Binary Search

Lesson Plan

PW SKILLS

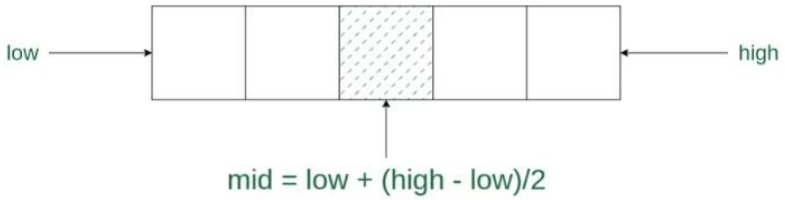
What & Why?
Binary search is an efficient algorithm used to search for a specific element within a sorted array or list. It works by repeatedly dividing the search interval in half until the target element is found or the interval becomes empty. Here's how it works:

Requirement: The array must be sorted in ascending or descending order for binary search to work effectively.

Algorithm:

In this algorithm,

1. Divide the search space into two halves by finding the middle index “mid”.
2. Compare the middle element of the search space with the key.
3. If the key is found at middle element, the process is terminated.
4. If the key is not found at middle element, choose which half will be used as the next search space.
5. If the key is smaller than the middle element, then the left side is used for next search.
6. If the key is larger than the middle element, then the right side is used for next search.
7. This process is continued until the key is found or the total search space is exhausted.



//Code

```
def binary_search(arr, x):
    l, r = 0, len(arr) - 1
    while l ⩽ r:
        m = l + (r - l) // 2

        # Check if x is present at mid
        if arr[m] == x:
            return m

        # If x is greater, ignore left half
        elif arr[m] < x:
 l = m + 1

        # If x is smaller, ignore right half
        else:
            r = m - 1

    # If we reach here, then the element was not present
    return -1
```

Ques : Binary Search [Leetcode 704]
Given an array of integers nums which is sorted in ascending order, and an integer target, write a function to search target in nums. If target exists, then return its index. Otherwise, return -1.

You must write an algorithm with O(log n) runtime complexity.

Input: nums = [-1,0,3,5,9,12], target = 9
Output: 4
Explanation: 9 exists in nums and its index is 4

Code:

```
def search(nums, target):
    left = 0  # initialize left pointer to 0
    right = len(nums) - 1  # initialize right pointer to
the last index of the array

    while left ⩽ right:  # continue the loop till the left
pointer is less than or equal to the right pointer
        mid = left + (right - left) // 2  # calculate the
middle index of the array

        if nums[mid] == target:  # check if the middle
element is equal to the target
            return mid  # return the middle index
        elif nums[mid] < target:  # check if the middle
element is less than the target
            left = mid + 1  # move the left pointer to the
right of the middle element
        else:  # if the middle element is greater than the
target
            right = mid - 1  # move the right pointer to
the left of the middle element
    return -1  # target not found in the array
```

Explanation: Simply use binary search as discussed above to find the target element

Time Complexity: O(logn) , because of binary search(each time we divide the array in half)
Space: O(1)
Ques: Given a sorted integer array and an integer 'x', find the lower bound of x.
Input : 4 6 10 12 18 18 20 20 30 45
Output : lower_bound for element 18 at index 4

```
def lower_bound(array, key):
    low = 0  # Initialize starting index
    high = len(array)  # Initialize ending index
    mid = 0
```

```
  # Iterate until high does not cross low
    while low < high:
        # Find the index of the middle element
        mid = low + (high - low) // 2

        # If key is less than or equal to array[mid], then
search in the left subarray
        if key ⩽ array[mid]:
            high = mid
        # If key is greater than array[mid], then search in
the right subarray
        else:
            low = mid + 1

    # If key is greater than the last element, increment
low
    if low < len(array) and array[low] < key:
        low += 1

    # Return the lower bound index
    return low
```

Explanation: Initialize the low as 0 and high as N.
Compare key with the middle element(arr[mid])
If the middle element is greater than or equal to the key then update the high as a middle index(mid).
Else update low as mid + 1.

Repeat step 2 to step 4 until low is less than high.
After all the above steps the low is the lower_bound of a key in the given array.

Time Complexity: O(logN),binary search
Auxiliary Space: O(1), no extra space being used

If you use C++, it has a build in builtin funciton to find lower bound using binary search
lower_bound(arr.begin() , arr.end() , target

Ques: Given a sorted integer array and an integer 'x', find the upper bound of x.
Input : 10 20 30 30 40 50
Output : upper_bound for element 30 is 40 at index 4

//code

```
def upper_bound(arr, key):
    N = len(arr)
    low = 0  # Initialize starting index
    high = N  # Initialize ending index
```

```
 while low < high and low ≠ N:
        mid = low + (high - low) // 2  # Find the index of
the middle element

        if key ≥ arr[mid]:  # If key is greater than or
equal to arr[mid], search in the right subarray
            low = mid + 1
        else:  # If key is less than arr[mid], search in
the left subarray
            high = mid

    if low == N:
        print("The upper bound of", key, "does not exist.")
        return
    else:
        print("The upper bound of", key, "is", arr[low],
"at index", low)
```

Explanation: Sort the array before applying binary search.
Initialize low as 0 and high as N.
Find the index of the middle element (mid)
Compare key with the middle element(arr[mid])
If the middle element is less than or equals to key then update the low as mid+1, Else update high as mid.
Repeat step 2 to step 4 until low is less than high.
After all the above steps the low is the upper_bound of the key

Time Complexity: O(logN),binary search
Auxiliary Space: O(1), no extra space being used

If you use C++, it has a build in builtin funciton to find upper bound using binary search
upper_bound(arr.begin() , arr.end() , target
Ques Given a sorted array of n elements and a target ‘x’. Find the first occurrence of ‘x’ in the array. If ‘x’ does not exist return -1.

```
def first_occurrence(array, key):
    x = lower_bound(array, key)
    if x < len(array) and array[x] == key:
        return x
    else:
        return -1
```

Explanation: Find the lower bound then check if the element at index lower bound equals the key the return the index, else it means the element doesn’t exits

Time Complexity: O(logN),binary search,in lower_bound
Auxiliary Space: O(1), no extra space being used

Ques : Peak index in mountain array(leetcode 852)

An array arr is a mountain if the following properties hold:
arr.length >= 3
There exists some i with 0 < i < arr.length - 1 such that:
arr[0] < arr[1] < ... < arr[i - 1] < arr[i]
arr[i] > arr[i + 1] > ... > arr[arr.length - 1]
Given a mountain array arr, return the index i such that arr[0] < arr[1] < ... < arr[i - 1] < arr[i] > arr[i + 1] > ... > arr[arr.length - 1].

You must solve it in O(log(arr.length)) time complexity.

Code:

```
class Solution:
    def peakIndexInMountainArray(self, arr: List[int]) →
int:
        start = 0
        end = len(arr) - 1

        while start < end:
            mid = start + (end - start) // 2
            if arr[mid] > arr[mid + 1]:
                end = mid
            else:
                start = mid + 1

        return start
```

Explanation:

- **Binary Search Approach:** Helps find the maximum element in a rotated sorted array.
- **Comparing Midpoints:** Check if the middle element is larger than its next element.
- **Deciding the Search Direction:**
  - If it's larger, explore the left side for a potentially larger element.
  - If not, explore the right side.
- **Refinement of Pointers:**
  - `start` and `end` pointers keep the best possible answer found so far.
  - When they converge to one element, it's likely the maximum.
- **Determining the Maximum:**
  - Return either `start` or `end` as they point to the maximum due to previous comparisons.

Time Complexity: O(logN),binary search
Auxiliary Space: O(1), no extra space being used

Ques: Search in Rotated Sorted Array (leetcode 33)
There is an integer array nums sorted in ascending order (with distinct values).
Prior to being passed to your function, nums is possibly rotated at an unknown pivot index k (1 <= k < nums.length) such that the resulting array is [nums[k], nums[k+1], ..., nums[n-1], nums[0], nums[1], ..., nums[k-1]] (0-indexed). For example, [0,1,2,4,5,6,7] might be rotated at pivot index 3 and become [4,5,6,7,0,1,2].

Given the array nums after the possible rotation and an integer target, return the index of target if it is in nums, or -1 if it is not in nums.

You must write an algorithm with O(log n) runtime complexity.
Input : arr[] = {5, 6, 7, 8, 9, 10, 1, 2, 3}, key = 3
Output : Found at index 8

//code

```
def search(arr, l, h, key):
    if l > h:
        return -1

    mid = (l + h) // 2
    if arr[mid] == key:
        return mid

    if arr[l] <= arr[mid]:
        if key >= arr[l] and key <= arr[mid]:
            return search(arr, l, mid - 1, key)
        return search(arr, mid + 1, h, key)

    if key >= arr[mid] and key <= arr[h]:
        return search(arr, mid + 1, h, key)
    return search(arr, l, mid - 1, key)
```

Explanation: The idea is to create a recursive function to implement the binary search where the search region is [l, r]. For each recursive call:

We calculate the mid value as mid = (l + h) / 2
Then try to figure out if l to mid is sorted, or (mid+1) to h is sorted
Based on that decide the next search region and keep on doing this till the element is found or l overcomes h.

Time Complexity: O(log N). Binary Search requires log n comparisons to find the element. So time complexity is O(log n).
Auxiliary Space: O(1). As no extra space is required.

Ques Find K Closest Elements (leetcode 658)
Given a sorted integer array arr, two integers k and x, return the k closest integers to x in the array. The result should also be sorted in ascending order.

An integer a is closer to x than an integer b if:

|a - x| < |b - x|, or
|a - x| == |b - x| and a < b

Input: arr = [1,2,3,4,5], k = 4, x = 3
Output: [1,2,3,4]

//Code:

```
class ClosestElements:
    def findClosestElements(self, arr, k, x):
        n = len(arr)
        R = self.lower_bound(arr, x)  # Find the lower
bound using a similar method as in C++
        L = R - 1

        # Expand the [L, R] window till its size becomes
equal to k
        while k > 0:
            if R ⩾ n or (L ⩾ 0 and x - arr[L] ⩽ arr[R] -
x):
                L -= 1  # Expand from left
            else:
                R += 1  # Expand from right
            k -= 1

        result = arr[L + 1:R]  # Slice the array to get the
closest elements
        return result

    # Implementing lower bound function
 def lower_bound(self, arr, target):
        low = 0
        high = len(arr)

        while low < high:
            mid = low + (high - low) // 2

            if arr[mid] < target:
                low = mid + 1
            else:
                high = mid

        return low

# Test the ClosestElements class
solution = ClosestElements()
arr = [1, 2, 3, 4, 5]
k = 4
x = 3
closest = solution.findClosestElements(arr, k, x)
print(f"Closest elements to {x} are: {closest}")
```

Explanation: We can take advantage of the fact that input array given to us is already sorted. We can use binary search to find the smallest element in arr which is greater or equal to x. Let's mark this index as R. Let's mark the index preceding to R as L and element at this index will be smaller than x.

Now, [L, R] forms our window of elements closest to x. We have to expand this window to fit k elements. We will keep picking either arr[L] or arr[R] based on whichever's closer to x and expand our window till it contains k elements.

Time Complexity : O(logn + k), We need O(logn) time complexity to find r at the start. Then we need another O(k) time to expand our window to fit k elements
Space Complexity : O(1)

Ques : Sum of Square Numbers (leetcode 633)
Given a non-negative integer c, decide whether there're two integers a and b such that a2 + b2 = c.

Input: c = 5
Output: true
Explanation: 1 * 1 + 2 * 2 = 5

//code

```
class Solution:
    def judgeSquareSum(self, c: int) -> bool:
        l = 0
        h = int(c ** 0.5)  # Using integer square root
function to get the floor value

        while l <= h:
            cur = l * l + h * h
            if cur < c:
                l += 1
            elif cur > c:
                h -= 1
            else:
                return True
        return False
```

Explanation: Binary Search the given element in which up to the given number if the square number is greater than the given number decrement the high and otherwise increment the low and in the else condition be true if there are equal, otherwise return false.

Time complexity : O(sqrt(c)log(c)), outer loop sqrt(c) , log(c) for binary search
Space: O(1)

Ques : Median of Two Sorted Arrays(leetcode 4)
Given two sorted arrays nums1 and nums2 of size m and n respectively, return the median of the two sorted arrays.

The overall run time complexity should be O(log (m+n)).

Input: nums1 = [1,3], nums2 = [2]
Output: 2.00000
Explanation: merged array = [1,2,3] and median is 2.

//Code:

```
def findMedianSortedArrays(nums1, nums2):
    n = len(nums1)
    m = len(nums2)
    if n > m:
        return findMedianSortedArrays(nums2, nums1)  #
Swapping to make nums1 the smaller array

    start = 0
    end = n
    real_mid_in_merged_array = (n + m + 1) // 2

    while start ⩽ end:
        mid = (start + end) // 2
        left_n_size = mid
        left_m_size = real_mid_in_merged_array - mid
        left_n = nums1[left_n_size - 1] if left_n_size > 0
else float('-inf')
        left_m = nums2[left_m_size - 1] if left_m_size > 0
else float('-inf')
        right_n = nums1[left_n_size] if left_n_size < n
else float('inf')
        right_m = nums2[left_m_size] if left_m_size < m
else float('inf')

        # Checking if the partition is correct
        if left_n ⩽ right_m and left_m ⩽ right_n:
            if (m + n) % 2 == 0:
                return (max(left_n, left_m) + min(right_n,
right_m)) / 2.0
            return max(left_n, left_m)

        elif left_n > right_m:
            end = mid - 1
        else:
            start = mid + 1

    return 0.0
```

Explanation: The given two arrays are sorted, so we can utilize the ability of Binary Search to divide the array and find the median.

Median means the point at which the whole array is divided into two parts. Hence since the two arrays are not merged so to get the median we require merging which is costly.

Hence instead of merging, we will use a modified binary search algorithm to efficiently find the median.

Time Complexity: O(min(log M, log N)): Since binary search is being applied on the smaller of the 2 arrays
Auxiliary Space: O(1),no extra space use

Ques: Given a sorted array of non-negative distinct integers, find the smallest missing non-negative element in it.
Input: {0, 1, 2, 6, 9}, n = 5, m = 10
Output: 3

Input: {4, 5, 10, 11}, n = 4, m = 12
Output: 0

//code

```
class SmallestMissingNumber:
    def findSmallestMissing(self, arr):
        left = 0
        right = len(arr) - 1

        while left ⩽ right:
            mid = left + (right - left) // 2

            if arr[mid] ≠ mid and (mid == 0 or arr[mid -
1] == mid - 1):
                return mid

            if arr[mid] == mid:
                left = mid + 1
            else:
                right = mid - 1

        return len(arr)


# Test the SmallestMissingNumber class
solution = SmallestMissingNumber()
arr = [0, 1, 2, 3, 4, 7, 10]
smallestMissing = solution.findSmallestMissing(arr)
print("Smallest missing number is:", smallestMissing)
```

Explanation: This solution utilizes a modified binary search algorithm:

It checks if the middle element matches its index and if the previous element is not missing (index-wise).
If the condition fails, the missing element is in the left half of the array; otherwise, it's in the right half.
The search continues in the respective half until the smallest missing number is found.

Time Complexity: O(Log n) ,for binary search
Auxiliary Space : O(1), no extra space used

Ques : Sqrt(x) (leetcode 69)
Given a non-negative integer x, return the square root of x rounded down to the nearest integer. The returned integer should be non-negative as well.

You must not use any built-in exponent function or operator.
Input: x = 4
Output: 2
Explanation: The square root of 4 is 2, so we return 2.

//code

```
def floorSqrt(x):
    # Base Cases
    if x == 0 or x == 1:
        return x

    # Do Binary Search for floor(sqrt(x))
    start = 1
    end = x // 2
    ans = 0

    while start <= end:
        mid = (start + end) // 2

        # If x is a perfect square
        if mid * mid == x:
            return mid

        # Since we need floor, we update answer when
        # mid*mid is smaller than x, and move closer to
        # sqrt(x)
        if mid * mid < x:
            start = mid + 1
            ans = mid
        else:  # If mid*mid is greater than x
            end = mid - 1

    return ans
```

Explanation: The idea is to find the largest integer i whose square is less than or equal to the given number. The values of i * i is monotonically increasing, so the problem can be solved using binary search.

Base cases for the given problem are when the given number is 0 or 1, then return X;
Create some variables, for storing the lower bound say l = 0, and for upper bound r = X / 2 (i.e, The floor of the square root of x cannot be more than x/2 when x > 1).

Run a loop until l <= r, the search space vanishes
Check if the square of mid (mid = (l + r)/2) is less than or equal to X, If yes then search for a larger value in the second half of the search space, i.e l = mid + 1, update ans = mid
Else if the square of mid is more than X then search for a smaller value in the first half of the search space, i.e r = mid – 1
Finally, Return the ans

Time Complexity: O(log(X)), Binary search
Auxiliary Space: O(1), no extra space being used

Ques : Capacity to ship packages within D days (leetcode 1011)
A conveyor belt has packages that must be shipped from one port to another within days days.
The ith package on the conveyor belt has a weight of weights[i]. Each day, we load the ship with packages on the conveyor belt (in the order given by weights). We may not load more weight than the maximum weight capacity of the ship.

Return the least weight capacity of the ship that will result in all the packages on the conveyor belt being shipped within days days.

Input: weights = [1,2,3,4,5,6,7,8,9,10], days = 5
Output: 15

Explanation: A ship capacity of 15 is the minimum to ship all the packages in 5 days like this:
1st day: 1, 2, 3, 4, 5
2nd day: 6, 7
3rd day: 8
4th day: 9
5th day: 10

Note that the cargo must be shipped in the order given, so using a ship of capacity 14 and splitting the packages into parts like (2, 3, 4, 5), (1, 6, 7), (8), (9), (10) is not allowed.

//code

```
def shipWithinDays(weights, D):
    left = max(weights)
    right = sum(weights)

    while left < right:
        mid = (left + right) // 2
        need = 1
        cur = 0

        for w in weights:
            if cur + w > mid:
                need += 1
                cur = 0
            cur += w

        if need > D:
            left = mid + 1
        else:
            right = mid

    return left
```

Explanation:

Given the number of bags,
return the minimum capacity of each bag,
so that we can put items one by one into all bags.

We binary search the final result.
The left bound is max(A),
The right bound is sum(A).

Time Complexity: O(n * log n). We perform log n analyses, and for each we process n packages.
Auxiliary Space: O(1), no extra space being used

Ques : Koko eating bananas (leetcode 875)
Koko loves to eat bananas. There are n piles of bananas, the ith pile has piles[i] bananas. The guards have gone and will come back in h hours.

Koko can decide her bananas-per-hour eating speed of k. Each hour, she chooses some pile of bananas and eats k bananas from that pile. If the pile has less than k bananas, she eats all of them instead and will not eat any more bananas during this hour.

Koko likes to eat slowly but still wants to finish eating all the bananas before the guards return.

Return the minimum integer k such that she can eat all the bananas within h hours.

Example 1:
Input: piles = [3,6,7,11], h = 8
Output: 4

Example 2:
Input: piles = [30,11,23,4,20], h = 5
Output: 30

//CODE

```
class Solution:
    def minEatingSpeed(self, piles, h):
        left = 1
        right = max(piles)

        while left < right:
            mid = (left + right) // 2
            if self.canEatAll(piles, mid, h):
                right = mid
            else:
                left = mid + 1
  return left

    def canEatAll(self, piles, speed, h):
        time = 0
        for pile in piles:
            time += (pile - 1) // speed + 1  # calculate
the time required to eat this pile
            if time > h:
                return False  # if the total time is
greater than h, return false
        return True  # if all piles can be eaten within h
hours, return true
```

Explanation: Intuition
We need to find the minimum integer k such that Koko can eat all the bananas within h hours. This means that we need to find the smallest value of k such that she can eat all the bananas within h hours.

Approach
Initialize left and right pointers as left = 1 and right = maximum number of bananas in any pile.

While the left pointer is less than the right pointer, repeat the following:
a. Calculate the middle pointer using mid = (left + right) / 2.
b. Check if Koko can eat all the bananas at the current speed (middle pointer) within h hours using the canEatAll method.
c. If Koko can eat all the bananas at the current speed, update the right pointer to the middle pointer using right = mid.
d. If Koko cannot eat all the bananas at the current speed, update the left pointer to mid + 1.

Once the left pointer is equal to the right pointer, return the left pointer as the minimum speed at which Koko can eat all the bananas within h hours.

The canEatAll method calculates the total time required to eat all the piles using the given speed. If the total time is greater than h, the method returns false, otherwise, it
returns true.

Time Complexity: O(n * log n).The binary search algorithm has a time complexity of O(logn), where n is the maximum number of bananas in a pile. The canEatAll function has a time complexity of O(n), where n is the number of piles We perform log n analyses, and for each we process n packages.
Auxiliary Space: O(1), no extra space being used

Ques : Minimum time to complete trips (leetcode 2187)
You are given an array time where time[i] denotes the time taken by the ith bus to complete one trip.

Each bus can make multiple trips successively; that is, the next trip can start immediately after completing the current trip. Also, each bus operates independently; that is, the trips of one bus do not influence the trips of any other bus.

You are also given an integer totalTrips, which denotes the number of trips all buses should make in total. Return the minimum time required for all buses to complete at least totalTrips trips.

Input: time = [1,2,3], totalTrips = 5
Output: 3

Explanation:
- At time t = 1, the number of trips completed by each bus are [1,0,0].
- The total number of trips completed is 1 + 0 + 0 = 1.
- At time t = 2, the number of trips completed by each bus are [2,1,0].
- The total number of trips completed is 2 + 1 + 0 = 3.
- At time t = 3, the number of trips completed by each bus are [3,1,1].
- The total number of trips completed is 3 + 1 + 1 = 5.
- So the minimum time needed for all buses to complete at least 5 trips is 3.

//Code:

```
def minimumTime(time, totalTrips):
    lo = 1
    hi = 100000000000000
    while lo < hi:
        mid = lo + (hi - lo) // 2
        if not f(mid, totalTrips, time):
            lo = mid + 1
        else:
            hi = mid
    return lo

def f(x, totalTrips, time):
    sum_val = 0
    for t in time:
        sum_val += x // t
    return sum_val ⩾ totalTrips
```

Explanation:
For any time x, we can have total trips = Σ(x / time[i]) where i in [0, time.size())
We need to minimize the above mentioned function total trips such that it is greater than or equal to the given variable totalTrips. We can use binary search.

During the contest I got away with keeping lo = 1 and hi = 10 ^ 15 On further inspection of the problem we can deduce that max value of x can be min(times) * totalTrips . So that can be used as hi

Time Complexity: O(nlog(min(time) * totalTrips)) //n for the boolean f funciton // log(min(time)*totaltrips) for binary search since max_x=min(time)*totaltrips
Auxiliary Space: O(1), no extra space being used