach

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ABSTRACT

Generalized synchronization is an emergent functional relationship between the states of the interacting dynamical systems. To analyze the stability of a generalized synchronization state, the auxiliary system technique is a seminal approach that is broadly used nowadays. However, a few controversies have recently arisen concerning the applicability of this method. In this study, we systematically analyze the applicability of the auxiliary system approach for various coupling configurations. We analytically derive the auxiliary system approach for a drive–response coupling configuration from the definition of the generalized synchronization state. Numerically, we show that this technique is not always applicable for two bidirectionally coupled systems. Finally, we analytically derive the inapplicability of this approach for the network of coupled oscillators and also numerically verify it with an appropriate example.

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For generalized synchronization, the functional dependence among the oscillators may have a complicated mathematical form even for simple coupled systems. It is still an unsolved problem to recognize the functional relationship from the time correlation of the state variables. Researchers used the auxiliary system approach as an indicator for the presence of such functional relationships among the interacting nodes, although this approach is unable to retrieve the exact correlation. This method was first proposed for drive–response coupled oscillators; later on, it then progressed to bidirectionally coupled oscillators and coupled networks. However, this method lacks a rigorous mathematical justification for its applicability in coupled systems. In this work, we theoretically analyze the precise applicability of this approach to detect generalized synchronization in coupled systems.

I. INTRODUCTION

The study of complex dynamical networks has become an interesting interdisciplinary research topic in the past few decades because of their potential real-world applications.[1] Despite the dynamics of each individual oscillator, different varieties of collective behavior emerge when they interact with each other. Among them, synchronization is a specific kind of emergent phenomenon that occurs due to the adjustment of their rhythm as a result of their interactions.[2,3] In 1665, Christiaan Huygens first observed this phenomenon as a form of conformity among the coupled pendulums. It is one of the classical phenomena, and researchers are still now investigating various natural systems in addition to man-made complex systems.[4–13]

Generalized synchronization (GS) is a robust form of synchronization in which a static functional relationship exists among the states of the interacting nodes.[14] It was first introduced for two coupled systems with a drive–response coupling configuration.[14] Evidence of GS was also pronounced in various numerical[15,16] and experimental[17–21] setups. Complete synchronization (CS) is a unique case of GS where the functional relationship is an identity function. However, CS is an ideal situation that takes place if the interacting oscillators are identical and either the coupling is bidirectional or the coupling function vanishes after reaching the CS state. The GS state is of significant importance owing to its robustness with respect to non-identical systems and any arbitrary coupling functions. Even if the drive and response are completely different systems, GS may occur.[22]

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The universal mechanism of the GS state in chaotic oscillators with the dissipative coupling has been described in Ref. 23. Here, the reasons behind the occurrence of this state have been clarified by means of a modified system approach. In Ref. 24, the authors investigated the GS behavior for mutually coupled and networking systems featuring complex topologies. Here, the GS regime is detected by the system's Lyapunov exponents, and the emergence of the GS state is verified by means of the nearest neighbor method. A novel synchronization scheme called complex generalized synchronization is introduced in Ref. 25 by extending the GS state from real space to complex space. Based on the Lyapunov stability theory, an adaptive controller and parameter update laws are designed to realize the complex generalized synchronization state. The approximate functional relationship of the GS state can be predicted by a generative statistical modeling method[26] using the truncated Volterra series. For the drive–response coupled system, a control law is designed to achieve the generalized synchronization of chaotic systems[27] by adopting the nonlinear control theory and based on the Lyapunov stability theory.

However, GS is a complicated fundamental phenomenon of nonlinear dynamics since the functional relationship is not recognizable in general. It is still in infancy to pick out its emergence and examine its stability in coupled systems. Such GS state up to now neither right way recognizable by means of a visual inspection from the time correlation of the state variables nor to decide the precise functional relationship among the oscillators. To get rid of this difficulty, an auxiliary system approach (ASA)[28] was proposed for the drive–response coupled system as an indirect manner to confirm the existence of this GS state. Later on, without proper mathematical justification, this method is extended for bidirectionally two coupled systems[29] and complex networks[30] by introducing a system consisting of an auxiliary node for each original node. Then, it is also extended to scale-free[31] and small-world networks.[32] Since its discovery, this technique has been extensively applied to determine the occurrence of GS in bidirectional complex networks.[33–35]

Two recent works[36,37] aroused controversies regarding the applicability of this approach for bidirectionally coupled systems. These results shed new light on the question of the applicability of ASA in coupled systems. These two papers also encourage us to investigate properly on the direction of whether the ASA approach is applicable irrespective of the coupling configuration. This method lacks a rigorous theoretical basis for its applicability in bidirectional complex networks and a network of coupled oscillators.

In this paper, we investigate the applicability of ASA to detect the GS state in coupled dynamical systems. We analytically derive the local and global stability analyses of the GS state in unidirectionally coupled systems and numerically verify the results using the Hindmarsh–Rose (HR) neuronal model. The onset of the GS regime in unidirectionally coupled systems is also confirmed by the calculation of conditional Lyapunov exponents. For bidirectionally coupled oscillators, we use the transverse Lyapunov exponent to show the inapplicability of the GS state. Finally, we analytically derive the inapplicability of this approach in a network of coupled oscillators. This analytical result is numerically demonstrated using Lorenz systems with directed and undirected networks.

II. DRIVE–RESPONSE COUPLING CONFIGURATIONS

Consider a coupled system with a drive–response coupling configuration as follows:

$$\dot{\mathbf{x}} = F_1(\mathbf{x}), \tag{1a}$$

$$\dot{\mathbf{y}} = F_2(\mathbf{y}) + \varepsilon H(\mathbf{x} - \mathbf{y}). \tag{1b}$$

Here, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ are the respective state variables of the drive and response systems with respective isolate evolution functions F_1 and F_2. H is the inner coupling matrix of order $d \times d$, and ε is the coupling parameter.

Definition 1. *The drive–response system (1) is said to be in the GS state if there exists a transformation $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, a subset $\mathscr{B} \subseteq \mathbb{R}^d \times \mathbb{R}^d$, and a manifold $\mathcal{S} = \left\{(\boldsymbol{x},\boldsymbol{y}) \subseteq \mathbb{R}^d \times \mathbb{R}^d : \Phi(\boldsymbol{x},\boldsymbol{y}) = 0\right\} \subseteq \mathscr{B}$ such that the trajectory of (1) with the initial condition from $\mathscr{B}$ approaches to $\mathcal{S}$ as t tends to ∞.*

Here, $\mathcal{S}$ is mathematically said to be the GS manifold, and the subset $\mathscr{B}$ is the basin of attraction of the GS state. The following lemma gives the necessary and sufficient condition for the emergence of the GS state of Eq. (1), and its proof can be found in Ref. 22.

Lemma 1. *The GS state occurs in the unidirectionally coupled system (1) if and only if the drive system (1b) is asymptotically stable. In other words, for any two arbitrary initial conditions $(\boldsymbol{x}_0, \boldsymbol{y}_{10})$ and $(\boldsymbol{x}_0, \boldsymbol{y}_{20})$ in $\mathscr{B}$, $\left\|\boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{y}_{10}) - \boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{y}_{20})\right\| \to 0$ as $t \to \infty$, where $\|\cdot\|$ denotes the Euclidean 2-norm.*

By this lemma (1), the equation of motion of its auxiliary counterpart[28] is described by

$$\dot{\mathbf{z}} = F_2(\mathbf{z}) + \varepsilon H(\mathbf{x} - \mathbf{z}), \tag{2}$$

where $\mathbf{z} \in \mathbb{R}^d$ is the state variable of the auxiliary system. As per the principle of ASA, the coupled system (1) is supposed to achieve the GS state if $\left\|\mathbf{y}(t) - \mathbf{z}(t)\right\| \to 0$ as $t \to \infty$. Here, the applicability of the ASA to detect the emergence of GS is based on the following theorem.

Theorem 1. *The response system (1b) is asymptotically stable if and only if the response system (1b) and the auxiliary system (2) asymptotically converge to each other.*

Proof. Note that the response system (1b) and the auxiliary system (2) are exactly identical; i.e., both exhibit identical solutions if they start from the same initial condition. Here, $\mathbf{z}$ plays a role as a dummy variable of $\mathbf{y}$.

Now assume that the response system is asymptotically stable. Then, for any two arbitrary initial conditions $(\mathbf{x}_0, \mathbf{y}_{10})$ and $(\mathbf{x}_0, \mathbf{y}_{20})$ from $\mathscr{B}$,

$$\left\|\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{10}) - \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{20})\right\| \to 0 \text{ as } t \to \infty. \tag{3}$$

For the initial condition $(\mathbf{x}_0, \mathbf{y}_{20}) \in \mathscr{B}$, the auxiliary system (2) also exhibits the solution $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_{20})$. Therefore, (3) immediately yields that Eqs. (1b) and (2) asymptotically converge to each other.

Conversely, assume that the response system (1b) and the auxiliary system (2) asymptotically converge to each other. If the initial condition of the entire systems (1) and (2) is $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, then $\left\|\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0) - \mathbf{z}(t, \mathbf{x}_0, \mathbf{z}_0)\right\| \to 0$ as $t \to \infty$. Thus, for the initial condition $(\mathbf{x}_0, \mathbf{z}_0)$, the response system also exhibits the solution

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$\mathbf{z}(t, \mathbf{x}_0, \mathbf{z}_0)$. Hence, it is asymptotically stable. This completes the proof. □

Using Lemma 1 and Theorem 1, we can conclude that the ASA can accurately detect the emergence of the generalized synchronization state for the drive–response system (1). It is an effective and simple mathematical method that is quite prevalent to detect the presence of a functional relationship among drive–response configurations. The stability properties of the associated GS manifold turned out to be the convergence of the response and auxiliary systems toward each other. The GS state is also emerged by designing a suitable coupling function based on the pre-assumed functional relationship between the interacting nodes.[38,39]

A. Local and global stability conditions

To study the GS state in the drive–response coupling configuration, a typical problem is to find conditions that guarantee the convergence of the solution toward the GS solution. As the ASA can detect the occurrence of the GS state for the drive–response coupling configuration, we now derive the local and global stability conditions of the GS state using ASA.

Theorem 2. *If the nodal dynamics F_2 of the response system (1b) is continuously differentiable with respect to its argument, then the transverse component $\delta\mathbf{z}(t)$ along the GS manifold satisfies the equation of motion $\delta\dot{\mathbf{z}} = [JF_2(\mathbf{y}) - \varepsilon H]\delta\mathbf{z}$, where $JF_2(\mathbf{y})$ denotes the Jacobian matrix of the isolate vector field F_2.*

Proof. Using ASA, the GS state of system (1) is said to be stable if the response system (1b) and the corresponding auxiliary system (2) evolve in unison.

Now, perturb the auxiliary system from the response system $\mathbf{y}(t)$ with a small value of $\delta\mathbf{z}(t)$. Then, the present state of the auxiliary system becomes $\mathbf{z}(t) = \mathbf{y}(t) + \delta\mathbf{z}(t)$. The perturbation term $\delta\mathbf{z}(t)$ dominates the equation of motion as

$$\delta\dot{\mathbf{z}}(t) = \dot{\mathbf{z}}(t) - \dot{\mathbf{y}}(t) = F_2(\mathbf{z}) - F_2(\mathbf{y}) - \varepsilon H(\mathbf{z} - \mathbf{y}). \quad (4)$$

Considering the term $F_2(\mathbf{z}) - F_2(\mathbf{y})$ with the Taylor series expansion near the GS manifold, we have $F_2(\mathbf{z}) - F_2(\mathbf{y}) = JF_2(\mathbf{y})\delta\mathbf{z} + R(\delta\mathbf{z})$. Here, $R : \mathbb{R}^d \to \mathbb{R}^d$ stands for the Taylor series remainder of the expansion of the vector field F_2, which satisfies $\|R(\delta\mathbf{z})\| = \mathcal{O}(\|\delta\mathbf{z}\|^2)$. As we are considering the local stability of the GS solution, we regard $\|R(\delta\mathbf{z})\|$ as being so small that we can neglect it.

Therefore, considering small perturbation $\delta\mathbf{z}(t)$, the perturbed variable $\delta\mathbf{z}$ satisfies the linearized evolution equation,

$$\delta\dot{\mathbf{z}} = [JF_2(\mathbf{y}) - \varepsilon H]\delta\mathbf{z}. \quad (5)$$

The GS solution of system (1) is locally stable whenever the maximum Lyapunov exponent of Eq. (5) turns out to be negative by tuning the interaction strength ε. □

We can measure the exponential contraction or expansion of the linearized variational equation by calculating its Lyapunov exponents. For the d-dimensional system (5), its d number of Lyapunov exponents is determined with respect to the reference trajectory $[\mathbf{x}(t), \mathbf{y}(t)]$ with the initial condition $\delta\mathbf{z}(0)$. Let $\mathbf{v}_i(0)$ be the orthonormal vector of $\delta\mathbf{z}(0)$ for $i = 1, 2, \ldots, d$. Then, the Lyapunov exponents are defined as $\Lambda_i = \lim_{t\to\infty} \frac{1}{t} \|\delta\mathbf{z}(t)\mathbf{v}_i(0)\|$. Among all the Lyapunov exponents of Eq. (5), the maximum one (say, Λ_{GS}) as a function of ε plays a key role in the stability analysis of the GS state. If the error dynamics evolves chaotically, then Λ_{GS} is greater than zero. However, if the error system (5) stabilizes to its trivial fixed point, then Λ_{GS} becomes negative. By adjusting the coupling parameter ε, we can trace out the synchronization region where the value of Λ_{GS} is negative. Such a linearization-based approach has theoretical and practical applications to determine whether the GS state is stable or not in terms of the sign of the maximum Lyapunov exponent Λ_{GS}.

Now, we will derive the global stability condition of the GS state by taking a few assumptions on the isolate node dynamics and the inner coupling matrix. The following theorem illustrates it in detail.

Theorem 3. *For the given drive–response coupled system (1), if the vector field F_2 of the response system is a Lipschitz function with the Lipschitz constant M, and the inner coupling matrix H is symmetric positive definite with a minimum eigenvalue $\lambda_{\min}[H]$. Then, the GS state is globally asymptotically stable for $\varepsilon > \frac{M}{\lambda_{\min}[H]}$.*

Proof. To establish the global stability of the GS state, define an error quantity between response and auxiliary systems as $\mathbf{e} = \mathbf{y} - \mathbf{z}$. This error system obeys the evolution equation,

$$\dot{\mathbf{e}} = F_2(\mathbf{y}) - F_2(\mathbf{z}) - \varepsilon H\mathbf{e}. \quad (6)$$

Then, consider a Lyapunov function as $V(t) = \frac{1}{2}\mathbf{e}^{tr}\mathbf{e}$, where tr denotes the transpose of a vector. The time derivative of $V(t)$ along the trajectory of systems (1) and (2) can be written as

$$\dot{V}(t) = \mathbf{e}^{tr}\dot{\mathbf{e}} = \mathbf{e}^{tr}[F_2(\mathbf{y}) - F_2(\mathbf{z}) - \varepsilon H\mathbf{e}]. \quad (7)$$

By hypothesis, if F_2 is Lipschitz, then there exists a positive constant M such that for all $\mathbf{y}, \mathbf{z} \in \mathbb{R}^d$,

$$\|F_2(\mathbf{y}) - F_2(\mathbf{z})\| \le M\|\mathbf{y} - \mathbf{z}\|. \quad (8)$$

From the linear algebra and using the Cauchy–Schwartz inequality, for all $\mathbf{y}, \mathbf{z} \in \mathbb{R}^d$, we can write

$$(\mathbf{y} - \mathbf{z})^{tr}[F_2(\mathbf{y}) - F_2(\mathbf{z})] \le \|\mathbf{y} - \mathbf{z}\|\,\|F_2(\mathbf{y}) - F_2(\mathbf{z})\|. \quad (9)$$

By virtue of the Lipschitz condition of F_2, we then have

$$(\mathbf{y} - \mathbf{z})^{tr}[F_2(\mathbf{y}) - F_2(\mathbf{z})] \le M(\mathbf{y} - \mathbf{z})^{tr}(\mathbf{y} - \mathbf{z}). \quad (10)$$

Now, the symmetric and positive definiteness of the matrix H implies[40]

$$\lambda_{\min}[H]\mathbf{e}^{tr}\mathbf{e} \le \mathbf{e}^{tr}H\mathbf{e}. \quad (11)$$

Using the inequalities (10) and (11) in Eq. (7), we obtain

$$\dot{V}(t) \le \left[M - \varepsilon\lambda_{\min}[H]\right]\mathbf{e}^{tr}H\mathbf{e}. \quad (12)$$

Therefore, by the Lyapunov stability theory,[41,42] the error system (6) is globally stable for $\varepsilon > \frac{M}{\lambda_{\min}[H]}$. Consequently, the GS state of system (1) also becomes globally stable. □

The above theorem guarantees the global convergence of the coupled oscillators (1) to the generalized synchronization manifold $\mathcal{S}$ irrespective of initial conditions. Theorem 3 also tells us about the persistence of the GS state for larger coupling strength in the drive–response coupling configurations.

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B. Numerical illustration for the drive–response coupling configuration

We now numerically illustrate the appearance of the GS state for two unidirectionally coupled HR oscillators through a chemical synaptic interaction. The study of GS in this coupling configuration has not yet been explored to the best of our knowledge. Therefore, systematic studies on such unnoticed phenomena deserve special attention. The evolution equation of this drive–response configuration can be written as

$$\begin{aligned}
\dot{x}_1 &= y_1 + (b - ax_1)x_1^2 - z_1 + I,\\
\dot{y}_1 &= c - dx_1^2 - y_1,\\
\dot{z}_1 &= r[s(x_1 - x_0) - z_1],\\
\dot{x}_2 &= y_2 + (b - ax_2)x_2^2 - z_2 + I + g_c\frac{(v_s - x_2)}{1 + \exp[-\lambda(x_1 - \theta_s)]},\\
\dot{y}_2 &= c - dx_2^2 - y_2,\\
\dot{z}_2 &= r[s(x_2 - x_0) - z_2].
\end{aligned} \tag{13}$$

We consider the values of the system parameters as $a = 1, b = 3, c = 1, d = 5, r = 0.005, s = 4, x_0 = -1.6$, and $I = 3.25$ for which the isolate system exhibits multi-time scale chaotic behavior. The coupling parameters are fixed at $v_s = 2, \theta_s = -0.25$, and $\lambda = 10$. The collective behavior of this coupled system is characterized by its Lyapunov spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_6$.

Since the coupling configuration is unidirectional and the coupling term does not vanish after achieving the CS state, CS is not an invariant solution. However, we find that the system possesses the GS state for a suitable value of g_c. Now, consider (x_2', y_2', z_2') as the state variable of the corresponding auxiliary system,

$$\begin{aligned}
\dot{x}'_2 &= y_2' + (b - ax_2')(x_2')^2 - z_2' + I + g_c\frac{(v_s - x_2')}{1 + \exp[-\lambda(x_1 - \theta_s)]},\\
\dot{y}'_2 &= c - d(x_2')^2 - y_2',\\
\dot{z}'_2 &= r[s(x_2' - x_0) - z_2'].
\end{aligned} \tag{14}$$

Then, the error term transverse to the GS state can be defined as $e_1 = x_2' - x_2, e_2 = y_2' - y_2$, and $e_3 = z_2' - z_2$. This error term dominates the equation of motion,

$$\begin{aligned}
\dot{e}_1 &= e_2 + (2b - 3ax_2)x_2e_1 - e_3 - g_c\frac{1}{1 + \exp[-\lambda(x_1 - \theta_s)]}e_1,\\
\dot{e}_2 &= -2dx_2e_1 - e_2,\\
\dot{e}_3 &= r[se_1 - e_3].
\end{aligned} \tag{15}$$

The asymptotic stability of this error term indicates the occurrence of the GS state among system (13). Thus, the maximum Lyapunov exponent Λ_{GS} of Eq. (15) plays a role as an indicator of this state. If $\Lambda_{GS} \geq 0$, the GS state is not stable, while $\Lambda_{GS} < 0$ gives the asymptotic stability of the GS state.

Figure 1(a) depicts the projection of the generalized synchronization manifold in the (x_1, x_2) phase space for the synaptic strength $g_c = 5.0$. In the inset, trajectories of the state variables x_1 and x_2 are plotted with time by red and green lines, respectively. Here, the drive and response systems are not completely synchronized, but a smooth functional relationship exists among themselves. The variations of Λ_{GS} with respect to g_c are presented in Fig. 1(b) for three different values of I. The blue circle curve is for $I = 3.25$ for which GS occurs from $g_c = 3.6$. The red square and green triangle curves are, respectively, for $I = 5$ and $I = 7$. For these two cases, the critical values of g_c are, respectively, 3.2 and 2.2. Thus, by increasing I, GS monotonically enhances with respect to g_c. For a more detailed investigation of this enhancement tendency, we plot the GS region in the (g_c, I) parameter space in Fig. 1(c), where the color bar depicts the variation of Λ_{GS}. Here, monotonic enhancement tendency is also observed with respect to both g_c and I.

The above ASA to detect GS is not applicable when the mathematical form of the dynamical system is unknown. Also, in an experiment, the construction of an auxiliary system identical to a response system is almost impossible. An alternative approach to detect the GS state in unidirectionally coupled systems is to find the conditional Lyapunov exponents.[23,24] When $g_c = 0$, the uncoupled systems have six Lyapunov exponents of Eq. (13); λ_1, λ_2 are positive; λ_3, λ_4 are zero; and λ_5, λ_6 are negative since each system is in a chaotic state. With increasing chemical synaptic coupling strength g_c, the negativity of the positive Lyapunov exponent λ_2 corresponding to the response system signifies the onset of the GS state.

Therefore, to validate the obtained GS regime, in parallel with the ASA, we use the dependency of the system Lyapunov exponents on the coupling strength g_c. The behavior of the four largest Lyapunov exponents of the coupled system (13) is shown in Fig. 2 for $I = 3.25$. It is clearly seen that two Lyapunov exponents of the drive system, λ_1 and λ_3, do not practically depend on the coupling parameter g_c. Initially from zero value, λ_4 becomes negative as soon as the coupling is turned on. At the same time, the Lyapunov exponent λ_2 strongly depends on the coupling parameter. At $g_c = 3.6$, λ_2 becomes negative. Thus, this fact agrees well with the auxiliary system based stability analysis.

III. TWO BIDIRECTIONALLY COUPLED OSCILLATORS

Now, we investigate the applicability of ASA for two mutually coupled oscillators. In the case of the drive–response coupling configuration, the foundation of ASA is asymptotic stability of the response system (see Theorem 1). This asymptotic stability proves a functional relationship among the drive and response systems. Nevertheless, the asymptotic stability of the response system may not occur in the case of bidirectional couplings. We apply the concept of ASA proposed by Zheng *et al.*[29] for such coupling configurations.

For this purpose, let us take an example. Consider two bidirectionally coupled Lorenz oscillators through an x-variable in the form

$$\begin{aligned}
\dot{x}_{1,2} &= \sigma(y_{1,2} - x_{1,2}) + \varepsilon(x_{2,1} - x_{1,2}),\\
\dot{y}_{1,2} &= x_{1,2}(\rho - z_{1,2}) - y_{1,2},\\
\dot{z}_{1,2} &= x_{1,2}y_{1,2} - \beta z_{1,2}.
\end{aligned} \tag{16}$$

Since the two oscillators are identical, the CS solution will also be an invariant solution for this system. The stability of GS and CS depends on the interaction strength ε.

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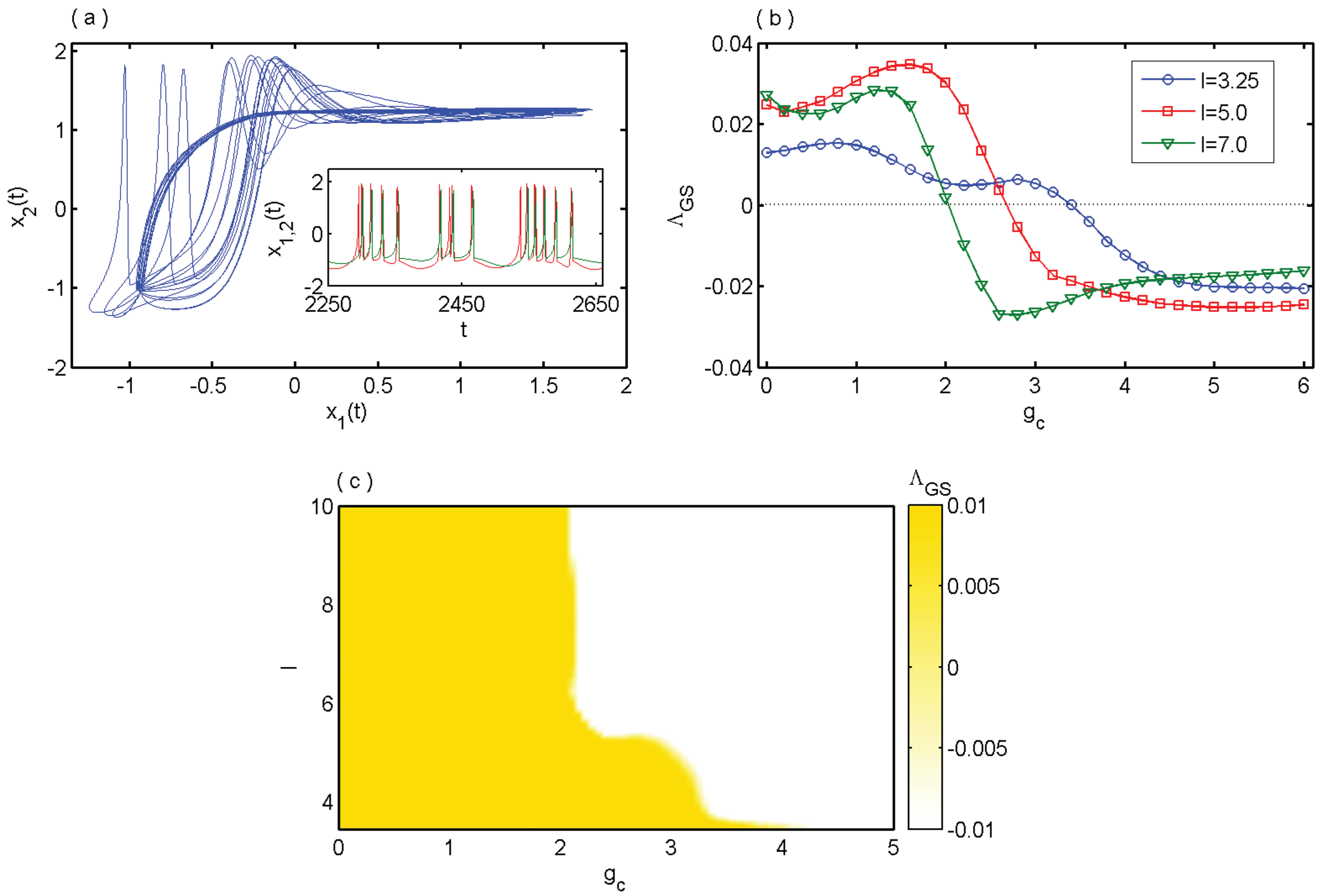

FIG. 1. (a) The projection of a generalized synchronized manifold in the (x_1, x_2) phase space for $g_c = 5.0$ and $I = 3.25$. The inset corresponds to the time evolution of $\mathbf{x}_1(t)$ (red curve) and $\mathbf{x}_2(t)$ (green curve) state variables. (b) Variation of Λ_{GS} with respect to g_c for $I = 3.25$ (blue circle curve), $I = 5.0$ (red square curve), and $I = 7.0$ (green triangle curve). (c) Region of the GS state (white region) in the two-dimensional (g_c, I) parameter plane; the color bar represents the variation of Λ_{GS}.

Now, we derive the error equations transverse to the CS and GS manifolds of two bidirectionally coupled oscillators. If (x_0, y_0, z_0) is the state variable of the CS state, then it satisfies the equation of motion,

$$\begin{aligned} \dot{x}_0 &= \sigma(y_0 - x_0), \\ \dot{y}_0 &= x_0(\rho - z_0) - y_0, \\ \dot{z}_0 &= x_0 y_0 - \beta z_0. \end{aligned} \tag{17}$$

The eigenvalues of the corresponding Laplacian matrix are 0 and 2. Then, the linearized equation of the transverse error component $\left[\xi^{(x)}, \xi^{(y)}, \xi^{(z)}\right]$ along the CS solution is

$$\begin{aligned} \dot{\xi}^{(x)} &= \sigma(\xi^{(y)} - \xi^{(x)}) - 2\varepsilon\xi^{(x)}, \\ \dot{\xi}^{(y)} &= (\rho - z_0)\xi^{(x)} - \xi^{(y)} - x_0\xi^{(z)}, \\ \dot{\xi}^{(z)} &= y_0\xi^{(x)} + x_0\xi^{(y)} - \beta\xi^{(z)}, \end{aligned} \tag{18}$$

where $\xi^{(x)} = x_2 - x_1, \xi^{(y)} = y_2 - y_1$, and $\xi^{(z)} = z_2 - z_1$. If the maximum Lyapunov exponent Λ_{CS} of the above error dynamics is negative, then the CS solution becomes asymptotically stable; otherwise, it is unstable.

To derive the stability of the GS state, let (u_i, v_i, w_i), $i = 1, 2$, be two auxiliary counterparts of system (16). Then, the auxiliary system takes the form[29]

$$\begin{aligned} \dot{u}_1 &= \sigma(v_1 - u_1) + \varepsilon(x_2 - u_1), & \dot{u}_2 &= \sigma(v_2 - u_2) + \varepsilon(x_1 - u_2), \\ \dot{v}_1 &= u_1(\rho - w_1) - v_1, & \dot{v}_2 &= u_2(\rho - w_2) - v_2, \\ \dot{w}_1 &= u_1 v_1 - \beta w_1, & \dot{w}_2 &= u_2 v_2 - \beta w_2. \end{aligned} \tag{19}$$

Let us define the error term for the GS state as $\left(\eta_i^{(x)}, \eta_i^{(y)}, \eta_i^{(z)}\right) = (u_i - x_i, v_i - y_i, w_i - z_i)$ for $i = 1, 2$. Then, according to the ASA and considering small perturbation from the GS manifold, the equation of motion of the error components of the GS solution of

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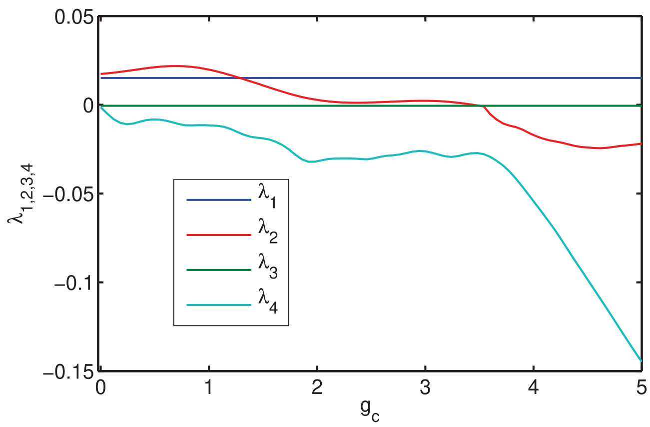

FIG. 2. The variation of four largest Lyapunov exponents of the coupled systems (13) by changing the chemical synaptic strength g_c as shown for $I = 3.25$.

Eq. (16) can be written as

$$\begin{aligned}\dot{\eta}_i^{(x)} &= \sigma(\eta_i^{(y)} - \eta_i^{(x)}) - \varepsilon\eta_i^{(x)},\\ \dot{\eta}_i^{(y)} &= (\rho - z_i)\eta_i^{(x)} - \eta_i^{(y)} - x_i\eta_i^{(z)},\\ \dot{\eta}_i^{(z)} &= y_i\eta_i^{(x)} + x_i\eta_i^{(y)} - \beta\eta_i^{(z)},\end{aligned} \tag{20}$$

where $i = 1, 2$ and (x_i, y_i, z_i) obeys Eq. (16). If Λ_{GS} is the maximum Lyapunov exponent, then the GS state of system (16) becomes stable if $\Lambda_{GS} < 0$; otherwise, it is unstable.

The variations of Λ_{CS} and Λ_{GS} as a function of ε are depicted in Fig. 3(a). It tells that for the coupled system (16), CS occurs for $\varepsilon \geq 4$, while GS for $\varepsilon \geq 8$. Therefore, for $\varepsilon \in [4, 8)$, the system is in the CS state, but not in the GS state (as per ASA). However, this is not true as the CS is a special type of GS for which the functional relationship is the identity function. Therefore, the principle of ASA gives the confusion result for the bidirectionally coupled systems.

From Fig. 3(a), the CS solution of system (16) becomes stable at $\varepsilon = 4$ and persists for further higher values of ε. Therefore, for $\varepsilon \geq 4$, both the two nodes of this coupled system evolve with the equation of motion as Eq. (17) and the state variables (x_1, y_1, z_1) and (x_2, y_2, z_2) become identical with (x_0, y_0, z_0). Then, both the components of Eq. (20) become identical to each other. Hence, for $\varepsilon \geq 4$, the two error components of the GS solution satisfy the equation

$$\begin{aligned}\dot{\eta}^{(x)} &= \sigma(\eta^{(y)} - \eta^{(x)}) - \varepsilon\eta^{(x)},\\ \dot{\eta}^{(y)} &= (\rho - z_0)\eta^{(x)} - \eta^{(y)} - x_0\eta^{(z)},\\ \dot{\eta}^{(z)} &= y_0\eta^{(x)} + x_0\eta^{(y)} - \beta\eta^{(z)}.\end{aligned} \tag{21}$$

Observe that the structural form of Eqs. (18) and (21) is the same except the effective coupling strength of the former one is 2ε and that for the latter one is ε. Therefore, as Eq. (18) becomes stable at $\varepsilon = 4$ for the stable CS state, Eq. (21) stabilizes at $\varepsilon = 8$ for the GS state. Due to the persistence of the stability of Eq. (18) for $\varepsilon \geq 4$, Eq. (21) remains stable for all $\varepsilon \geq 8$.

To strengthen our hypothesis, we again consider another instance of two bidirectionally coupled Hindmarsh–Rose oscillators through the chemical synaptic interaction. The evolution equation of the coupled system is of the form

$$\begin{aligned}\dot{x}_{1,2} &= y_{1,2} + (b - ax_{1,2})x_{1,2}^2 - z_{1,2} + I + \frac{g_c(v_s - x_{1,2})}{1 + \exp[-\lambda(x_{2,1} - \theta_s)]},\\ \dot{y}_{1,2} &= c - dx_{1,2}^2 - y_{1,2},\\ \dot{z}_{1,2} &= r[s(x_{1,2} - x_0) - z_{1,2}].\end{aligned} \tag{22}$$

Here, $I = 1.5$, and all other system parameters are the same as the unidirectional case in Eq. (13). Here, we also consider two identical oscillators; hence, both CS and GS are invariant for this coupled system. Let $e_i^{(x)} = u_i - x_i, e_i^{(y)} = v_i - y_i, e_i^{(z)} = w_i - z_i$ $(i = 1, 2)$ be the transverse error components along the GS manifold (using ASA), where $(u_{1,2}, v_{1,2}, w_{1,2})$ are the auxiliary counterparts of $(x_{1,2}, y_{1,2}, z_{1,2})$. Then, the error equations become

$$\begin{aligned}\dot{e}_{1,2}^{(x)} &= e_{1,2}^{(y)} + (2b - 3ax_{1,2})x_{1,2}e_{1,2}^{(x)} - e_{1,2}^{(z)} - \frac{g_c e_{1,2}^{(x)}}{1 + \exp[-\lambda(x_{2,1} - \theta_s)]},\\ \dot{e}_{1,2}^{(y)} &= -2dx_{1,2}e_{1,2}^{(x)} - e_{1,2}^{(y)},\\ \dot{e}_{1,2}^{(z)} &= r[se_{1,2}^{(x)} - e_{1,2}^{(z)}].\end{aligned} \tag{23}$$

Also, the transverse error component $(\delta x, \delta y, \delta z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ of the CS manifold satisfies the equation of motion,

$$\begin{aligned}\delta\dot{x} &= \delta y + (2b - 3ax)x\delta x - \delta z - \frac{g_c\delta x}{1 + \exp[-\lambda(x - \theta_s)]}\\ &\quad - g_c(v_s - x)\frac{\lambda\exp[-\lambda(x - \theta_s)]}{[1 + \exp\{-\lambda(x - \theta_s)\}]^2}\delta x,\\ \delta\dot{y} &= -2dx\delta x - \delta y,\\ \delta\dot{z} &= r[s\delta x - \delta z],\end{aligned} \tag{24}$$

where (x, y, z) is the synchronization solution that satisfies the equation

$$\begin{aligned}\dot{x} &= y + (b - ax)x^2 - z + I + g_c(v_s - x)\frac{1}{1 + \exp[-\lambda(x - \theta_s)]},\\ \dot{y} &= c - dx^2 - y,\\ \dot{z} &= r[s(x - x_0) - z].\end{aligned} \tag{25}$$

Figure 3(b) shows the variations of Λ_{CS} and Λ_{GS} as a function of g_c. For this system, CS occurs for all $g_c \geq 1.1$ except for $g_c \in (1.4, 1.6)$, while according to ASA,[29] GS occurs for $g_c \geq 1.8$. Therefore, for all $g_c \in [1.1, 1.4] \cap [1.6, 1.8)$, system (22) is in CS, but not in GS, once again that is absurd. Thus, the critical value of coupling strength for GS is greater than CS due to the inapplicability of ASA in mutually coupled systems. Our justification for the above two examples is that the coupled oscillators reach GS as they are in CS. However, this also contradicts with the result obtained by the auxiliary system method.

By the above two counterexamples, it is clear that ASA is inapplicable for two bidirectionally coupled oscillators.

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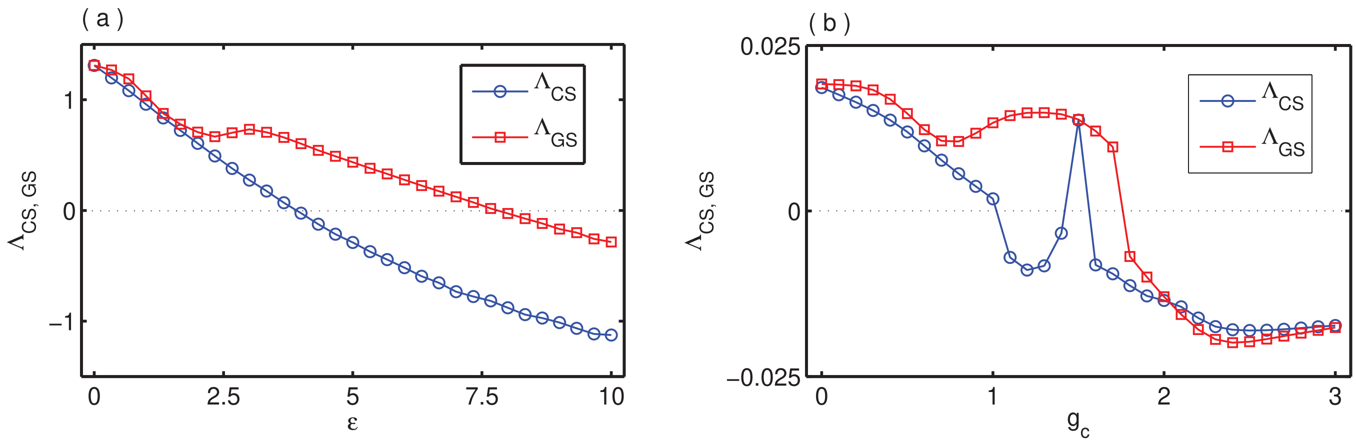

FIG. 3. Variations of maximum transverse Lyapunov exponents Λ_{CS} (blue circle) and Λ_{GS} (red square) for identifying the critical coupling strength in bidirectionally coupled (a) Lorenz system (16) with parameters $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$ and (b) Hindmarsh–Rose systems (22) with parameters $a = 1, b = 3, c = 1, d = 5, r = 0.005, s = 4, x_0 = -1.6, I = 1.5, v_s = 2, \theta_s = -0.25$, and $\lambda = 10$. In both these cases, CS occurs before GS according to the ASA.

IV. NETWORK OF COUPLED OSCILLATORS

Finally, we now check whether the ASA is applicable in a coupled network or not. We consider a network model of N oscillators in the form

$$\dot{\mathbf{x}}_i = F_i(\mathbf{x}_i) + \varepsilon \sum_{j=1}^{N} \mathscr{A}_{ij} H(\mathbf{x}_j - \mathbf{x}_i), \tag{26}$$

where $i = 1, 2, \ldots, N$. Here, $\mathscr{A} = [\mathscr{A}_{ij}]_{N\times N}$ is the adjacency matrix of the underlying network. If there is a link from node j to node i, then $\mathscr{A}_{ij} = 1$ or else $\mathscr{A}_{ij} = 0$. $F_i : \mathbb{R}^d \to \mathbb{R}^d$ is the evolution function of node i.

Definition 2. *The network (26) is said to realize the GS state if there exists a smooth mapping $\Phi : \mathbb{R}^{dN} \to \mathbb{R}^d$ such that $\Phi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = 0$.*

The GS regime implies the presence of an implicit functional relation between the system's state vectors. It implies the arousal of a generic manifold in the phase space wherein the overall trajectory is lying during the time evolution. Consequently, the GS manifold can be defined as $\mathcal{S} = \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \subset \mathbb{R}^{dN} : \Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = 0\}$. The GS state can be observed physically if this manifold is stable with respect to the perturbation in its transverse subspace.

The generic ith oscillator node dynamics of the auxiliary system[30,33–35] corresponding to Eq. (26) can be written as

$$\dot{\mathbf{y}}_i = F_i(\mathbf{y}_i) + \varepsilon \sum_{j=1}^{N} \mathscr{A}_{ij} H(\mathbf{x}_j - \mathbf{y}_i), \tag{27}$$

where $\mathbf{y}_i \in \mathbb{R}^d$. According to the ASA, the dynamical network (26) is supposed to achieve the GS state if $\lim_{t\to\infty} \frac{1}{N} \sum_{i=1}^{N} \|\mathbf{x}_i(t) - \mathbf{y}_i(t)\| = 0$ for any initial conditions from its basin of attraction and $\mathbf{x}_i(0) \neq \mathbf{y}_i(0), i = 1, 2, \ldots, N$.

For the sake of our analytical results, we need the following two assumptions on the dynamics of each isolate node and inner coupling matrix.

Assumption 1. *Assume that the isolate evolution function F_i of the ith oscillator is a Lipschitz function. Then, for each $i = 1, 2, \ldots, N$, there exists a non-negative constant L_i such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,*

$$\|F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})\| \le L_i \|\boldsymbol{x} - \boldsymbol{y}\|.$$

This Lipschitz criterion actually ensures the upper bound of the rate of change of the state variables in the phase space.

Assumption 2. *Consider that the inner coupling matrix H is a symmetric positive definite matrix and $\lambda_{\min}[H]$ is its minimum eigenvalue. Hence, $\lambda_{\min}[H]$ is real and positive.*

Now, we calculate the global stability condition of the GS state in the network equation (26) based on ASA in the following lemma.

Lemma 2. *If we assume that the auxiliary system approach is valid, then the generalized synchronization state for the coupled system (26) will be globally stable for $\varepsilon > \frac{L}{k_{\min}\lambda_{\min}[H]}$, where $k_{\min}$ denotes the minimum degree of the underlying network and $L = \max\{L_i : i = 1, 2, \ldots, N\}$.*

Proof. Let us consider the error term as $\mathbf{e}_i(t) = \mathbf{y}_i(t) - \mathbf{x}_i(t)$ for $i = 1, 2, \ldots, N$ at any instance t. Then, the error dynamics becomes

$$\dot{\mathbf{e}}_i = \dot{\mathbf{y}}_i - \dot{\mathbf{x}}_i = F_i(\mathbf{y}_i) - F_i(\mathbf{x}_i) - \varepsilon k_i H \mathbf{e}_i. \tag{28}$$

Consider a Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{N} \mathbf{e}_i^{tr}\mathbf{e}_i$. The time derivative of $V(t)$ along the GS manifold becomes

$$\dot{V} = \sum_{i=1}^{N} \mathbf{e}_i^{tr}\big[F_i(\mathbf{y}_i) - F_i(\mathbf{x}_i)\big]\mathbf{e}_i - \varepsilon k_i \sum_{i=1}^{N} \mathbf{e}_i^{tr} H \mathbf{e}_i. \tag{29}$$

By virtue of the Lipschitz criterion of the isolate node dynamics, we have the upper bound of $\dot{V}$ as

$$\dot{V} \le \sum_{i=1}^{N} L_i \mathbf{e}_i^{tr}\mathbf{e}_i - \varepsilon k_i \sum_{i=1}^{N} \mathbf{e}_i^{tr} H \mathbf{e}_i. \tag{30}$$

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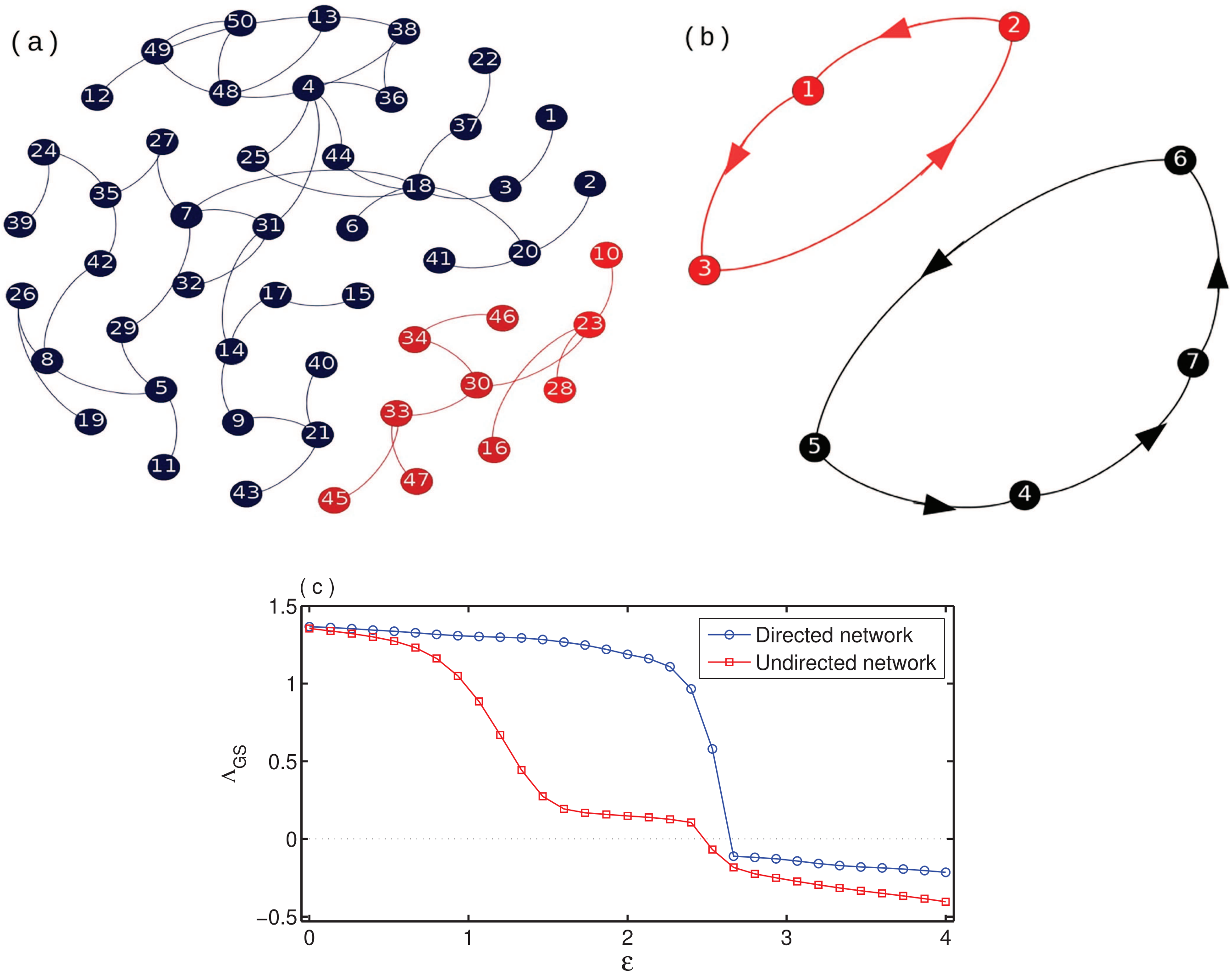

FIG. 4. (a) Network consisting of two disconnected components where nodes are bidirectionally connected. (b) Undirected network of seven nodes with two components. (c) Variation of maximum Lyapunov exponent Λ_{GS} with respect to ε for directed (blue circle) and undirected (red square) networks.

If we assume $L = \max\{L_i : i = 1, 2, \ldots, N\}$, then the above inequality immediately transforms as

$$\dot{V} \leq L \sum_{i=1}^{N} \mathbf{e}_i^{tr} \mathbf{e}_i - \varepsilon k_i \sum_{i=1}^{N} \mathbf{e}_i^{tr} H \mathbf{e}_i. \tag{31}$$

Now, the symmetric positive definiteness of the matrix H immediately implies

$$\dot{V} \leq \left[L - \varepsilon k_i \lambda_{\min}[H]\right] \sum_{i=1}^{N} \mathbf{e}_i^{tr} \mathbf{e}_i. \tag{32}$$

Based on the Lyapunov method for stability,[41,42] the error system (28) will be asymptotically stable if and only if $\dot{V} < 0$ whenever $\|\mathbf{e}\| \neq 0$. Thus, for the global stability of the GS state, we have $L - \varepsilon k_i \lambda_{\min}[H] < 0$ for all $i = 1, 2, \ldots, N$.

Therefore, if $k_{\min}$ is the minimum degree of the underlying graph, then the global stability condition can be written as $\varepsilon > \frac{L}{k_{\min}\lambda_{\min}[H]}$. □

For the above derivation, only one assumption on the underlying network is required and which is $k_{\min} > 0$; i.e., the minimum degree of the underlying network is greater than zero. In other words, there should not exist any isolate node in the network for the

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emergence of the GS state. No other restriction on the underlying network is required.

Theorem 4. *Consider an underlying network G for the dynamical network (26), which has two components, each one is fully connected and size greater than one and $k_{\min}$ is the minimum degree of the entire network G. Then, by the principle of ASA, the GS is globally stable for $\varepsilon > \frac{L}{k_{\min}\lambda_{\min}[H]}$.*

Proof. By the hypothesis, the network $G = (V, E)$ has two components, say, G_1 and G_2. The size of the two sub-networks G_1 and G_2 is greater than one. Thus, G has no vertex with degree zero; i.e., the minimum degree of G is $k_{\min} > 0$. In this network, we associate the dynamical system $\dot{\mathbf{x}}_i = F_i(\mathbf{x}_i)$ with the ith node and considering diffusive coupling among the neighboring nodes with inner coupling matrix H. Then, by Lemma 2, this coupled dynamical network possesses the GS state if $\varepsilon > \frac{L}{k_{\min}\lambda_{\min}[H]}$. □

However, the underlying network is disconnected, and it has two isolate components, i.e., in the absence of communication between a node of one component and a node in another component. Then, Lemma 2 yields some kind of correlation (in the form of GS) among themselves. Therefore, the ASA is also not always valid for the network of coupled oscillators.

To find the local stability condition for the GS state using ASA, linearizing Eq. (28) around the GS solution of Eq. (26), we get the transverse error equation as

$$\dot{\mathbf{e}}_i = \left[JF_i(\mathbf{x}_i) - \varepsilon k_i H\right]\mathbf{e}_i, \quad i = 1, 2, \ldots, N. \tag{33}$$

According to the ASA, the maximum Lyapunov exponent (Λ_{GS}) of Eq. (33) reveals the stability of the GS state. The ASA claims that if $\Lambda_{GS} < 0$, the GS state is locally asymptotically stable but otherwise unstable. Let us now illustrate the above fact numerically.

Figure 4(a) represents a bidirectional disconnected network of size 50, which has two connected components of respective sizes 40 and 10. The nodes and edges of the component of size 40 are represented by black color and those of the component of size 10 are red. Incorporating this graph as the underlying network, we associate the Lorenz oscillator in each node. Then, the equation of motion of the entire coupled network becomes

$$\begin{aligned}
\dot{x}_i &= \sigma(y_i - x_i) - \varepsilon\sum_{j=1}^{N}\mathscr{L}_{ij}x_j,\\
\dot{y}_i &= x_i(\rho_i - z_i) - y_i - \varepsilon\sum_{j=1}^{N}\mathscr{L}_{ij}y_j,\\
\dot{z}_i &= x_i y_i - \beta z_i - \varepsilon\sum_{j=1}^{N}\mathscr{L}_{ij}z_j, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{34}$$

The values of the system parameters are chosen as $\sigma = 10$, $\beta = \frac{8}{3}$, and ρ_i ($i = 1, 2, \ldots, N$) are chosen uniformly random from the interval $[25, 30]$ for which all the subsystems exhibit chaotic nonidentical oscillations.

The corresponding evolution equation of the perturb system $\mathbf{e}_i = \left(e_i^{(x)}, e_i^{(y)}, e_i^{(z)}\right)$ transverse to the GS manifold can be written as

$$\begin{aligned}
\dot{e}_i^{(x)} &= \sigma(e_i^{(y)} - e_i^{(x)}) - \varepsilon k_i e_i^{(x)},\\
\dot{e}_i^{(y)} &= (\rho_i - z_i)e_i^{(x)} - e_i^{(y)} - x_i e_i^{(z)} - \varepsilon k_i e_i^{(y)},\\
\dot{e}_i^{(z)} &= y_i e_i^{(x)} + x_i e_i^{(y)} - \beta e_i^{(z)} - \varepsilon k_i e_i^{(z)}, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{35}$$

ASA yields that the negative maximum Lyapunov exponent Λ_{GS} of the above error equation gives the stable GS state.

The variation of the corresponding maximum Lyapunov exponent Λ_{GS} transverse to the GS subspace with respect to ε is shown in Fig. 4(c) by a red square line. As the coupling strength gradually increases from zero, the value of Λ_{GS} decreases. At $\varepsilon \simeq 2.533$, the values of Λ_{GS} first become negative and persist for further higher values of ε. This indicates that the dynamical network (34) is in the GS state, and a functional relationship exists among the state variables for $\varepsilon \geq 2.533$. Here, the underlying network has two disconnected components, but the ASA based stability analysis yields a stable GS.

We also get almost similar results for a directed graph. For such a case, we consider a directed network with seven nodes [Fig. 4(b)] consisting of two components. One component contains three vertices (red filled circle) and the other contains four vertices (black filled circle). With this disconnected underlying network, the variation of Λ_{GS} is plotted in Fig. 4(c) by a blue circle curve. In this case, the ASA also yields a stable GS state for $\varepsilon \geq 2.667$.

Now the question arises: *How some correlations of their rhythms are possible in these disconnected networks?* Even the critical transition point getting from this approach for the connected network may be wrong, as we have already proved it for two mutually coupled systems (cf. Fig. 3).

V. CONCLUSION

In conclusion, we have investigated the criteria for the use of an auxiliary system approach to detect a generalized synchronization state in a coupled system. The ASA is an appropriate technique for the GS state in two unidirectionally coupled oscillators. Surprisingly, for two bidirectionally coupled oscillators, it gives an unstable GS state, while the system already is in a CS state. Therefore, this technique is not always applicable for two bidirectionally coupled systems. Additionally, we analytically confirmed that this approach is not always applicable for the network of coupled systems whether it is unidirectional or bidirectional. In such case, ASA gives a stable GS state, while the underlying networks are disconnected. Similar to the visual inspection of the measured time series for the CS state, the emergence of the GS state has no further verification as the functional relationship is completely unknown. No inference until now has been drawn regarding the cross-checking of the stability of the GS state using the ASA. In this context, an interesting open problem is to obtain an analytical testing method (such as the master stability function for CS[43]) to detect the occurrence and analyze the stability of the GS state in the coupled system. This aspect is critically important since ASA is of great interest to scientists in the context of generalized synchronization.

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DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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