ADA 2024 Tutorial 1

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1 Recurrences

Solve the following recurrences:

a. $A(n) = 2A(n/4) + \sqrt{n}$

b. $B(n) = B(n/2) + B(n/3) + B(n/6) + n$

c. $C(n) = \sqrt{n}C(\sqrt{n}) + n$

d. $T(n) = T(n/8) + T(3n/5) + 2n$

Solution.

a. Use master theorem, identify the constants and check conditions. We get $O(\sqrt{n}\log n)$.

b. Construct the recurrence tree, and notice that the total work at each level is exactly n. Why? First, the work at a node is just the problem size at that node. The sum of the problem sizes for the different nodes at each level is n. Prove this by induction. The problem size at level zero is n. Then, suppose the sum of problem sizes at level k is n for $k \geq 1$. The key observation is that the problem size of any node at level k is exactly the sum of the problem sizes of its children at level $k+1$. This will immediately imply that the sum of the problem sizes at level $k+1$ is the same as at level k. To prove the observation, any node of level k of problem size m has children of problem sizes $m/2, m/3, m/6$ at level $k+1$. Then, the sum of these problem sizes of the children $m/2 + m/3 + m/6$ is also m. QED.

   Hence, $T(n) = n \times \text{number of levels} = O(n \log n)$

c. Can we apply Master's Theorem? Why or why not? Construct the recurrence tree and notice the same observation as above! The total work at each level is exactly n. First, the work at a node is just the problem size at that node. Now, the sum of the problem sizes for the different nodes at each level is n. Again prove it by induction. The problem size at level zero is n. Then, suppose the sum of problem sizes at level k is n for $k \geq 1$. The key observation is that the problem size of any node at level k is exactly the sum of the problem sizes of its children at level $k+1$. This will immediately imply that the sum of the problem sizes at level $k+1$ is the same as at level k. To prove the observation, any node of level k of problem size m has $\sqrt{m}$ children of problem sizes $\sqrt{m}$. Then the sum of these problem sizes of the children $\sqrt{m} \times \sqrt{m}$ is also m. QED.

Hence, $T(n) = n \times$ number of levels. How many levels are there? To determine this, notice that the problem size at level $k+1$ is the square-root of the problem size at level k. Hence, the problem size at level k is just $n^{1/2^k}$, because the problem size at level 0 is n. To determine the number of levels, just set the problem size to c for some constant c in the base case i.e. solve for k, $n^{1/2^k} = c$. Taking log of both sides, $2^k = \log n / \log c$. Taking log again, $k = O(\log \log n)$.

Hence, $T(n) = O(n \log \log n)$.

d. We use the substitution method here. We guess $T(n) \leq 20n$ for all $n \geq 1$.
Base Case: $n = 1$ and $T(1) = 1 \leq 20n$.
Induction Hypothesis: We assume this guess is correct for all values $1 \leq n \leq k-1$.
Proof for $n = k$.

$$T(k) = T(k/8) + T(3k/5) + 2n$$

Using induction hypothesis we get

$$T(k/8) + T(3k/5) + 2k \leq \frac{5k}{2} + 12k + 2k \leq \frac{33k}{2}$$

Therefore $T(n)$ is $O(n)$.

2 Induction fallacies

Consider the following counting problem: you are hosting a party and you decide to invite n couples. At your party, there are $2n+1$ chairs arranged in a row for you and the $2n$ guests. To keep things fun, you want to arrange the $2n+1$ attendees on the chairs so no couple sits together. Let's call such an arrangement a 'fun' arrangement.

Question: How many fun arrangements are there for the host and the n couples?

Here's an attempt. *Claim:* The number of fun arrangements of n couples and the host is $(2n)!$.

Let's 'prove' this by induction.

Let $P(n)$ be the proposition that the number of fun arrangements of n couples and the host is $(2n)!$.

Base case: For $n = 1$, the couple must sit on the opposite sides of the host. This can be done in two ways. And indeed, $(2 \times 1)! = 2$.

Induction hypothesis: Let us assume $P(n)$ i.e. n couples and the host can be arranged in $(2n)!$ fun ways.

Induction step: Let us take any fun arrangement for n couples and the host (i.e., one of the $2n!$ sequences), then try to add one more couple. So out of $2n+2$ seats where we can place the additional couple (since the additional couple must not sit together). Hence, to obtain all possible fun arrangements for the additional couple, it is enough to find all pairs of seats among these possible $2n+2$ seats and multiply that by 2 (since for any two selected seats, the additional couple can be seated in two ways on these two seats). The number of pairs is just $\binom{2n+2}{2}$.

Hence, the number of arrangements of $n+1$ couples is

$$(2n)! \times \binom{2n+2}{2} \times 2 = (2(n+1))!$$

This proves $P(n+1)$.

Is the above proof correct? Try to take small values of n and try to verify what's going on. If it is not correct, what is the mistake?

Solution: The issue with the above argument is the following. The calculation assumes that the *only* way to find a fun arrangement for $n+1$ couple plus host is to start with a fun arrangement for n couple plus the host and then extend it by placing the remaining couple in the gaps. This is a false assumption. Here is the proof. Consider a fun arrangement for $n+1$ couple plus host. Now remove the last couple. However, this *does not* guarantee that you will be left with a fun arrangement for the remaining couple. For instance, it might be the case that one of the person in the final couple was sitting between another couple and upon removing him/her.

In a nutshell, the calculation in the problem shows that the number of fun arrangement for $n+1$ couples is at least the given quantity but is actually more.

3 n-Stack Game

This game is one-player, to be played by you. You will be given n tiles stacked on each other in a column (so that the height of the stack is n).

The game move for you is as follows: break a stack of height at least 2 into two stacks. Whenever you make this move, you get a score equal to the product of the heights of the two new stacks.

The game ends when all the stacks are of height 1.

Question: Prove that no matter what moves you make over the course of the game, your total score will be the same.

Solution The answer is that no matter what your strategy is at any point in the game, you will end up with the same score $\frac{n(n-1)}{2}$. One can prove this using strong induction. Let $P(n)$ be the following proposition.

$P(n)$: The total score for the game is always $n(n-1)/2$ no matter what the strategy is to break any intermediate stack. Base case is easy. Suppose the hypothesis holds for *every* $k = 1, 2, \cdots k-1$. Now consider $P(k)$. Let the stack be broken at a tiles from below and $b = k - a$. Now, apply strong induction on the remaining stacks of heights a and b, respectively, to obtain scores $a(a-1)/2$ and $b(b-1)/2$. Finally, we add the score $a \cdot b$ for the first split. This gives us that the final score is

$$a \cdot b + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{(a+b)(a+b-1)}{2} = \frac{k(k-1)}{2}$$