Assignment 6

Due: 16.07.2024, 23:59

Points: 11

The solutions have to be handed in via Moodle. We do not accept late submissions.

We would recommend using LaTeX for writing your submission but also accept handwritten solutions, but please note that if we can not read or understand it, we cannot grade it.

To get full points, always provide the steps in your derivation/proofs and make clear when/how you use known results, for example, from the lecture (e.g. already proven concentration inequalities).

Exercise 6.1: Feature maps of universal kernels are injective

Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a universal kernel. Denote by $\mathcal{H}$ its RKHS, and by $\phi : \mathbb{R}^d \to \mathcal{H}$ its feature map. Recall the reproducing property: For every $f \in \mathcal{H}$ and any $x \in \mathbb{R}^d$, we have $f(x) = \langle \phi(x), f \rangle$.

Prove that ϕ is injective.

(5 points)

Exercise 6.2: Solving Machine Learning (?)

Consider a neural network $h : [0,1] \to \mathbb{R}$ with just one hidden neuron:

$$h(x) = \sigma(ax + b)$$

for some activation σ and $a, b \in \mathbb{R}$.

1. Show that there exists an activation function σ such that for any polynomial $f : [0,1] \to \mathbb{R}$ and all $\epsilon > 0$, there exists some a, b satisfying

   $$\sup_{x \in [0,1]} |f(x) - h(x)| \leq \epsilon$$

   Hint: The set of all polynomials with coefficients in $\mathbb{Q}$ is countable (and dense in the set of all polynomials with coefficients in $\mathbb{R}$).

2. Junior data scientist Alex is thrilled about this observation and wants to use it for all his machine learning models. What do you tell him?

(3+3=6 points)