Assignment 05

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1 Probabilistic Reasoning

1.1 From Logical to Probabilistic Reasoning

We define the binary random variables as follows:

- A (Frost): Frost occurs.
- B (Slippery Road): The road is slippery.
- C (Accident): An accident happens.

a. English-language Interpretations

Interpretations of the probabilistic statements:

a) $P(b|\neg a) \leq P(b)$: The probability of the road being slippery given that no frost occurs is less than or equal to the prior probability of the road being slippery.

b) $P(c|b) \geq P(c)$: The probability of an accident occurring given a slippery road is greater than or equal to the prior probability of an accident.

c) $P(a|b) \geq P(a)$: Given that the road is slippery, the probability that frost occurred is greater than or equal to the prior probability of frost.

d) $P(\neg a|\neg b) \geq P(\neg a)$: Given that the road is not slippery, the probability that no frost occurred is greater than or equal to the prior probability of no frost.

b. Generalization of Logical Implication

Assume logical implication $(a \Rightarrow b)$ is represented as $P(b|a) = 1$.

a) Show $P(b|\neg a) \leq P(b)$

Using the Law of Total Probability for $P(b)$:

$$P(b) = P(b|a)P(a) + P(b|\neg a)P(\neg a)$$

Substitute $P(b|a) = 1$ and $P(\neg a) = 1 - P(a)$:

$$P(b) = 1 \cdot P(a) + P(b|\neg a)(1 - P(a))$$

$$P(b) = P(a) + P(b|\neg a) - P(b|\neg a)P(a)$$

To show $P(b|\neg a) \leq P(b)$, we evaluate the difference:

$$P(b) - P(b|\neg a) = P(a) - P(b|\neg a)P(a) = P(a)(1 - P(b|\neg a))$$

Since $P(a) \geq 0$ and $(1 - P(b|\neg a)) \geq 0$, the difference is non-negative, meaning $P(b) \geq P(b|\neg a)$.

b) Show $P(a|b) \geq P(a)$

Using Bayes' Theorem:

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Substitute $P(b|a) = 1$:

$$P(a|b) = \frac{P(a)}{P(b)}$$

Since $P(b) \leq 1$, it follows that $\frac{P(a)}{P(b)} \geq P(a)$, thus $P(a|b) \geq P(a)$.

c) Show $P(\neg a|\neg b) = 1$

By Bayes' Theorem:

$$P(\neg a|\neg b) = \frac{P(\neg b|\neg a)P(\neg a)}{P(\neg b)}$$

From the Law of Total Probability, since $P(b|a) = 1$, then $P(\neg b|a) = 0$:

$$P(\neg b) = 0 \cdot P(a) + P(\neg b|\neg a)P(\neg a) = P(\neg b|\neg a)P(\neg a)$$

Substituting this into the numerator:

$$P(\neg a|\neg b) = \frac{P(\neg b|\neg a)P(\neg a)}{P(\neg b|\neg a)P(\neg a)} = 1$$

c. Bayesian Network Computations

Given: $P(a) = 0.6, P(b|a) = 0.7, P(b|\neg a) = 0.5, P(c|b) = 0.6, P(c|\neg b) = 0.3$.

a) $P(b)$

Using marginalization:

$$P(b) = P(b|a)P(a) + P(b|\neg a)P(\neg a) = (0.7 \cdot 0.6) + (0.5 \cdot 0.4) = \mathbf{0.62}$$

b) $P(c)$

Using marginalization:

$$P(c) = P(c|b)P(b) + P(c|\neg b)P(\neg b) = (0.6 \cdot 0.62) + (0.3 \cdot 0.38) = \mathbf{0.486}$$

c) $P(a|c)$

Using Bayes' Theorem:

$$P(a|c) = \frac{P(c|a)P(a)}{P(c)}$$

Where $P(c|a) = P(c|b)P(b|a) + P(c|\neg b)P(\neg b|a) = (0.6 \cdot 0.7) + (0.3 \cdot 0.3) = 0.51$.

$$P(a|c) = \frac{0.51 \cdot 0.6}{0.486} = \frac{0.306}{0.486} \approx \mathbf{0.6296}$$