Integración

Ejercicio 1 (teórico): Hacer pasos intermedios de la regla del trapecio simple.

Partiendo de $p_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b)$

$$p_1(x) = \frac{x-b}{a-b}f(a) - \frac{x-a}{a-b}f(b)$$

$$= \frac{(x-b)f(a) - (x-a)f(b)}{(a-b)}$$

$$= \frac{xf(a) - bf(a) - xf(b) + af(b)}{(a-b)}$$

$$= \frac{x(f(a)-f(b)) - bf(a) + af(b)}{(a-b)}$$

$$\int_a^b P_1\,dx = \int_a^b \frac{x(f(a)-f(b)) - bf(a) + af(b)}{a-b}\,dx$$

$$= \int_a^b \frac{f(a)-f(b)}{a-b}\,x\,dx + \int_a^b \frac{af(b)-bf(a)}{a-b}\,dx$$

$$= \frac{f(a)-f(b)}{a-b}\int_a^b x\,dx + \frac{af(b)-bf(a)}{a-b}\int_a^b dx$$

$$= \frac{f(a)-f(b)}{a-b}\,\frac{x^2}{2}\Big|_a^b dx + \frac{af(b)-bf(a)}{a-b}\cdot x\Big|_a^b$$

$$= \frac{f(a)-f(b)}{2(a-b)}(b^2-a^2) + \frac{af(b)-bf(a)}{a-b}(b-a)$$

$$= \frac{(f(a)-f(b))(b-a)(b+a)}{2(a-b)} + \frac{bf(a)-af(b)}{\cancel{(b-a)}}\cancel{(b-a)}$$

$$= \frac{(f(b)-f(a))\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} + bf(a) - af(b)$$

$$= \frac{(f(b)-f(a))(b+a)}{2} + \frac{2(bf(a)-af(b))}{2}$$

$$= \frac{(f(b)-f(a))(b+a) + 2bf(a) - 2af(b)}{2}$$

$$= \frac{bf(b) - \cancel{bf(a)} + \cancel{af(b)} - af(a) + \cancel{2}bf(a) - \cancel{2}af(b)}{2}$$

$$= \frac{1}{2}\left(bf(b) - af(a) + bf(a) - af(b)\right)$$

$$= \frac{1}{2}\left(b(f(b)+f(a)) - a(f(b)+f(a))\right)$$

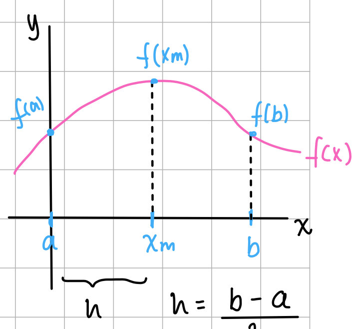
$$= \frac{1}{2}\left[(b-a)(f(b)+f(a))\right]$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$\boxed{\int_a^b P_1\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)}$$

Ejercicio 3 (teórico): Hacer pasos intermedios de la regla de Simpson simple.

$$\int_a^b f(x)\,dx \approx \int_a^b P_2(x)\,dx$$



$$h = \frac{b-a}{2}$$

Partiendo de $P_2(x) = \underbrace{\frac{(x-b)(x-x_m)}{(a-b)(a-x_m)}}_{L_0} f(a) + \underbrace{\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)}}_{L_1} f(x_m) + \underbrace{\frac{(x-a)(x-x_m)}{(b-a)(b-x_m)}}_{L_2} f(b)$

Los puntos a, x_m y b se pueden reescribir de la forma $x = a + th$,

de modo que:

$$\begin{aligned} a &= a + (0)h \\ x_m &= a + (1)h \\ b &= a + (2)h \end{aligned} \qquad \text{y} \qquad \begin{aligned} x - a &= th \\ x - x_m &= x - (a+h) = x - a - h = th - h \\ x - b &= x - (a+2h) = x - a - 2h = th - 2h \end{aligned}$$

Así, podemos reescribir L_0, L_1 y L_2 en términos de t como:

$$\begin{aligned} L_0 &= \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} \\ &= \frac{(th-2h)(th-h)}{(a-(a+2h))(a-(a+h))} \\ &= \frac{h(t-2)\,h(t-1)}{(\cancel{a}-\cancel{a}-2h)(\cancel{a}-\cancel{a}-h)} \\ &= \frac{\cancel{h^2}(t-2)(t-1)}{2\cancel{h^2}} \\ &= \frac{(t-1)(t-2)}{2} \end{aligned}$$

$$\begin{aligned} L_1 &= \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} \\ &= \frac{(th)(th-2h)}{(a+h-a)(a+h-(a+2h))} \\ &= \frac{(th)\,h(t-2)}{h(a+h-a-2h)} \\ &= \frac{t\cancel{h^2}(t-2)}{-\cancel{h^2}} \\ &= \frac{t(t-2)}{-1} \end{aligned}$$

$$\begin{aligned} L_2 &= \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} \\ &= \frac{(th)(th-h)}{(a+2h-a)(a+2h-(a+h))} \\ &= \frac{(th)\,h(t-1)}{2h\,(a+2h-a-h)} \\ &= \frac{t\cancel{h^2}(t-1)}{2\cancel{h^2}} \\ &= \frac{t(t-1)}{2} \end{aligned}$$

Ahora, estableciendo la integral:

$$\begin{aligned} \int_a^b P_2(x)\,dx &= \int_a^b L_0 f(a)\,dx + \int_a^b L_1 f(x_m)\,dx + \int_a^b L_2 f(b)\,dx \\ &= f(a)\int_a^b L_0(x)\,dx + f(x_m)\int_a^b L_1(x)\,dx + f(b)\int_a^b L_2(x)\,dx \end{aligned}$$

Cambiando los límites a términos de t y utilizando $x = a + th$:

→ Si x tiende a a,

entonces:
$$\begin{aligned} a &= a + th \\ th &= a - a \\ th &= 0 \\ t &= 0 \end{aligned}$$

→ Si x tiende a b,

entonces:
$$\begin{aligned} b &= a + th \\ th &= b - a \\ th &= a + 2h - a \\ th &= 2h \\ t &= 2 \end{aligned}$$

Hallando el dx en téminos de dt

$$x = a + th$$

$$\frac{dx}{dt} = \frac{d(a+th)}{dt}$$

$$\frac{dx}{dt} = h$$

$$dx = h\,dt$$

Reescribiendo las integrales en términos de t y realitándolas:

$$\int_a^b P_2(x)\,dx = f(a)\int_a^b L_0(x)\,dx + f(X_m)\int_a^b L_1(x)\,dx + f(b)\int_a^b L_2(x)\,dx$$

$$= f(a)\int_0^2 L_0(t)\,h\,dt + f(X_m)\int_0^2 L_1(t)\,h\,dt + f(b)\int_0^2 L_2(t)\,h\,dt$$

$$= hf(a)\int_0^2 \frac{(t-1)(t-2)}{2}\,dt + h\,f(X_m)\int_0^2 \frac{t(t-2)}{-1}\,dt + h\,f(b)\int_0^2 \frac{t(t-1)}{2}\,dt$$

$$= \frac{h}{2}f(a)\int_0^2 (t^2-3t+2)\,dt - h\,f(X_m)\int_0^2 (t^2-2t)\,dt + \frac{h}{2}f(b)\int_0^2 (t^2-t)\,dt$$

$$= \frac{h}{2}f(a)\left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_0^2 - hf(X_m)\left(\frac{t^3}{3} - t^2\right)\Big|_0^2 + \frac{h}{2}f(b)\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_0^2$$

$$= \frac{h}{2}f(a)\left(\frac{2^3}{3} - \frac{3\cdot 2^2}{2} + 2(2)\right) - hf(X_m)\left(\frac{2^3}{3} - 2^2\right) + \frac{h}{2}f(b)\left(\frac{2^3}{3} - \frac{2^2}{2}\right)$$

$$= \frac{h}{\cancel{2}}f(a)\left(\frac{\cancel{2}}{3}\right) - hf(X_m)\left(\frac{-4}{3}\right) + \frac{h}{\cancel{2}}f(b)\left(\frac{\cancel{2}}{3}\right)$$

$$= \frac{hf(a)}{3} + \frac{4hf(X_m)}{3} + \frac{hf(b)}{3}$$

$$= \frac{h}{3}\Big(f(a) + 4f(X_m) + f(b)\Big)$$

$$\boxed{\int_a^b P_2(x)\,dx = \frac{h}{3}\Big(f(a) + 4f(X_m) + f(b)\Big)}$$