Sample Problems 1

To be discussed on 26.04.2024

Sample Problem 1.1: Probability Bounds 1

Remind yourself of the **Markov inequality** to prove the following. If Z is a random variable that takes values in the interval $[0,1]$, then for $a \in (0,1)$

$$\frac{\mathbb{E}[Z]-a}{1-a} \leq \mathbb{P}(Z > a) \leq \frac{\mathbb{E}[Z]}{a}$$

Sample Problem 1.2: Probability Bounds 2

Remind yourself of the **Cauchy-Schwarz inequality** to prove the following. If Z is a non-negative random variable with finite variance, then

$$\mathbb{E}\big[Z\mathbf{1}\{Z>0\}\big] \leq \sqrt{\mathbb{E}[Z^2]\mathbb{P}(Z>0)}.$$

From the above relation, show that

$$\mathbb{P}(Z>0) \geq \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}$$

Sample Problem 1.3: Probability Bounds 3

We now look at a probability bound defined by the first two moments. Assume that $Z \geq 0$ is a random variable with finite variance and $0 \leq \theta \leq 1$. Show that:

$$\mathbb{P}(Z > \theta\mathbb{E}[Z]) \geq (1-\theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}\left[Z^2\right]}$$

Sample Problem 1.4: Bayes Risk for 2 Gaussians

Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{Y} = \{\pm 1\}$. Assume $Y \sim \text{Bernoulli}\left(\frac{1}{2}\right)$, and

$$X \sim \mathcal{N}(\mu, \sigma^2) \text{ , if } Y = 1$$
$$X \sim \mathcal{N}(-\mu, \sigma^2) \text{ , if } Y = -1$$

for some $\mu, \sigma^2 > 0$. Derive $\eta(x)$, the Bayes classifier h^* and the Bayes risk. Express everything in terms of the probability density ϕ and the cumulative distribution function Φ of Gaussian random variables.