Self-Study Notes on Process Matrices and Indefinite Causal Order

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1 Introduction

These notes grew out of a very simple question I asked myself while reading about the foundations of quantum theory:

Why do we assume that physical events always happen in a definite causal order?

In everyday physics, the idea that *A happens before B* or *B happens before A* is so deeply ingrained that it becomes invisible. Quantum states can be uncertain, measurements can be probabilistic, spacetime can curve—but the *order* of events is usually taken as fixed. Even in quantum information, where almost everything is allowed to be superposed or entangled, the temporal structure of the circuit remains classical.

The first time I encountered the process matrix framework of Oreshkov, Costa, and Brukner, it felt like someone had pulled a hidden assumption out from under quantum theory and replaced it with something far more flexible and intriguing. What if the flow of information between two quantum laboratories is not constrained by any predefined direction? What if the universe permits correlations that cannot be embedded into a single, global causal order? And what would such correlations *look like* operationally?

These questions are not only philosophically exciting—they also lead to concrete, testable predictions. The framework predicts situations where two parties can succeed at communication tasks better than any causally ordered world would allow. This is not due to quantum tricks like entanglement or superposition (although they are present), but because the *order* of events itself is not well defined.

To me, this was the most striking part: the theory does not say that the order is "unknown" or "random." It says that in certain processes, the notion of a definite "before" or "after" simply does not apply.

My goal in these notes is to build intuition for this idea. I try to explain the framework as I understood it—not as a polished textbook, but as a curious learner trying to make sense of a new and beautiful concept. If you're reading this, I hope it sparks the same sense of curiosity and conceptual freedom that it sparked in me.

Outline

These notes are organized as follows:

1. What "causal order" usually means in physics
2. Why one might want to question this assumption
3. Local operations in the process matrix picture
4. The definition of a process matrix (with intuition)
5. A simple causal game and the causal inequality
6. How a particular process violates the inequality
7. A small toy model for simulation (conceptual overview)
8. What indefinite causal order *really* means

These are "understanding-first" notes: I tried to write down the things I wish someone had told me when I started reading about causal nonseparability.

2 The usual idea of causal order

In everyday quantum theory, even in quantum information, we assume:

$$\text{Either } A \text{ happens before } B, \qquad \text{or } B \text{ happens before } A.$$

In a circuit diagram this looks like:

$$A \longrightarrow B \quad \text{or} \quad B \longrightarrow A.$$

This is a very deep assumption: even when probabilities or states are uncertain, the *temporal order* of events is assumed to be fixed.

Why question this assumption?

Quantum theory already challenges classical intuitions. So one can ask: is fixed causal order also an assumption that can be relaxed? Especially in regimes involving quantum gravity or indefinite spacetime structures.

The goal of Oreshkov–Costa–Brukner was:

> Not to break quantum mechanics, but to remove the assumption of a global causal order and see what remains.

This is similar in spirit to how general relativity removes the assumption of a fixed background spacetime.

3 Local operations: the basic building blocks

Each party—say A—has:

- an *input system* A_I - an *output system* A_O

This is a quantum “laboratory”. Something comes in, A performs an operation, and something goes out.

A local operation is represented by a Choi matrix M_A, with conditions:

$$M_A \geq 0, \qquad \mathrm{Tr}_{A_O}(M_A) = \mathbb{I}_{A_I}.$$

These are simply the mathematical translation of:

- positivity - complete positivity - trace preservation

For now, the main idea is:

$$M_A \quad \text{encodes everything that party } A \text{ does.}$$

4 What is a process matrix?

A process matrix W tells us the probability of local operations:

$$p(M_A, M_B) = \mathrm{Tr}[W(M_A \otimes M_B)]\,.$$

This is the single most important formula in the framework. It generalizes both:

- density matrices (when there are no outputs) - quantum channels (when there is a definite order)

Intuition

Think of W as the “environment” connecting the laboratories of A and B. But this environment:

- does *not* assume a causal direction - might allow information to flow in ways incompatible with $A \prec B$ or $B \prec A$

A nice mental picture:

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A_I -----> [    W    ] -----> A_O
B_I -----> [           ] -----> B_O
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But W doesn’t have arrows drawn in; the direction emerges from W itself.

5 A causal inequality (the "game")

This is the simplest way to *test* whether the world has a definite causal order.

Two players A and B receive inputs $x, y \in \{0,1\}$ and output bits a, b.

A random variable c determines the task:

$$\begin{aligned} c = 0: \quad & a = y \quad \text{(A must guess B's input)} \\ c = 1: \quad & b = x \quad \text{(B must guess A's input)} \end{aligned}$$

If the world has a definite order, then:

- If $A \prec B$, then A cannot know y in time. - If $B \prec A$, then B cannot know x in time.

Mixing the two orders with classical randomness still cannot help.

As a result, one proves:

$$P_{\text{succ}} \leq \frac{3}{4}.$$

This is the *causal inequality.* If you beat it, then the correlations being generated cannot arise from any definite causal order.

6 The OCB process that violates the inequality

The famous noncausal process is:

$$W = \frac{1}{4}\Big(\mathbb{I} + \frac{1}{\sqrt{2}}\sigma_z^{A_O}\sigma_z^{B_I} + \frac{1}{\sqrt{2}}\sigma_z^{B_O}\sigma_z^{A_I}\Big).$$

This contains two "cross terms":

- one from A_O to B_I (looks like $A \to B$) - one from B_O to A_I (looks like $B \to A$)

But they appear *together*, in a coherent superposition-like way.

Using simple measurement operations one obtains:

$$P_{\text{succ}} = \frac{2+\sqrt{2}}{4} \approx 0.853 > 0.75.$$

Thus the process *cannot* arise from $A \prec B$ or $B \prec A$, or any mixture of the two.

We call such a process **causally nonseparable**.

7 Toy simulation idea

To simulate this:

1. Represent W as a 16×16 matrix.
2. Choose simple CPTP maps M_A, M_B (e.g. Pauli instruments).
3. Compute $p(M_A, M_B)$ with $\mathrm{Tr}[W(M_A \otimes M_B)]$.
4. Combine probabilities appropriately to estimate P_{succ}.

This is conceptually straightforward and matches the original OCB result.

8 What does indefinite causal order *mean*?

Here are the two biggest misconceptions I had when learning this:

1. It does not mean time travel. No paradoxes, no sending information to your past self.

2. It does not mean the order is "random". Classical randomness between $A \prec B$ and $B \prec A$ *cannot* violate the causal inequality.

3. It means the order is not definable. Not even in principle. Not even given the full underlying description.

The process is genuinely incompatible with any embedding into a definite temporal structure.

Final Thoughts

If you are reading these notes, I hope they give you the same feeling of clarity they gave me while putting them together. As I worked through the process matrix framework, I realized that the key challenge is not the algebra (once the objects are defined, the calculations follow quite cleanly). The real conceptual leap is understanding that *causality itself can be an operational property rather than a built-in axiom of the theory.*

This shift in perspective makes the framework surprisingly powerful. It opens the door to physical situations where the usual notion of "who comes before whom" simply does not apply. For me, this was the most exciting part: the idea that quantum mechanics might allow scenarios where causal order is not just unknown or random, but *undefined.*

If you are coming from quantum information, this viewpoint connects naturally to ideas in computation and communication (like advantages in communication complexity), and possibly even to learning frameworks where information flow is itself a resource. If you come from the more foundational side, it offers an elegant way to generalize quantum theory without breaking any of its core principles.

These notes are far from complete, but they represent my first attempt at building intuition for indefinite causal order. If you have comments, suggestions, corrections, or simply want to discuss the topic, I would genuinely enjoy hearing from you.

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References used while preparing these notes:

1. O. Oreshkov, F. Costa, and Č. Brukner, *Quantum correlations with no causal order*, Nature Communications **3**, 1092 (2012).

2. C. Branciard, *Witnesses of causal nonseparability*, Physical Review Letters **119**, 180408 (2017).

3. P. Wechs et al., *Quantum circuits with indefinite causal structure*, arXiv:1909.05658.