19.02.2018.

Функције

$f: \underset{\text{домен}}{A} \to \underset{\text{кодомен}}{B}$ — сваком ел. скупа A додељује тачно 1 ел. скупа B

$D_f = \{x \in \mathbb{R} \mid f(x) \text{ дефинисано}\}$

* f је инјективна ("1-1") ако: $(\forall a_1, a_2 \in A)\ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
* f је сурјективна ("на") ако: $(\forall b \in B)(\exists a \in A)\ f(a) = b$.
* f је бијекција ако је и "1-1" и "на"

$f: A \to B$ бијекција $\Rightarrow$ постоји инверзна ф-ја $f^{-1}: B \to A$ (исто бијекција)

* $f^{-1}(b) := a$ $\qquad f^{-1} \circ f(a) = a = \mathbb{1}_A(a)$ идентичко пресликавање

* $\left.\begin{array}{l} f: A \to B \text{ бијекција} \\ A, B \text{ - коначни скупови} \end{array}\right\}$ (кардиналност) $|A| = |B|$

Деф A и B су скупови. $|A| = |B|$ ако постоји бијекција између њих $f: A \to B$.

* $\left.\begin{array}{l} \mathbb{N}, \mathbb{Z}, \mathbb{Q} \text{ - пребројиви} \\ \mathbb{R} \text{ - непребројив} \end{array}\right\}$ $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

•

$\mathbb{N}$	1	2	3	4	5	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	1	-1	2	-2	...

$f: \mathbb{Z} \to \mathbb{N}$, $f(0) = 1$

$f(n) = 2n$, $n \in \mathbb{N}$

$f(n) = -2n + 1$, $n < 0$

f бијекција $\Rightarrow |\mathbb{N}| = |\mathbb{Z}|$

• $|\mathbb{N}| = |\mathbb{Q}| \Rightarrow$ бијекција са $\mathbb{N}$

0	1	2	3	4	5	...
-1	-2	-3	-4	-5	...	
$1/2$	$3/2$	$5/2$	$7/2$	...		
$-1/2$	$-3/2$	$-5/2$	...			

1 $|A| = m$, $|B| = n$, $m, n \in \mathbb{N}$

А) Колико има ф-ја $f: A \to B$? – $\underset{1)}{n} \, \underset{2)}{n} \cdots \underset{m)}{n} \to n^m$

Б) Колико има "1-1" ф-ја $f: A \to B$? – $n, n-1, \ldots, n-(m-1) \to n(n-1)\cdots(n-m+1)$

В) Колико има бијекција $f: A \to B$? – $n!$ $(m = n)$

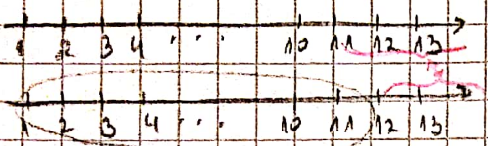
2 Да ли постоји "НА" ф-ја $f: \mathbb{N} \to \mathbb{N}$ т.д. $f(n+5) \geq 2n$, $\forall n \in \mathbb{N}$?

$m = n + 5 \geq 6$ $\quad f(m) \geq 2(m-5)$

$n = m - 5$ $\quad f(m) \geq 2m - 10$, $\forall m \geq 6$

Пп да постоји такво f које је "НА".

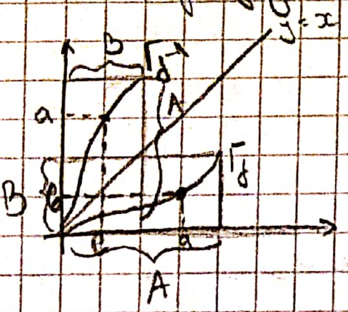
$f(m) \geq 2m - 10$ када је $> m \Rightarrow$ за $m > 10$ тј $m \geq 11 \to f(m) > 11$



→ у њих се сликају само $1, \ldots, 10 \Rightarrow$ није "НА" ↯

* Графиk $f: A \to B$, $\Gamma_f = \{(a,b) \in A \times B \mid b = f(a)\} = \{(a, f(a)) \mid a \in A\} \subset A \times B$

* $f: A \to B$ бијекција $\Gamma_f \rightsquigarrow \exists f^{-1}: B \to A$ $\quad \Gamma_{f^{-1}}$? $\quad (A, B \subset \mathbb{R})$



$(a, b) \in \Gamma_f \leftrightarrow (b, a) \in \Gamma_{f^{-1}} \subset B \times A$

$f^{-1}(b) = a$

• Γ_f и $\Gamma_{f^{-1}}$ су симетрични у односу на $y = x$.

⊛ Монотоност

• $f: A \to B$ је растућа ако $(\forall a_1, a_2 \in A)\ a_1 < a_2 \Rightarrow f(a_1) \leq f(a_2)$

• $f: A \to B$ је опадајућа ако $(\forall a_1, a_2 \in A)\ a_1 < a_2 \Rightarrow f(a_1) \geq f(a_2)$

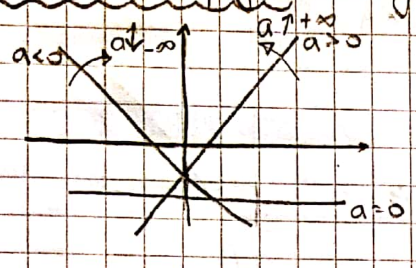
Једина ф-ја која је и растућа и опадајућа је const. ф-ја

Парност / непарност

- $f: A \to B$ и A - симетричан у односу на нулу
- f је **парна** ако $(\forall x \in A)\ f(-x) = f(x)$ (Γ_f је симетричан у одн. на y-осу)
- f је **непарна** ако $(\forall x \in A)\ f(-x) = -f(x)$ (Γ_f је сим. у одн на коорд. почетак)

~ Основне функције ~

I Линеарна ф-ја – $y = f(x) = ax + b$



II Квадратна ф-ја – $f(x) = ax^2 + bx + c,\ a \neq 0$

$f: \mathbb{R} \to \mathbb{R}$ $\quad D = b^2 - 4ac$ (дискриминанта) $\quad a>0 \quad a<0$

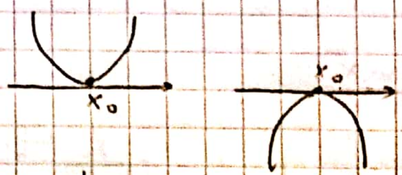
$T\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

1° $D > 0$  $\quad x_1, x_2 = \frac{-b \pm \sqrt{D}}{2a}$ – 2 реалне нуле

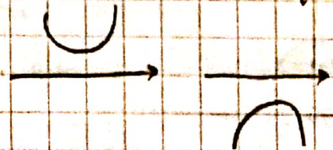
$$f(x) = a(x - x_1)(x - x_2)$$

2° $D = 0$ – 1 двострука нула $\quad x_0 = \frac{-b}{2a}$

$$f(x) = a(x - x_0)^2$$



3° $D < 0$ – 2 коњуговано компл. нуле

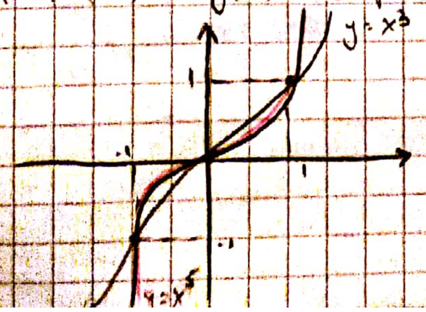


* $f(x) = x^2$ $\quad f: \mathbb{R} \to \mathbb{R}$ није ни "на" ни "1-1"

III $f(x) = x^n$, $n \in \mathbb{N} \setminus \{1\}$

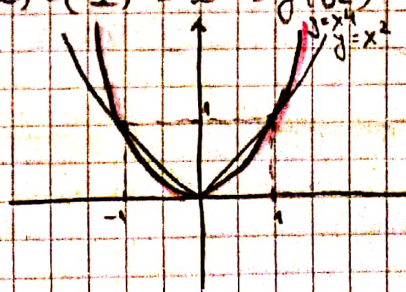
1° n-непарно

$f(-x) = (-x)^n = -f(x)$ – непарна ф-ја



2° n-парно

$f(-x) = (-x)^n = x^n = f(x)$ парна ф-ја



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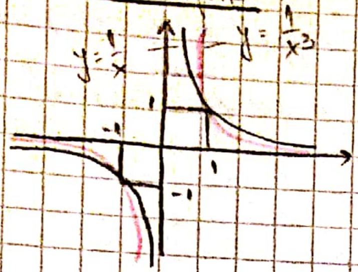
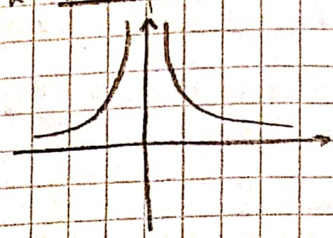
IV $f(x) = \frac{1}{x^n}$, $n \in \mathbb{N}$, $x \neq 0$

1° n-непарно



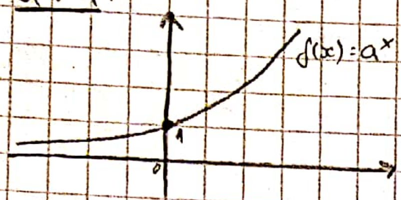
2° n-парно



V Експоненцијална и логаритамска ф-ја – $f(x) = a^x$, $f: \mathbb{R} \to \mathbb{R}$

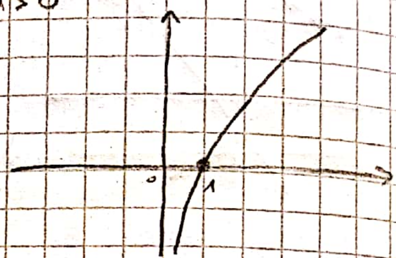
$a \neq 1$
$a > 0$

• $a > 1$:

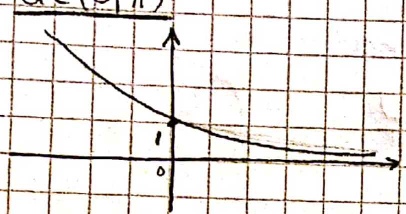


$\mathbb{R} \mapsto (0, +\infty)$
бијекција

$\xRightarrow{\text{инверз}} \log_a : (0, +\infty) \to \mathbb{R}$

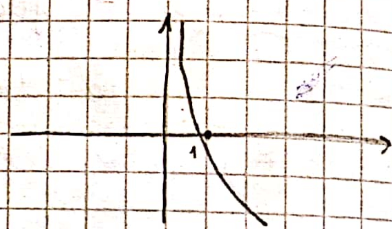


• $a \in (0, 1)$



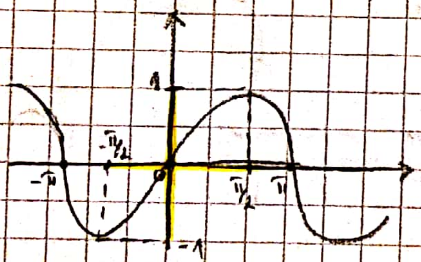
$\mathbb{R} \to (0, +\infty)$
бијекција

$\xRightarrow{\text{инверз}} \log_a : (0, +\infty) \to \mathbb{R}$



VI Тригонометријске ф-је

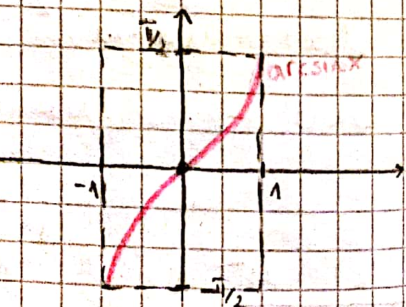
• $f(x) = \sin x$



$\mathbb{R} \to [-1, 1]$ "на"

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \xrightarrow{\sin} [-1, 1]$
бијекција

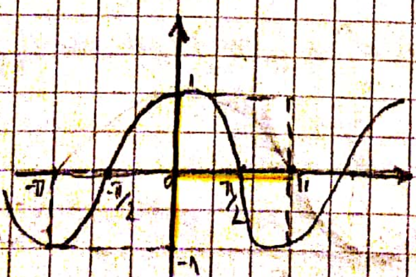
$\boxed{\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$



• $t \in [-1, 1] \to \sin(\arcsin t) = t$

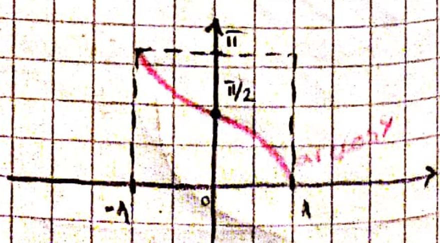
• $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \arcsin(\sin \alpha) = \alpha$ } важи само за ове услове

• $f(x) = \cos x$

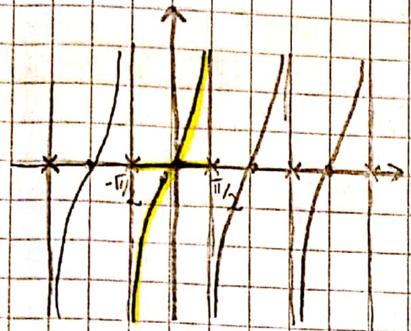


$[0, \pi] \xrightarrow{\cos} [-1, 1]$
бијекција

$\boxed{\arccos : [-1, 1] \mapsto [0, \pi]}$



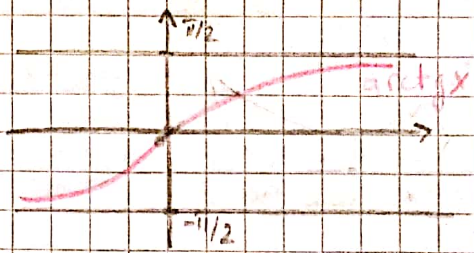
• $\underline{f(x) = tg\, x} = \frac{\sin x}{\cos x}$ – $D_f = \{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$; јесте "на" биј



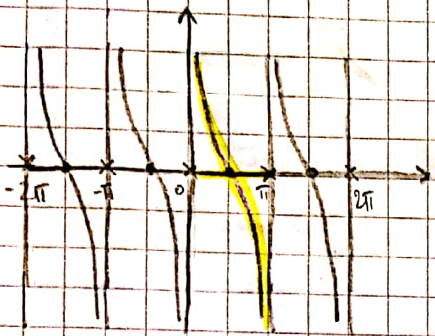
$(-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{tg} \mathbb{R}$

бијекција

$\boxed{arctg : \mathbb{R} \mapsto (-\frac{\pi}{2}, \frac{\pi}{2})}$



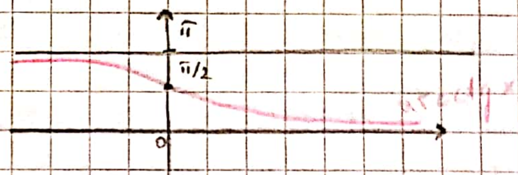
• $\underline{f(x) = ctg\, x} = \frac{\cos x}{\sin x}$ – $D_f = \{x \in \mathbb{R} \mid x \neq k\pi,\ k \in \mathbb{Z}\}$

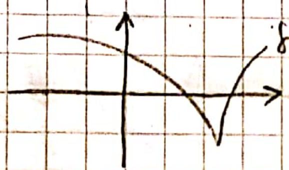


$(0, \pi) \xrightarrow{ctg} \mathbb{R}$

бијекција

$\boxed{arcctg : \mathbb{R} \mapsto (0, \pi)}$



$*$ f, Γ_f (нпр.: )

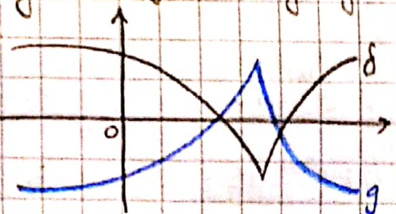
1° $g(x) = f(x) + c$

$\uparrow c > 0$ $\downarrow c < 0$

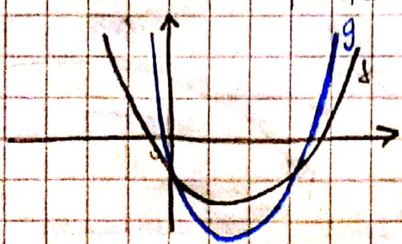
2° $g(x) = f(x+a)$

$\leftarrow a > 0$ $\rightarrow a < 0$

3° $g(x) = -f(x)$ (у одн. на x-осу)

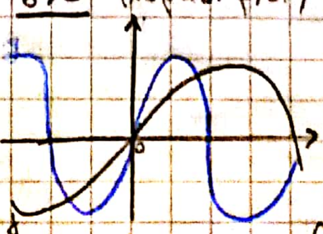


4° $g(x) = c \cdot f(x)$ (вертикално раст./сабиј.)

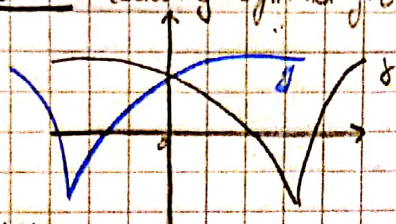


5° $g(x) = f(b \cdot x)$

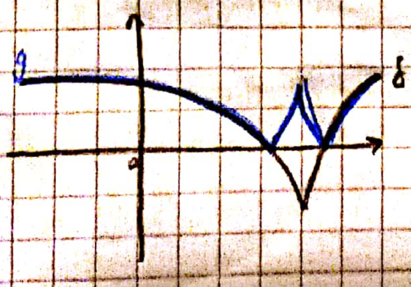
$\underline{b > 0}$ (хориз. р/с.)



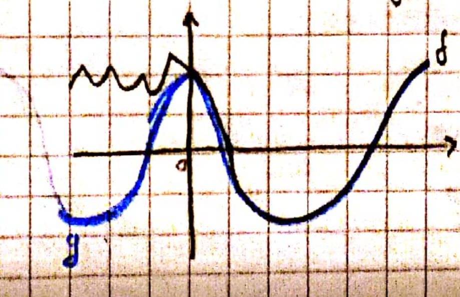
$\underline{b < 0}$ (сим. у одн. на y-осу)



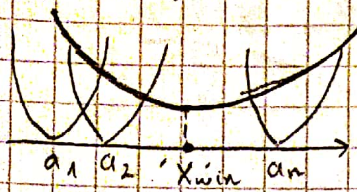
6° $g(x) = |f(x)|$



7° $g(x) = f(|x|) = \begin{cases} f(x), & x > 0 \\ f(-x), & x < 0 \end{cases}$ (у одн. на y-осу)



$\boxed{1}$ $a_1, a_2, \dots, a_n \in \mathbb{N}$, $f: \mathbb{R} \to \mathbb{R}$, $f(x) = (x-a_1)^2 + \dots + (x-a_n)^2$. Наћи тачку у којој ф-ја f достиже минимум.

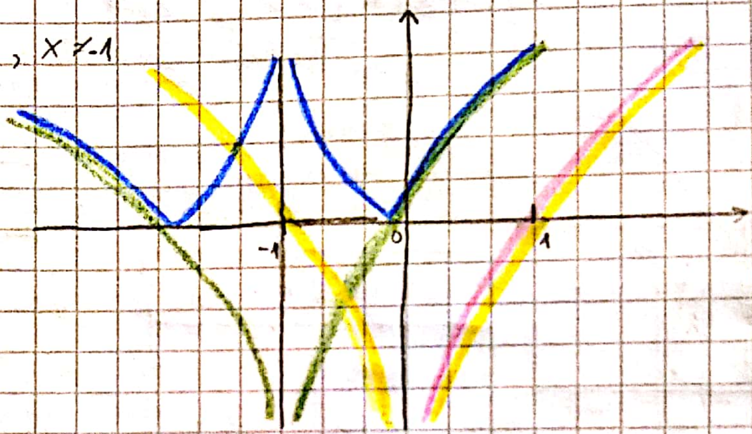


$$f(x) = x^2 - 2a_1x + a_1^2 + \dots + x^2 - 2a_nx + a_n^2 =$$
$$= n \cdot x^2 - 2(a_1 + \dots + a_n)x + (a_1^2 + \dots + a_n^2)$$

$$x_{min} = \frac{2(a_1 + \dots + a_n)}{2n} = \frac{a_1 + \dots + a_n}{n} \quad \text{(аритметичка средина)}$$

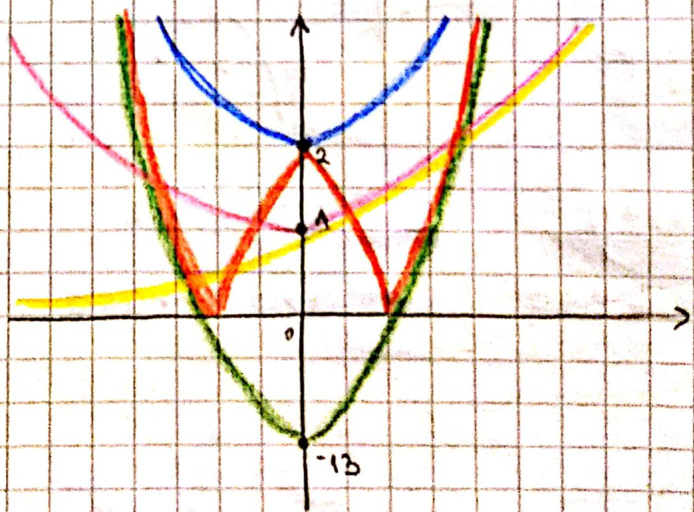
$\boxed{2}$ $f(x) = |\ln|x+1||$, $|x+1| > 0$, $x \neq -1$

- $f_1(x) = \ln x$
- $f_2(x) = |x|$
- $f_3(x) = \ln|x+1|$
- $f_4(x) = |\ln|x+1||$



$\boxed{3}$ $f(x) = |2 \cdot 5^{|x|} - 15|$

- $f_1(x) = 5^x$
- $f_2(x) = 5^{|x|}$
- $f_3(x) = 2 \cdot 5^{|x|}$
- $f_4(x) = 2 \cdot 5^{|x|} - 15$
- $f_5(x) = |2 \cdot 5^{|x|} - 15|$



$\boxed{4}$ $f(x) = ||x| - 3| - 1$

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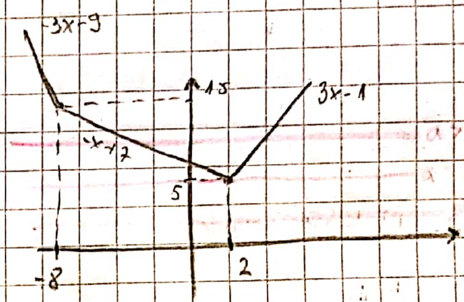
5. $f(x) = 2 \cdot \sin\left(\frac{4}{3}x + \frac{\pi}{3}\right)$

6. $f(x) = |x+8| + 2|x-2| - 5$, $a \in \mathbb{R}$. Одреди број решења ј-не $f(x) = a$ у зависности од a.

	$(-\infty, -8)$	$(-8, 2)$	$(2, +\infty)$
$\lvert x-8 \rvert$	$-x-8$	$x-8$	$x-8$
$2\lvert x-2 \rvert$	$-2(x-2)$	$-2(x-2)$	$2(x-2)$
$f(x)$	$-3x-9$	$-x+7$	$3x-1$

$f(-8) = 15$

$f(2) = 5$



- $a<5$: нема решења
- $a=5$: 1 решење ($x=2$)
- $a>5$: тачно 2 решења увек

26.02.2018.

Математичка индукција

1° база $P(1)$ (n_0)

2° индуктивни корак: $n \mapsto n+1$; $\underset{\text{и.х.}}{P(n)} \Rightarrow P(n+1)$, $\forall n \in \mathbb{N}$ $(\forall n \geq n_0)$

$\xRightarrow{1^\circ, 2^\circ} P(n)$ важи за $\forall n \in \mathbb{N}$ $(\forall n \geq n_0)$

* $\forall n \in \mathbb{N}$ важи:

а) $1+2+\ldots+n = \frac{n(n+1)}{2}$

б) $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

в) $1^3+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

г) $1+3+\ldots+(2n-1) = n^2$

д) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$

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[1] Докажи: $133 \mid 11^{n+2} + 12^{2n+1}$, $\forall n \in \mathbb{N}$

1° $P(1)$: $133 \mid 11^3 + 12^3$ $\quad 133 \mid 3059$ ✓

2° И.Х.: $133 \mid 11^{n+2} + 12^{2n+1}$. За $n+1$: $11^{n+3} + 12^{2n+3} = 11^{n+2} \cdot 11 + 144 \cdot 12^{2n+1} =$

$= 11(\underbrace{11^{n+2} + 12^{2n+1}}_{✓}) + \underbrace{133 \cdot 12^{2n+1}}_{✓} \Rightarrow$ важи за $n+1$

[2] $A_n = (n+1)(n+2)\ldots(n+n)$. Докажи: $2^n \mid A_n$ и $2^{n+1} \nmid A_n$ ($2^n \| A_n$ "точно дели")

1° $P(1)$: $A_1 = 2$ $\quad 2 \mid 2$ $\quad 2^2 \nmid 2$ ✓

2° И.Х.: $2^n \mid A_n$, $2^{n+1} \nmid A_n$. За $n+1$: $A_{n+1} = (n+2)(n+3)\ldots(2n+2) =$

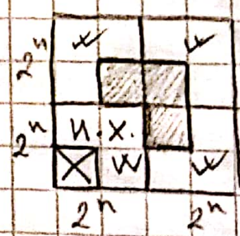
$= 2\underbrace{(n+1)(n+2)(n+3)\ldots(n+n)}_{A_n = 2^n \cdot B \text{ (B-нечетно)}}(2n+1)$

$A_{n+1} = 2 \cdot 2^n \cdot B \cdot (2n+1) = 2^{n+1} \cdot$ нечетно $\quad 2^{n+1} \mid A_{n+1}$, $2^{n+2} \nmid A_{n+1}$ ✓

[3] $2^n \times 2^n$ табло без 1 поле. Докажи да се може покриват са  и  без препокриване.

1° $P(1)$: 2×2  ✓

2° И.Х. важи за n. За $n+1$: $2^{n+1} \times 2^{n+1}$ − 1 поле:



$\Rightarrow$ важи за $n+1$ $\xRightarrow{1^\circ, 2^\circ}$ тм

28.02.2018.

Биномни коефициенти

$k \in \mathbb{N}_0$, n: $\binom{n}{k} = \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$, за $n \in \mathbb{N}_0$ $\quad 0! = 1$

$\binom{n}{0} = 1 \quad \binom{n}{n} = 1 \quad \binom{n}{k}$ – бр. начини да од n елементи изабереме k одеднаш

1) $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

2) $\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$

Доказ

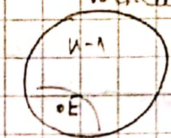
I начин

$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} =$

$= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{n-k}\right) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

II начин

n елементи → бираме k



$\binom{n}{k} = \underbrace{\binom{n-1}{k}}_{\text{ако E не је изабран}} + \underbrace{\binom{n-1}{k-1}}_{\text{E је изабран}}$

~ Њутнова биномна формула ~

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + b^n$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

Доказ

• **I начин:** 1° $n=1$: $(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{0}a^1 + \binom{1}{1}b^1 = a + b$ ✓

2° $n \mapsto n+1$: И.Х.: важи за n. За n+1:

$(a+b)^{n+1} = (a+b)(a+b)^n \overset{\text{И.Х.}}{=} (a+b)\cdot\left(a^n + \binom{n}{1}a^{n-1}b + \ldots + b^n\right) =$

$= a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + \ldots + \binom{n}{k}a^{n+1-k}b^k + \ldots + \binom{n}{n-1}a^2 b^{n-1} + ab^n +$

$+ \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \ldots + \binom{n}{k-1}a^{n+1-k}b^k + \ldots + \binom{n}{n-2}a^2 b^{n-1} + \ldots + b^{n+1} =$

$= a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+2}{2}a^{n-1}b^2 + \ldots + b^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k$ ✓

• **II начин:** $(a+b)^n = \underbrace{(a+b)\cdot\ldots\cdot(a+b)}_{n\text{ пута}} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\cdot b^k$

$\binom{n}{k}$ – бр. начина да од n заграда изаберемо k из којих узимамо b

$\boxed{1}$ Доказати да је $\cos\frac{\pi}{2^n}$ ирационалан за $n \geq 2$.

1° $\underline{n=2}$: $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \notin \mathbb{Q}$ ✓

2° $\underline{n \mapsto n+1}$: И. Х.: $\cos\frac{\pi}{2^n} \notin \mathbb{Q}$. За $n+1$:

$\cos\frac{\pi}{2^{n+1}} \Rightarrow \left(\cos\frac{\pi}{2^{n+1}}\right)^2 = \underbrace{\frac{1-\cos\frac{\pi}{2^n}}{2}}_{\text{их}\,\notin\mathbb{Q}} \Rightarrow \cos\frac{\pi}{2^{n+1}} \notin \mathbb{Q}$ ✓

$\boxed{2}$ Дат је низ: $x_1 = 1,\ x_2 = 2,\ x_n = (n-1)(x_{n-1} + x_{n-2}),\ \forall n > 3$. Доказати да је $x_n = n!$ за $\forall n \in \mathbb{N}$

• Индукција са више претпоставки:

1° база: $n=1$ $x_1 = 1 = 1!$ ✓
$n=2$ $x_2 = 2 = 2!$ ✓

2° $\underline{n-1, n-2 \mapsto n}$ И.Х.: Важи за $n-1$ и $n-2$: $x_{n-1} = (n-1)!$ и $x_{n-2} = (n-2)!$. За n:

$x_n = (n-1)(x_{n-1} + x_{n-2}) \overset{\text{их}}{=} (n-1)((n-1)! + (n-2)!) = (n-1)(n-2)!(n-\cancel{1}+\cancel{1}) = \underline{n!}$ ✓

$\boxed{3}$ $x_0 = 1,\ x_n = x_{n-1} + x_{n-2} + \ldots + x_2 + x_1 + 2x_0$. Доказати да је $x_n = 2^n$.

• Трансфинитна индукција:

1° $\underline{n=0}$: $x_0 = 1 = 2^0$ ✓

2° $\underline{0, 1, \ldots, n-1 \mapsto n}$: И.Х.: $x_0 = 2^0, x_1 = 2^1, \ldots, x_{n-1} = 2^{n-1}$. За n:

$x_n = x_{n-1} + x_{n-2} + \ldots + x_1 + 2x_0 \overset{\text{их}}{=} 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 2\cdot 1 = 1 + \frac{2^n - 1}{2-1} = 1 + 2^n - 1 = \underline{\underline{2^n}}$

$\boxed{4}$ Доказати:

а) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1),\ \forall n \in \mathbb{N}$

б) $\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n} \leq \sqrt{\frac{1}{3n+1}},\ \forall n \in \mathbb{N}$

в) $2^n > n^2,\ n \geq 5$

г) $\frac{4^n}{n+1} \leq \frac{(2n)!}{(n!)^2},\ \forall n \geq 2$

д) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n},\ \forall n \in \mathbb{N}$

А) 1° $n=1$: $\frac{1}{\sqrt{1}} > 2(\sqrt{2}-1) \Rightarrow 1 > 2\sqrt{2}-2 \Rightarrow 3 > 2\sqrt{2}$ ✓

2° $n \mapsto n+1$: И.Х.: Важи за n. За $n+1$:

$$\underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}_{\overset{\text{их}}{>} 2(\sqrt{n+1}-1)} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2}-1)$$

$$2(\sqrt{n+1}-1) + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2}-1) \quad / \cdot \sqrt{n+1}$$

$$2(n+1)+1 > 2\sqrt{(n+1)(n+2)} \quad /^2$$

$$\cancel{4n^2} + \cancel{12n} + 9 > \cancel{4n^2} + \cancel{12n} + 8 \quad ✓$$

Д) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \geq n \cdot \underset{\overset{\|}{1/\sqrt{n}}}{\text{најмањи}} = n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}$ ✓

Г) 1° $n=2$: $\frac{4^2}{3} \leq \frac{4!}{(2!)^2} \Rightarrow \frac{16}{3} \leq \frac{24}{4}$ ✓

2° $n \mapsto n+1$: И.Х.: Важи за n. За $n+1$:

$$\frac{4^{n+1}}{n+2} < \frac{(2(n+1))!}{((n+1)!)^2}$$

$$\frac{4^n}{n+1} \cdot \frac{n+1}{n+2} \cdot 4 \overset{\text{их}}{<} \cancel{\frac{(2n)!}{(n!)^2}} \cdot \frac{n+1}{n+2} \cdot 4 < \frac{(2(n+1))!}{((n+1)!)^2}$$

$$\Rightarrow \frac{n+1}{n+2} \cdot 4 < \frac{(2n+1)\,2\cancel{(n+1)}}{(n+1)^{\cancel{2}}} \Rightarrow \frac{n+1}{n+2} \cdot 4 < \frac{2(2n+1)}{n+1}$$

$$\Rightarrow 4(n+1)^2 < (4n+2)(n+2)$$

$$\cancel{4n^2} + 8n + 4 < \cancel{4n^2} + 10n + 4 \quad ✓$$

Лема $n, k, l \in \mathbb{N}$, $\max(k,l) = n \Rightarrow k = l = n$.

* **Доказ** *

1° $n=1$: $\max(k,l) = 1 \Rightarrow k = l = 1$

2° $n \mapsto n+1$: И.Х.: Важи за n. За $n+1$: $\max(k,l) = n+1 \Rightarrow k = l = n+1$, $k, l \in \mathbb{N}$

$\max(\underbrace{k-1, l-1}_{\Downarrow}) = n \overset{\text{их}}{\Rightarrow} k-1 = l-1 = n$

$\Rightarrow k = l = n+1$ ✓

Неједнакости

* **Бернулијева неједнакост:** $n\in\mathbb{N},\ x>-1$ $\quad \boxed{(1+x)^n \geq 1+n\cdot x}$

• За $x>0$: $(1+x)^n = 1+nx+\underbrace{\binom{n}{2}x^2+\dots+\binom{n}{n-1}x^{n-1}+x^n}_{\geq 0} \geq 1+nx$

• још $x>-1$: индукцијом по n:

$n=1$: $(1+x)^1 \geq 1+x$ ✓

$n\to n+1$: ИХ: $(1+x)^n \geq 1+nx$. За $n+1$:

$(1+x)^{n+1} = (1+x)^n(1+x) \overset{\text{ИХ}}{\geq} (1+nx)(1+x)$, $1+x>0$, $x>-1$

$= 1+x+nx+nx^2 = 1+(n+1)x+\underbrace{nx^2}_{\geq 0} \geq 1+(n+1)x$ ✓

1 $a,b,c\in\mathbb{R}$

А) $a^2+b^2 \geq 2ab$

доказ

$(a-b)^2 \geq 0$

$a^2-2ab+b^2 \geq 0$

$a^2+b^2 \geq 2ab$ ✓

(= ако $a=b$)

Б) $a^2+b^2+c^2 \geq ab+bc+ca$

$a^2+b^2 \geq 2ab$

$b^2+c^2 \geq 2bc$ $\quad +$

$a^2+c^2 \geq 2ac$

$2(a^2+b^2+c^2) \geq 2(ab+bc+ac)$

$a^2+b^2+c^2 \geq ab+bc+ac$ ✓

(= ако $a=b=c$)

В) ако $a+b\geq 0$, $a^3+b^3 \geq a^2b+ab^2$

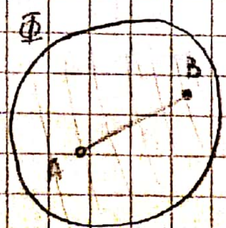
$a^3-a^2b-b^2a+b^3 \geq 0$

$a^2(a-b)+b^2(b-a) \geq 0$

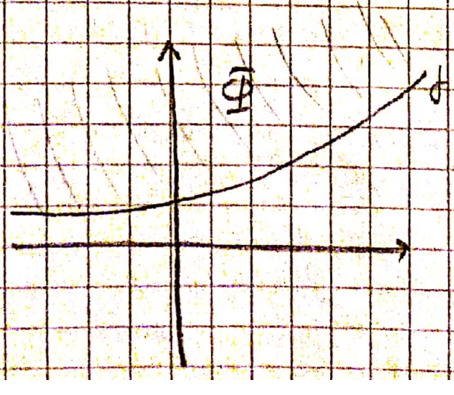
$(a-b)(a^2-b^2) \geq 0$

$\underbrace{(a-b)^2}_{\geq 0}\underbrace{(a+b)}_{\geq 0} \geq 0$ ✓

~ Конвексна фигура ~

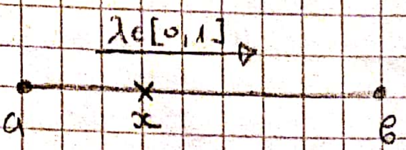


$\Phi\subset\mathbb{R}^n$, Φ је конвексно ако $\forall A,B\in\Phi$ дуж $[AB]\subset\Phi$



Φ - изнад графика

Φ конв. $\Rightarrow f$ је конв.



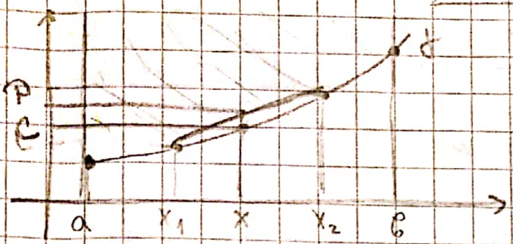
$x=(1-\lambda)a+\lambda b$, $\lambda\in[0,1]$

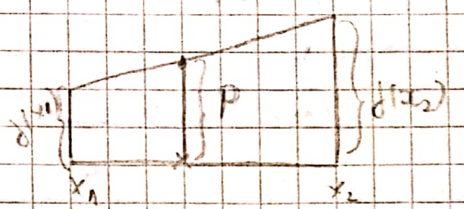
$\lambda=0$: $x=a$

$\lambda=1$: $x=b$

Деф $f:[a,b]\to\mathbb{R}$ је конвексна на $[a,b]$ ако $\forall x_1,x_2\in[a,b]$, $\forall\lambda\in[0,1]$,

$x=(1-\lambda)x_1+\lambda x_2$: $\boxed{f((1-\lambda)x_1+\lambda x_2)\le(1-\lambda)f(x_1)+\lambda f(x_2)}$





слично $\Rightarrow P=(1-\lambda)f(x_1)+\lambda f(x_2)$

- $\lambda=\frac{1}{2}$: $f\left(\frac{x_1+x_2}{2}\right)\le\frac{f(x_1)+f(x_2)}{2}$, f-конвексна

* Јенсенова једнакост $f:[a,b]\to\mathbb{R}$ конвексна, $x_1,\dots,x_n\in[a,b]$. Важи:

$$\boxed{f\left(\frac{x_1+\dots+x_n}{n}\right)\le\frac{f(x_1)+\dots+f(x_n)}{n}}$$

* Неједнакости средина $x_1,\dots,x_n\in\mathbb{R}>0$. Важи:

$$\boxed{\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}}\le\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}\le\frac{x_1+x_2+\dots+x_n}{n}\le\sqrt{\frac{x_1^2+x_2^2+\dots+x_n^2}{n}}}$$

H_n - хармонијска средина, G_n - геометријска средина, A_n - аритметичка средина, K_n - квадратна средина

- $A_n\le K_n$, $n\in\mathbb{R}$ $\qquad a,b>0$: $\frac{2}{\frac{1}{a}+\frac{1}{b}}\le\sqrt{ab}\le\frac{a+b}{2}\le\sqrt{\frac{a^2+b^2}{2}}$

Доказ

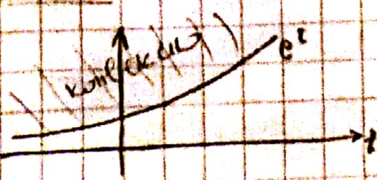
$A_n\le K_n$, $f(x)=x^2$

$f:\mathbb{R}\to\mathbb{R}$ је конвексно. $x_1,\dots,x_n\in\mathbb{R}$

$\xRightarrow{\text{Јенсен}}\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^2\le\frac{x_1^2+\dots+x_n^2}{n}$. $t^2\le c\Rightarrow t\le\sqrt{c}\Rightarrow\frac{x_1+\dots+x_n}{n}\le\sqrt{\frac{x_1^2+\dots+x_n^2}{n}}$

- $=$ (за све средине) када је: $x_1=x_2=\dots=x_n$
- $G_n\le A_n$, $n\in\mathbb{R}$

Доказ



$f:\mathbb{R}\to\mathbb{R}$, $f(t)=e^t$, f-конвексна.

(Јенсен) $t_1,\dots,t_n$: $f\left(\frac{t_1+\dots+t_n}{n}\right)\le\frac{f(t_1)+\dots+f(t_n)}{n}$

$e^{\frac{t_1+\dots+t_n}{n}}\le\frac{e^{t_1}+\dots+e^{t_n}}{n}$

$x_i=e^{t_i}$, $t_i\in\mathbb{R}\to x_i>0$

$\sqrt[n]{e^{t_1}\cdot e^{t_2}\cdots e^{t_n}}=\sqrt[n]{x_1\cdots x_n}\Rightarrow\sqrt[n]{x_1\cdots x_n}\le\frac{x_1+\dots+x_n}{n}$

• $H_n \leq G_n$

доказ

$G_n \leq A_n$ за $\frac{1}{x_1}, \dots, \frac{1}{x_n}$ • $\sqrt[n]{\frac{1}{x_1} \cdot \dots \cdot \frac{1}{x_n}} \leq \frac{\frac{1}{x_1} + \dots + \frac{1}{x_n}}{n}$

$\sqrt[n]{x_1 \cdot \dots \cdot x_n} \geq \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$ ✓

$\boxed{1}$ Доказати: $(x_1 + \dots + x_n)\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) \geq n^2$, $x_1, \dots, x_n > 0$.

$H_n \leq A_n$: $\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \leq \frac{x_1 + \dots + x_n}{n}$

$(x_1 + \dots + x_n)\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) \geq n^2$ ✓

$\boxed{2}$ Доказати:

А) $1 \cdot 2^2 \cdot 3^3 \cdot \dots \cdot n^n \leq \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$.

$1 + 2 + \dots + n = \frac{n(n+1)}{2}$. $G_{\frac{n(n+1)}{2}} \leq A_{\frac{n(n+1)}{2}}$

$$\sqrt[\frac{n(n+1)}{2}]{1 \cdot 2 \cdot 2 \cdot \dots \cdot \underbrace{n \cdot \dots \cdot n}_{n}} \leq \frac{1 + 2 + 2 + \dots + \overbrace{n + \dots + n}^{n}}{\frac{n(n+1)}{2}} = \frac{1 + 2^2 + \dots + n^2}{\frac{n(n+1)}{2}} = \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}} = \frac{2n+1}{3}$$

$1 \cdot 2^2 \cdot \dots \cdot n^n \leq \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$ ✓

Б) $1 \cdot 2^2 \cdot \dots \cdot n^n \geq \left(\frac{n+1}{2}\right)^{\frac{n(n+1)}{2}}$ (преко $G_n \geq H_n$)

В) $n! \leq \left(\frac{n+1}{2}\right)^n$ (преко $G_n \leq A_n$)

$\boxed{3}$ $x, y > 0$. Доказати: $\left(\frac{x+y}{2}\right)^3 \leq \frac{x^3 + y^3}{2}$

$\frac{x^3 + 3x^2y + 3xy^2 + y^3}{\cancel{8}\, 4} \leq \frac{x^3 + y^3}{\cancel{2}\, 1}$ $/ \cdot 4$

$\cancel{x^3} + 3x^2y + 3xy^2 + \cancel{y^3} \leq \cancel{4}3x^3 + \cancel{4}3y^3$ $/ : 3$

$x^2y + xy^2 \leq x^3 + y^3$

$x^3 - x^2y + y^3 - xy^2 \geq 0$

$\underbrace{(x-y)^2}_{\geq 0}\underbrace{(x+y)}_{> 0} \geq 0$ ✓

Коши-Шварц – $a_1,\dots,a_n, b_1,\dots,b_n \in \mathbb{R}$. Важи:

$$\boxed{(a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2) \geq (a_1b_1+\dots+a_nb_n)^2} \quad (*)$$

Доказ

$f:\mathbb{R}\to\mathbb{R}$, $f(x) = (a_1x+b_1)^2+\dots+(a_nx+b_n)^2 \geq 0$

$D \leq 0$: $f(x) = a_1^2x^2+2a_1b_1x+b_1^2+\dots+a_n^2x^2+2a_nb_nx+b_n^2 =$

$$= (a_1^2+\dots+a_n^2)x^2+(2a_1b_1+\dots+2a_nb_n)x+(b_1^2+\dots+b_n^2).$$

$$D = 4(a_1b_1+\dots+a_nb_n)-4(a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2) \leq 0 \Rightarrow \text{важи } (*)$$

- Када важи једнакост?

$D=0 \to \exists x \; f(x)=0$, $\exists x$: $a_1x+b_1=0 \to b_1=\overset{\lambda}{(-x)}a_1$, $\vdots$, $a_nx+b_n=0 \to b_n=(-x)a_n$ $\Big\}$ $(b_1,\dots,b_n)=\lambda(a_1,\dots,a_n)$ пропорционалност

Пример (1. задатак)

$$\big((\sqrt{x_1})^2+\dots+(\sqrt{x_n})^2\big)\cdot\left(\left(\tfrac{1}{\sqrt{x_1}}\right)^2+\dots+\left(\tfrac{1}{\sqrt{x_n}}\right)^2\right) \overset{\text{К.Ш.}}{\geq} (\underbrace{1+\dots+1}_{n})^2 = n^2 \;\checkmark$$

$a_i=\sqrt{x_i}$, $b_i=\frac{1}{\sqrt{x_i}}$

* Неједнакост троугла – $x,y\in\mathbb{R}$. $\boxed{|x|+|y| \geq |x+y|}$

Доказ

$|x|+|y| \geq x+y \geq -(|x|+|y|)$

$-|x| \leq x \leq |x|$

$-|y| \leq y \leq |y|$ $\Big|\,(+) \Rightarrow -|x|-|y| \leq x+y \leq |x|+|y| \;\checkmark$

Пример $||x|-|y|| \leq |x-y|$

$-|x-y| \overset{(2)}{\leq} |x|-|y| \overset{(1)}{\leq} |x-y|$: (1) $|\underset{x+y-y}{x}| \leq |x-y|+|y|$ ✓

(2) $|y| \leq |x|+|x-y|$

$|\underset{x+y-x}{y}| \leq |x|+|y-x|$ ✓

- Када важи једнакост?

– Ако су $x,y\geq 0 \vee x,y\leq 0$.

4. $x_1, \dots, x_n \in [0, \pi]$. Доказати: $\left|\sin \sum_{k=1}^{n} x_k\right| \le \sum_{k=1}^{n} \sin x_k$. (*)

$\sin x \ge 0$

• Индукцијом по n:

1° $n=1$: $|\sin x_1| \le \sin x_1$ ✓

2° $n \to n+1$: И.Х. Важи (*) за $n+1$:

$$\left|\sin\sum_{k=1}^{n+1} x_k\right| = \Big|\sin\Big(\underbrace{\sum_{k=1}^{n} x_k}_{\alpha} + \underbrace{x_{n+1}}_{\beta}\Big)\Big| = |\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta|$$

$$\overset{\text{нј.}\Delta}{\le} |\sin\alpha|\cdot\underbrace{|\cos\beta|}_{\le 1} + \underbrace{|\cos\alpha|}_{\le 1}\cdot|\sin\beta| \le |\sin\alpha| + |\sin\beta| \overset{\text{ИХ}}{\le} \sum_{k=1}^{n} \sin x_k + \sin x_{n+1} = \sum_{k=1}^{n+1} \sin x_k$$

05.03.2018

Супремуми и инфимуми

$A \neq \emptyset$, $A \subset \mathbb{R}$

$\inf A$ $\sup A$

* $\bar{a} \in \mathbb{R}$ је горње ограничење скупа A ако $\boxed{(\forall a \in A)\ a \le \bar{a}}$

* $\boxed{\bar{a} = \sup A} \iff \begin{cases} 1^\circ\ (\forall a \in A)\ a \le \bar{a} & \text{(горње ограничење)} \\ 2^\circ\ (\forall b \in \mathbb{R})\ ((\forall a \in A)\ a \le b) \Rightarrow \bar{a} \le b & \text{(најмање)} \end{cases}$

$\iff \begin{cases} 1^\circ\ (\forall a \in A)\ a \le \bar{a} \\ 2^\circ\ (\forall \varepsilon > 0)(\exists a \in A)\ a \ge \bar{a} - \varepsilon \end{cases}$

* $\boxed{\underline{a} = \inf A} \iff \begin{cases} 1^\circ\ (\forall a \in A)\ \underline{a} \le a \\ 2^\circ\ (\forall \varepsilon > 0)(\exists a \in A)\ a \le \underline{a} + \varepsilon \end{cases}$

* $\max A$, $\min A$: • $\bar{a} \in A$, $\bar{a} = \max A$ ако $(\forall a \in A)\ a \le \bar{a}$

• $\underline{a} \in A$, $\underline{a} = \min A$ ако $(\forall a \in A)\ \underline{a} \le a$

* $\sup A \notin A \to$ не постоји $\max A$

* $\inf A \notin A \to$ не постоји $\min A$

* Аксиома супремума:

• $A \neq \emptyset$ и A - ограничен одозго $\Rightarrow \exists \sup A$

• $A \neq \emptyset$ и A - ограничен одоздо $\Rightarrow \exists \inf A$

(E) Архимедова аксиома: $(\forall \varepsilon>0)(\forall M>0)(\exists n\in\mathbb{N})\ n\cdot\varepsilon>M$

- $M=1$: $(\forall\varepsilon>0)\ \exists n\in\mathbb{N}\ \ n\cdot\varepsilon>1$, $\boxed{\varepsilon>\frac{1}{n}}$

[1] $A=\{\frac{m}{n}\mid m,n\in\mathbb{N},\ m<n\}$, $\sup A, \inf A, \max A, \min A$?

$\frac{m}{n}<1$ и $\frac{m}{n}>0$

- $0=\inf A$: 1° $0\le\frac{m}{n}$ ✓

2° $(\forall\varepsilon>0)(\exists\frac{m}{n}\in A)\ \frac{m}{n}<\varepsilon$?, $\varepsilon>0$ произвољно

$m=1$: $\exists n: \frac{1}{n}<\varepsilon$ ✓ (из Архимедове аксиоме)

- $0=\inf A\notin A \Rightarrow \neg\exists \min A$

- $1=\sup A$: 1° $\frac{m}{n}<1$ ✓ $(m<n)$

2° $\varepsilon>0$ произвољно, ? $\exists\frac{m}{n}\in A: \frac{m}{n}>1-\varepsilon$. Изаберемо $m=n-1$

$\exists n$? т.д. $\frac{n-1}{n}>1-\varepsilon \Leftrightarrow 1-\frac{1}{n}>1-\varepsilon \Leftrightarrow \varepsilon>\frac{1}{n}$ ✓

- $1=\sup A\notin A \Rightarrow \neg\exists\max A$

* Својства: $A,B\neq\emptyset$, $A,B\subset\mathbb{R}$: $A+B=\{a+b\mid a\in A, b\in B\}$

$A\cdot B=\{a\cdot b\mid a\in A, b\in B\}$

$-A=\{-a\mid a\in A\}$

$c\in\mathbb{R}$: $A>c \Leftrightarrow (\forall a\in A)\ a>c$

1° $\sup(A+B)=\sup A+\sup B$

2° $\inf(A+B)=\inf A+\inf B$

3° $\sup(-A)=-\inf A$

4° $\inf(-A)=-\sup A$

5° Ако $A,B>0$: $\sup(A\cdot B)=\sup A\cdot\sup B$

Докази

(1) $\bar a=\sup A$, $\bar b=\sup B$, $\bar a+\bar b=\sup(A+B)$?

1° $x\in A+B$ ($x=a+b$)

$\bar a=\sup A\Rightarrow a\le\bar a$

$\bar b=\sup B\Rightarrow b\le\bar b$ (+)

$a+b\le\bar a+\bar b$ ✓

2° $\varepsilon>0$ - произвољно.

$\bar a=\sup A$ за $\frac{\varepsilon}{2}>0$ $(\exists a\in A)$ $a>\bar a-\frac{\varepsilon}{2}$

$\bar b=\sup B$ за $\frac{\varepsilon}{2}>0$ $(\exists b\in B)$ $b>\bar b-\frac{\varepsilon}{2}$ (+)

$a+b>\bar a+\bar b-\varepsilon$ ✓

(4) $\inf(-A) = -\sup A$?, $\bar{a} = \sup A$

1° $-a \in -A$

$-\bar{a} \le -a$ јер је $\bar{a} \ge a, \forall a \in A$ ✓

2° $\varepsilon > 0$ произвољно, ? $\exists a \in A: -a \le -\bar{a} + \varepsilon \Leftrightarrow a \ge \bar{a} - \varepsilon$
знамо да постоји јер $\bar{a} = \sup A$ ✓

• Да ли (5) може без услова?

нпр. $\left.\begin{matrix} A = [-3,-1] \\ B = [0,1] \end{matrix}\right\}$ $\left.\begin{matrix} \sup A = -1 \\ \sup B = 1 \end{matrix}\right\}$ $-1 \cdot 1 = -1 \ne \underbrace{\sup(A \cdot B)}_{0}$

$\boxed{2}$ supA, infA, minA, maxA?

а) $A = \{\sqrt{n+1} - \sqrt{n} \mid n \in \mathbb{N}\}$.

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \Rightarrow \underbrace{\frac{1}{\sqrt{2}-1}}_{\max A \text{ и } \sup A} > \frac{1}{\sqrt{3}-\sqrt{2}} > \dots$$

$0 = \inf A$: 1° $0 \le \frac{1}{\sqrt{n+1}+\sqrt{n}}$ ✓

2° $\varepsilon > 0$ произвољно. ? $\exists n \in \mathbb{N}$ т.д. $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon$. Довољно је наћи

n тако да $\frac{1}{2\sqrt{n}} < \varepsilon \Leftrightarrow \frac{1}{4\varepsilon^2} < n$ ✓

$0 = \inf A \notin A \Rightarrow \neg\exists \min A$

б) $A = \{\frac{m}{n} + \frac{n}{m} \mid m, n \in \mathbb{N}\}$

$\frac{m}{n} + \frac{n}{m} \ge 2\sqrt{\frac{m}{n} \cdot \frac{n}{m}} = 2$, $m = n$: $\frac{m}{n} + \frac{n}{m} = 2 \in A \Rightarrow \min A = 2$, $\inf A = 2$

$n = 1$: $\frac{m}{1} + \frac{1}{m} > m$ произвољно $\Rightarrow A$ није ограничено одозго $\Rightarrow \neg\exists \sup A, \neg\exists \max A$

в) $A = \{\frac{1}{k} + \frac{1}{k+n} \mid k, n \in \mathbb{N}\}$

$\frac{1}{k} + \frac{1}{k+n} \le \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \Rightarrow \sup A = \frac{3}{2}$, $\max A = \frac{3}{2}$

1° $\frac{1}{k} + \frac{1}{k+n} > 0, \forall k, n \in \mathbb{N}$ ✓

2° $\varepsilon > 0$ произвољно. ? $\exists n, k \in \mathbb{N}$ т.д. $\frac{1}{k} + \frac{1}{k+n} < \varepsilon$. нпр. $k = n$

Желимо да $\exists n$ т.д. $\frac{1}{n} + \frac{1}{2n} < \varepsilon \Leftrightarrow \frac{3}{2n} < \varepsilon \Leftrightarrow n > \frac{3}{2\varepsilon} \Rightarrow \exists n$

$\inf A = 0 \notin A \Rightarrow \min A$ не постоји

3. sup, inf, min, max?

А) $A = \left([2,\pi] \cup [17, 20e]\right) \cap \mathbb{Q}$

$\inf A = 2 \in \mathbb{Q} \Rightarrow \min A = 2$

$\sup A = 20e \notin \mathbb{Q} \Rightarrow \nexists \max A$

б) $A = \{x \in \mathbb{R} \mid x^2 \cdot \log_2 |x-2| \le 0\}$

$x^2 \cdot \log_2 |x-2| \le 0$, $\boxed{x \neq 2}$

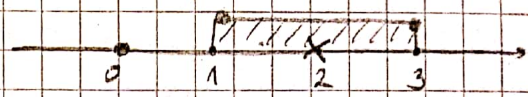
$x^2 = 0 \quad \vee \quad (\log_2 |x-2| \le 0 \quad \wedge \quad x^2 > 0)$

$x = 0 \quad \vee \quad \log_2 |x-2| \le 0$

$|x-2| \le 1$

$-1 \le x-2 \le 1$

$1 \le x \le 3$

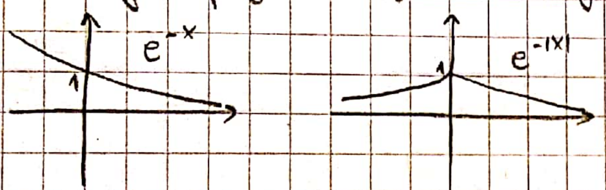


$A = \{0\} \cup [1,2) \cup (2,3]$

$\inf A = 0 = \min A$

$\sup A = 3 = \max A$

в) А скуп вредности $f: \mathbb{R} \to \mathbb{R}$: $f(x) = e^{-|x|}$



тј. $e^{-|x|} = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$

$A = \{e^{-|x|} \mid x \in \mathbb{R}\} = (0,1]$

$\inf A = 0 \notin A \ \nexists \min A$

$\sup A = 1 = \max A$

г) $A = \left\{ \frac{n}{4} - \left[\frac{n}{4}\right] \mid n \in \mathbb{N} \right\}$

$\underline{n = 4k}$: $\frac{4k}{4} - \left[\frac{4k}{4}\right] = 0$

$\underline{n = 4k+1}$: $\frac{4k+1}{4} - \underbrace{\left[\frac{4k+1}{4}\right]}_{k} = \frac{1}{4}$

$\underline{n = 4k+2}$: $\frac{4k+2}{4} - \underbrace{\left[\frac{4k+2}{4}\right]}_{k} = \frac{1}{2}$

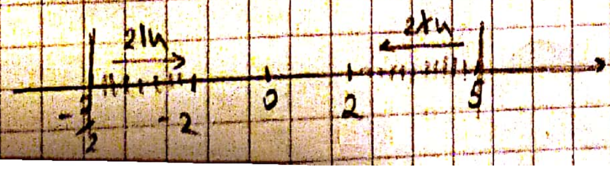
$\underline{n = 4k+3}$: $\frac{4k+3}{4} - \left[\frac{4k+3}{4}\right] = \frac{3}{4}$ $\qquad A = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$

$\inf A = 0 = \min A$, $\sup A = \frac{3}{4} = \max A$

д) $A = \left\{ (-1)^{n-1} \left(2 + \frac{3}{n}\right) \mid n \in \mathbb{N} \right\}$

$\underline{2 \nmid n}$: $(-1)^{n-1} = 1 \to$ бројеви: $2 + \frac{3}{n}$: $2 + \frac{3}{1}, 2 + \frac{3}{3}, 2 + \frac{3}{5}, \dots, 2 + \frac{3}{n}$ (n непарно) (тежи ка 2)

$\underline{2 \mid n}$: $(-1)^{n-1} = -1 \to$ бројеви: $-\left(2 + \frac{3}{n}\right)$: $-\left(2 + \frac{3}{2}\right), -\left(2 + \frac{3}{4}\right), \dots$ (n-парно) (тежи ка −2)



$\sup A = 5 = \max A$

$\inf A = -\frac{7}{2} = \min A$

[4] $x>0$ фиксиран број. Доказати да је $x = \sup\left\{ \frac{[n \cdot x]}{n} \mid n\in\mathbb{N} \right\} = A$. Да ли је $x = \max A$?

$A = \left\{ [x], \frac{[2x]}{2}, \frac{[3x]}{3}, \dots \right\}$

1° $\frac{[nx]}{n} \le x, \ \forall n\in\mathbb{N} \iff [nx] \le nx$

2° $\varepsilon>0$, $? \exists n_0\in\mathbb{N}$ т.д. $\frac{[n_0x]}{n_0} \ge x-\varepsilon \ / \cdot n_0 \iff [n_0x] \ge n_0x - n_0\varepsilon \iff [n_0x]+n_0\varepsilon \ge n_0x$

Знамо да $\exists n_0\in\mathbb{N}$ т.д. $n_0\varepsilon > 1 \Rightarrow [n_0x]+n_0\varepsilon > [n_0x]+1 > n_0x$

$\overset{1^\circ, 2^\circ}{\Longrightarrow} x = \sup A$

• $x = \max A \ ? \iff x\in A \iff (\exists n_0\in\mathbb{N})$ т.д. $\frac{[n_0x]}{n_0} = x \iff (\exists n_0\in\mathbb{N})\ [n_0x] = n_0x$

$\iff (\exists n_0\in\mathbb{N})\ n_0x\in\mathbb{Z} \iff x\in\mathbb{Q}$

19.03.2018

Домен функције

$D_f = \{x\in\mathbb{R} \mid f(x) \text{ дефинисано}\}$

[1] $f(x) = \mathrm{ctg}(\pi x) + \arccos 2^x$

• ctg: $\sin(\pi x) \ne 0$

$\pi x \ne k\pi \Rightarrow x \ne k, k\in\mathbb{Z}$

$\Rightarrow \boxed{x\notin\mathbb{Z}}$

• arccos: $2^x \in [-1,1]$

$2^x > 0 \quad 2^x \le 1$

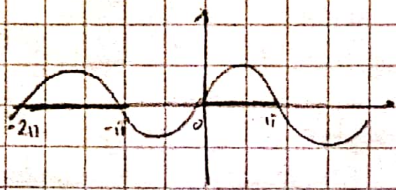
$\boxed{x\le 0}$

$D_f = (-\infty, 0)\setminus\mathbb{Z}$

[2] $f(x) = \sqrt{\sin\sqrt{x}}$

• $\sqrt{\cdot}$: $\boxed{x\ge 0}$, • $\sin\sqrt{x} \ge 0$: $2k\pi \le \sqrt{x} \le 2k\pi+\pi \ /^2$

$4k^2\pi^2 \le x \le (2k\pi+\pi)^2$, $k\in\mathbb{N}_0$



$D_f = \bigcup_{k\in\mathbb{N}_0} [4k^2\pi^2, (2k\pi+\pi)^2]$

[3] $f(x) = \mathrm{arctg}\left(\sqrt{\frac{2-x}{2x+1}} \cdot \ln\left(\frac{x+1}{1-x}\right)\right)$

• $1-x\ne 0$, $\boxed{x\ne 1}$; $2x+1 \ne 0$, $\boxed{x\ne -1/2}$; • $\frac{2-x}{2x+1} \ge 0$

	$-1/2$		2	
$2-x$	+	+		−
$2x+1$	−	+		+
$f(x)$	−	+		−

$\boxed{x\in(-\frac{1}{2}, 2]}$

• $\frac{x+1}{1-x} > 0$

	-1		1	
$x+1$	−	+		+
$1-x$	+	+		−
$f(x)$	−	+		−

$\boxed{x\in(-1, 1)}$

• arctg ⇒ увек дефинисан

$D_f = (-\frac{1}{2}, 1)$

4 $f(x) = \arcsin\sqrt{x^2+x+1}$

• $x^2+x+1 \geq 0 \Rightarrow \forall x$ • arcsin: $-1 \leq \sqrt{x^2+x+1} \leq 1 \;/^2$

$x^2+x+1 \leq 1$

$x(x+1) \leq 0$

$+ \; (-) \; +$, -1, 0

$\boxed{x \in [-1, 0]}$

$D_f = [-1, 0]$

5 $f(x) = \arcsin\left(\ln \frac{x+3}{x+1}\right)$

• $\frac{x+3}{x+1} > 0$:

	-3		-1	
$x+3$	$-$	$+$		$+$
$x+1$	$-$	$-$		$+$
$f(x)$	$+$	$-$		$+$

• $x+1 \neq 0$

$\boxed{x \neq -1}$

$\boxed{x \in (-\infty, -3) \cup (-1, +\infty)}$

• $-1 \leq \ln \frac{x+3}{x+1} \leq 1$

$e^{-1} \leq \frac{x+3}{x+1} \leq e$

$\frac{x+3-e(x+1)}{x+1} \leq 0$

$\frac{(1-e)x+3-e}{x+1} \leq 0 \;/ : \underbrace{(1-e)}_{<0}$

$\frac{x+\frac{3-e}{1-e}}{x+1} \geq 0$

...

19.03.2018

Граничне вредности функције

$f: A \to \mathbb{R}$, a - тачка нагомилавања скупа A

* Околина:

1) $a \in \mathbb{R} \Rightarrow U$ је околина од a ако $\exists \varepsilon > 0 \; (a-\varepsilon, a+\varepsilon) \subset U$

2) $\pm\infty \Rightarrow U$ је околина од $+\infty$ ако $\exists M \in \mathbb{R} \; (M, +\infty) \subset U$

* Деф $a, b \in \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$), a - тачка нагомилавања од A.

$b = \lim_{x\to a} f(x)$ ако $\forall$ околину V тачке b $\exists$ околина U тачке a, $f(U \setminus \{a\} \cap A) \subset V$.

искључујемо тачку a

• $a, b \in \mathbb{R}$: $\lim_{x\to a} f(x) = b \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; (0 < |x-a| < \delta \Rightarrow |f(x) - b| < \varepsilon)$

• $\lim_{x\to +\infty} f(x) = b \Leftrightarrow \forall \varepsilon > 0 \; \exists M \in \mathbb{R} \; \forall x \in A \; (x > M \Rightarrow |f(x) - b| < \varepsilon)$

* **Особине:**

1° ако $\exists \lim\limits_{x\to a} f(x) = b$ онда је јединствен

2° ако $\lim\limits_{x\to a} f(x) = b$ и $\lim\limits_{x\to a} g(x) = c$ онда:

- $\lim\limits_{x\to a} (\beta f(x) \pm \gamma g(x)) = \beta b \pm \gamma c$
- $\lim\limits_{x\to a} (f(x)\cdot g(x)) = b\cdot c$
- $c \neq 0$: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{b}{c}$

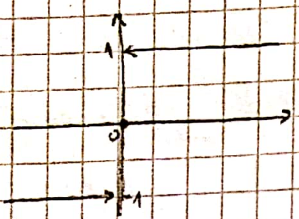
3° Теорема о лимесу сложене ф-је:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & b \arrow[r, "g"] & c \end{tikzcd}$$

* **Једнострани лимеси:**

- $\lim\limits_{x\to a^-} f(x) := \lim\limits_{(-\infty, a)\cap A \ni x\to a} f(x)$
- $\lim\limits_{x\to a^+} f(x) := \lim\limits_{(a, +\infty)\cap A \ni x\to a} f(x)$

Пример $\operatorname{sgn} x$



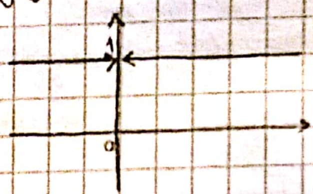
$\lim\limits_{x\to 0^-} \operatorname{sgn} x = -1$

$\lim\limits_{x\to 0^+} \operatorname{sgn} x = 1$

* За $\lim\limits_{x\to a} f(x)$ није битна вредност $f(a)$!

* $\exists \lim\limits_{x\to a} f(x) \Leftrightarrow \exists \lim\limits_{x\to a^-} f(x) \wedge \exists \lim\limits_{x\to a^+} f(x) \wedge$ једнаки су

Пример $|\operatorname{sgn} x|$ $\quad \lim\limits_{x\to 0} |\operatorname{sgn} x| = 1$



* **Лимеси:**

1. $\lim\limits_{x\to +\infty} \left(1+\dfrac{1}{x}\right)^x = e$

   $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

2. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$

   $\lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$

3. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

   $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln a$

4. $\lim\limits_{x\to 0} \dfrac{(1+x)^\alpha - 1}{x} = \alpha$, $\alpha \in \mathbb{R}$

5. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

6. $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x}{x} = 1$

7. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

* Све елементарне ф-је су непрекидне у свакој тачки свог домена.

* Неодређени изрази: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 1^∞, $0\cdot\infty$, 0^0, ∞^0

$\boxed{1}$ a) $\lim\limits_{x\to 0}\frac{\sin 7x-\sin 2x}{\sin x} = \lim\limits_{x\to 0}\frac{\frac{\sin 7x}{7x}\cdot 7x-\frac{\sin 2x}{2x}\cdot 2x}{\frac{\sin x}{x}\cdot x} = \underline{\underline{5}}$

б) $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a} = \lim\limits_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a} = \lim\limits_{x\to a}\frac{\sin\frac{x-a}{2}\cdot\cos\frac{x+a}{2}}{\frac{x-a}{2}} = \underline{\cos a}$

в) $\lim\limits_{x\to 1}\frac{1-x^2}{\sin\pi x} \underset{x=1+t}{=} \lim\limits_{t\to 0}\frac{1-(1+t)^2}{\underbrace{\sin(\pi+\pi t)}_{-\sin\pi t}} = \lim\limits_{t\to 0}\frac{(1+1+t)\cdot t}{\sin\pi t} = \lim\limits_{t\to 0} 2\cdot\frac{t}{\sin\pi t}\cdot\frac{\pi}{\pi} = \underline{\underline{\frac{2}{\pi}}}$

г) $\lim\limits_{x\to 5}\frac{\operatorname{tg}\pi x}{x-\sqrt{3}} = \underline{\underline{0}}$ $\quad(\operatorname{tg}\pi x\to 0,\ x-\sqrt3\to 5-\sqrt3)$

д) $\lim\limits_{x\to+\infty} x(\ln(x+1)-\ln x) = \lim\limits_{x\to+\infty} x\cdot\ln\frac{x+1}{x} = \lim\limits_{x\to+\infty}\ln\left(\frac{1+x}{x}\right)^x = \lim\limits_{x\to+\infty}\ln\underbrace{\left(1+\frac{1}{x}\right)^x}_{e} = \underline{\underline{1}}$

ђ) $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x} = \lim\limits_{x\to 0}\left(\frac{e^{ax}-1}{ax}\cdot a-\frac{e^{bx}-1}{bx}\cdot b\right) = \underline{a-b}$

$a,b\neq 0$

$\boxed{2}$

$\lim\limits_{x\to+\infty}\frac{a_m x^m+\dots+a_1x+a_0}{b_nx^n+\dots+b_1x+b_0} = \lim\limits_{x\to+\infty}\underbrace{\frac{x^m}{x^n}}_{x^{m-n}}\cdot\underbrace{\frac{a_m+\frac{a_{m-1}}{x}+\dots+\frac{a_0}{x^m}}{b_n+\frac{b_{n-1}}{x}+\dots+\frac{b_0}{x^n}}}_{a_m/b_n} = \begin{cases} a_m/b_n, & \text{за } m=n\\ 0, & \text{за } m<n\\ +\infty, & \text{за } m>n,\ a_m\cdot b_n>0 \text{ (исти)}\\ -\infty, & \text{за } m>n,\ a_m\cdot b_n<0 \text{ (разл. знака)}\end{cases}$

$a_m, b_n\neq 0$

• за $x\to-\infty$ пазити и на знак $\frac{x^m}{x^n}$. Пример:

$\lim\limits_{x\to-\infty}\frac{-5x^{18}+7x}{9x^7-4x^3} = +\infty$: $\lim\limits_{x\to-\infty}\underbrace{\frac{x^{18}}{x^7}}_{x^{11}\to-\infty}\cdot\underbrace{\frac{-5+\frac{7}{x^{17}}}{9-\frac{4}{x^4}}}_{(-)5/9} = +\infty$

$\boxed{3}$ за $x\to 0$, кад $b_0\neq 0$

$\lim\limits_{x\to 0}\frac{a_mx^m+\dots+a_1x+a_0}{b_nx^n+\dots+b_1x+b_0} = \frac{a_0}{b_0}$ $\quad$ (да је $b_0=0 \Rightarrow \frac{a_0}{0}=\infty$; ако је $a_0=0\to 0$)

• за $b_0=0$? Пример: $\lim\limits_{x\to 0}\frac{2x^7-5}{3x^8+x}$ ∄

• $x\to 0^+$: $\lim\limits_{x\to 0^+}\frac{2x^7-5}{3x^8+x} = \lim\limits_{x\to 0^+}\frac{2x^7-5}{x(3x^7+1)} = -\infty$

• $x\to 0^-$: $\lim\limits_{x\to 0^-}\frac{2x^7-5}{x(3x^7+1)} = +\infty$

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Пример $\lim\limits_{x\to 0}\dfrac{2x^{10}-x^{9}}{7x^{15}+x^{13}} = \lim\limits_{x\to 0}\dfrac{x^9(\overbrace{2x-1}^{\to -1})}{x^{13}(\underbrace{7x^2+1}_{\to 1})} = -\infty$

$\dfrac{1}{x^4}\to+\infty$

Пример $\lim\limits_{x\to 0}\dfrac{2x^{10}-x^{9}}{7x^{5}-x^{3}} = \lim\limits_{x\to 0}\dfrac{x^9(\overbrace{2x-1}^{-1})}{x^{3}(\underbrace{7x^2-1}_{-1})} = 0$

$x^6\to 0$

$\boxed{4}$ $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1} \underset{\substack{x=1+t\\ t\to 0}}{=} \lim\limits_{t\to 0}\dfrac{(1+t)^m-1}{(1+t)^n-1} = \lim\limits_{t\to 0}\dfrac{(1+t)^m-1}{t}\cdot\dfrac{t}{(1+t)^n-1} = \dfrac{m}{n}$

• ако искаме [illegible], ако $m,n\in\mathbb{N}$

$\Rightarrow x^k-y^k=(x-y)(x^{k-1}+x^{k-2}y+\dots+xy^{k-2}+y^{k-1}),\ k\in\mathbb{N}$

$x^k-1=(x-1)(x^{k-1}+x^{k-2}+\dots+x+1)$

$\Rightarrow \lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1} = \lim\limits_{x\to 1}\dfrac{\cancel{(x-1)}(\overbrace{x^{m-1}+\dots+x+1}^{\to m})}{\cancel{(x-1)}(\underbrace{x^{n-1}+\dots+x+1}_{\to n})} = \dfrac{m}{n}$

$\boxed{5}$ $\lim\limits_{x\to 1}\dfrac{(1+x)^x-2}{x-1} \underset{\substack{x=1+t\\ t\to 0}}{=} \lim\limits_{t\to 0}\dfrac{(2+t)^{1+t}-2}{t} = \lim\limits_{t\to 0}\dfrac{e^{(1+t)\ln(2+t)}-2}{t} =$

$= \lim\limits_{t\to 0}\dfrac{e^{t\ln(2+t)}\cdot e^{\ln(2+t)}-2}{t} = \lim\limits_{t\to 0}\dfrac{2\cdot e^{t\ln(2+t)}+t\cdot e^{t\ln(2+t)}-2}{t} =$

$= \lim\limits_{t\to 0}\overbrace{\dfrac{2(e^{t\ln(2+t)}-1)}{t\cdot\ln(2+t)}}^{\to 1}\cdot\underbrace{\ln(2+t)}_{\to\ln 2}+\underbrace{e^{t\ln(2+t)}}_{\to 1} = 2\ln 2+1$

$\boxed{6}$ $\lim\limits_{x\to -\infty}\left(\dfrac{3x^2+x+1}{3x^2-x+1}\right)^{5x} = \lim\limits_{x\to -\infty}\left(1+\dfrac{2x}{3x^2-x+1}\right)^{5x} = \lim\limits_{x\to -\infty}\Bigg(\underbrace{\left(1+\dfrac{1}{\frac{3x^2-x+1}{2x}}\right)^{\frac{3x^2-x+1}{2x}}}_{e}\Bigg)^{\frac{2x\cdot 5x}{3x^2-x+1}}$

$= e^{10/3}$

$\boxed{7}$ $\lim\limits_{\substack{x\to -\infty\\ a,b\neq 0}}\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+b^2}+x} \underset{\substack{t=-x\\ t\to+\infty}}{=} \lim\limits_{t\to +\infty}\dfrac{\sqrt{t^2+a^2}-t}{\sqrt{t^2+b^2}-t} = \lim\limits_{t\to +\infty}\dfrac{\sqrt{1+\left(\frac{a}{t}\right)^2}-1}{\sqrt{1+\left(\frac{b}{t}\right)^2}-1} = \lim\limits_{t\to +\infty}\underbrace{\dfrac{\left(1+\left(\frac{a}{t}\right)^2\right)^{\frac12}-1}{\left(\frac{a}{t}\right)^2}}_{\frac12}\cdot\Big($

$\cdot\underbrace{\dfrac{\left(\frac{b}{t}\right)^2}{\left(1+\left(\frac{b}{t}\right)^2\right)^{1/2}-1}}_{1/2}\cdot\left(\dfrac{t}{b}\right)^2 = \dfrac{a^2}{b^2}$

$\boxed{8}$ $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}-1}{\ln(1-x)} = \lim\limits_{x\to 0}\underbrace{\dfrac{(1-x)^{1/3}-1}{-x}}_{1/3}\cdot\dfrac{-x}{-x}\cdot\underbrace{\dfrac{-x}{\ln(1-x)}}_{1} = \dfrac{1}{3}$

$\boxed{9}$ $\lim\limits_{x\to 0} \frac{1}{x}\cdot \ln\sqrt{\frac{1+x}{1-x}} = \lim\limits_{x\to 0} \frac{1}{x}\cdot \underbrace{\frac{1}{2}\frac{\ln\left(1+\frac{2x}{1-x}\right)}{\frac{2x}{1-x}}}_{1}\cdot\frac{2x}{1-x} = 1$

$\boxed{10}$ $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\frac{\ln(\cos x)}{x}} = e^{\lim\limits_{x\to 0}\frac{\ln(\cos x)}{x}} = e^{\lim\limits_{x\to 0}\underbrace{\frac{\ln(1+(\cos x-1))}{\cos x-1}}_{1}\cdot\underbrace{\frac{\cos x-1}{x^2}}_{-1/2}\cdot x}$

$= e^0 = 1$ Граф-ка на ф-ду → двек е но нещо ✓

$\boxed{11}$ $\lim\limits_{x\to 0} \frac{\sqrt[m]{\cos\alpha x}-\sqrt[n]{\cos\beta x}}{x^2} = \lim\limits_{x\to 0}\left(\underbrace{\frac{(1+(\cos\alpha x-1))^{1/m}-1}{\cos\alpha x-1}}_{1/m}\cdot\underbrace{\frac{\cos\alpha x-1}{x^2}}_{-\frac{\alpha^2}{2}} - \underbrace{\frac{(1+(\cos\beta x-1))^{1/n}-1}{\cos\beta x-1}}_{1/n}\cdot\underbrace{\frac{\cos\beta x-1}{x^2}}_{-\frac{\beta^2}{2}}\right)$

$= -\frac{\alpha^2}{2m}+\frac{\beta^2}{2n}$

$\boxed{12}$ $\lim\limits_{x\to\pi} \frac{\sqrt{1-\operatorname{tg}x}-\sqrt{1+\operatorname{tg}x}}{\sin 2x} = \lim\limits_{x\to\pi}\frac{(1-\operatorname{tg}x)-(1+\operatorname{tg}x)}{\sin 2x\left(\sqrt{1-\operatorname{tg}x}+\sqrt{1+\operatorname{tg}x}\right)} = \lim\limits_{x\to\pi}\frac{-2\frac{\sin x}{\cos x}}{2\sin x\underbrace{\cos x}_{-1}\underbrace{(\dots)}_{2}} = -\frac{1}{2}$

$\boxed{13}$ А) $x\to 1^+, 1^-$ $\quad f(x) = \operatorname{arctg}\frac{1}{1-x}$

$\lim\limits_{x\to 1^+}\operatorname{arctg}\underbrace{\frac{1}{1-x}}_{\to 0^-,\ -\infty} = -\frac{\pi}{2}$ $\qquad \lim\limits_{x\to 1^-}\operatorname{arctg}\underbrace{\frac{1}{1-x}}_{\to 0^+,\ +\infty} = \frac{\pi}{2}$

Б) $x\to 0^+, 0^-$ $\quad f(x) = \frac{1}{1+e^{1/x}}$

$\lim\limits_{x\to 0^+}\frac{1}{1+\underbrace{e^{1/x}}_{+\infty}} = 0$ $\qquad \lim\limits_{x\to 0^-}\frac{1}{1+\underbrace{e^{1/x}}_{0}} = 1$

Дом.

А) $\lim\limits_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$

Б) $\lim\limits_{x\to 0}\frac{\ln(1+\sin x)}{\operatorname{tg}2x}$

В) $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\frac{\pi x}{2}$

Г) $\lim\limits_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$

~~Д)~~ $\lim\limits_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

Е) $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$

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Асимптотске релације $o, O, \sim$

26.03.

* $\lim\limits_{x\to a} \alpha(x)=0$, α је бесконачно мала кад $x\to a$

Деф. f је бесконачно мала у односу на g код $x\to a$ ако $\exists$ U ок. од a и ф-ја $\alpha: U\to\mathbb{R}$ т.д.:

$$f(x)=\alpha(x)\,g(x),\ x\in U\setminus\{a\},\quad \lim_{x\to a}\alpha(x)=0$$

ознака: $f=o(g),\ x\to a$

* $\lim\limits_{x\to a}\dfrac{o(f)}{f}=0$

Пример $f(x)=x^3,\ g(x)=x^5$

$x\to+\infty$: $x^3=\underbrace{\frac{1}{x^2}}_{\to 0}\cdot x^5 \Rightarrow x^3=o(x^5),\ x\to+\infty$ $\quad\boxed{m>n \Rightarrow x^n=o(x^m),\ x\to\pm\infty}$

$x\to 0$: $x^5=\underbrace{x^2}_{\to 0}\cdot x^3 \Rightarrow x^5=o(x^3),\ x\to 0$ $\quad\boxed{m>n \Rightarrow x^m=o(x^n),\ x\to 0}$

* **Особине:**

1° $o(f)+o(f)=o(f),\ x\to a$

2° $o(f)\cdot o(g)=o(f\,g),\ x\to a$

3° $f\cdot o(g)=o(f\cdot g),\ x\to a$

4° $o(o(f))=o(f),\ x\to a$

5° $\underset{c\neq 0}{o(c\cdot f)}=o(f),\ x\to a$

Деф $f=O(g),\ x\to a$ ако $\exists$ околина U од a и $\exists$ ограничена ф-ја $\beta: U\to\mathbb{R}$ т.д.

$$f(x)=\beta(x)\cdot g(x),\quad x\in U\setminus\{a\}$$

Пример $f(x)=\dfrac{1+x}{1-x}\,g(x)$, $f=O(g),\ x\to+\infty$

Деф $f \sim g,\ x \to a$ ако $\exists$ околност U на a и ф-я $h: U \to \mathbb{R}$ т.ч.

$$f(x) = h(x)\, g(x),\quad x \in U \setminus \{a\},\quad \lim_{x \to a} h(x) = 1.$$

$$f \sim g,\ x \to a \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 1 \iff f = g + o(g),\ x \to a$$

$$f - g = o(g)$$

Доказ

$(\Rightarrow)$ $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = 1 \Rightarrow \frac{f(x)}{g(x)} = 1 + \alpha(x) \Big|_{\cdot g(x)}$, $\alpha(x) \xrightarrow[x \to a]{} 0 \Rightarrow f(x) = g(x) + \underbrace{\alpha(x) \cdot g(x)}_{} \Rightarrow$

$$\Rightarrow f(x) = g(x) + o(g)$$

* $f \sim g,\ x \to a \Rightarrow o(f) = o(g),\ x \to a$ ([illegible])

$\frac{f}{g} \xrightarrow[x \to a]{} 1 \Rightarrow f = g + o(g),\ x \to a$

* $\frac{e^x - 1}{x} \to 1,\ x \to 0 \Rightarrow e^x - 1 = x + o(x) \Rightarrow \underline{e^x = 1 + x + o(x),\ x \to 0}$

$\frac{a^x - 1}{x} \to \ln a,\ x \to 0 \Rightarrow a^x - 1 = x \cdot \ln a + o(\ln a \cdot x) \Rightarrow \underline{a^x = 1 + x \cdot \ln a + o(x),\ x \to 0}$

$\frac{\ln(1+x)}{x} \to 1 \Rightarrow \ln(1+x) \sim x \Rightarrow \underline{\ln(1+x) = x + o(x),\ x \to 0}$

$(1+x)^\alpha - 1 \sim \alpha x \Rightarrow \underline{(1+x)^\alpha = 1 + \alpha x + o(x)}$

$\sin x \sim x,\ x \to 0 \Rightarrow \underline{\sin x = x + o(x),\ x \to 0}$

$\frac{1 - \cos x}{\frac{1}{2}x^2} \to 1,\ x \to 0 \Rightarrow \underline{\cos x = 1 - \frac{1}{2}x^2 + o(x^2),\ x \to 0}$

$\boxed{1}$ $a, b, c > 0$. $\underbrace{\lim\limits_{x \to 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}}_{L} = \sqrt[3]{abc}$

$$L = \lim_{x \to 0} e^{\frac{1}{x} \cdot \ln\left(\frac{a^x + b^x + c^x}{3}\right)} = e^{\underbrace{\lim_{x \to 0} \frac{1}{x} \ln\left(\frac{a^x + b^x + c^x}{3}\right)}_{L_1}}$$

$$L_1 = \lim_{x \to 0} \ln\left(\frac{1}{3}\left(1 + \ln a \cdot x + o(x) + 1 + \ln b \cdot x + o(x) + 1 + \ln c \cdot x + o(x)\right)\right) \cdot \frac{1}{x} =$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \ln\left(\frac{3 + \ln(abc) \cdot x + o(x)}{3}\right) = \lim_{x \to 0} \frac{1}{x} \ln\left(1 + \underbrace{\ln\sqrt[3]{abc}\ x + o(x)}_{t}\right) =$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \left(\underbrace{\ln\sqrt[3]{abc} \cdot x + o(x)}_{t} + \underbrace{o\left(\ln\sqrt[3]{abc} \cdot x + o(x)\right)}_{o(t)} \right) =$$

$$= \lim_{x \to 0} \left(\ln\sqrt[3]{abc} + \frac{o(x)}{x}^{\to 0} + \frac{o\left(\ln\sqrt[3]{abc} \cdot x + o(x)\right)}{x}^{\to 0} \right) = \ln\sqrt[3]{abc}$$

$$L = e^{L_1} = e^{\ln\sqrt[3]{abc}} = \sqrt[3]{abc}$$

$\boxed{2}$ $L = \lim\limits_{x\to 0} \dfrac{1-(\cos x)^{\sin x}}{x}$

$$\bullet\ (\cos x)^{\sin x} = e^{\sin x \cdot \ln(\cos x)} = e^{(x+o(x))\cdot \ln\big(\underbrace{1-\frac{1}{2}x^2+o(x^2)}_{}\big)} = e^{(x+o(x))\cdot\big(-\frac{1}{2}x^2+o(x^2)+\underbrace{o(-\frac{1}{2}x^2+o(x^2))}_{o(x^2)}\big)} =$$

$$= e^{(x+o(x))\cdot(-\frac{1}{2}x^2+o(x^2))} =$$

$$= e^{-\frac{1}{2}x^3+\underbrace{x\cdot o(x^2)}_{o(x^3)}+\underbrace{o(x)\cdot(-\frac{1}{2}x^2)}_{o(x^3)}+\underbrace{o(x)\cdot o(x^2)}_{o(x^3)}} = e^{\overbrace{-\frac{1}{2}x^3+o(x^3)}^{\to 0}} =$$

$$= 1 + \frac{-1}{2}x^3 + o(x^3) + o\left(-\frac{1}{2}x^3+o(x^3)\right) = 1-\frac{1}{2}x^3+o(x^3)$$

$$L = \lim_{x\to 0}\frac{1-(1-\frac{1}{2}x^3+o(x^3))}{x^3} = \lim_{x\to 0}\left(\frac{1}{2}+\frac{o(x^3)}{x^3}\right) = \underline{\underline{\frac{1}{2}}}$$

$\boxed{3}$ Одредити $c\in\mathbb{R}$ и $n\in\mathbb{N}$ т.д. $f(x)\sim c\cdot x^n$, $n\to 0$

а) $f(x) = 2x-3x^2+x^5$

$$f(x) = 2x\big(\underbrace{1-\frac{3}{2}x+\frac{x^4}{2}}_{\to 1,\ x\to 0}\big) \Rightarrow f(x)\sim 2x,\ x\to 0$$

б) $f(x) = \sqrt{1+x}-\sqrt[3]{1-x} = (1+x)^{1/2}-(1-x)^{1/3} = 1+\frac{1}{2}x+o(x)-\left(1+\frac{1}{3}(-x)+o(x)\right)$

$$= \frac{5}{6}x+o(x) \Rightarrow f(x)\sim\frac{5}{6}x,\ x\to 0$$

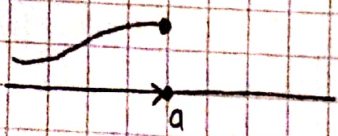
Непрекидност

02.04.2018

$f: A\to\mathbb{R}$, f непр. у $a \Leftrightarrow \forall\varepsilon>0\ \exists\delta>0\ \forall x\in A\ (|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon)$

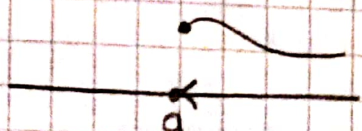
$\Leftrightarrow \lim\limits_{x\to a} f(x) = f(a)$

* f непрекидна слева у a



$\lim\limits_{x\to a_-} f(x) = f(a)$

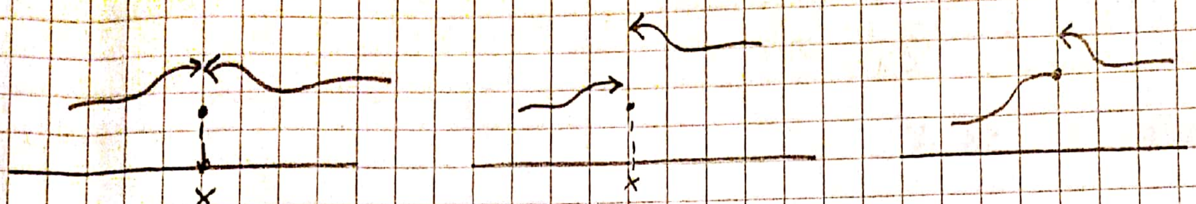
* f непрекидна здесна у a



$\lim\limits_{x\to a_+} f(x) = f(a)$

$$\boxed{f \text{ непр. у } a \Leftrightarrow \lim_{x\to a_-} f(x) = f(a) = \lim_{x\to a_+} f(x)}$$

* f није непрекидна у a:



(отклоњив прекид)

• прекид I врсте: • ако $\exists \lim\limits_{x\to a_-} f(x)$ и $\exists \lim\limits_{x\to a_+} f(x)$

• ако $\lim\limits_{x\to a_-} f(x) = \lim\limits_{x\to a_+} f(x) \neq f(a)$ $\Rightarrow$ отклоњив прекид

$$\tilde{f}(x) = \begin{cases} f(x), & x \neq a \\ \lim\limits_{x\to a} f(x), & x = a \end{cases}$$, $\tilde{f}(x)$ - непрекидна у a

• ако f није I врсте $\Rightarrow$ II врсте је

пример 1) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$:

• $x \neq 0$: $\frac{\sin x}{x}$ → непрекидна

• $x = 0$: $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$ ✓

2) $f(x) = \begin{cases} \frac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$:

• $x \neq 0$: $\frac{\sin x}{|x|}$ → непрекидна

• $x = 0$: $\lim\limits_{x\to 0_+} \frac{\sin x}{|x|} = \lim\limits_{x\to 0_+} \frac{\sin x}{x} = 1$

$\neq$

$\lim\limits_{x\to 0_-} \frac{\sin x}{|x|} = \lim\limits_{x\to 0_-} \frac{\sin x}{-x} = -1$

$\Rightarrow f(x)$ је непр. сдесна у 0, али није непр. у 0

$\Rightarrow f$ непр. на $\mathbb{R}\setminus\{0\}$

$\boxed{1}$

$$f(x) = \begin{cases} \frac{1}{1+e^{\frac{1}{x-1}}}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

• $x \neq 1$: f је непрекидна

• $x = 1$: $\lim\limits_{x\to 1} \frac{1}{1+e^{\frac{1}{x-1}}} = f(1) = 1$?

$x \to 1_+$: $\lim\limits_{x\to 1_+} \frac{1}{1+e^{\frac{1}{x-1}}} = 0 \neq f(1)$ $\quad \left(\frac{1}{x-1} \to \frac{1}{0_+} \to +\infty\right)$

$x \to 1_-$: $\lim\limits_{x\to 1_-} \frac{1}{1+e^{\frac{1}{x-1}}} = 1 = f(1)$ $\quad \left(\frac{1}{x-1} \to \frac{1}{0_-} \to -\infty\right)$

} f непр. слева у 1

$\Rightarrow f$ непр. на $\mathbb{R}\setminus\{1\}$

$\boxed{2}$ $f(x) = [x]$, $g(x) = \underbrace{x - [x]}_{\substack{\geq 0 \\ < 1}} = \{x\}$ разломљени део

- $[x]$ је непр. за $x \notin \mathbb{Z}$
- $x \in [0,1): \quad x - \underbrace{[x]}_{0} = x$

  $x \in [1,2): \quad x - \underbrace{[x]}_{1} = x - 1$

  $\vdots$

  $x \in [n, n+1): \quad x - [x] = x - n \quad \Rightarrow g(x)$ непр. у $x \Leftrightarrow x \notin \mathbb{Z}$

$\boxed{3}$ Да ли важи импликација?

А) $\left.\begin{array}{l} f \text{ непр. у } x_0 \\ g \text{ прекид у } x_0 \end{array}\right\} \overset{?}{\Longrightarrow} f+g$ има прекид у x_0 $\boxed{T}$

Б) $\left.\begin{array}{l} f \text{ има прекид у } x_0 \\ g \text{ има прекид у } x_0 \end{array}\right\} \overset{?}{\Longrightarrow} f+g$ има прекид у x_0 $\boxed{\perp}$

$\boxed{4}$ $f(x) = [x] \cdot \left([x] - (-1)^{[x]} \cdot \cos \pi x\right)$, на $\mathbb{R}$

- $\underline{x \notin \mathbb{Z}}$: $[x] = n$, $x \in (n, n+1) \Rightarrow f(x) = \underbrace{n(n-(-1)^n}_{\text{конст.}} \underbrace{\cos \pi x}_{\text{непр.}}) \to$ непр у x
- $\underline{x \in \mathbb{Z}}$: $x = n$, $\lim\limits_{x \to n_-} [x] = n-1$, $\lim\limits_{x \to n_+} [x] = n$

$$\lim_{x \to n_-} \overset{?}{=} f(n) \overset{?}{=} \lim_{x \to n_+} f(x)$$

$$f(n) = [n] \cdot \left([n] - (-1)^{[n]} \underbrace{\cos(n\pi)}_{(-1)^n}\right) = n\left(n - (-1)^n (-1)^n\right) = n(n-1)$$

$$\lim_{x \to n_-} f(x) = \lim_{x \to n_-} [x]\left([x] - (-1)^{[x]} \cos x\pi\right) = (n-1)\left(n-1-(-1)^{n-1}(-1)^n\right) =$$
$$= (n-1) \cdot n = f(n) \checkmark$$

$$\lim_{x \to n_+} f(x) = \lim_{x \to n_+} [x]\left([x] - (-1)^{[x]} \cos x\pi\right) = n\left(n - (-1)^n \underbrace{\cos n\pi}_{(-1)^n}\right) = n(n-1) = f(n)$$

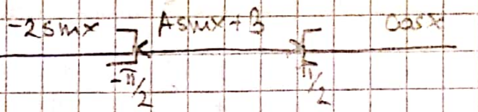
$\Rightarrow f$ непр у n

<u>пример</u> $f(x) = \operatorname{sgn} x$ $\quad g(x) = 1 + x - [x]$

- $f \circ g(x) = \operatorname{sgn}(\underbrace{1 + x - [x]}_{>0}) = 1$
- $g \circ f(x) = 1 + \cancel{\operatorname{sgn} x} - \underbrace{[\cancel{\operatorname{sgn} x}]}_{[\operatorname{sgn} x]} = 1$

$\Rightarrow f \circ g$ непрекидна $\quad\quad \Rightarrow g \circ f$ непрекидна

5 $f(x) = \begin{cases} -2\sin x, & x \le -\pi/2 \\ A\sin x + B, & |x| < \pi/2 \\ \cos x, & x \ge \pi/2 \end{cases}$ $A, B \in \mathbb{R} = ?$ ако је f непр. на $\mathbb{R}$



$x \neq -\pi/2, \pi/2$: f непр. на x

$x = -\pi/2$: $\lim_{x\to -\pi/2-} f(x) = f(-\pi/2) = \lim_{x\to -\pi/2+} f(x)$ $\quad f(-\pi/2) = -2\sin(-\pi/2) = 2$

$\lim_{x\to -\pi/2-} (-2\sin x) = -2\sin(-\pi/2) = 2$ ✓

$\lim_{x\to -\pi/2+} (A\sin x + B) = A\sin(-\pi/2) + B = -A + B$

$\Rightarrow$ f непр. у $-\pi/2$ акко $\boxed{-A+B=2}$

$x = \pi/2$: $\lim_{x\to \pi/2-} f(x) = f(\pi/2) = \lim_{x\to \pi/2+} f(x)$ $\quad f(\pi/2) = \cos\pi/2 = 0$

$\lim_{x\to \pi/2-} (A\sin x + B) = A\sin\pi/2 + B = A + B$

$\lim_{x\to \pi/2+} (\cos x) = \cos\pi/2 = 0$ ✓

$\Rightarrow$ f непр. у $\pi/2$ акко $\boxed{A+B=0}$

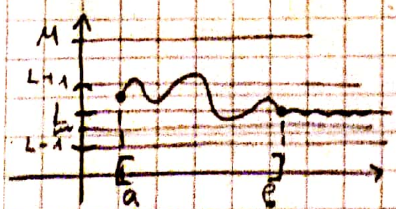
$\begin{array}{l} A+B=0 \\ -A+B=2 \end{array}$ /(+) $\quad \left.\begin{array}{l} B=1 \\ A=-1 \end{array}\right\}$ f је непр. на $\mathbb{R}$

Теорема (Вајерштрас) $f:[a,b]\to\mathbb{R}$ непр. $\Rightarrow$ f огр. на $[a,b]$ и $\exists\, x_{min}$ и $\exists\, x_{max}$ т.д. $f(x_{min}) = \min_{x\in[a,b]} f(x)$, $f(x_{max}) = \max_{x\in[a,b]} f(x)$.

Теорема (Болцано-Коши) $f:[a,b]\to\mathbb{R}$ непр.; C између $f(a)$ и $f(b)$ $\Rightarrow$ $\exists\, c\in[a,b]$ т.д. $f(c)=C$.

Последица Б.К. $f:[a,b]\to\mathbb{R}$ непр. и $f(a)$ и $f(b)$ различитог знака, $f(a)\cdot f(b)\le 0$ $\Rightarrow$ $\exists\, c\in[a,b]$ т.д. $f(c)=0$.

6 $f:[a,+\infty)\to\mathbb{R}$ непр. $\exists \lim_{x\to+\infty} f(x) = L$, $L\in\mathbb{R}$ $\Rightarrow$ f ограничена на $[a,+\infty)$



$\lim_{x\to+\infty} f(x) = L$, $\varepsilon = 1 \Rightarrow \exists\, b>a\ \forall x>b$

$|f(x)-L|<1$, $L-1<f(x)<L+1$

f непр. на $[a,b] \Rightarrow f$ огр. на $[a,b]$ т.д. $\exists\, m, M\in\mathbb{R}$

$\forall x\in[a,b]$, $m\le f(x)\le M$ $\Rightarrow \forall x\ge a$: $\min\{L-1, m\} \le f(x) \le \max\{L+1, M\}$

$\Rightarrow f$ огр. на $\mathbb{R}$

$\boxed{7}$ $f:\mathbb{R}\to\mathbb{R}$ непр. и огр. $\Rightarrow \exists x_0\in\mathbb{R}$, $f(x_0)=x_0^3$

$F(x)=f(x)-x^3$ непр. $\quad ?\exists x_0\in\mathbb{R}$ т.д. $F(x_0)=0$

$\lim\limits_{x\to+\infty} F(x)=\lim\limits_{x\to+\infty}(\underbrace{f(x)}_{\text{огр.}}-\underbrace{x^3}_{+\infty})=-\infty \Rightarrow \exists b>0\ f(b)<0$

$\lim\limits_{x\to-\infty} F(x)=\lim\limits_{x\to-\infty}(f(x)-x^3)=+\infty \Rightarrow \exists a<0\ f(a)>0$

$\Bigg\}\ \exists x_0\in[a,b] \Rightarrow f(x_0)=x^3$

$\boxed{8}$ $f,g:[a,b]\to[a,b]$ непр., g је "на" $\Rightarrow \exists x_0\in[a,b]$ т.д. $f(x_0)=g(x_0)$

$F:[a,b]\to\mathbb{R}$, $F(x)=f(x)-g(x)$, F-непр.

g "на" $\Rightarrow \exists x_1,x_2\in[a,b]$ т.д. $g(x_1)=b$, $g(x_2)=a$

$F(x_1)=\underbrace{f(x_1)}_{\le b}-\underbrace{g(x_1)}_{b}\le 0 \qquad F(x_2)=\underbrace{f(x_2)}_{\ge a}-\underbrace{g(x_2)}_{a}\ge 0$

$\overset{БК}{\Longrightarrow}$ F непр., $\exists x_0$ т.д. $F(x_0)=0$, $f(x_0)=g(x_0)$.

$\boxed{9}$ $f:[a,b]\to[a,b]$ непр. $\Rightarrow \exists x_0\in[a,b]$ $f(x_0)=x_0$

(претх. зад.) $g(x)=x$, $g:[a,b]\to[a,b]$ непр., "на"

$\Rightarrow (\exists x_0\in[a,b])\ f(x_0)=g(x_0)=x_0$

$\boxed{10}$ $f:[0,1]\to[0,1]$ непр. $?\exists A,B\subseteq[0,1]$, $A,B\neq\emptyset$, $A\cap B=\emptyset$, $A\cup B=[0,1]$ т.д. $f(A)\subseteq B$, $f(B)\subseteq A$

Пп, $\exists A,B \Rightarrow \exists x_0\in[0,1]$ т.д. $f(x_0)=x_0$

ако $x_0\in A \Rightarrow f(x_0)\in B \Rightarrow x_0\in B$ ↯

Слично, за $x_0\in B$. ↯

$\Rightarrow \neg\exists A,B$

$\boxed{11}$ $f:\mathbb{R}\to\mathbb{R}$ непр. Знамо да $\exists c\in\mathbb{R}$ т.д. $f(f(c))=c$. Доказати:

$\exists d$ $f(d)=d$.

$F(x)=f(x)-x$ непр., $F(c)=f(c)-c$

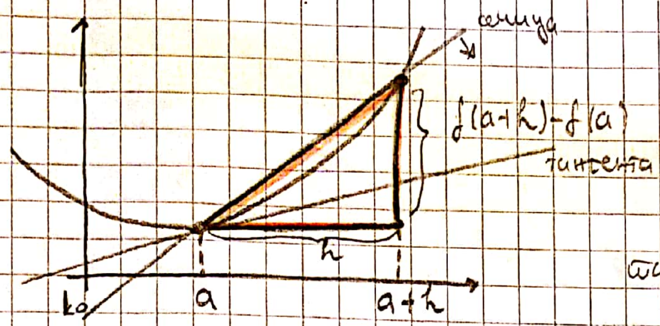
$F(f(c))=f(f(c))-f(c)=c-f(c)$

$F(c)\cdot F(f(c))\le 0$

$\overset{БК}{\Longrightarrow}$ $\exists$ између тачка d т.д. $F(d)=d \Rightarrow f(d)=d$

Диференцијабилност

02.04.2018



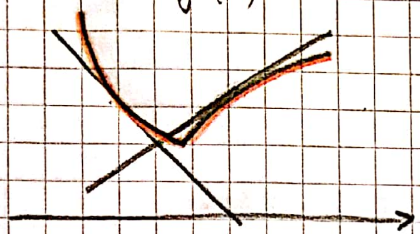
$k = tg\,\alpha$ - коефицијент правца

$$tg\,\alpha = \frac{f(a+h)-f(a)}{h}$$

тангента: $k_0 = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Деф $f: A \to \mathbb{R}$, $a \in \operatorname{int} A$. Ако постоји коначан лимес:

$$\underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}_{f'(a)} \in \mathbb{R}$$, кажемо да је f диференцијабилна у a



$$f'_+(a) = \lim_{h\to 0_+} \frac{f(a+h)-f(a)}{h}$$

$$f'_-(a) = \lim_{h\to 0_-} \frac{f(a+h)-f(a)}{h}$$

f диференцијабилна у a $\Longleftrightarrow$ $f'_+(a) = f'_-(0) \in \mathbb{R}$

Теорема f диф. у a $\Rightarrow$ f непрекидна у a

Својства:

1° $(\alpha f + \beta g)' = \alpha f' + \beta g'$, α, β - const

2° $(\alpha)' = 0$, α - const

3° $(fg)' = f'g + g'f$

4° $\left(f/g\right)' \ (g \neq 0) = \frac{f'g - fg'}{g^2}$

5° $(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0)$, $y_0 = f(x_0)$

Пример $f(x) = |x|$

- $x > 0$: $f(x) = x$, $f'(x) = 1$
- $x < 0$: $f(x) = -x$, $f'(x) = -1$
- $x = 0$: $f'_+(0) = \lim\limits_{h\to 0_+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0_+} \frac{\overset{h}{|h|} - 0}{h} = 1$

$$f'_-(0) = \lim_{h\to 0_-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0_+} \frac{\overset{-h}{|h|} - 0}{h} = -1$$

$f'_+(0) \neq f'_-(0) \Rightarrow |x|$ није диф. у тачки $x = 0$

$\boxed{1}$ Ако постоји, наћи $a, b \in \mathbb{R}$ т.д. f буде диф. на $\mathbb{R}$:

$$f(x) = \begin{cases} x \cdot e^{-bx} + a, & x < 0 \\ x^3 + bx, & x \geq 0 \end{cases}$$

$\boxed{\text{непр.}}$

- $x \neq 0$ ✓
- $x = 0$: $f(0) = 0$

$$\left.\begin{aligned} \lim_{x \to 0+} f(x) &= \lim_{x \to 0+} (x^3 + bx) = 0 \\ \lim_{x \to 0-} f(x) &= \lim_{x \to 0-} (x e^{-bx} + a) = a \end{aligned}\right\} f \text{ је непр. у } \underline{\underline{a = 0}}$$

$\boxed{\text{диф.}}$

$$a = 0 \Rightarrow f(x) = \begin{cases} x \cdot e^{-bx}, & x < 0 \\ x^3 + bx, & x \geq 0 \end{cases}$$

- $x < 0$: $f'(x) = (x \cdot e^{-bx})' = e^{-bx} + x \cdot e^{-bx} \cdot (-b) = e^{-bx} - bx\, e^{-bx}$
- $x > 0$: $f'(x) = (x^3 + bx)' = 3x^2 + b$
- $x = 0$:

$$\left.\begin{aligned} f'_+(0) &= \lim_{h \to 0+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0+} \frac{h^3 + bh - 0}{h} = b \\ f'_-(0) &= \lim_{h \to 0-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0-} \frac{h \cdot e^{-bh} - 0}{h} = \lim_{h \to 0-} e^{-bh} = 1 \end{aligned}\right\} \underline{\underline{b = 1}}$$

$*$ $f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$f'_+(a)$ $h \to 0+$, $f'_-(a)$ $h \to 0-$

f диф. у $a \iff \exists f'(a) \in \mathbb{R}$

$\iff \exists f'_+(a) = f'_-(a) \in \mathbb{R}$

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$\boxed{2}$ Испитати диф. на D_f: $f(x)=\sqrt{1-e^{-x^2}}$.

$\boxed{\text{Решен}}$

- $1-e^{-x^2}\geq 0$

$1\geq e^{-x^2}$

$0\geq -x^2 \Rightarrow \underline{x\in\mathbb{R}} \Rightarrow f$ $\boxed{\text{непрекидна}}$ на $\mathbb{R}$

$$f'(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\left(1-e^{-x^2}\right)'=$$

$$=\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot\left(-e^{-x^2}\right)\cdot\left(-\cancel{2}x\right)=$$

$1-e^{-x^2}\neq 0$, $\underline{\underline{x\neq 0}}$

$$=\frac{x\cdot e^{-x^2}}{\sqrt{1-e^{-x^2}}},\quad x\neq 0$$

- $\underline{x=0}$: $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1-e^{-h^2}}}{h}$ (ако $\exists$)

$$f'_+(0)=\lim_{h\to 0+}\frac{\sqrt{1-e^{-h^2}}}{h}=\lim_{h\to 0+}\sqrt{\frac{1-e^{-h^2}}{h^2}}=\lim_{h\to 0+}\sqrt{\frac{e^{-h^2}-1}{-h^2}}=1$$

$$f'_-(0)=\lim_{h\to 0-}\ominus\sqrt{\frac{1-e^{-h^2}}{h^2}}=\lim_{h\to 0-}-\sqrt{\frac{e^{-h^2}-1}{-h^2}}=-1$$

јер је $-\sqrt{h^2}$

$\Rightarrow f'_+(0)\neq f'_-(0)\Rightarrow f$ није диф у 0

* Таблица извода

1° $(x^\alpha)'=\alpha\cdot x^{\alpha-1}$

2° $(a^x)'=a^x\cdot\ln a$

3° $(e^x)'=e^x$

4° $(\ln x)'=\dfrac{1}{x}$

5° $(\log_a x)'=\dfrac{1}{x\ln a}$

6° $(\sin x)'=\cos x$

7° $(\cos x)'=-\sin x$

8° $(\operatorname{tg} x)'=\dfrac{1}{\cos^2 x}$

9° $(\operatorname{ctg} x)'=-\dfrac{1}{\sin^2 x}$

10° $(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$

11° $(\operatorname{arctg} x)'=\dfrac{1}{1+x^2}$

$\boxed{3}$ $f' = ?$

A) $f(x) = \ln x \cdot \sqrt{2+x^2} \cdot \sqrt[3]{3+x^3}$

$f'(x) = (\ln x \sqrt{2+x^2})' \cdot \sqrt[3]{3+x^3} + (\ln x \cdot \sqrt{2+x^2}) \cdot (\sqrt[3]{3+x^3})' =$

$= \frac{1}{x} \cdot \sqrt{2+x^2} \cdot \sqrt[3]{3+x^3} + \ln x \cdot \frac{1}{2} \cdot \frac{2x}{2+x^2} \cdot \sqrt[3]{3+x^2} + \ln x \cdot \sqrt{2+x^2} \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(3+x^3)^2}} \cdot \ldots$

Б) $f(x) = \sqrt[3]{1+\sqrt[4]{1+x^4}}$

$f'(x) = \frac{1}{3} \cdot \frac{1}{(1+\sqrt[4]{1+x^4})^{2/3}} \cdot \underbrace{(1+\sqrt[4]{1+x^4})'}_{\frac{1}{4} \cdot \frac{1}{(1+x^4)^{3/4}} \cdot 4x^3} =$

$= \frac{1}{3} \cdot \frac{1}{(1+\sqrt[4]{1+x^4})^{2/3}} \cdot \frac{x^3}{(1+x^4)^{3/4}}$

В) $f(x) = (\operatorname{arctg} x)^x$

$f'(x) = (e^{\ln(\operatorname{arctg} x) \cdot x})' = e^{\ln(\operatorname{arctg} x) \cdot x} \cdot (\ln(\operatorname{arctg} x) \cdot x)' =$

$= (\operatorname{arctg} x)^x \cdot \left(\frac{1}{\operatorname{arctg} x} \cdot \frac{1}{1+x^2} \cdot x + \ln(\operatorname{arctg} x)\right)$

$f(x) = g(x)^{h(x)} = \underbrace{e^{\ln(g(x) \cdot h(x))}}_{\text{извод доста}}$

$\boxed{4}$ $f(x) = \underbrace{\sin x \cdot |\sin x|}_{\text{непр. ф-ја}}$, испитати диф.

$f(x) = \begin{cases} \sin^2 x, & x \geq 0 & x \in [2k\pi, 2k\pi + \pi] \\ -\sin^2 x, & x < 0 & x \in (2k\pi + \pi, 2k\pi + 2\pi) \end{cases}, \ k \in \mathbb{Z}$

• $x \neq k\pi$: $f'(x) = \begin{cases} 2\sin x \cos x, & x \in (2k\pi, 2k\pi + \pi) \\ -2\sin x \cos x, & x \in (2k\pi + \pi, 2k\pi + 2\pi) \end{cases}, \ k \in \mathbb{Z}$

f диф. у x, $x \neq \pi k$

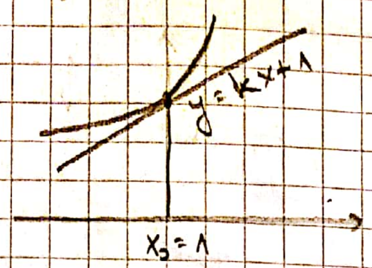
• $x = k\pi$: $f'(k\pi) \overset{\text{ако} \exists}{=} \lim_{h \to 0} \frac{f(k\pi + h) - f(k\pi)}{h} = \lim_{h \to 0} \frac{\sin(k\pi + h) \cdot |\sin(k\pi + h)| - \sin k\pi \cdot |\sin k\pi|}{h}$

$= \lim_{h \to 0} \frac{(-1)^k \sin h}{h} \cdot |(-1)^k \sin h| = 0$

$\Rightarrow f$ је диф. на $\mathbb{R}$

Дом. 1. $f(x) = x \cdot |x|$ (доказати да је диф. на $\mathbb{R}$)

$\boxed{5}$ $f(x) = 3x^2 + e^x - x$. Нађи тангенту на криву у тачки $x_0 = 1$.



$f'(x) = 6x + e^x - 1$

$k = f'(1) = 6 + e - 1 = \underline{5 + e}$

$(x_0, f(x_0)) \in t$

$f(1) = 3 + e - 1 = 2 + e$

$t: \ y = kx + n \rightarrow f(1) = k + n$

$n = (2+e) - (5+e) = 2 + \cancel{e} - 5 - \cancel{e} = \underline{-3}$

$\underline{t: \ y = (5+e)x - 3}$

Лопиталова правила

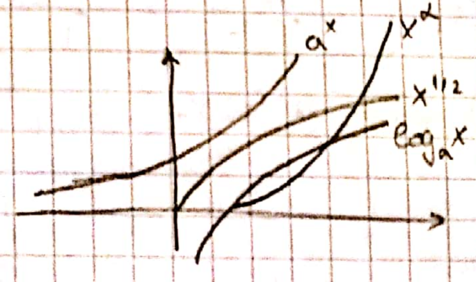
04.04.2018

<u>Теорема</u> $f, g : (a,b) \rightarrow \mathbb{R}$; f и g диф на (a,b) ; $g'(x) \neq 0$; $x \in (a,b)$:

$\frac{\pm\infty}{\pm\infty}$ $\lim\limits_{x\to a+} f(x) = +\infty = \lim\limits_{x\to\ldots} g(x)$

Ако $\exists \lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$, онда $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = \lim\limits_{x\to a+} \frac{f'(x)}{g'(x)}$.

$\frac{f \to 0}{g \to 0}$: $\lim\limits_{x\to \square} \frac{f(x)}{g(x)} \underset{\text{ако } \exists}{=} \underbrace{\lim\limits_{x\to\square} \frac{f'(x)}{g'(x)}}_{*}$



• $\log_a x, \ x^\alpha, \ a^x \longrightarrow +\infty$
$a>1, \ \alpha>0, \ a>1 \quad x \to +\infty$

$\boxed{1}$ $\lim\limits_{x\to+\infty} \frac{x}{a^x} \overset{\infty}{\underset{\infty}{=}} \lim\limits_{x\to+\infty} \frac{x'}{(a^x)'} = \lim\limits_{x\to+\infty} \frac{1}{\underbrace{a^x}_{+\infty} \ln a} = 0$
$(a>1)$

$\boxed{2}$ $\lim\limits_{x\to+\infty} \frac{x^{10}}{a^x} \overset{\infty}{\underset{\infty}{=}} \lim\limits_{x\to+\infty} \frac{10x^9}{a^x \ln a} \overset{\infty}{\underset{\infty}{=}} \lim\limits_{x\to+\infty} \frac{10\cdot 9\, x^8}{a^x \ln a \ln a} \overset{\infty}{\underset{\infty}{=}} \ldots \overset{\infty}{\underset{\infty}{=}} \lim\limits_{x\to+\infty} \frac{10\cdot 9 \cdot \ldots \cdot 2 \cdot 1}{\underbrace{a^x}_{+\infty} (\ln a)^{10}} = 0$

$*$ $\lim\limits_{x\to+\infty} \frac{x^k}{a^x} = 0$, $k \in \mathbb{N}$, $x^k \ll a^x$, $x \to +\infty$

3] $$\lim_{\substack{x\to+\infty\\(\alpha>0,\ a>1)}} \frac{\log_a x}{x^\alpha} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to+\infty} \frac{\frac{1}{x\ln a}}{\alpha x^{\alpha-1}} = \lim_{x\to+\infty} \frac{1}{\underbrace{x^\alpha}_{\to+\infty}\alpha \ln a} = 0$$

$$\log_a x << x^\alpha << a^x, \quad x\to+\infty$$
$$a>1,\ \alpha>0$$

4] $$\lim_{x\to+\infty} \frac{(\log_5 x)^{1000} + x^{700} - \left(\sqrt{\frac{3}{2}}\right)^x}{x - 3\log_2 x + (\sqrt{2})^x} = \lim_{x\to+\infty} \frac{\frac{(\log_5 x)^{1000}}{(\sqrt{2})^x} + \frac{x^{700}}{(\sqrt{2})^x} - \left(\frac{\sqrt{3/2}}{\sqrt{2}}\right)^x}{\frac{x}{(\sqrt{2})^x} - 3\frac{\log_2 x}{(\sqrt{2})^x} + 1} = 0$$

<u>Лема</u> f непр. у a; f диф. у некој околини a $(\setminus\{a\})$.

$\Rightarrow f'(a) = \lim_{x\to a} f'(x)$

$f'_+(a) = \lim_{x\to a+} f'(x)$, $\quad f'_-(a) = \lim_{x\to a-} f'(x)$

5] Испитати диф. ф-је $f(x) = \begin{cases} \operatorname{arctg} x, & |x|\le 1 \\ \frac{\pi}{4}\operatorname{sgn} x + \frac{x-1}{2}, & |x|>1 \end{cases}$

$-\frac{\pi}{4} + \frac{x-1}{2}$, $f' = 1/2$ | -1 [$\operatorname{arctg} x$] 1 | $\frac{\pi}{4} + \frac{x-1}{2}$, $f' = 1/2$

$\boxed{\text{Непр.}}$

• <u>$x=-1$</u>: $f(-1) = \operatorname{arctg}(-1) = -\pi/4$

$\lim_{x\to-1-} f(x) = \lim_{x\to-1-}\left(-\frac{\pi}{4} + \frac{x-1}{2}\right) = -\frac{\pi}{4} - 1$ $\Big\}$ f није непр. у -1

$\Rightarrow$ није ни диф. у -1

• <u>$x=1$</u>: $f(1) = \operatorname{arctg}(1) = \pi/4$

$\lim_{x\to1-} \operatorname{arctg} x = \pi/4$

$\lim_{x\to1+} f(x) = \lim_{x\to1+}\left(\frac{\pi}{4} + \frac{x-1}{2}\right) = \frac{\pi}{4}$ $\Big\}$ f је непр у 1

$\Rightarrow f$ је непр на $\mathbb{R}\setminus\{-1\}$.

$\boxed{\text{Диф.}}$

• <u>$x=-1$</u> $\Rightarrow$ ф-ја је прекидна $\Rightarrow$ није диф.

$$f'(x) = \begin{cases} \frac{1}{1+x^2}, & x\in(-1,1) \\ \frac{1}{2}, & x\in(-\infty,-1)\cup(1,+\infty) \end{cases}$$

• $\underline{x = 1}$: (f је непр)

$$\left.\begin{aligned} f'_+(1) &= \lim_{x\to 1+} f'(x) = 1/2 \\ f'_-(1) &= \lim_{x\to 1-} f'(x) = \lim_{x\to 1-} \frac{1}{1+x^2} = 1/2 \end{aligned}\right\} f \text{ је диф. у } 1$$

$\Rightarrow$ f диф. на $\mathbb{R}\setminus\{-1\}$.

$\boxed{6}$ $$\lim_{x\to 0+} \sqrt{x}\cdot e^{1/x} \underset{\substack{t=1/x \\ t\to+\infty}}{=} \lim_{t\to+\infty} \sqrt{\frac{1}{t}}\cdot e^t = \lim_{t\to+\infty} \frac{e^t}{\sqrt{t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to+\infty} \frac{e^t}{\frac{1}{2}\cdot\frac{1}{\sqrt{t}}} = \lim_{t\to+\infty} 2e^t\sqrt{t} = \underline{\underline{+\infty}}$$

$\boxed{7}$ $$\lim_{x\to+\infty} \left(\frac{2}{\pi}\operatorname{arctg} x\right)^x = \lim_{x\to+\infty} e^{\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)^x} = e^{\underbrace{\lim_{x\to+\infty} \ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)^x}_{L}}$$

• $$L = \lim_{x\to+\infty} \frac{\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)}{1/x} \overset{\frac{0}{0}}{=} \frac{\frac{1}{\frac{2}{\pi}\operatorname{arctg} x}\cdot\frac{2/\pi}{1+x^2}}{-1/x^2} = \lim_{x\to+\infty} \underbrace{\frac{-x^2}{1+x^2}}_{-1} \frac{2/\pi}{\frac{2}{\pi}\operatorname{arctg} x} = -\frac{2}{\pi}$$

$\Rightarrow$ $e^{-2/\pi}$

$\boxed{8}$ а) $$\lim_{x\to 0+} x\cdot\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+}(-x) = \underline{\underline{0}}$$

$$\boxed{\lim_{x\to 0+} x\cdot\ln x = 0}$$

б) $$\lim_{x\to 0+} x\ln^2 x = \lim_{x\to 0} \frac{\ln^2 x}{1/x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0} \frac{2\ln x\cdot\frac{1}{x}}{-1/x^2} = \lim_{x\to 0} \frac{2\ln x}{-1/x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0} \frac{\frac{2}{x}}{\frac{1}{x^2}} = \underline{\underline{0}}$$

$\boxed{9}$ $$\lim_{x\to 0+} x^{x^x-1} = \lim_{x\to 0+} e^{\ln x(x^x-1)} = e^{\underbrace{\lim_{x\to 0+}\ln x(x^x-1)}_{L}} \Rightarrow e^0 = \underline{\underline{1}}$$

• $$L = \lim_{x\to 0+} \ln x\left(e^{x\cdot\ln x}-1\right) = \lim_{x\to 0+} \ln x\cdot\underbrace{\frac{e^{x\ln x}-1}{x\cdot\ln x}}_{1}\cdot\underbrace{x\ln^2 x}_{0} = 0$$

$\boxed{10}$ $$\lim_{x\to 0}\left(\operatorname{ctg} x - \frac{1}{x}\right) = \lim_{x\to 0} \frac{\cos x\cdot x - \sin x}{x\sin x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\sin x\cdot x + \cancel{\cos x} - \cancel{\cos x}}{\sin x + x\cdot\cos x} \overset{\frac{0}{0}}{=}$$

$$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\cos x\cdot \overset{0}{x} - \overset{0}{\sin x}}{\underbrace{\cos x + \cos x}_{2} - \underbrace{x\sin x}_{0}} = \frac{0}{2} = 0$$

$\boxed{11}$ $\lim\limits_{\substack{x\to 0\\(a>0)}} \frac{(a+x)^x - a^x}{x^2} = L$

• $((a+x)^x)' = (e^{\ln(a+x)\cdot x})' = e^{\ln(a+x)\cdot x}\left(\frac{1}{a+x}\cdot x + \ln(a+x)\right) =$

$= (a+x)^x\left(\frac{x}{a+x} + \ln(a+x)\right)$

$$L \overset{0/0}{=} \lim_{x\to 0} \frac{(a+x)^x\left(\frac{x}{a+x} + \ln(a+x)\right) - a^x \ln a}{2x} \overset{0/0}{=}$$

$$\overset{0/0}{=} \lim_{x\to 0} \frac{1}{2}\left((a+x)^x\left(\frac{x}{a+x}+\ln(a+x)\right)^2 + (a+x)^x\left(\frac{a+x-x}{(a+x)^2} + \frac{1}{a+x}\right) - a^x \ln^2 a\right) =$$

$$= \frac{1}{2}\left(\cancel{(\ln a)^2} + \frac{1}{a} + \frac{1}{a} - \cancel{\ln^2 a}\right) = \frac{1}{a}$$

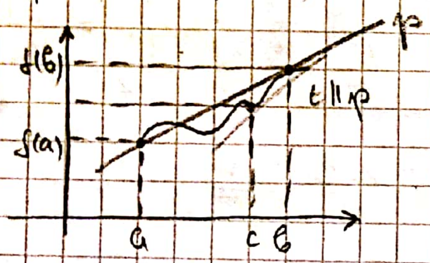
* Важне теореме

Теорема

$f \in \underset{\text{непр-}[a,b]}{C[a,b]} \cap \underset{\text{диф-}(a,b)}{D(a,b)}$

$f(a) = f(b) \Rightarrow \exists c \in (a,b)$ т.д. $f'(c) = 0$

Теорема (Лагранж)



$f \in C[a,b] \cap D(a,b)$

$\Rightarrow \exists c \in (a,b)$ т.д. $f(b) - f(a) = f'(c)\cdot(b-a)$

$f'(c) = \frac{f(b) - f(a)}{b-a} = k$

$\boxed{12}$ $a_0, a_1, \dots, a_n \in \mathbb{R}$ т.д. $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0$. Доказати да полином $p(x) = a_0 + a_1 x + \dots + a_n x^n$ има нулу у $\mathbb{R}$ ($\exists x_0 \in \mathbb{R}$ т.д. $p(x_0) = 0$)

$g: \mathbb{R} \to \mathbb{R}$ полином т.д. $g'(x) = p(x)$

$g(x) := a_0 x + a_1 \frac{x^2}{2} + \dots + a_n \frac{x^{n+1}}{n+1}$

$$\left.\begin{aligned} &g(0) = 0 \\ &g(1) = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{a_{n+1}} = 0 \\ &g \in C[0,1] \wedge D(0,1) \end{aligned}\right\} \overset{T1}{\Rightarrow} \exists x_0 \in (0,1) \text{ т.д. } g'(x_0) = 0$$

$\to p(x_0) = 0$

$\boxed{13}$ Доказати:

а) $\forall x,y>0: \left|\frac{1}{1+x}-\frac{1}{1+y}\right| \le |x-y|$

б) $\forall x,y \in \mathbb{R}: |\operatorname{arctg} x - \operatorname{arctg} y| \le |x-y|$

$x=y$ ✓

б.у.о: $x<y$

Лагранж: $[x,y]$

$$\left.\begin{aligned} &f(t)=\operatorname{arctg} t \\ &f\in C[x,y] \wedge D(x,y) \end{aligned}\right\} \Rightarrow \exists c\in(x,y) \text{ т.д. } f(y)-f(x)=f'(c)\cdot(y-x)$$

$$\operatorname{arctg} y-\operatorname{arctg} x=\underbrace{f'(c)}_{\frac{1}{1+c^2}}\cdot(y-x)$$

$$\Rightarrow |\operatorname{arctg} y-\operatorname{arctg} x| = \underbrace{\frac{1}{1+c^2}}_{\le 1}|y-x| \le |y-x|$$

Баш када је апс. вр. <1

$\boxed{14}$ $f\in C[0,1] \wedge D(0,1)$

Док: $\exists c\in(0,1)$ т.д. $f(1)-f(0)=\frac{1}{2}\overbrace{\left(f'(c)+f'(1-c)\right)}^{F'(c)}$.

$F:[0,1]\to\mathbb{R} \quad F(x):=\frac{1}{2}\left(f(x)-f(1-x)\right)$

$F\in C[0,1] \wedge D(0,1)$

$F(0)=\frac{1}{2}\left(f(0)-f(1)\right) \quad F(1)=\frac{1}{2}\left(f(1)-f(0)\right)$

$\xRightarrow{\text{Лаг.}} \exists c\in(0,1)$ т.д. $F(1)-F(0) = F'(c)\cdot(1-0)$

$= f(1)-f(0) = \frac{1}{2}\left(f'(c)+f'(1-c)\right)$

$\boxed{15}$ $f:\mathbb{R}\to\mathbb{R}$ дифер. $\exists a\in\mathbb{R}, a\neq 0$, $f(a)=1$. Доказати $\exists c\in\mathbb{R}$ $cf'(c)+f(c)=1$.

$\Leftrightarrow \underbrace{c\cdot f'(c)+f(c)-1}_{F'(c)}=0$.

$F:\mathbb{R}\to\mathbb{R}, \quad F(x):=xf(x)-x$

$F'(x)=f(x)+xf'(x)-1$

$$\left.\begin{aligned} &F(0)=0 \\ &F(a)=a\cdot f(a)-a=0 \\ &f\in D(\mathbb{R}) \Rightarrow F\in D(\mathbb{R}) \end{aligned}\right\} \begin{aligned} &[0,a], a>0; [a,0], a<0 \\ &\Rightarrow \exists c \text{ између } 0 \text{ и } a \text{ т.д. } F'(c)=0. \end{aligned}$$

* $f \in C[a,b] \cap D(a,b)$

$f(a) = f(b) = 0$

$g(a) \cdot g(b) \neq 0$

$f'(x)\, g(x) \neq f(x) \cdot g'(x)$, $\forall x \in (a,b)$

16.04.18.

Изводи вишег реда

$f: D_f \to \mathbb{R}$, $f'(x) = \ldots$, $f''(x) = (f'(x))'$, ... , $f^{(n)}(x) := \left(f^{(n-1)}(x)\right)'$

Пример $f(x) = a^x$

$f'(x) = a^x \ln a$, $f''(x) = a^x (\ln a)^2$, ... , $f^{(n)}(x) = a^x (\ln a)^n$

Пример $f(x) = \ln x$

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, ... , $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$

* $(u \cdot v)^{(n)} = ?$, $(u \cdot v)' = u' \cdot v + u \cdot v'$

$(u \cdot v)'' = u'' v + u'v' + u'v' + u v'' = u''v + 2u'v' + u v''$ ...

* Лајбницова формула:

$$(u(x) \cdot v(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot u^{(k)}(x) \cdot v^{(n-k)}(x) = u(x)\, v^{(n)}(x) + \binom{n}{1} u'(x)\, v^{(n-1)}(x) + \ldots + \binom{n}{n-1} u^{(n-1)}(x)\, v'(x) + u^{(n)}(x)\, v(x)$$

$\boxed{1}$ $f(x) = \underbrace{(5x^2 - 2x + 1)}_{u(x)} \cdot \underbrace{e^{-3x}}_{v(x)}$. $f^{(n)}(x)$, $f^{(n)}(1) = ?$

$u'(x) = 10x - 2$ $\qquad v'(x) = -3e^{-3x}$

$u''(x) = 10$ $\qquad v^{(k)}(x) = (-3)^k \cdot e^{-3x}$

$u'''(x) = 0$

$u^{(k)}(x) = 0$, $k \geq 3$

$\overset{\text{Л.ф.}}{\Longrightarrow}$ $f^{(n)}(x) = u(x) \cdot v^{(n)}(x) + \binom{n}{1} \cdot u'(x)\, v^{(n-1)}(x) + \binom{n}{2} u''(x)\, v^{(n-2)}(x) + \overbrace{0 + \ldots + 0}^{k \geq 3}$

$f^{(n)}(x) = (5x^2 - 2x + 1)(-3)^n \cdot e^{-3x} + n(10x - 2)(-3)^{n-1} e^{-3x} + \binom{n}{2} 10 (-3)^{n-2} \cdot e^{-3x}$

$f^{(n)}(1) = 6(-3)^n e^{-3x} + 8n(-3)^{n-1} e^{-3x} + \binom{n}{2} 10 (-3)^{n-2} e^{-3x}$

* **Тејлорова формула** – f има непрекидне изводе до n-тог реда у некој околини $N(a)$ и $\exists$ $(n+1)$-ви извод у $N(a)$.

$$\boxed{f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\text{остатак}}$$

$T_n(x,a)$ – Тејлоров развој

у Пеановом облику: $o((x-a)^n),\ x\to a$

Пример $f(x)=\operatorname{arctg}x,\ a=5,\ n=2$ (Тејлоров развој)

$$f'(x)=\frac{1}{1+x^2},\quad f'(5)=\frac{1}{26}$$

$$f''(x)=\frac{-2x}{(1+x^2)^2},\quad f''(5)=\frac{-10}{26^2}$$

$$\Rightarrow \operatorname{arctg}x=\operatorname{arctg}5+\frac{1}{26}(x-5)-\frac{5}{26^2}(x-5)^2+o((x-5)^2),\ x\to 5$$

* **Маклоренов развој** (Т.ф. када је $a=0$)

$$\boxed{f(x)=f(0)+\frac{f'(0)}{1!}\cdot x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}\cdot x^n+o(x^n),\ x\to 0}$$

- $e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+o(x^n),\ x\to 0$
- $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2}),\ x\to 0$
- $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\cdot\frac{x^{2n}}{(2n)!}+o(x^{2n+1}),\ x\to 0$
- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+o(x^n),\ x\to 0$
- $(1+x)^\alpha=1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\ldots+\binom{\alpha}{n}x^n+o(x^n),\ x\to 0$

за $\alpha\in\mathbb{R},\ k\in\mathbb{N}_0 \Rightarrow \binom{\alpha}{k}:=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$

$\underline{\alpha=-1}$: $\frac{1}{1+x}=(1+x)^{-1}=1+\binom{-1}{1}x+\binom{-1}{2}x^2+\ldots+\binom{-1}{n}x^n+o(x^n),\ x\to 0$

$=1-x+x^2-x^3+\ldots+(-1)^n x^n+o(x^n),\ x\to 0$

$\frac{1}{1-x}=(1-x)^{-1}=1+x+x^2+x^3+\ldots+(-1)^n(-x)^n+o(x^n),\ x\to 0$

$\boxed{2}$ $f(x) = (3x+1)\cdot\ln(1+2x)$, $f^{(n)}(0) = ?$

$$f(x) = (3x+1)\left(2x - \frac{(2x)^2}{2} + \ldots + (-1)^{n-2}\cdot\frac{(2x)^{n-1}}{n-1} + (-1)^{n-1}\cdot\frac{(2x)^n}{n} + o(x^n)\right) =$$

$$= f(0) + \frac{f'(0)}{1}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n),\ x \to 0$$

y3 x^n: $3x(-1)^{n-2}\cdot\frac{(2x)^{n-1}}{n-1} + 1\cdot(-1)^{n-1}\cdot\frac{(2x)^n}{n} =$

$$= x^n\left(\frac{3(-1)^{n-2}\cdot 2^{n-1}}{n-1} + \frac{(-1)^{n-1}\cdot 2^n}{n}\right)$$

$$f^{(n)}(0) = n!\left(\frac{3(-1)^{n-2}\cdot 2^{n-1}}{n-1} + \frac{(-1)^{n-1}\cdot 2^n}{n}\right)$$

$\boxed{3}$ (Маклоренів розклад)

А) $f(x) = \sin(x + \pi/3)$, $n = 5$

$$\sin(x+\pi/3) = \sin x\cdot\cos\pi/3 + \cos x\cdot\sin\pi/3 = \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) + \frac{\sqrt3}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)\right) =$$

$$= \frac{\sqrt3}{2} + \frac{x}{2} - \frac{\sqrt3 x^2}{4} - \frac{x^3}{12} + \frac{\sqrt3 x^4}{48} + \frac{x^5}{240} + o(x^5),\ x\to 0$$

Б) $f(x) = \ln(2e + x)$, $n = 3$

$$\ln(2e+x) = \ln\left(2e\left(1+\frac{x}{2e}\right)\right) = \ln 2e + \ln\left(1+\frac{x}{2e}\right) = \overbrace{\ln 2 + 1}^{\ln 2e} + \frac{x}{2e} - \frac{1}{2}\left(\frac{x}{2e}\right)^2 + \frac{1}{3}\left(\frac{x}{2e}\right)^3 + \ldots$$

$\boxed{4}$ $f(x) = \operatorname{tg} x$ $(x\to 0)$, $n = 5$ (Маклоренів розклад)

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \sin x\,(\cos x)^{-1}$$

$$(\cos x)^{-1} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right)^{-1} =$$

$$= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right) + \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)\right)^2 - \left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right)^3 + o\left(\left(-\frac{x^2}{2} + \frac{x^4}{4!} + \ldots\right)\right)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{4!} + o(x^5) + \frac{x^4}{4} + o(x^5) + o(x^6) + o(x^6) =$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^5),\ x \to 0$$

$$\operatorname{tg} x = \sin x\,(\cos x)^{-1} =$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^5)\right) =$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 + o(x^6) - \frac{x^3}{3!} - \frac{x^5}{3!\cdot 2} + o(x^5) + \frac{x^5}{5!} + o(x^7) + o(x^6) =$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5),\ x \to 0$$

5 $\lim\limits_{x\to 0} \dfrac{e^{\sin x}\cdot\cos(\sin x) - 1 - x}{x^3} = ?$

• $\cos(\sin x) = \cos\left(x - \frac{x^3}{3!} + o(x^4)\right) = 1 - \frac{1}{2}\left(x - \frac{x^3}{3} + o(x^4)\right)^2 + o(x^3) = 1 - \frac{1}{2}\left(x^2 + o(x^4)\right) + o(x^3) =$

$= 1 - \frac{1}{2}x^2 + o(x^3)$

• $e^{\sin x} = e^{x - \frac{x^3}{3!} + o(x^4)} = 1 + \left(x - \frac{x^3}{3!} + o(x^4)\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + o(x^4)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + o(x^4)\right)^3 + o(x^3)$

$= 1 + x - \frac{x^3}{3!} + o(x^4) + \frac{1}{2}x^2 + o(x^3) + \frac{1}{6}x^3 + o(x^3) =$

$= 1 + x + \frac{1}{2}x^2 + o(x^3)$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{\left(1 + x + \frac{1}{2}x^2 + o(x^3)\right)\left(1 - \frac{1}{2}x^2 + o(x^3)\right) - 1 - x}{x^3} =$

$= \lim\limits_{x\to 0} \dfrac{\cancel{1} - \cancel{\frac{1}{2}x^2} + o(x^3) + \cancel{x} - \frac{1}{2}x^3 + o(x^4) + \cancel{\frac{1}{2}x^2} + o(x^5) + o(x^3) - \cancel{1} - \cancel{x}}{x^3} =$

$= \lim\limits_{x\to 0} \dfrac{-\frac{1}{2}x^3 + o(x^3)}{x^3} = \lim\limits_{x\to 0}\left(-\frac{1}{2} + \frac{o(1)}{1}\right) = -\frac{1}{2}.$

6 Испитати дифер. у зависности од $a\in\mathbb{R}$, $f(x) = \begin{cases} \frac{1}{x} - \frac{1}{e^x - 1}, & x \neq 0 \\ a, & x = 0 \end{cases}$.

$\boxed{\text{Непр}}$

• $x \neq 0$ ✓

• $x = 0$: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x(e^x - 1)} \overset{ЛП}{\underset{0}{\overset{0}{=}}} \lim\limits_{x\to 0} \dfrac{e^x - 1}{e^x + xe^x - 1} \overset{ЛП}{\underset{0}{\overset{0}{=}}}$

$= \lim\limits_{x\to 0} \dfrac{e^x}{e^x + e^x + xe^x} = \underline{\underline{\frac{1}{2}}}$

$\Rightarrow$ ф-ја f је непр у $x = 0$ за $a = \frac{1}{2}$.

$\boxed{\text{Дифер}}$

• $x \neq 0$: $f'(x) = -\frac{1}{x^2} + \frac{e^x}{(e^x - 1)^2}$ ✓

• $x = 0$: мора $a = \frac{1}{2} \to$ непр.

$f'(0) = \lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} \dfrac{e^x x^2 - (e^x - 1)^2}{x^2(e^x - 1)^2} = \lim\limits_{x\to 0} \dfrac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)x^2 - \left(x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)^2}{x^2\left(x + \frac{x^2}{2} + o(x^2)\right)^2} =$ $:x^2$

$= \lim\limits_{x\to 0} \dfrac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) - \left(1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2)\right)^2}{\left(x + \frac{x^2}{2} + o(x^2)\right)^2} =$

$= \lim\limits_{x\to 0} \dfrac{1 + x + \frac{x^2}{2} + o(x^2) - 1 - \frac{x^2}{4} - 2\cdot\frac{x}{2} - 2\frac{x^2}{6} + o(x^2)}{x^2 + o(x^2)} = \lim\limits_{x\to 0} \dfrac{\frac{x^2}{2} - \frac{x^2}{3} - \frac{x^2}{4} + o(x^2)}{x^2 + o(x^2)} =$

$= \lim\limits_{x\to 0} \dfrac{-x^2/12 + o(x^2)}{x^2 + o(x^2)} = -\frac{1}{12} = f'(0)$. За $a = \frac{1}{2} \Rightarrow f$ је дифер на $\mathbb{R}$

$\boxed{7}$ $\lim\limits_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right)=\lim\limits_{x\to+\infty}\left(\sqrt[3]{x^3}\cdot\sqrt[3]{1+3/x}-\sqrt{x^2}\cdot\sqrt{1-2/x}\right)=$

$=\lim\limits_{x\to+\infty}\left(x\left((1+3/x)^{1/3}-(1-2/x)^{1/2}\right)\right)=$

$=\lim\limits_{x\to+\infty}\left(x\left(1+\binom{1/3}{1}\cdot\frac{3}{x}+o\left(\frac{3}{x}\right)-\left(1+\binom{1/2}{1}\cdot\left(\frac{-2}{x}\right)+o\left(\frac{2}{x}\right)\right)\right)\right)=$

$=\lim\limits_{x\to+\infty}\left(x\left(\frac{1}{x}+o(1/x)+1/x+o(1/x)\right)\right)=\lim\limits_{x\to+\infty}\left(2+\underbrace{o\left(x\cdot\frac{1}{x}\right)}_{0}\right)=2$

$\boxed{8.}$ Наћи $a,b,c\in\mathbb{R}$ т.д. $f(x)=(1+x)^{1/x}=a+bx+cx^2+o(x^2)$, $x\to0$.

- $(1+x)^{1/x}=e^{\frac{1}{x}\ln(1+x)}=e^{\frac{1}{x}\left(x-\frac{x^2}{2}+o(x^2)\right)}=e^{1-\frac{x}{2}+o(x)}$ → није довољно!

- $(1+x)^{1/x}=e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)}=e^{1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)}=e\cdot e^{-\frac{x}{2}+\frac{x^2}{3}+o(x^2)}=$ јер претх. корак није то

$=e\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}+o(x^2)\right)+\frac{1}{2}\left(-\frac{x}{2}+\frac{x^2}{3}+o(x^2)\right)^2+o(x^4)\right)=$

$=e\left(1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)+\frac{1}{8}x^2+o(x^2)+o(x^2)\right)=$

$=e-\frac{e}{2}x+\frac{11e}{24}x^2+o(x^2)\Rightarrow a=e\quad b=-\frac{e}{2}\quad c=\frac{11e}{24}$

Асимптоте

16.04.2018.

=> говоре о понашању у ∞

Деф $y=ax+b$ је асимптота ф-је f кад $x\to+\infty$ ако $f(x)=ax+b+o(1)$, $x\to+\infty$

$\Longleftrightarrow$ $a=\lim\limits_{x\to+\infty}\frac{f(x)}{x}$ и $b=\lim\limits_{x\to+\infty}(f(x)-ax)$, $a,b\in\mathbb{R}$

• слично важи и за $x\to-\infty$

* $a\neq0$: $y=ax+b$ је коса асимптота f, $x\to\infty$
* $a=0$: $y=ax+b$ је хоризонтална асимптота

Пример $f(x)=\sqrt[3]{x^3+2x^2}$. Наћи асимптоте ако постоје за $x\to+\infty$ и $x\to-\infty$ и одр. са које стране ф-је су

$f(x)=\underbrace{ax+b}_{\text{асимп.}}+\underbrace{c/x+o(1/x)}_{\text{са које стр.}}$, $x\to+\infty$

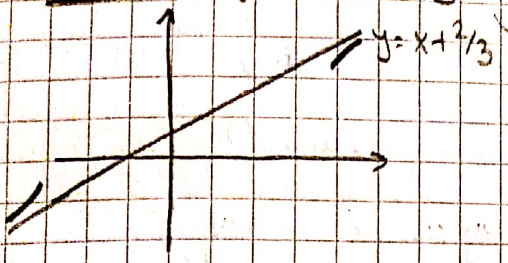
$$f(x) = \sqrt[3]{x^3}\cdot\sqrt[3]{1+\frac{2}{x}} = x\left(1+\frac{2}{x}\right)^{1/3} = x\left(1+\binom{1/3}{1}\frac{2}{x}+\binom{1/3}{2}\left(\frac{2}{x}\right)^2+o\left(\frac{1}{x^2}\right)\right)=$$

$$= \underbrace{x+\frac{2}{3}}_{K.A.}\ \underbrace{-\frac{4}{9}x + o(1/x)}_{o(1)}, \quad x\to\pm\infty$$

$y = x+2/3, \ x\to\pm\infty$

• $\underline{x\to+\infty}$: $f(x) = x+2/3 \underbrace{- 4/9 x + o(1/x)}_{<0} \Rightarrow$ f је испод асимптоте

• $\underline{x\to-\infty}$: $f(x) = x+2/3 \underbrace{- 4/9 x + o(1/x)}_{>0} \Rightarrow$ f је изнад асимптоте



$\boxed{1}$ $f(x) = \sqrt{9x^2+6x+1}\cdot e^{\frac{1}{x-2}}$.

$$f(x) = \sqrt{(3x+1)^2}\cdot e^{\frac{1}{x(1-2/x)}} = |3x+1|\cdot e^{\frac{1}{x}\left(1-\frac{2}{x}\right)^{-1}} =$$

$$= |3x+1|\cdot e^{\frac{1}{x}\left(1+\frac{2}{x}-\frac{4}{x^2}+o\left(\frac{1}{x^2}\right)\right)} = |3x+1|\cdot e^{\frac{1}{x}+\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)},$$

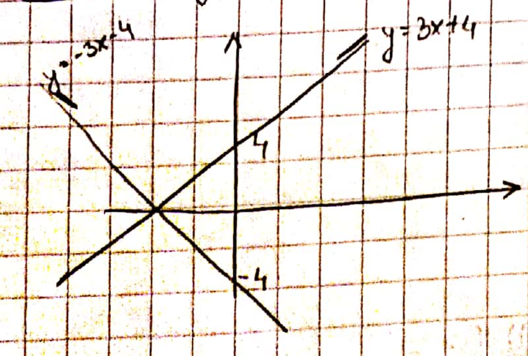
$$= |3x+1|\cdot\left(1+\left(\frac{1}{x}+\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right)+\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right)^2+o\left(\frac{1}{x^2}\right)\right)=$$

$$= |3x+1|\cdot\left(1+\frac{1}{x}+\frac{2}{x^2}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right) = |3x+1|\left(1+\frac{1}{x}+\frac{5/2}{x^2}+o\left(\frac{1}{x^2}\right)\right), \quad x\to\pm\infty$$

• $\underline{x\to+\infty}$: $f(x) = (3x+1)\left(1+\frac{1}{x}+\frac{5/2}{x^2}+o\left(\frac{1}{x^2}\right)\right) = 3x+3+\frac{15/2}{x}+o\left(\frac{1}{x}\right)+1+\frac{1}{x}+\frac{5/2}{x^2}+o\left(\frac{1}{x^2}\right) =$

$$= \underbrace{3x+4}_{K.A.,\ x\to+\infty} + \underbrace{\frac{17/2}{x}+o\left(\frac{1}{x}\right)}_{>0 \ \ f \text{ је изнад}}$$

• $\underline{x\to-\infty}$: $f(x) = -(3x+1)\left(1+\frac{1}{x}+\frac{5/2}{x^2}+o\left(\frac{1}{x^2}\right)\right) = \underbrace{-3x-4}_{K.A.,\ x\to-\infty} \underbrace{-\frac{17/2}{x}+o\left(\frac{1}{x^2}\right)}_{>0}$ f је изнад



23.04.2018

Испитивање ф-ја

1° D_f

2° **парност**: $f(x) = f(-x)$ или $f(x) = -f(x)$ (D_f мора бити симетричан)
парна ... непарна

периодичност: $f(x) = f(x+T)$

3° **нуле и знак**

4° **асимптоте**: 1) $x \to \pm\infty$: $\lim\limits_{x\to\pm\infty} f(x) = b$ (хоризонтална асимптота)

$\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$ (да ли има косу асимпт.?)

2) $x_0 \in \mathbb{R}$: $\lim\limits_{x\to x_0} f(x) = \pm\infty$, $x = x_0$ је вертикална асимпт.
x_0 - на рубу домена

5° **I извод**: • $f'(x) > 0$, $x \in (a,b) \Rightarrow f \uparrow$ на (a,b)

• $f'(x) < 0$, $x \in (a,b) \Rightarrow f \downarrow$ на (a,b)

• $f'(c) = 0 \Rightarrow c$ је локални min/max ако f' мења знак при преласку кроз c

6° **II извод**: • $f''(x) > 0$, $x \in (a,b) \Rightarrow f$ је конвексна на (a,b) $\cup$

• $f''(x) < 0$, $x \in (a,b) \Rightarrow f$ је конкавна на (a,b) $\cap$

• $f''(c) = 0 \Rightarrow$ у c се мења конвексност / конкавност
$\Rightarrow c$ је преводна тачка

7° **скица** Γ_f

1] $f(x) = (x+2)e^{1/x}$

1° $x \in \mathbb{R} \setminus \{0\} \Rightarrow$ $\boxed{D_f = (-\infty, 0) \cup (0, +\infty)}$

2° ~~нп~~ - ~~нн~~

3° $x+2$: $-$ (-2) $+$ (0) $+$

$f(x) < 0$ за $x \in (-\infty, -2)$

$\boxed{x = -2}$

$f(x) > 0$ за $x \in (-2, 0) \cup (0, +\infty)$

4° $f(x) = (x+2)\left(1+\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right) = x+1+\frac{1}{2x}+o\left(\frac{1}{x}\right)+2+\frac{2}{x}+\frac{1}{x^2}+o\left(\frac{1}{x^2}\right) =$

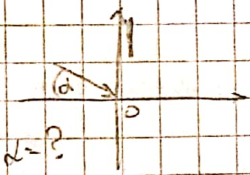
$= x+3+\frac{5/2}{x}+o\left(\frac{1}{x}\right)$

• $\underline{x\to+\infty}$: $\boxed{y=x+3}$ К.А. ; f је изнад

• $\underline{x\to-\infty}$: $\boxed{y=x+3}$, К.А. ; f је испод

• $\underline{x=0}$ из D_f: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(x+2)e^{1/x}$

$\lim\limits_{x\to 0_+} f(x) = +\infty \qquad \lim\limits_{x\to 0_-} f(x) = 0$



5° $f'(x) = e^{1/x}+(x+2)e^{\frac{1}{x}}\cdot\frac{-1}{x^2} = \underbrace{\frac{1}{x^2}e^{1/x}}(x^2-x-2) = \underbrace{\frac{1}{x^2}e^{1/x}}_{>0}(x-2)(x+1)$, $x\neq 0$

	-1		0		2	
$x-2$	$-$		$-$			$+$
$x+1$	$-$			$+$		
f'	$+$		$+$			$+$
	↗		↘			↗

f ↗ на $x\in(-\infty,-1)\cup(2,+\infty)$

f ↘ на $x\in(-1,0)\cup(0,2)$

$\boxed{x=-1}$ је max $\qquad \boxed{x=2}$ је min

$\alpha = \lim\limits_{x\to 0-} f'(x) = \lim\limits_{x\to 0-}\left(\frac{1}{x^2}e^{1/x}\underbrace{(x-2)}_{-2}\underbrace{(x+1)}_{1}\right) = -2\lim\limits_{x\to 0}\frac{e^{1/x}}{x^2} \underset{\substack{t=-1/x\\ t\to+\infty}}{=} -2\lim\limits_{t\to+\infty}\frac{e^{-t}}{1/t^2} = \boxed{0} = \alpha$ (хоризонтални углом)

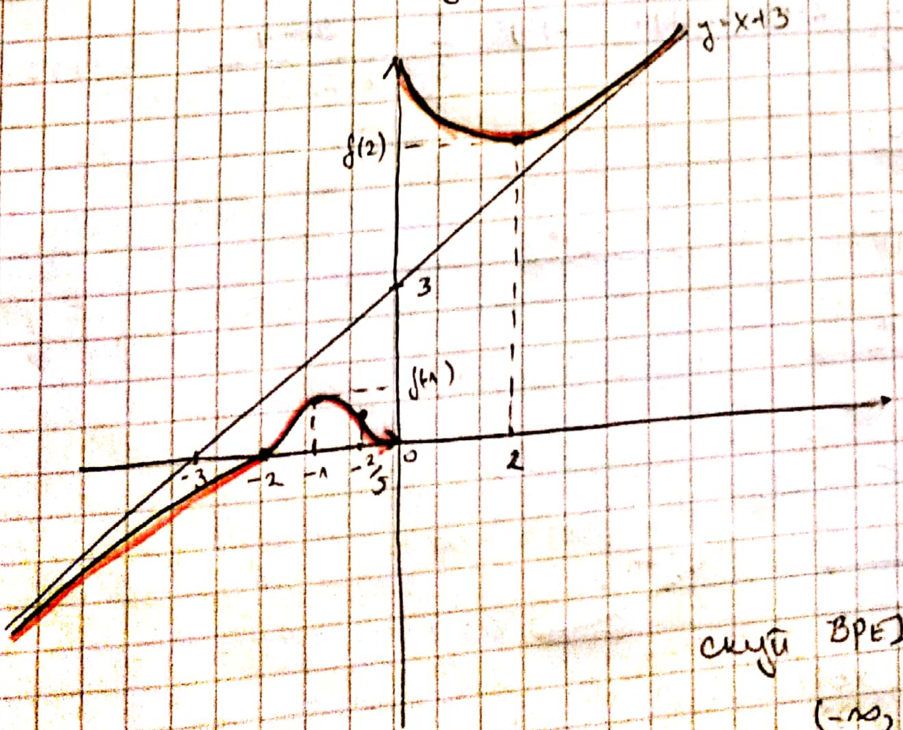
6° $f''(x) = \ldots = \underbrace{\frac{e^{1/x}}{x^4}}_{>0}(5x+2)$, $x\neq 0$

$5x+2$: $-$, $-\frac{2}{5}$, $+$, 0 , $+$

∩ $\left(-\frac{2}{5}\right)$ ∪ 0 ∪

$-\frac{2}{5}$ → превојна тачка

7°



скуп вредности:

$(-\infty, f(-1))\cup(f(2),+\infty)$

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$\boxed{2}$ $f(x) = \sqrt[3]{(x-2)^2|x-1|}$

1° $D_f = \mathbb{R}$

2° НН-НН

3° нуле: $\boxed{x=2}$ $\vee$ $\boxed{x=1}$ $f(x)>0$ за $(-\infty,1)\cup(1,2)\cup(2,+\infty)$

4° $\left.\begin{array}{l}\lim\limits_{x\to+\infty} f(x) = +\infty \\ \lim\limits_{x\to-\infty} f(x) = +\infty\end{array}\right\}$ нема Х.А. у $\pm\infty$

$\underline{x\to+\infty}$: $f(x) = \sqrt[3]{(x-2)^2(x-1)} = (x-2)^{2/3}(x-1)^{1/3} = x^{2/3}\left(1-\frac{2}{x}\right)^{2/3}\cdot x^{1/3}\left(1-\frac{1}{x}\right)^{1/3} =$

$= x\big(1\underbrace{-\tfrac{2}{x}}_{\to 0}\big)^{2/3}\big(1\underbrace{-\tfrac{1}{x}}_{\to 0}\big)^{1/3} =$

$= x\left(1+\binom{2/3}{1}\frac{-2}{x}+\binom{2/3}{2}\frac{4}{x^2}+o\left(\frac{1}{x^2}\right)\right)\left(1+\binom{1/3}{1}\frac{-1}{x}+\binom{1/3}{2}\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right) =$

$= x\left(1-\frac{4/3}{x}-\frac{4}{9x^2}+o\left(\frac{1}{x^2}\right)\right)\left(1-\frac{1/3}{x}-\frac{1}{9x^2}+o\left(\frac{1}{x^2}\right)\right) =$

$= x\left(1+\frac{-4/3}{x}-\frac{1/3}{x}-\frac{1}{9x^2}+\frac{4}{9x^2}-\frac{4}{9x^2}+o\left(\frac{1}{x^2}\right)\right) =$

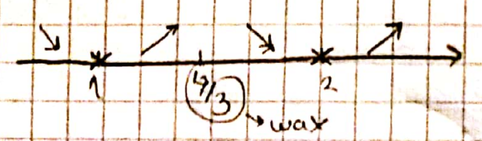
$= \underbrace{x-\frac{5}{3}}_{\text{К.А.}}\underbrace{-\frac{1}{9x}+o\left(\frac{1}{x}\right)}_{<0 \text{ ф је испод}}, \quad x\to+\infty$

$\underline{x\to-\infty}$: $f(x) = \underbrace{-x+\frac{5}{3}}_{\text{К.А.}}\underbrace{+\frac{1}{9x}+o\left(\frac{1}{x}\right)}_{<0}, \quad x\to-\infty$ f је испод

5° $\underline{x>1}$: $f'(x) = \frac{1}{3}\cdot\frac{1}{\left(\sqrt[3]{(x-2)^2(x-1)}\right)^2}\left(2(x-2)(x-1)+(x-2)^2\right) = \frac{1}{3}\cdot\frac{3x-4}{(x-2)^{1/3}(x-1)^{2/3}}, \quad x\neq 2$

$x\in(1,2)\cup(2,+\infty)$

$\underline{x<1}$: $f'(x) = -\frac{1}{3}\cdot\frac{3x-4}{(x-2)^{1/3}(x-1)^{2/3}}, \quad x\in(-\infty,1)$



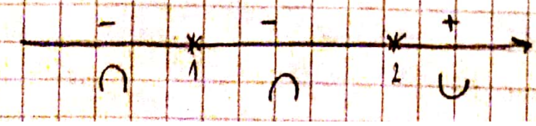
$\lim\limits_{x\to1-} f'(x) = -\infty$

$\lim\limits_{x\to1+} f'(x) = +\infty$

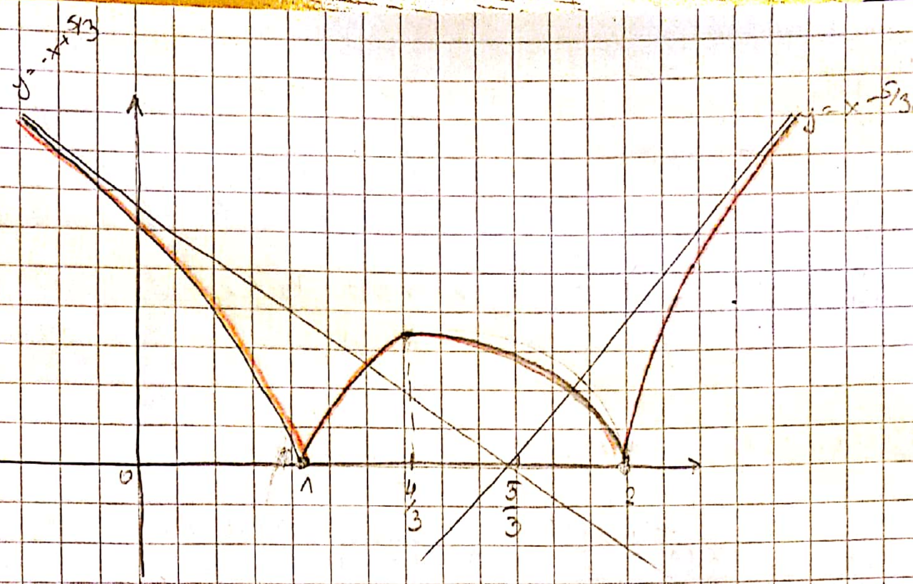
$\lim\limits_{x\to2-} f'(x) = -\infty$

$\lim\limits_{x\to2+} f'(x) = +\infty$

6° $f''(x) = \ldots = \begin{cases} \dfrac{-2}{9(x-2)^{4/3}(x-1)^{5/3}}, & x\in(1,2)\cup(2,+\infty) \\ \dfrac{2}{9(x-2)^{4/3}(x-1)^{5/3}}, & x\in(-\infty,1) \end{cases}$



7°



$\boxed{3}$ Доказати $\forall x>0$: $\operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \pi/2$

$F(x) = \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x}$, $F: (0, +\infty) \to \mathbb{R}$

$F'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

$F'(x) = 0$, $\forall x \in (0, +\infty)$ $\Rightarrow$ $F \equiv const$ на (a, b)

$\Rightarrow F(x) = c$, $\forall x > 0$ $\qquad c = ?$ $\Rightarrow F(x) \equiv F(1) = \operatorname{arctg} 1 + \operatorname{arctg} 1 = \pi/2$ ✓

$\boxed{4}$ $f(x) = \frac{-1}{|x|} + \operatorname{arctg} \frac{2x}{x^2-1}$

1° D_f: $x \neq 0, \pm 1$ $\qquad D_f = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, +\infty)$

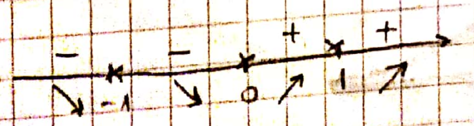
2° нн-ни

3° КАСНИЈЕ

4° • $\left(\operatorname{arctg} \frac{2x}{x^2-1}\right)' = \ldots = \frac{-2}{x^2+1}$, $x \neq \pm 1$

• $\left(\frac{1}{|x|}\right)' = \frac{1}{x \cdot \operatorname{sgn} x} = \frac{1}{\operatorname{sgn} x} \cdot \frac{-1}{x^2} = \frac{-\operatorname{sgn} x}{x^2}$, $x \neq 0$ $\Rightarrow$ $\frac{1-x^2}{x^2(x^2+1)}$

$\Rightarrow f'(x) = \begin{cases} \frac{1}{x^2} - \frac{2}{x^2+1}, & x>0, x \neq 1 \\ -\frac{1}{x^2} - \frac{2}{x^2+1}, & x<0, x\neq -1 \end{cases} \Rightarrow -\frac{1}{x^2} - \frac{2}{x^2+1}$



5° • $\lim\limits_{x\to-\infty}\left(\frac{-1}{|x|}+\operatorname{arctg}\frac{2x}{x^2-1}\right)=0 \Rightarrow y=0$ Х.А., $x\to-\infty$

• $\lim\limits_{x\to+\infty}\left(\frac{-1}{|x|}+\operatorname{arctg}\frac{2x}{x^2-1}\right)=0 \Rightarrow y=0$ Х.А., $x\to+\infty$

• $\lim\limits_{x\to0}\left(-\frac{1}{|x|}+\underbrace{\operatorname{arctg}\frac{2x}{x^2-1}}_{0}\right)=-\infty \Rightarrow x=0$ В.А.

• y 1: $\lim\limits_{x\to1+}f(x)=-1+\pi/2$ $\lim\limits_{x\to1-}f(x)=-1-\pi/2$

(угао између трунке): $\lim\limits_{x\to1}f'(x)=\lim\limits_{x\to1}\frac{x^2-1}{x^2(x^2+1)}=0=\alpha$

• y -1: $\lim\limits_{x\to-1+}f(x)=-1+\pi/2$ $\lim\limits_{x\to-1-}f(x)=-1-\pi/2$

$\lim\limits_{x\to-1}f'(x)=\lim\limits_{x\to-1}\left(-\frac{1}{x^2}-\frac{2}{x^2+1}\right)=-2$

6° $f''(x)=\ldots=\begin{cases}\frac{2}{x^3}+\frac{4x}{(x^2+1)^2}, & x<0,\ x\neq-1\\ -\frac{2}{x^3}+\frac{4x}{(x^2+1)^2}, & x>0,\ x\neq1\end{cases}$

$-\ \ -1\ \ -\ \ 0\ \ 1$ ($\cap$ $\cap$)

• $x>0$: f'' знак: $-\frac{2}{x^3}+\frac{4x}{(x^2+1)^2}>0$ $/\cdot x^3(x^2+1)^2$

$-2(x^2+1)^2+4x^4>0$

$-2x^4-4x^2-2+4x^4>0$

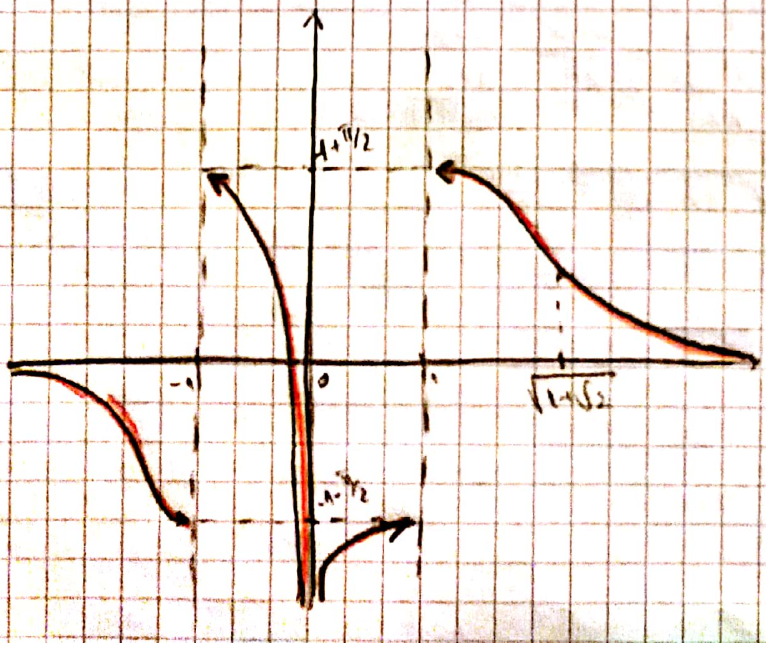
$x^4-2x^2-1>0$

$(t=x^2)$ $t_{1,2}=1\pm\sqrt{2} \Rightarrow x^2>1+\sqrt{2} \Rightarrow x>\sqrt{1+\sqrt{2}}$

f'': $-$ 0 $-$ 1 $-$ $\sqrt{1+\sqrt{2}}$ $+$

f: $\searrow$ -1 $\searrow$ 0 $\nearrow$ 1 $\searrow$

7°



$\boxed{5}$ $f(x) = \ln(e^{2x} - 3e^{x} + 3)$

1° D_f: $t = e^x$, $t^2 - 3t + 3 > 0$, $D_f \in \mathbb{R}$

2° ~~Нп+Нп~~

3° $\ln(e^{2x} - 3e^{x} + 3) = 0$

$e^{2x} - 3e^{x} + 3 = 1$

$(t = e^x)$ $t_{1,2} = 2, 1$ $e^x = 2$, $e^x = 1$ $f(x) = 0 \Leftrightarrow \underline{x = 0} \vee \underline{x = \ln 2}$ нуле

$f(x) > 0$ $e^{2x} - 3e^x + 3 > 1$, $x \in (0, \ln 2)$ $x < 0$

$$\begin{array}{cccccc} + & & - & & + & \\ \hline & 0 & & \ln 2 & & \rightarrow \end{array}$$

$f \rightarrow \exists \alpha \in (-1, 0)$ $f(\alpha) = 0$, $x < 0$ $x \in (-\infty, -1) \cup (\alpha, 0) \cup (0, 1)$

$x > 0$ $x \in (-1, \alpha) \cup (1, +\infty)$

4° $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \ln(\cancel{e^{2x}}^{\,0} - \cancel{3e^{x}}^{\,0} + 3) = \ln 3 \Rightarrow y = \ln 3$ Х.А, $x \to -\infty$

$\lim\limits_{x\to+\infty} f(x)$; $a = \lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty} \ln(e^{2x}) + \ln\left(1 - \dfrac{3}{e^x} + \dfrac{3}{e^{2x}}\right) = \underline{2}$

$b = \lim\limits_{x\to+\infty} f(x) - 2x = \lim\limits_{x\to+\infty} \ln\left(1 - \dfrac{3}{e^x} - \dfrac{3}{e^{2x}}\right) = \underline{0}$

5° $f'(x) = \ldots = e^x(2e^x - 3) / e^{2x} - 3e^x + 3$

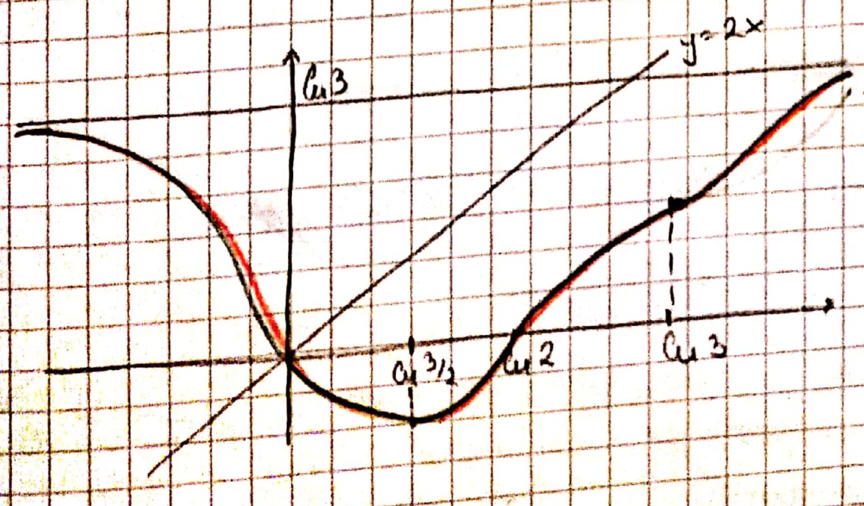
$$f' \quad \begin{array}{ccc} - & & + \\ \hline \searrow & \ln \frac{3}{2} & \nearrow \end{array}$$

min

6° $f''(x) = \ldots = -3 \cdot \dfrac{e^x(e^x - 1)(e^x - 3)}{(e^{2x} - 3e^x + 3)^2}$

$$f'' \quad \begin{array}{cccccc} - & & - & & + & \\ \hline \cap & 0 & \cap & \ln 3 & \cup & \end{array}$$

7°



Вежбање

1 Доказати да ј-на $\ln x = \frac{1}{x^2}$ има јединствено решење.

$D: \ x \in (0, +\infty)$

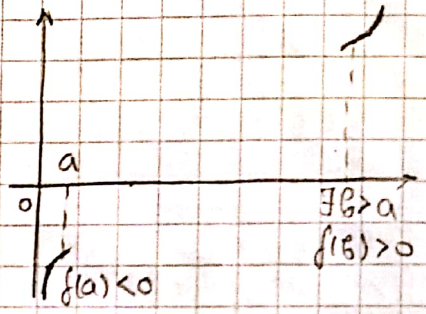
$F(x) = \ln x - \frac{1}{x^2}, \quad F: (0,+\infty) \to \mathbb{R}$

- $\lim\limits_{x\to 0+} F(x) = \lim\limits_{x\to 0+} (\underbrace{\ln x}_{-\infty} - \underbrace{\frac{1}{x^2}}_{+\infty}) = -\infty$
- $\lim\limits_{x\to +\infty} F(x) = \lim\limits_{x\to +\infty} (\underbrace{\ln x}_{+\infty} - \underbrace{\frac{1}{x^2}}_{0}) = +\infty$

$\overset{F \text{ непр}}{\underset{\text{Б.К.}}{\Longrightarrow}} \ \exists x_0 \text{ т.д. } \underline{F(x_0) = 0}$

$F'(x) = \frac{1}{x} + \frac{2}{x^3} > 0 \Rightarrow \underline{F \nearrow \text{ на } (0,+\infty)}$

$\Rightarrow$ Решење је јединствено

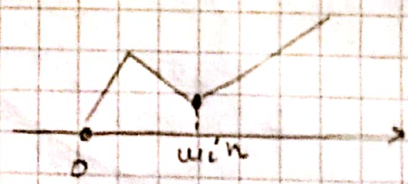


2 Доказати: $\forall x \geq 0$ важи $\arctg\sqrt{x} \geq \frac{\sqrt{x}(5x+3)}{3(x+1)^2}$.

$\underline{x=0}: \ \arctg x \geq 0 \Rightarrow 0 \geq 0 \ \checkmark$

$\underbrace{\arctg\sqrt{x} - \frac{\sqrt{x}(5x+3)}{3(x+1)^2}}_{F(x)} \geq 0, \quad \forall x \geq 0$

$F(0) = 0, \ F$ непр. на $[0,+\infty)$



Ако је $F(\min) \geq 0 \Rightarrow F(x) \geq 0, \ \forall x \geq 0$

$F'(x) = \frac{4x^2}{3\sqrt{x}(x+1)^3} > 0, \ \forall x > 0 \Rightarrow F \nearrow$ на $(0,+\infty)$

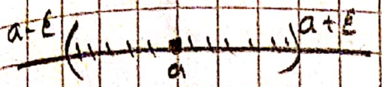
$\Rightarrow F \nearrow$ на $[0,+\infty)$ јер је непр на $[0,+\infty) \Rightarrow F(x) \geq 0, \ \forall x \geq 0$

Низови

14.05.2018.

$x: \mathbb{N} \to \mathbb{R}$, $\underbrace{x(1)}_{x_1}, \underbrace{x(2)}_{x_2}, \dots, \underbrace{x(n)}_{x_n}$

~ Конвергентни низови ~



* $a \in \mathbb{R}$, $\lim\limits_{n\to\infty} x_n = a$ тј. $x_n \xrightarrow[n\to\infty]{} a$, ако за $(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})$ т.д. за $(\forall n \in \mathbb{N})$ важи $n \geq n_0 \Rightarrow |x_n - a| < \varepsilon$.

- $\boxed{+\infty}$: $\lim\limits_{n\to\infty} x_n = +\infty$ ако $(\forall M>0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})$ $n \geq n_0 \Rightarrow x_n > M$
- $\boxed{-\infty}$: Слично, $(\forall M < 0)$
- Ако $\exists$ коначан $\lim\limits_{n\to\infty} x_n \Rightarrow (x_n)$ конвергира, иначе дивергира (ако је $\lim = \pm\infty$ или не постоји $\lim$)

* $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, $a, b \in \mathbb{R}$ (низови конвергирају у коначне вредности)

1) $\lim\limits_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha a + \beta b$
2) $\lim\limits_{n\to\infty} (x_n \cdot y_n) = a \cdot b$
3) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$, $b \neq 0$

* $\lim\limits_{n\to\infty} x_n = +\infty$, $\lim\limits_{n\to\infty} y_n = b \in \mathbb{R}$

1) $x_n + y_n \xrightarrow[n\to\infty]{} +\infty$
2) $x_n \cdot y_n \xrightarrow[n\to\infty]{} \begin{cases} +\infty, & b > 0 \\ -\infty, & b < 0 \\ \text{неодређено}, & b = 0 \end{cases}$

Теорема f непр. у $a \Leftrightarrow$ за сваки низ x_n такав да је $\lim\limits_{n\to\infty} x_n = a$ важи:

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Пример $\lim\limits_{n\to\infty} \frac{3n^2-2}{-4n^2+7n} = \lim\limits_{n\to\infty} f(n) = \lim\limits_{x\to\infty} f(x) = -\frac{3}{4}$, $f: (10, +\infty) \to \mathbb{R}$ непр.

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$$* \lim_{n\to\infty} \frac{a_k n^k + \dots + a_1 n + a_0}{b_m n^m + \dots + b_1 n + b_0} = \begin{cases} \frac{a_k}{b_m}, & k = m \\ 0, & k < m \\ +\infty, & k > m \text{ и } a_k, b_m \text{ иог знака} \\ -\infty, & k > m \text{ и } a_k, b_m \text{ разл. знака} \end{cases}$$

$a_k, b_m \neq 0$

Пример $\lim_{n\to\infty} \log_2\left(1+\frac{1}{\sqrt{n+1}}\right) = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \log_2\left(1+\frac{1}{\sqrt{x+1}}\right) = \underline{\underline{0}}$

шора + јер је x произвољно, а не низ

$f:(0,+\infty)\to\mathbb{R}$ нейр.

$\boxed{1}$ $\lim_{n\to\infty} \sqrt[n]{a} = 1$, $a>0$

1° $\underline{a>1}$: $\sqrt[n]{a} \xrightarrow[n\to\infty]{} 1 \iff \underbrace{(\sqrt[n]{a}-1)}_{h_n} \xrightarrow[n\to\infty]{} 0$

$\varepsilon>0$ произвољно. $? \exists n_0 \in \mathbb{N}$ $n \geq n_0 \Rightarrow |h_n - 0| < \varepsilon$ тј $|h_n| < \varepsilon$.

$a = (\sqrt[n]{a})^n = (1+(\sqrt[n]{a}-1))^n = (1+h_n)^n = 1+\binom{n}{1}h_n + \underbrace{\binom{n}{2}h_n^2 + \dots}_{\geq 0} \geq 1+n h_n$

$(a>1 \Rightarrow \sqrt[n]{a}>1 \Rightarrow h_n>0)$

$\Rightarrow a \geq 1+n h_n \Rightarrow h_n \leq \frac{a-1}{n}$, ако $\underbrace{\frac{a-1}{n}}_{\text{бити за } n > \frac{a-1}{\varepsilon}} < \varepsilon \Rightarrow h_n < \varepsilon$, $n_0 = \left[\frac{a-1}{\varepsilon}\right]+1$

$\Rightarrow$ за $n \geq n_0$ $h_n < \varepsilon \Rightarrow h_n > 0 \Rightarrow |h_n| < \varepsilon$ ✓

$\Rightarrow \lim_{n\to\infty} h_n = 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$

2° $\underline{a=1}$: $\sqrt[n]{a} = 1$, $\forall n$ ✓

3° $\underline{a\in(0,1)}$: $\frac{1}{a}>1$, из $1^\circ \Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$ $\lim_{n\to\infty} \frac{1}{\sqrt[n]{a}} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$ ✓

II начин: $\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{1/n} = \lim_{x\to+\infty} a^{\overset{0}{1/x}} = \underline{\underline{1}}$

$\boxed{2}$ $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

$\lim_{n\to\infty} n^{1/n} = \lim_{x\to+\infty} x^{1/x} = \lim_{x\to+\infty} e^{\overset{0}{\frac{1}{x}\ln x}} = \underline{\underline{1}}$

3 А) $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+\dots+n^2}{n^3} = \lim\limits_{n\to\infty} \dfrac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \lim\limits_{n\to\infty} \dfrac{\frac{2}{6}n^3 + \text{нижи степени}}{n^3} = \underline{\dfrac{1}{3}}$

Б) $\lim\limits_{n\to\infty} \dfrac{1^2+3^2+5^2+\dots+(2n-1)^2}{n^3} = \lim\limits_{n\to\infty} \dfrac{1^2+2^2+3^2+\dots+(2n)^2 - \overbrace{(2^2+4^2+\dots+(2n)^2)}^{2^2\sum_{k=1}^{n} k^2}}{n^3} =$

$= \lim\limits_{n\to\infty} \dfrac{\frac{2n(2n+1)(4n+1)}{6} - 4\frac{n(n+1)(2n+1)}{6}}{n^3} = \lim\limits_{n\to\infty} \dfrac{n^3\left(\frac{16}{6}-\frac{4}{3}\right)+\text{нижи}}{n^3} = \underline{\dfrac{4}{3}}$

4 (x_n) $\quad S_n = x_1+\dots+x_n = \sqrt{n^2-n+1}$, $\quad \lim\limits_{n\to\infty} x_n = ?$

$S_{n+1} = S_n + x_{n+1}$, $\quad x_{n+1} = S_{n+1} - S_n$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} (S_{n+1}-S_n) = \lim\limits_{n\to\infty} \left(\sqrt{(n+1)^2-(n+1)+1} - \sqrt{n^2-n+1}\right) =$

$= \lim\limits_{n\to\infty} \left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right) = \lim\limits_{n\to\infty} \dfrac{\overbrace{(n^2+n+1)-(n^2-n+1)}^{2n}}{\underbrace{\sqrt{n^2+n+1}}_{n} + \underbrace{\sqrt{n^2-n+1}}_{n}} = \underline{1}$

~ Геометријски низ ~

$a_0 = 1,\ a_1 = q,\ a_2 = q^2, \dots, a_n = q^n,\ a_{n+1} = q\cdot a_n$

$$\lim_{n\to\infty} q^n = \begin{cases} 1, & q = 1 \\ 0, & |q| < 1 \\ +\infty, & q > 1 \\ \nexists, & q \le -1 \ \text{(нема граничне вр.)} \end{cases}$$

1 А) $\lim\limits_{n\to\infty} \dfrac{2^{n+1}+3^{n+1}}{2^n+3^n} \ \Big/ :3^n = \lim\limits_{n\to\infty} \dfrac{2\overbrace{\left(\frac{2}{3}\right)^n}^{\to 0}+3}{\underbrace{\left(\frac{2}{3}\right)^n}_{\to 0}+1} = \underline{3}$

Б) $\lim\limits_{n\to\infty} \dfrac{2^{\frac{1}{n+1}}+3^{\frac{1}{n+1}}}{2^{1/n}+3^{1/n}} = 1$ $\quad$ (сви сабирци $\to 1$)

~Неједнакости~

Теорема $(a_n), (b_n)$

1° $(\forall n \in \mathbb{N})\ a_n \le b_n \Rightarrow \lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$

2° $(\forall n \in \mathbb{N})\ a_n < b_n \Rightarrow \lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$

Теорема (о два жандарма) $(a_n), (b_n), (c_n)$

1° $a_n \le b_n \le c_n$, $\forall n \ge n_0$

2° $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L \Rightarrow \lim\limits_{n\to\infty} b_n = L$, $L \in \overline{\mathbb{R}}$

$\boxed{1}$ $a_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \ldots + \dfrac{1}{\sqrt{n^2+n}}$. Наћи $\lim\limits_{n\to\infty} a_n = ?$

• $n \cdot \min \le a_n \le n \cdot \max$ •

$$\frac{n}{\sqrt{n^2+n}} \overset{/:n}{\le} a_n \le \frac{n}{\sqrt{n^2+1}} \ {}^{/:n}$$

$$\underbrace{\frac{1}{\sqrt{1+1/n}}}_{\to 1} \le a_n \le \underbrace{\frac{1}{\sqrt{1+1/n^2}}}_{\to 1} \Rightarrow \lim_{n\to\infty} a_n = \underline{1}$$

$\boxed{2}$ $a_n = \sqrt[n]{3^n + 6^n + 11^n + 15^n}$

$$\sqrt[n]{15^n} \le a_n \le \sqrt[n]{4 \cdot 15^n} = 15 \underbrace{\sqrt[n]{4}}_{\to 1}$$

$$15 \le a_n \le 15 \Rightarrow \lim_{n\to\infty} a_n = \underline{15}$$

Тврђење $a_1, a_2, \ldots, a_k > 0$,

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \ldots + a_k^n} = \max\{a_1, \ldots, a_k\}$$

$$\ge \underline{\max} \qquad \le \sqrt[n]{k \cdot \max^n} = \max \underbrace{\sqrt[n]{k}}_{\to 1} = \underline{\max}$$

$\boxed{3}$ $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} \dfrac{\binom{n}{k}}{n^k} = ?$

$$\lim_{n\to\infty} \frac{\frac{n(n-1)\ldots(n-k+1)}{k!}}{n^k} = \frac{\frac{1}{k!}}{1} = \underline{\underline{k!}}$$

* Да ли важе сл. импликације?

1) $a_n \xrightarrow[n\to\infty]{} a \Rightarrow |a_n| \xrightarrow[n\to\infty]{} |a|$ $\boxed{\top}$

2) $|a_n| \xrightarrow[n\to\infty]{} |a| \Rightarrow a_n \xrightarrow[n\to\infty]{} a$ $\boxed{\perp}$ (важи $\top$ за $a=0$)

3) $|a_n| \xrightarrow[n\to\infty]{} 0 \Leftrightarrow a_n \xrightarrow[n\to\infty]{} 0$ $\boxed{\top}$

~ Монотони низови ~

* (a_n) <u>монотон</u>: $(a_n)\uparrow$ за $a_n \leq a_{n+1}$, $\forall n$
  $(a_n)\downarrow$ за $a_n \geq a_{n+1}$, $\forall n$

* (a_n) <u>ограничен</u>: $(\exists M>0)(\forall n)\ |a_n|<M$
  - одозго: $(\exists M>0)\ \forall n\ a_n<M$
  - одоздо: $(\exists m>0)\ \forall n\ a_n>m$

* (a_n) <u>конвергентан</u> $\Rightarrow$ (a_n) <u>ограничен</u>

* (a_n) конвергентан $\not\Rightarrow$ (a_n) монотон

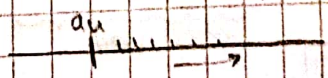
* (a_n) ограничен $\not\Rightarrow$ (a_n) конвергентан

* (a_n) монотон $\not\Rightarrow$ (a_n) конвергентан

<u>Теорема</u> (a_n) монотон + ограничен $\Rightarrow$ (a_n) конвергентан

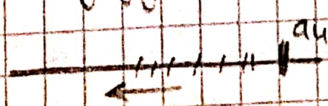
<u>Теорема 1</u> Ако је (a_n) <u>растући</u> низ за $n\geq n_0$ и (a_n) ограничен одозго

$\Leftrightarrow$ (a_n) конвергира



<u>Теорема 2</u> Ако је (a_n) <u>опадајући</u> низ за $n\geq n_0$ и (a_n) ограничен одоздо

$\Leftrightarrow$ (a_n) конвергира



[1] Испитати конвергенцију: $b_n = \underbrace{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}_{n \text{ корена}}$

1° <u>ограничен</u>: инд: $b_n \in (0,2)$, $\forall n \in N$

1) база: $n=1 \Rightarrow b_1 = \sqrt{2} \in (0,2)$ ✓

2) $n \to n+1$: их $\to b_n \in (0,2)$. За $n+1$: $b_{n+1} = \sqrt{2+b_n}$, $b_{n+1} > 0$ ✓

$b_{n+1} = \sqrt{2+\underbrace{b_n}_{<2}} \overset{их}{<} \sqrt{2+2} = 2$ ✓

$\Rightarrow b_n \in (0,2)$, $\forall n \in N$ $\to$ ограничен је

2° <u>растућа</u>: инд. $b_n \leq b_{n+1}$

1) база: $n=1 \Rightarrow b_1 < b_2$ ✓

2) $n \to n+1$: $b_n < b_{n+1} \Rightarrow b_{n+1} < b_{n+2}$.

$\sqrt{2+b_n} \overset{их}{<} \sqrt{2+b_{n+1}}$ ✓

$\to b_n \uparrow$

$\overset{1^\circ, 2^\circ}{\Longrightarrow}$ $\exists b \in R$ т.д. $\lim\limits_{n\to\infty} b_n = b = ?$,

$b_{n+1} = \sqrt{2+b_n} \;/\lim\limits_{n\to\infty} \Rightarrow b = \sqrt{2+b} \;/^2$, $b \geq 0$

$b^2 = 2+b$, $\boxed{b_1 = 2}$ $b_2 = -1$ ⊥

[2] $a_n = \dfrac{c^n}{n!}$, $c>0$ const.

* Ако $\exists \lim\limits_{n\to\infty} a_n = a \in R$, шта је a?

$a_{n+1} = \dfrac{c^{n+1}}{(n+1)!} = \dfrac{c^n}{n!} \cdot \dfrac{c}{n+1} \Rightarrow \underbrace{a_{n+1} = \dfrac{c}{n+1} \cdot a_n}_{(*)} \;/\lim\limits_{n\to\infty}$

$a = \lim\limits_{n\to\infty} \dfrac{c}{n+1}^{\to 0} \cdot a \Rightarrow \boxed{a=0}$

1° $a_n > 0$, $\forall n$ ✓ огр. одоздо

2° $(a_n) \downarrow$ за $n \geq n_0$

$a_{n+1} \overset{?}{\leq} a_n \Leftrightarrow \dfrac{a_{n+1}}{a_n} \leq 1 \overset{(*)}{\Leftrightarrow} \dfrac{c}{n+1} \leq 1 \Leftrightarrow n \geq c-1 \to$ а одатле за $n \geq n_0$: $a_{n+1} \leq a_n$, $n_0 = [c-1]+1$

$\overset{1^\circ,2^\circ}{\underset{T2}{\Longrightarrow}}$ $\exists \lim\limits_{n\to\infty} a_n = a \in R \Rightarrow a = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\boxed{3}$ $a>0$ const., $x_1>0$ дат, $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$

* Ако $\exists \lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$

$$\underbrace{x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)}\quad /\lim_{n\to\infty}$$

$$x=\frac{1}{2}\left(x+\frac{a}{x}\right) \Rightarrow x=\frac{a}{x} \Rightarrow \boxed{x=\sqrt{a}} \text{ или } x=-\sqrt{a} \perp$$

1° $\underline{n\geq 1}$: $x_{n+1}=\dfrac{x_n+\frac{a}{x_n}}{2} \overset{A_n \geq G_n}{\geq} \sqrt{\cancel{x_n}\cdot\frac{a}{\cancel{x_n}}}=\sqrt{a}$, $\underline{x_n\geq\sqrt{a},\ \forall n\geq 2}$

2° $(x_n)\downarrow$ за $n\geq n_0$

$$x_{n+1} \overset{?}{\leq} x_n$$

$$x_{n+1}-x_n=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)-x_n=\frac{1}{2}\left(\frac{a}{x_n}-x_n\right)=\frac{1}{2}\,\frac{\overbrace{a-x_n^2}^{\leq 0 \text{ jep } x_n\geq\sqrt{a}}}{\underbrace{x_n}_{\geq 0}}\leq 0$$

$\Rightarrow (x_n)\downarrow$

$\overset{1^\circ,2^\circ}{\underset{T2}{\Longrightarrow}}$ $\exists \lim\limits_{n\to\infty} x_n = x\in\mathbb{R} \Rightarrow \lim\limits_{n\to\infty} x_n=\sqrt{a}$

$\boxed{4}$ $0<c<1$ const, $a_1=0$, $a_{n+1}=a_n+(a_n-c)^2$

* Ако $\exists \lim\limits_{n\to\infty} a_n = a\in\mathbb{R}$

$$\underbrace{a_{n+1}=a_n+(a_n-c)^2}\quad /\lim_{n\to\infty}$$

$$a=a+(a-c)^2 \Rightarrow \boxed{a=c}$$

1° $a_{n+1}=a_n+\underbrace{(a_n-c)^2}_{\geq 0}\geq a_n$, $\forall n \Rightarrow (a_n)\uparrow$

2° Доказујемо инд.: $a_n\leq c$, $\forall n$

1) база, $n=1 \Rightarrow a_1=0\leq c$ ✓

2) $n\to n+1$: ИХ: $a_n\leq c$, за $n+1$:

$$a_{n+1}\overset{?}{\leq} c$$

$$a_{n+1}-c=a_n+(a_n-c)^2-c=\underbrace{(a_n-c)}_{\leq 0}\underbrace{(a_n-c+1)}_{>0}\leq 0 \Rightarrow a_{n+1}\leq c$$

$\Rightarrow$ Низ је огр одозго са c

$\overset{1^\circ,2^\circ}{\underset{T1}{\Longrightarrow}}$ $\exists \lim\limits_{n\to\infty} a_n = a\in\mathbb{R} \Rightarrow a=c$.

Број e

15.05.2018

$a_n = \left(1+\frac{1}{n}\right)^n \nearrow$, огр. одозго $\longrightarrow \exists \lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$

$b_n = \left(1+\frac{1}{n}\right)^{n+1} \searrow$, огр. одоздо $\longrightarrow \exists \lim\limits_{n\to\infty} b_n = b \in \mathbb{R}$

$b_n = a_n\left(1+\frac{1}{n}\right) \Big/ \lim\limits_{n\to\infty}$ $\qquad e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right)$

$e := b = a$

$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$, $\forall n\in\mathbb{N}$ јер низ је e гр. бр.

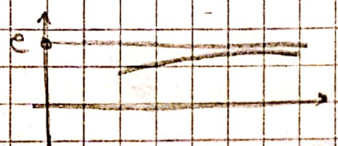
$\boxed{1}$ Доказати $\forall n\in\mathbb{N}$: $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$.

$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \Big/ \ln$

$n\cdot\ln\left(1+\frac{1}{n}\right) < 1 < (n+1)\ln\left(1+\frac{1}{n}\right)$

$\frac{1}{n} > \ln\left(1+\frac{1}{n}\right) \qquad \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right)$.

* $e = \lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$



<u>Лема</u> (p_n) - низ $\lim\limits_{n\to\infty} p_n = +\infty$, (q_n) - низ $\lim\limits_{n\to\infty} q_n = -\infty$

$\Rightarrow \lim\limits_{n\to\infty}\left(1+\frac{1}{p_n}\right)^{p_n} = e = \lim\limits_{n\to\infty}\left(1+\frac{1}{q_n}\right)^{q_n}$

исто као код дја

$\boxed{2}$ А) $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{-n}\right)^{-n(-1)} = e^{-1} = \frac{1}{e}$

б) $\lim\limits_{n\to\infty}\left(\frac{n^2+n+1}{n^2-n+1}\right)^{\frac{n^2+1}{3n}} = \lim\limits_{n\to\infty}\left(1+\frac{2n}{n^2-n+1}\right)^{\frac{n^2+1}{3n}} = \lim\limits_{n\to\infty}\underbrace{\left(1+\frac{1}{\frac{n^2-n+1}{2n}}\right)^{\frac{n^2-n+1}{2n}}}_{e}{}^{\cdot\frac{2n}{n^2-n+1}\cdot\frac{n^2+1}{3n}} =$

$= e^{2/3}$

$\boxed{3}$ Доказати:

А) $\lim\limits_{n\to\infty}\frac{n^n}{(n!)^2} = 0 \qquad (n^n << (n!)^2,\ n\to+\infty\ n\in\mathbb{N})$

уводимо низ. $a_n = \frac{n^n}{(n!)^2}$

1° $a_n > 0, \forall n\in\mathbb{N}$

2° $(a_n)\downarrow$?

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{((n+1)!)^2}\cdot\frac{(n!)^2}{n^n} = \frac{(n+1)^n}{n^n}\cdot\frac{1}{n+1} = \underbrace{\left(1+\frac{1}{n}\right)^n}_{<e}\cdot\frac{1}{n+1} < e\cdot\frac{1}{n+1} < 1$$

за $n > e-1$
за $n \geq 2$

$a_{n+1} < a_n \ , \ \forall n \geq 2$

1°, 2° $\underset{\text{ТМК2}}{\Rightarrow}$ $\exists \lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$

Веза: $\frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n+1}$ ($\lim\limits_{n\to\infty} \longrightarrow$ „$\frac{a}{a} = e\cdot 0$" $1 = 0$ $\Rightarrow$ не смемо да радимо lim јер не знамо да a није 0)

$\Rightarrow a_{n+1} = a_n\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n+1}$ $/\lim\limits_{n\to\infty}$ $\to$ сад смемо (нема сумњивог дељења)

$a = a\cdot e\cdot 0$

$\underline{\underline{a = 0}}$

~ Поређење ~

$\log_a x \ll x^\alpha \ll c^x \ , \ x\to+\infty$
$a>0, a\neq 1 \quad \alpha>0 \quad c>1$

+ за $n\in\mathbb{N}$: $\boxed{\log_a n \ll \underset{\alpha>0}{n^\alpha} \ll \underset{c>1}{c^n} \ll n! \ll n^n \ll (n!)^2}$

$\boxed{1}$ $\lim\limits_{n\to\infty} \frac{(\sin(n+1)^3)^{14} + n! - 12n^n + \log_{200}(n^2+12)}{n^{50} - n + \log_{20} n - 13n^n + [n^2+15]} \overset{:n^n}{=}$

$= \lim\limits_{n\to\infty} \frac{\frac{(\sin(n+1)^3)^{14}}{n^n}\to 0 + \frac{n!}{n^n}\to 0 - 12 + \frac{\log_{200}(n^2+12)}{n^n}\to 0}{\frac{n^{50}}{n^n}\to 0 - \frac{n}{n^n}\to 0 + \frac{\log_{20} n}{n^n}\to 0 - 13 + \frac{[n^2+15]}{n^n}\to 0} = \underline{\underline{\frac{12}{13}}}$

Теорема f непр у a $\Leftrightarrow$ $\forall$низ (x_n)

$\lim\limits_{n\to\infty} x_n = a \Leftrightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

$\boxed{2}$ Да ли се ф-ја $f(x)=\sin\frac{1}{x}$ може додефинисати у $x=0$ т.д. буде непр на $[0,1]$?

$$f(x)=\begin{cases}\sin\frac{1}{x}, & x\in[0,1]\\ a, & x=0\end{cases}$$

Пп $\exists a\in\mathbb{R}$ т.д. f непр на $[0,1]$. $a=f(0)=\lim\limits_{x\to 0+}\sin\frac{1}{x}$

Непр. $x_n\xrightarrow[n\to\infty]{}0 \Rightarrow f(x_n)\xrightarrow[n\to\infty]{}a$ $\quad \sin\frac{1}{x_n}\xrightarrow[n\to\infty]{}a$

$x_n=1/_{2n\pi}\to 0 \Rightarrow a=0$ (из $\sin 2n\pi\xrightarrow[n\to\infty]{}0$)

$y_n=\dfrac{1}{2n\pi+\pi/2}\to 0 \Rightarrow a=1$ (из $\sin(2n\pi+\pi/2)\xrightarrow[n\to\infty]{}1$)

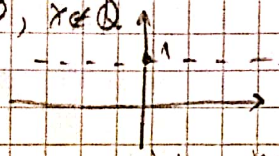
$\Big\}$ ↯ f се не може додеф.

$\boxed{3}$ $f(x)=x\cdot\chi(x)$, Испитати непр ф-је f.

χ – дирихлеова ф-ја

$$\chi(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$$

χ-непр?



χ - није непр ни у једној тачки $x\in\mathbb{R}$

$$f(x)=x\cdot\chi(x)=\begin{cases}x, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$$

• $\underline{x_0\neq 0}$: $\exists$ низ (q_n): $\lim\limits_{n\to\infty}q_n=x_0$, $q_n\in\mathbb{Q}$, $\forall n$

$\exists$ низ (y_n): $\lim\limits_{n\to\infty}y_n=x_0$, $y_n\notin\mathbb{Q}$, $\forall n$

Ако би f била непр у $x_0 \Rightarrow \underbrace{\lim\limits_{n\to\infty}\underbrace{f(q_n)}_{q_n}}_{x_0}=\underset{x_0}{\underset{\parallel}{f(x_0)}}$ и $\underbrace{\lim\limits_{n\to\infty}\underbrace{f(y_n)}_{0}}_{0}=f(x_0)=0$ ↯

$\Rightarrow f$ није непр у x_0, $\forall x_0\neq 0$

• $\underline{x_0=0}$: f непр у 0: $(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})\ |x-0|<\delta\Rightarrow|f(x)-f(0)|<\varepsilon$

$|x|<\delta\Rightarrow|f(x)|<\varepsilon$.

$\varepsilon>0$ фиксирано. $\delta=\varepsilon$.

$|x|<\underset{\varepsilon}{\delta}\overset{?}{\Rightarrow}|f(x)|<\varepsilon$ ✓ $\Rightarrow f$ непр. у 0

$\underline{Дом}$ а) $f(x)=\lim\limits_{n\to\infty}\dfrac{x+x^2e^{nx}}{1+e^{nx}}$, $x\in\mathbb{R}$

б) $f(x)=\lim\limits_{n\to\infty}\dfrac{n^x-n^{-x}}{n^x+n^{-x}}$, $x\in\mathbb{R}$

Тачке нагомилавања

15.05.2016.

$(-1)^n \to T(a_n)=\{-1,1\}$

* $a\in\mathbb{R}$ тачка нагомилавања низа (x_n) ако $\exists$ подниз (x_{n_k}) т.д. $\lim\limits_{k\to\infty} x_{n_k}=a$
  • $T(a_n)$ - скуп свих т.н. низа (a_n)
* (a_n) ограничен $\Rightarrow T(a_n)\neq\emptyset$ (у $\mathbb{R}$)
* (a_n) не мора да буде ограничен $\Rightarrow T(a_n)\neq\emptyset$ у $\overline{\mathbb{R}}$ (т.н. су $\pm\infty$)

Теорема $T(a_n)$ има $\underbrace{\min}_{\underline{\lim}\, a_n}$ и $\underbrace{\max}_{\overline{\lim}\, a_n}$ у $\overline{\mathbb{R}}$

($\lim\inf(a_n)$) ($\lim\sup(a_n)$)

* $\lim\limits_{n\to\infty} a_n = a\in\overline{\mathbb{R}} \Leftrightarrow \underline{\lim}\, a_n=\overline{\lim}\, a_n=a$

Пример $a_n=n^{(-1)^n}=\begin{cases} n, & 2\mid n \\ 1/n, & 2\nmid n\end{cases}$

$n=2k,\ a_{2k}=2k\xrightarrow[k\to\infty]{}+\infty$

$n=2k+1,\ a_{2k+1}=\frac{1}{2k+1}\xrightarrow[k\to\infty]{}0 \qquad T(a_n)=\{0,+\infty\}$

$\underline{\lim}\, a_n=0,\ \overline{\lim}\, a_n=+\infty$

$\boxed{1}$ Наћи низ (a_n) т.д. $T(a_n)=\overline{\mathbb{R}}$.

(ми хоћемо да кад узмемо било коју тачку a онда можемо да кажемо да увек постоји подниз од низа које је т.н. a)

$\mathbb{Q}$ пребројив (бесконачан скуп)

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \ \dots \to (a_n)\text{ низ} \\ -1 & -2 & -3 & -4 & \dots \\ 1/2 & 3/2 & 5/2 & \dots & \\ -1/2 & -3/2 & \dots & & \end{array}$$

низ свих рац. бр. који се добија овим поступком је добро дат и скуп свих т.н. је $T(a_n)=\overline{\mathbb{R}}$

$\boxed{2}$ $a_n = \sqrt[n]{1 + 2^{n(-1)^n}}$, $T(a_n) = ?$ $\underline{\lim}\, a_n, \overline{\lim}\, a_n = ?$

$\underline{n=2k}$: $a_{2k} = \sqrt[2k]{1 + 2^{2k}} \xrightarrow[k\to\infty]{} 2$

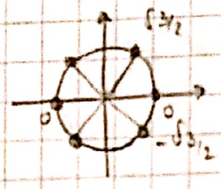
$\underline{n=2k+1}$: $a_{2k+1} = \sqrt[2k+1]{1 + 2^{-(2k+1)}} = \sqrt[2k+1]{1^{2k+1} + \left(\frac{1}{2}\right)^{2k+1}} \xrightarrow[k\to\infty]{} 1$

$T(a_n) = \{1, 2\}$, $\underline{\lim}\, a_n = 1$, $\overline{\lim}\, a_n = 2$

*!

$\boxed{3}$ $a_n = \left(1 + \frac{1}{n}\right)^n (-1)^n + \sin\frac{n\pi}{3}$. $T(a_n)$, $\underline{\lim}\, a_n$, $\overline{\lim}\, a_n = ?$

$\left.\begin{array}{l} (-1)^n \to \bmod 2 \\ \sin\frac{n\pi}{3} \to \bmod 6 \end{array}\right\} \to \underline{\underline{\bmod 6}}$



$2\pi : \pi/3 = 6$

$\underline{n=6k}$: $\left(1 + \frac{1}{6k}\right)^{6k} \cdot (-1)^{6k} + \sin\frac{6k\pi}{3} \to \underline{e}$ ($\left(1 + \frac{1}{6k}\right)^{6k} \to e$, $(-1)^{6k} = 1$, $\sin\frac{6k\pi}{3} = 0$)

$\underline{n=6k+1}$: $\underbrace{\left(1 + \frac{1}{6k+1}\right)^{6k+1}}_{e} \cdot \underbrace{(-1)^{6k+1}}_{-1} + \underbrace{\sin(2k\pi + \pi/3)}_{\sin\pi/3 = \sqrt{3}/2} \longrightarrow \underline{-e + \frac{\sqrt{3}}{2}}$

$\underline{n=6k+2}$: $\underbrace{\left(1 + \frac{1}{6k+2}\right)^{6k+2}}_{e} \cdot \underbrace{(-1)^{6k+2}}_{1} + \underbrace{\sin\left(2k\pi + \frac{2\pi}{3}\right)}_{\sin\frac{2\pi}{3} = \sqrt{3}/2} \longrightarrow \underline{e + \frac{\sqrt{3}}{2}}$

$\underline{n=6k+3}$: $-e + 0 = \underline{-e}$

$\underline{n=6k+4}$: $\underline{e - \sqrt{3}/2}$

$\underline{n=6k+5}$: $\underline{-e - \sqrt{3}/2}$

$T(a_n) = \{\underbrace{-e - \frac{\sqrt{3}}{2}}_{\underline{\lim}\, a_n}, -e + \frac{\sqrt{3}}{2}, -e, e - \frac{\sqrt{3}}{2}, e, \underbrace{e + \frac{\sqrt{3}}{2}}_{\overline{\lim}\, a_n}\}$

__Дом:__ $a_n = 1 + \frac{n}{n+1}\cos\frac{n\pi}{2}$ (померамо за $\pi/2$ што је 4 по тачки $\bmod 4$)

__Теорема__ (Штолцова тм.) (за велике суме и %)

(a_n) - произвољан, (b_n) - строго $\uparrow$ од неког n, $\lim\limits_{n\to\infty} b_n = +\infty$. Ако $\exists \lim\limits_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$, онда $\exists \lim\limits_{n\to\infty} \frac{a_n}{b_n}$ и они су једнаки

$$\overset{(1)}{\lim_{n\to\infty} \frac{a_n}{b_n}} = \overset{(2)}{\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}}$$

(ако (2) $\nexists$, (1) ипак може $\exists$ само га не решујемо овако)

4. $$\lim_{n\to\infty}\frac{\overbrace{1^k+2^k+\dots+n^k}^{a_n}}{\underbrace{n^{k+1}}_{b_n\uparrow\infty}} \overset{\text{ШТ.}}{=} \lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \lim_{n\to\infty}\frac{n^k}{n^{k+1}-(n-1)^{k+1}} =$$

$$=\lim_{n\to\infty}\frac{n^k}{\cancel{n^{k+1}}-\cancel{n^{k+1}}+(k+1)n^k\cdot n+\text{мањи ст.}} = \lim_{n\to\infty}\frac{\cancel{n^k}}{(k+1)\cancel{n^k}+\text{мањи}} = \frac{1}{k+1}$$

5. $$\lim_{n\to\infty}\left(\frac{1^k+2^k+\dots+n^k}{n^k}-\frac{n}{k+1}\right) = \lim_{n\to\infty}\frac{\overbrace{(1^k+\dots+n^k)(k+1)-n^{k+1}}^{a_n}}{\underbrace{n^k(k+1)}_{b_n\uparrow\infty}} \overset{\text{ШТ}}{=} \lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}} =$$

$$=\lim_{n\to\infty}\frac{(\cancel{1^k}+\dots+n^k)(k+1)-n^{k+1}-\left(\cancel{(1^k+\dots+(n-1)^k)}(k+1)-(n-1)^{k+1}\right)}{(k+1)n^k-(k+1)(n-1)^k} =$$

$$=\lim_{n\to\infty}\frac{n^k(k+1)-n^{k+1}+(n-1)^{k+1}}{(k+1)\left(\cancel{n^k}-(\cancel{n^k}-k\,n^{k-1}+\text{мањи})\right)} = \lim_{n\to\infty}\frac{\cancel{n^k(k+1)}-\cancel{n^{k+1}}+\cancel{n^{k+1}}-\cancel{\binom{k+1}{1}n^k}+\binom{k+1}{2}n^{k-1}+\text{мањи}}{(k+1)k\cdot n^{k-1}+\text{мањи}}$$

$$=\lim_{n\to\infty}\frac{\frac{(k+1)k\,n^{k-1}}{2}+\text{мањи}}{(k+1)k\,n^{k-1}+\text{мањи}} = \frac{1}{2}$$

Теорема (Кошијева т-ма)

Ако $\lim_{n\to\infty}a_n=a \Rightarrow \lim_{n\to\infty}\frac{a_1+a_2+\dots+a_n}{n}=a$

доказ:

$$\lim_{n\to\infty}\frac{\overbrace{a_1+a_2+\dots+a_n}^{x_n}}{\underbrace{n}_{y_n}} \overset{\text{ШТ}}{=} \lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}} = \lim_{n\to\infty}\frac{a_n}{n-(n-1)} = \lim_{n\to\infty}a_n = a.$$

6. $a_n>0,\ \forall n \quad \lim_{n\to\infty}a_n=a \Rightarrow \lim_{n\to\infty}\underbrace{\frac{n}{\frac{1}{a_1}+\dots+\frac{1}{a_n}}}_{H_n} = a = \lim_{n\to\infty}\underbrace{\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}}_{G_n}$

доказ

$A_n=\frac{a_1+\dots+a_n}{n}\xrightarrow[n\to\infty]{}a$. Довољно је доказати за H_n јер тада важи и за G_n.

1° $\underline{a=0}$: $\quad 0<H_n\le G_n\le \underset{\to 0}{A_n} \Rightarrow \lim_{n\to\infty}H_n=\lim_{n\to\infty}G_n=0$ ✓

2° $\underline{a\neq 0}$ $(a>0)$: $\quad a_n\to a \Rightarrow \frac{1}{a_n}\to\frac{1}{a}$

$\overset{\text{коши}}{\Longrightarrow} \lim_{n\to\infty}\underbrace{\frac{\frac{1}{a_1}+\dots+\frac{1}{a_n}}{n}}_{1/H_n} \longrightarrow \frac{1}{a} \Rightarrow H_n\xrightarrow[n\to\infty]{}a$

$\underset{\to a}{H_n}\le G_n\le\underset{\to a}{A_n} \Rightarrow \lim_{n\to\infty}G_n=a$ ✓

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Теорема $a_n>0, n\in\mathbb{N}$. Ако $\exists \lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}}$, онда $\exists \lim\limits_{n\to\infty}\sqrt[n]{a_n}$ и вони да су они једнаки.

доказ

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\underbrace{\frac{a_n}{a_{n-1}}}_{b_n}\cdot\underbrace{\frac{a_{n-1}}{a_{n-2}}}_{b_{n-1}}\cdots\underbrace{\frac{a_2}{a_1}}_{b_2}\cdot\underbrace{\frac{a_1}{1}}_{b_1}}\xrightarrow[n\to\infty]{\text{аритм. злад.}\ b_n}\beta$$

$$b_n=\frac{a_n}{a_{n-1}}\xrightarrow[n\to\infty]{\text{ако }\exists}\beta$$

$\boxed{7}$ $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=1/e$, $\lim\limits_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=1/e$ $\qquad \frac{n!}{n}\xrightarrow[n\to\infty]{}0$, $\left(\frac{1}{2}\right)^n\to 0$

$\sqrt[n]{\frac{n!}{n^n}}\to 1/e$, $1/2 \not\to 0$

доказ

$$a_n=\frac{n!}{n^n},\ \lim_{n\to\infty}\sqrt[n]{a_n}\underset{\text{ако }\exists}{=}\lim_{n\to\infty}\frac{a_n}{a_{n-1}}=\lim_{n\to\infty}\frac{\frac{n!}{n^n}}{\frac{(n-1)!}{(n-1)^{n-1}}}=\lim_{n\to\infty}\frac{(n-1)^{n-1}}{n^{n-1}}=$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1}=\lim_{n\to\infty}\left(1+\frac{1}{-n}\right)^{-n\cdot\frac{n-1}{-n}}=e^{-1}=\frac{1}{e}$$

Дом.

А) $\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0$

Б) $\lim\limits_{n\to\infty}\frac{1\cdot2\cdot3+2\cdot3\cdot4+\ldots+n(n+1)(n+2)}{n^4}=1/4$

В) $\lim\limits_{\substack{n\to\infty\\k\in\mathbb{N}}}\frac{1^k+3^k+\ldots+(2n+1)^k}{n^{k+1}}=\frac{2^k}{k+1}$.

$\boxed{8}$ $a_0>0$, $a_{n+1}=a_n+\frac{1}{3a_n^2}$. А) $\lim\limits_{n\to\infty}a_n=?$ Б) $\lim\limits_{n\to\infty}\frac{a_n^3}{n}=?$

А) 1° (низ неопадајућ $\nearrow$) $a_{n+1}>a_n, \forall n\in\mathbb{N}$ $(a_n)\nearrow$

ако је огр. одозго $\Longrightarrow \exists\lim\limits_{n\to\infty}a_n=a\in\mathbb{R}$ $(a>0$ јер $a_0>0$ и $(a_n)\nearrow)$

$a_{n+1}=a_n+\frac{1}{3a_n^2}\ /\lim\limits_{n\to\infty}$

$a=a+\frac{1}{3a^2}\Rightarrow\frac{1}{3a^2}=0$ ↯ $\Rightarrow$ није огр. одозго

• низ који расте и није огр. одозго: $\lim\limits_{n\to\infty}a_n=+\infty$

б) $\lim\limits_{n\to\infty} \frac{a_n^3}{n} \overset{Š.T.}{=} \lim\limits_{n\to\infty} \frac{a_n^3 - a_{n-1}^3}{n-(n-1)} = \lim\limits_{n\to\infty} \left(\left(a_{n-1} + \frac{1}{3a_{n-1}^2}\right)^3 - a_{n-1}^3 \right) =$

$= \lim\limits_{n\to\infty} \Big(\cancel{a_{n-1}^3} + \underbrace{3a_{n-1} \cdot \frac{1}{(3a_{n-1}^2)^2}}_{1/3a^3 \to 0} + \underbrace{3a_{n-1}^2 \cdot \frac{1}{3a_{n-1}^2}}_{1} + \underbrace{\frac{1}{27a_{n-1}^6}}_{\to 0 \text{ jer } a_{n-1}\to\infty} - \cancel{a_{n-1}^3} \Big) = 1$

$\boxed{3}$ $x_1 > 0$, $x_{n+1} = \frac{x_n}{1+x_n^2}$. А) $\lim\limits_{n\to\infty} x_n = ?$ б) $\lim\limits_{n\to\infty} n \cdot x_n^2 = ?$

А) 1° $x_n > 0$, $\forall n$

2° $x_{n+1} = \frac{x_n}{\underbrace{1+x_n^2}_{>1}} < x_n$ $(x_n)\downarrow$ $\Big\} \Rightarrow \exists \lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$

$x_{n+1} = \frac{x_n}{1+x_n^2}$ $/\lim\limits_{n\to\infty}$ $\Rightarrow$ $x = \frac{x}{1+x^2}$ $\Rightarrow$ $x + x^3 = x$ $\Rightarrow$ $x = 0$

$\lim\limits_{n\to\infty} x_n = \underline{\underline{0}}$

б) $\lim\limits_{n\to\infty} \underset{\infty}{n} \cdot \underset{0}{x_n^2} = ?$

$\lim\limits_{n\to\infty} \frac{1}{n x_n^2} = \lim\limits_{n\to\infty} \frac{\frac{1}{x_n^2}}{n} \overset{Š.T.}{=} \lim\limits_{n\to\infty} \frac{\frac{1}{x_n^2} - \frac{1}{x_{n-1}^2}}{\cancel{n} - (\cancel{n}-1)} = \lim\limits_{n\to\infty} \left(\frac{1}{x_n^2} - \frac{1}{x_{n-1}^2} \right) =$

$= \lim\limits_{n\to\infty} \left(\frac{(1+x_{n-1}^2)^2}{x_{n-1}^2} - \frac{1}{x_{n-1}^2} \right) = \lim\limits_{n\to\infty} \frac{x_{n-1}^4 + 2x_{n-1}^2}{x_{n-1}^2} = \lim\limits_{n\to\infty} (\underset{\to 0}{x_{n-1}^2} + 2) = \underline{\underline{2}}$

$\Rightarrow \lim\limits_{n\to\infty} n x_n^2 = \underline{\underline{1/2}}$

<u>Dom.</u> 1. $x_1 \in (0,1)$, $x_{n+1} = x_n - x_n^2$ (pek. cp-ja). А) $\lim\limits_{n\to\infty} x_n$ Б) $\lim\limits_{n\to\infty} n x_n^2$

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Линеарне диференцне ј-не

- $x_1, x_2 \to$ дати

$x_{n+2} = px_{n+1} + qx_n \to$ карак. ј-на: $t^2 = pt + q \begin{cases} \to t_1 \\ \to t_2 \end{cases}$

1° $t_1 \neq t_2 \to x_n = c_1 t_1^n + c_2 t_2^n$, c_1, c_2 из x_1, x_2

2° $t_1 = t_2 \to x_n = c_1 t_0^n + c_2 n \cdot t_0^n$, -||-

Пример $a_{n+2} = 2a_{n+1} + 3a_n$, $a_0 = 3$ $a_1 = 1$

$t^2 = 2t + 3$

$t^2 - 2t - 3 = 0$

$t_1 = 3$ $t_2 = -1$

$a_n = c_1 \cdot 3^n + c_2(-1)^n$

$\left.\begin{array}{l} 3 = c_1 + c_2 \\ 1 = c_1 3 - c_2 \end{array}\right\} c_1 = 1,\ c_2 = 2 \Rightarrow a_n = 3^n + 2(-1)^n$

Пример $x_{n+2} = 6x_{n+1} - 9x_n$, $x_0 = 2$ $x_1 = 3$

$t^2 - 6t + 9 = 0$

$t_1 = t_2 = 3$

$x_n = c_1 \cdot 3^n + c_2 \cdot n \cdot 3^n$

$\left.\begin{array}{l} c_1 = 2 \\ 3c_1 + 3c_2 = 3 \end{array}\right\} c_2 = -1 \Rightarrow x_n = 3^n(2-n)$

$\boxed{1}$ $x_{n+1} = 2x_n - y_n$, $x_0 = 2$

$y_{n+1} = x_n + 4y_n$ $y_0 = 1$

$y_n = 2x_n - x_{n+1}$

$\Rightarrow 2x_{n+1} - x_{n+2} = x_n + 4(2x_n - x_{n+1})$

$x_{n+2} - 6x_{n+1} + 9x_n = 0$, $x_0 = 2 \Rightarrow x_1 = 2x_0 - y_0 = 3 \Rightarrow x_n = 3^n(2-n)$

$y_n = 2 \cdot 3^n(2-n) - 3^{(n+1)}(2-(n+1)) = \ldots$

$\boxed{2}$ $a_1 = 1$, $a_{n+1} = \dfrac{1-4a_n}{1-6a_n}$

$a_1 = a$, $a_{n+1} = \dfrac{pa_n + q}{ra_n + s}$, $a_n = \dfrac{x_n}{y_n}$ $\dfrac{x_{n+1}}{y_{n+1}} = \dfrac{p\frac{x_n}{y_n} + q}{r \cdot \frac{x_n}{y_n} + s}$

$\dfrac{x_{n+1}}{y_{n+1}} = \dfrac{px_n + qy_n}{rx_n + sy_n}$

$a_1 = 1 = \dfrac{x_1}{y_1}$, $x_1 = y_1 = 1$

$a_n = \dfrac{x_n}{y_n}$ Веза $\to \dfrac{x_{n+1}}{y_{n+1}} = \dfrac{1 - \frac{4x_n}{y_n}}{1 - \frac{6x_n}{y_n}} = \dfrac{y_n - 4x_n}{y_n - 6x_n} = \dfrac{4x_n - y_n}{6x_n - y_n}$

$x_{n+1} = 4x_n + y_n$

$y_{n+1} = 6x_n - y_n$

(1) $y_n = 4x_n - x_{n+1}$

(2) $4x_{n+1} - x_{n+2} = 6x_n - 4x_n + x_{n+1}$

$x_{n+2} - 3x_{n+1} + 2x_n = 0$

$t^2 - 3t + 2 = 0$

$t_1 = 1, \quad t_2 = 2$

$x_1 = 1, \quad x_2 = 3 \qquad x_n = c_1 \cdot 2^n + c_2$

$\left. \begin{array}{l} 1 = x_1 = c_1 2 + c_2 \\ 3 = x_2 = c_1 \cdot 4 + c_2 \end{array} \right\} \Rightarrow c_1 = 1, \; c_2 = -1$

$\underline{x_n = 2^n - 1} \qquad y_n = 4(2^n - 1) - (2^{(n+1)} - 2) = \underline{2^{n+1} - 3}$

$\Rightarrow a_n = \dfrac{2^n - 1}{2^{n+1} - 3}$

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