Homework 02

**Problem 1**

Consider using a gradient algorithm to minimize the function. With the initial guess as

* 1. To initialize the line-search apply bracketing procedure along the line starting at in the direction of negative gradient. Use = 0.075
  2. Apply the golden section method to reduce the width of uncertainty region to 0.01. Organize results of your computation in table format.
  3. Repeat the above using Fibonacci method.

**Solution:**

*Matlab code – problem1.m*

clc; clear all; close all;

% Declare the symbolic variables

syms x1 x2 alph;

% Declare function in terms of x1 x2

f = @(x1, x2) x1.^2+x2.^2+x1.\*x2;

X\_init = [0.8,-.25];

epsilon = 0.075;

k = 1;

[a0, b0] = Bracketing(f, X\_init, epsilon); %run bracketing on phia to find points

[s,t, dat] = GoldenSection(a0,b0,f);

[s1,t1, dat1] = FibonacciSeq(a0,b0,f);

xlswrite('GoldenSearch.xlsx',dat);

xlswrite('Fibonacci.xlsx', dat1);

Bracketing procedure is applied as below. The output obtained from the bracketing points is then used as an input to the Golden search and the Fibonacci method algorithms.

In our case the range of bracketing is [0.35,-0.7], [0.8,-0.25]

*Matlab function – Brackting.m*

function [a0,b0] = Bracketing(f, x\_init, epsilon)

%BRACKETING Summary of this function goes here

% Detailed explanation goes here

x0 = x\_init; x1 = x0+epsilon; x2 = x1+2\*epsilon;

n = 1; r = 1;

%bracketing procedure

if((f(x0(1), x0(2))>f(x1(1), x1(2))) && (f(x1(1), x1(2))<f(x2(1), x2(2))))

a0 = x0;

b0 = x2;

else

epsi = 2\*epsilon;

while not((f(x0(1), x0(2))>f(x1(1), x1(2))) && (f(x1(1), x1(2))<f(x2(1), x2(2))))

if((f(x0(1), x0(2))>f(x1(1), x1(2))) && (f(x1(1), x1(2))>f(x2(1), x2(2))))

x0 = x1;

x1 = x2;

x2 = x2+(2^(n+1))\*epsilon;

% epsi = 2\*epsi;

% x2 = x2+epsi;

n = n+1;

elseif((f(x0(1), x0(2))<f(x2(1), x2(2))) && (f(x1(1), x2(2))<f(x2(1), x2(2))))

x2 = x1;

x1 = x0;

x0 = x0-(2^r)\*epsilon;

r = r+1;

% epsi = 2\*epsi;

% x0 = x0-epsi;

end

% n = n+1;

end

a0 = x0;

b0 = x2;

end

end

*Matlab code – GoldenSection.m*

function [s,t, dat] = GoldenSection(a0, b0, fcn)

%GOLDENSECTION Summary of this function goes here

% Detailed explanation goes here

%golden section

epsi = 0.075;

N = ceil(log(0.01/norm(b0-a0))/log(0.6180));

rho = 0.382;

a = a0;

b = b0;

s = a+rho\*(b-a);

t = a+(1-rho)\*(b-a);

f1= fcn(s(1), s(2));

f2= fcn(t(1), t(2));

dat = {'Iteration','rhok','ak','bk','f(ak)','f(bk)','new int'};

for n = 1:N

if fcn(s(1), s(2))< fcn(t(1), t(2))

b = t;

t = s;

s = a+rho\*(b-a);

f2 = f1;

f1 = fcn(s(1), s(2));

dat{n+1,5} = f1;

dat{n+1,6} = f2;

elseif fcn(s(1), s(2))> fcn(t(1), t(2))

a = s;

s = t;

t = a+(1-rho)\*(b-a);

f1 = f2;

f2 = fcn(t(1), t(2));

dat{n+1,5} = f1;

dat{n+1,6} = f2;

else

break;

end

% dat{n,:} = {n,rho,s,t,f1,f2,[s,t]};

dat{n+1,1} = n;

dat{n+1,2} = rho;

dat{n+1,3} = mat2str(s);

dat{n+1,4} = mat2str(t);

dat{n+1,7} = mat2str([s;t]);

end

end

*Matlab code – Fibonacci.m*

function [s,t, dat] = FibonacciSeq(a0, b0, fcn)

%FIBONACCISEQ Summary of this function goes here

% Detailed explanation goes here

% rho =

N = ((1+(2\*0.01))/(0.01/norm(b0-a0)));

n=1;

while (fibonacci(n)<=N)

fiboNum(n) = fibonacci(n+1);

n=n+1;

end

N = length(fiboNum);

for i = 1:N-1

if i == N-1

rho(i) = 1- (fiboNum(N-i)/fiboNum(N-i+1))- 0.01;

else

rho(i) = 1- (fiboNum(N-i)/fiboNum(N-i+1));

end

end

a = a0;

b = b0;

s = a+rho(1)\*(b-a);

t = a+(1-rho(1))\*(b-a);

f1= fcn(s(1), s(2));

f2= fcn(t(1), t(2));

dat = {'Iteration','rhok','ak','bk','f(ak)','f(bk)','new int'};

for n = 1:length(fiboNum)-1

if fcn(s(1), s(2))< fcn(t(1), t(2))

b = t;

t = s;

s = a+rho(n)\*(b-a);

f2 = f1;

f1 = fcn(s(1), s(2));

else

a = s;

s = t;

t = a+(1-rho(n))\*(b-a);

f1 = f2;

f2 = fcn(t(1), t(2));

end

% dat{n,:} = {n,rho,s,t,f1,f2,[s,t]};

dat{n+1,1} = n;

dat{n+1,2} = rho(n);

dat{n+1,3} = mat2str(s);

dat{n+1,4} = mat2str(t);

dat{n+1,5} = f1;

dat{n+1,6} = f2;

dat{n+1,7} = mat2str([s;t]);

end

end

The results were recorded in a excel sheet for every iteration in the process for Golden section and Fibonacci methods as mentioned in the problem.

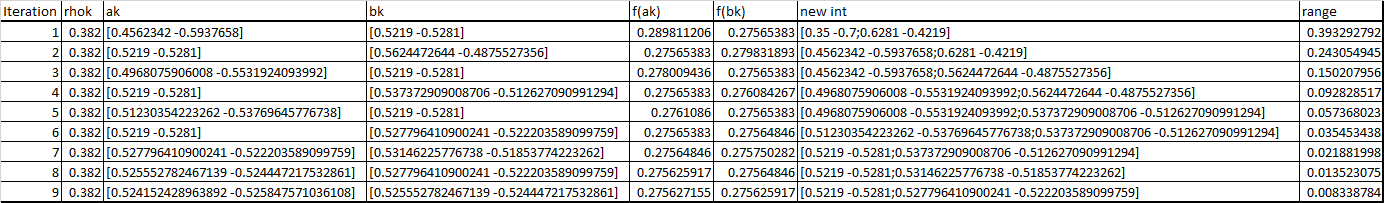
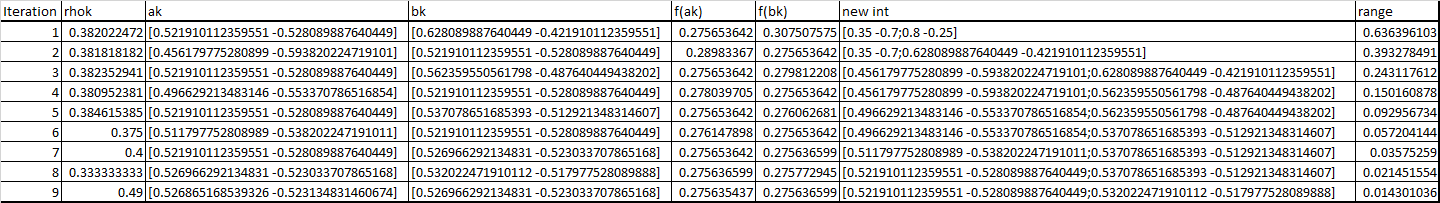
Table 1. Results for Golden Search Method. First iteration is the first improvement over the points given by Bracketing

Table 2. Results for Fibonacci Method. First iteration is the first improvement over the points given by Bracketing

**Problem 2**

For the function  
f(x1, x2) = (x2 - x1)4 + 12x1x2 - x1 + x2 - 3,

1. Use MATLAB’s commands meshgrid and mesh to generate its 3D plot. The range of  
   x1 and x2 is the same and it should be equal to [-1, 1]. Set the box on.
2. A close up of a map

   Description automatically generatedUse the command contour to generate 20 contours. Use the same range for x1 and x2 as in (a)

A picture containing table

Description automatically generatedFig 1. 3D plot of the function using the mesh function in MATLAB

Fig 2. Contours of the function generated in MATLAB

*MATLAB Code – Problem2.m*

%problem 2

clc; close all;

%set range of x from -1 to 1

x = linspace(-1,1,50);

y = x;

% create a meshgrid for plotting the function

[x1,x2] = meshgrid(x,y);

% define the given function for 3d plotting

f = (x2-x1).^4+12.\*x1.\*x2-x1+x2-3;

box on; mesh(x1,x2,f); %plot the function

figure;

% plot contours of the function

contour(x1,x2,f, 20)

**Problem 3**

Minimize the above function using the method of the gradient descent when α = 0.02 and  
locate these points on the level sets of f. Connect the successive points with lines or lines  
with arrows to show clearly the progression of the optimization process. Use two staring  
points,  
 and

Obtain the sequence of points using the steepest descent method and locate these points on  
the level sets of f

**Solution:** *MATLAB Code – Problem3.m*

clc; clear all; close all;

%Gradient descent

syms x1 x2;

f = @(x1, x2) (x2-x1).^4+12.\*x1.\*x2-x1+x2-3; %declare the function in terms of x1, x2

%Initializations for the sequence of points

X\_init\_array1{1} = [0.55, 0.7]; %starting point 1

X\_init\_array1{2} = [0,0];

X\_init\_array2{1} = [-0.9 -0.5]; %starting point 2

X\_init\_array2{2} = [0,0];

%Gradient descent from the two starting points

X\_init\_array1 = gradDesc(f, X\_init\_array1);

X\_init\_array2 = gradDesc(f, X\_init\_array2);

%Steepest gradient method

X\_init\_array3{1} = [0.55, 0.7]; %starting point 1

X\_init\_array3{2} = [0,0];

X\_init\_array3 = steepestGrad(f, X\_init\_array3);

X\_init\_array4{1} = [-0.9,-0.5]; %starting point 1

X\_init\_array4{2} = [0,0];

X\_init\_array4 = steepestGrad(f, X\_init\_array4);

%plotting the contours/ level sets for the function to plot sequence later

x = linspace(-1,1,50);

y = x;

[x1,x2] = meshgrid(x,y);

f = (x2-x1).^4+12.\*x1.\*x2-x1+x2-3;

% box on; mesh(x1,x2,f); %plotting the function

contour(x1,x2,f, 20); hold on;

for i = 1:length(X\_init\_array1)

px(i) = X\_init\_array1{i}(1);

py(i) = X\_init\_array1{i}(2);

end

hold on;

plot(px, py, '^-'); %plot sequence of points starting from starting point1

figure;

contour(x1,x2,f, 20); hold on;

for i = 1:length(X\_init\_array2)

px1(i) = X\_init\_array2{i}(1);

py1(i) = X\_init\_array2{i}(2);

end

hold on;

plot(px1, py1, 'x-'); %plot sequence of points starting from starting point2

figure;

contour(x1,x2,f, 20); hold on;

for i = 1:length(X\_init\_array3)

px2(i) = X\_init\_array3{i}(1);

py2(i) = X\_init\_array3{i}(2);

end

hold on;

plot(px2, py2, 'x-'); %plot sequence of points starting from starting point1 using sd

figure;

contour(x1,x2,f, 20); hold on;

for i = 1:length(X\_init\_array4)

px3(i) = X\_init\_array4{i}(1);

py3(i) = X\_init\_array4{i}(2);

end

hold on;

plot(px3, py3, 'x-'); %plot sequence of points starting from starting point1 using sd

*MATLAB Code – gradDescent.m*

function [X\_init\_array1] = gradDesc(f, X\_init\_array1)

%GRADDESC Summary of this function goes here

% Detailed explanation goes here

syms x1 x2;

grad = {};

alpha = 0.02;

epsilon = .001;

% Initialize first gradients and 2nd element

grad{1} = (double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(1))))';

X\_init\_array1{2}= X\_init\_array1{1}-alpha\*grad{1};

k = 2;

% Iterate till the norm of consecutive points becomes less than epsilon

while(norm((X\_init\_array1{k-1}-X\_init\_array1{k}),2)>epsilon)

grad{k} = (double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(k))))';

X\_init\_array1{k+1}= X\_init\_array1{k}-alpha\*grad{k};

k = k+1;

end

end

*MATLAB Code – steepestGrad.m*

function [X\_init\_array1] = steepestGrad(f, X\_init\_array1)

%Steepest Gradient - Used to calculate the minima using the steepest

%gradient method

% The function takes f - The objective function and the Initial point

% from the sequence as the input and gives out an array of sequence of

% points to reach the minima as output. At each step Line search is used

% to find the optimum step size

syms x1 x2;

epsilon = .001;

% Initialize first gradients and 2nd element

grad{1} = double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(1)))';

syms alph;

searchDir = -gradient(f(x1, x2), [x1 x2]); %search dir in direction of negative grad

fx\_ag = f(x1+alph\*searchDir(1), x2+alph\*searchDir(2)); %f(x-alpha\*g)

f\_alph = subs(fx\_ag, [x1, x2], {X\_init\_array1(1)}); %f(x1-alph\*g1)

f\_dash\_alph = diff(f\_alph, alph); %f'(x1-alph\*g1)

f\_ddash\_alph = diff(f\_dash\_alph, alph); %f''(x1-alph\*g1)

%initialize alpha values to arbitrary levels

alpha{1} = 0.02;

alpha{2} = alpha{1}-double(subs((f\_dash\_alph/f\_ddash\_alph), alph, alpha{1}));

%find 2nd point based on initial alpha

X\_init\_array1{2}= X\_init\_array1{1}-alpha{1}\*grad{1};

k = 2;

% Iterate till the norm of consecutive points becomes less than epsilon

while(k<100&& norm((X\_init\_array1{k}-X\_init\_array1{k-1}),2)>epsilon)

syms alph;

%find the kth gradient, f(xk-alph\*gk), f'(xk-alph\*gk),

%f''(xk-alph\*gk)

grad{k} = (double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(k))))';

f\_alph = subs(fx\_ag, [x1, x2], {X\_init\_array1(k)});

f\_dash\_alph = diff(f\_alph, alph);

f\_ddash\_alph = diff(f\_dash\_alph, alph);

%initialize alpha for line search using NM

alpha\_new{1} = alpha{k-1};

alpha\_new{2} = alpha{k};

i = 2;

% while loop to evalulate optimum alpha to minimize f(x-alpha\*g)

% using Newtons method (Line search)

while(norm(alpha\_new{i}-alpha\_new{i-1})>0.001)

% this while loop can be removed and the statement below can be

% used to get alpha after one iteration of NM. And the point

% converges to minima using this approach. Not sure if it is the

% correct way.

% alpha{k} = alpha{k-1}- double(subs((f\_dash\_alph/f\_ddash\_alph), alph, alpha{k-1}));

alpha\_new{i+1} = alpha\_new{i}- double(subs((f\_dash\_alph/f\_ddash\_alph), alph, alpha\_new{i}));

i = i+1;

end

alpha{k+1} = alpha\_new{i}; %update alpha with the one found using NM

X\_init\_array1{k+1}= X\_init\_array1{k}-alpha{k+1}\*grad{k}; %update point using the new found alpha

k = k+1;

end

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Description automatically generatedFig 3. The sequence of points plotted on level set of the function for starting point [0.55, 0.7]TFig 4. The sequence of points plotted on level set of the function for starting point [-0.9,-0.5]TA picture containing table

Description automatically generatedA picture containing table, indoor

Description automatically generatedFig 5. The sequence of points plotted on level set of the function for starting point [.55,.7]T using steepest descent

Fig 6. The sequence of points plotted on level set of the function for starting point [-0.9,-0.25]T using steepest descent

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Fig 7. Sequence of points for [0.55,0.77] starting point. Alpha calculated using Golden section.

While using Newtons method to find optimum alpha, As we can see for the first starting point, while using the Steepest descent method we get caught in a ‘narrow valley’ region due to which, it becomes difficult to reach the minimizer.

For the second point, due to the optimum step size selection, the function reaches the local minimizer very quickly as we can see in Figure 5.

However if we use Golden section to find alpha, the point converges to the minima as shown in Fig. 7.

*Golden Section function to calculate optimum alpha.*

function [a,b, dat] = GoldenSection(a0, b0, fcn)

%GOLDENSECTION Summary of this function goes here

% Detailed explanation goes here

%golden section

syms alph;

epsi = 0.075;

N = ceil(log(0.01/norm(b0-a0))/log(0.6180));

rho = 0.382;

a = a0;

b = b0;

s = a+rho\*(b-a);

t = a+(1-rho)\*(b-a);

f1= double(subs(fcn, alph, s));

f2= double(subs(fcn, alph, t));

dat = {'Iteration','rhok','ak','bk','f(ak)','f(bk)','new int', 'range'};

for n = 1:N

if double(subs(fcn, alph, s))< double(subs(fcn, alph, t))

b = t;

t = s;

s = a+rho\*(b-a);

f2 = f1;

f1 = double(subs(fcn, alph, s));

dat{n+1,5} = f1;

dat{n+1,6} = f2;

elseif double(subs(fcn, alph, s))> double(subs(fcn, alph, t))

a = s;

s = t;

t = a+(1-rho)\*(b-a);

f1 = f2;

f2 = double(subs(fcn, alph, t));

dat{n+1,5} = f1;

dat{n+1,6} = f2;

else

break;

end

% dat{n,:} = {n,rho,s,t,f1,f2,[s,t]};

dat{n+1,1} = n;

dat{n+1,2} = rho;

dat{n+1,3} = mat2str(s);

dat{n+1,4} = mat2str(t);

dat{n+1,7} = mat2str([a;b]);

dat{n+1,8} = norm(a-b);

end

end

*Steepest Gradient function to calculate minima using the alpha obtained by golden section*

function [X\_init\_array1] = steepestGrad(f, X\_init\_array1)

%Steepest Gradient - Used to calculate the minima using the steepest

%gradient method

% The function takes f - The objective function and the Initial point

% from the seqeuence as the input and gives out an array of sequence of

% points to reach the minima as output. At each step Line search is used

% to find the optimum step size

syms x1 x2;

epsilon = .001;

% Initialize first gradients and 2nd element

grad{1} = double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(1)))';

syms alph;

searchDir = -gradient(f(x1, x2), [x1 x2]); %search dir in direction of negative grad

fx\_ag = f(x1+alph\*searchDir(1), x2+alph\*searchDir(2)); %f(x-alpha\*g)

f\_alph = subs(fx\_ag, [x1, x2], {X\_init\_array1(1)}); %f(x1-alph\*g1)

[s,t,~] = GoldenSection(0,0.5,f\_alph);

alpha{1} = (s+t)/2;

%find 2nd point based on initial alpha

X\_init\_array1{2}= X\_init\_array1{1}-alpha{1}\*grad{1};

k = 2;

% Iterate till the norm of consecutive points becomes less than epsilon

while(k<100 && norm((X\_init\_array1{k}-X\_init\_array1{k-1}),2)>epsilon)

syms alph;

grad{k} = (double(subs(gradient(f(x1, x2), [x1 x2]), {x1,x2}, X\_init\_array1(k))))';

f\_alph = subs(fx\_ag, [x1, x2], {X\_init\_array1(k)});

[s,t,~] = GoldenSection(0,0.5, f\_alph);

alpha{k} = (s+t)/2;

X\_init\_array1{k+1}= X\_init\_array1{k}-alpha{k}\*grad{k}; %update point using the new found alpha

k = k+1;

end

end

As we can see choosing and optimum alpha lets us reach the minimizer in fewer steps than that as compared to normal gradient descent

**Problem 4**

Minimize the above function using Newton’s method. Locate the points on Level sets of f.

The given function has the very first Hessian which is not pd. So, the Newtons method won’t converge for this function.   
The code is written to handle this condition. However, we can still run the code to visualize the sequence of points on the level sets

Solution:

% problem4;

clear all; clc; close all;

% Use newtons method to minimize the given function

syms x1 x2;

%define the function in terms of symbolic variables x1 and x2

f = (x2-x1).^4+12.\*x1.\*x2-x1+x2-3;

X\_init\_array1{1} = [1,-.8];

X\_init\_array1{2} = [0,0]; %initialize second point in the sequence

% alpha = 0.02;

epsilon = .001; %set epsilon to some small value

%calculate the gradient and hessian at the initial point x0

grad{1} = (double(subs(gradient(f, [x1 x2]), {x1,x2}, X\_init\_array1(1))));

hess{1} = double(subs(hessian(f, [x1 x2]), {x1,x2}, X\_init\_array1(1)));

%The conditions for the positive definiteness can be removed to check

%the updated sequence we get through the newtons method. But as this

%example has no pd matrix the sequence obtained by using the Hessian as the

%step size does not converge to the minimizer.

if (eig(hess{1})>0) %check for the kth Hessian to be pd

% update point 2 with hess{1} as step size

X\_init\_array1{2}= X\_init\_array1{1}-(pinv(hess{1})\*grad{1})';

k = 2;

%loop to update the points while calculating the hessian at every step

%to set the step size

while(norm((X\_init\_array1{k-1}-X\_init\_array1{k}),2)~=0)

grad{k} = (double(subs(gradient(f, [x1 x2]), {x1,x2}, X\_init\_array1(k))));

hess{k} = double(subs(hessian(f, [x1 x2]), {x1,x2}, X\_init\_array1(k)));

if eig(hess{k})>0 %check if hess{k} is pd

% update kth point with hess{k} as step size

X\_init\_array1{k+1}= X\_init\_array1{k}-(pinv(hess{k})\*grad{k})';

k = k+1;

else

sprintf('Hessian at the %d th is not pd. Descent not guaranteed.', k);

end

end

else

disp('Hessian at the first point is not pd. Descent not guaranteed.');

end

% plotting the contours/ level sets for the function to plot sequence later

x = linspace(-1,1,50);

y = x;

[x1,x2] = meshgrid(x,y);

f = (x2-x1).^4+12.\*x1.\*x2-x1+x2-3;

% box on; mesh(x1,x2,f); %plotting the function

contour(x1,x2,f, 20); hold on;

for i = 1:length(X\_init\_array1)

px(i) = X\_init\_array1{i}(1);

py(i) = X\_init\_array1{i}(2);

end

hold on;

plot(px, py, 'x-'); %plot sequence of points starting from starting point1

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Description automatically generatedFigure 8. Sequence of point obtained by ignoring the fact that the Hessian is pd. Init pt – [0.55,0.7]

Figure 9. Sequence of point obtained by ignoring the fact that the Hessian is pd. Init pt – [0-0.9,-0.5]

The plots shown above shows that if we try to apply the Newtons method to a point away from the minimizer and to a function whose Hessian is not positive definite, the method fails to converge to a minimizer. We can see that the method reaches the saddle point and stops there.

However, for sanity check if we check a starting point near to minimizer, the Hessian comes out to be pd and converges to the minima.

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Description automatically generatedFigure 10. Sequence of points for a point near the minimizer