HW4

$P_K(k) = \begin{cases} \frac{5^k e^{-5}}{k!} & k=0,1,2 \\ 0 & \text{otherwise} \end{cases}$ $\quad \boxed{\alpha=5}$ $T = 10$ sec $\Rightarrow \alpha = \lambda T \Rightarrow \lambda = \frac{\alpha}{T} = \frac{5}{10} \Rightarrow \underline{(\lambda = 0.5 \text{ queries/sec})}$

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a) $P_K[K=0] = P_K(0) = e^{-5} \Rightarrow$ prob. that no queries in 10 sec interval

b) In 2 sec interval $T=2$, $\alpha = 2\lambda \Rightarrow \alpha = 2(0.5) \Rightarrow \boxed{\alpha = 1}$

$\Rightarrow P_N(n) = \begin{cases} \frac{e^{-1}(1)}{n!} & n=0,1,2\ldots \\ 0 & \text{otherwise} \end{cases}$

[no of queries : N]

$P[N \geq 2] = 1 - P[N=0] - P[N=1] = 1 - e^{-1} - e^{-1} \approx \underline{0.26}$

prob of atleast 2 queries being processed in T=2 sec interval.

2. prob of caller failing thrice $= (1-p)^3$. For 95% callers to get through,

$\therefore (1-p)^3 \leq 1 - \frac{95}{100} \Rightarrow (1-p)^3 \leq 0.05$

$\Rightarrow 1-p \leq (0.05)^{1/3} \Rightarrow 1-(0.05)^{1/3} \leq p$

min val of $p = 1 - (0.05)^{1/3} \approx \underline{0.63}$

3. Here $p = 0.2$, $n \to$ no. of agents. If all n agents are busy, call attempt fails.

$\therefore$ 3 failed attempts prob. $= (1-p)^{3n} = (0.8)^{3n}$. To serve the 95% customer goal, $(0.8)^{3n} \leq 1 - 0.95 \Rightarrow 3n \log_{0.8}(0.8) \leq \log_{0.8}(0.05) \Rightarrow 3n \geq 13.44$

$\Rightarrow n \geq 4.48$

$\Rightarrow$ We need $\underline{5 \text{ operators}}$. $\underline{(\text{min val of } n = 5)}$

4. a) PMF $\Rightarrow$ $P_Y(y) = \begin{cases} 1/11 & y = 5,6,\ldots,15 \\ 0 & \text{otherwise} \end{cases}$

b) $P[Y < 10] = P_Y(5) + P_Y(6) + P_Y(7) + P_Y(8) + P_Y(9) = \left(\frac{1}{11}\right)5 = \frac{5}{11}$

c) $P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = \frac{3}{11}$

d) $P[8 \leq Y \leq 12] = P_Y(8) + P_Y(9) + P_Y(10) + P_Y(11) + P_Y(12) = \frac{5}{11}$

5. a) $P_K(k) = \begin{cases} {}^nC_k \, p^k (1-p)^{n-k} & k=0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}$

The no. of times the pager receives the same msg (K) = no. of successes in n Bernoulli trials.

b) Let X: event that paging msg was recd. atleast once.

$P[X] = P[B > 0] = 1 - P[B = 0] = \underline{1 - (1-p)^n}$

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To get min val of n s.t. prob. of receiving msg is atleast one $\boxed{P[X] \geq 0.95}$

$\Rightarrow P[X] \geq 0.95 \Rightarrow 1 - (1-p)^n \geq 0.95 \Rightarrow (1-p)^n \leq 0.05$

$\Rightarrow n \ln(1-p) \leq \ln(0.05)$

$\Rightarrow n \leq \dfrac{\ln(0.05)}{\ln(1-p)}$ $\quad p = 0.8$ (given) $\Rightarrow n \leq \dfrac{\ln(0.05)}{\ln(0.2)}$

$\Rightarrow \cancel{n \leq 1.86} \Rightarrow n \geq 1.86 \Rightarrow$ min val of $n = 2$

6. K = no. of fish hooked on a single cast of line

m = no. of hooks attached to line (prob = h)

PMF of $K \Rightarrow$ Binomial PMF

$$P_K(k) = \begin{cases} {}^mC_k \, h^k (1-h)^{m-k} & k = 0, 1, \ldots m \\ 0 & \text{otherwise} \end{cases}$$

7. When the dog catches frisbee, he runs away w/ it and the frisbee throw

a) game ends. So, the exp has geometric PMF, $\because$ the game ends when the desired outcome (dog running away w/ the frisbee) is achieved

$$P_X(x) = \begin{cases} (1-p)^{x-1} \cdot p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$p = 0.2$

b) Child throws the frisbee > 4 times iff there r 4 failures initially.

$\therefore$ ~~Prob~~ Let X = child throws frisbee > 4 times

$\Rightarrow P[X] = (1-p)^4 = (1-0.2)^4 = (0.8)^4 = 0.4096.$

8a) The paging msg is sent again if the paging system doesn't receive the ACK. ⇒ This msg is sent again & again till the pager receives the msg (desired outcome)

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95) ⇒ exp has geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1} p & n=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

b) $P[N \leq 3]$ → prob. that max of 3 paging attempts are required.

$\Rightarrow P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3$

• $P[N \leq 3]$ should be ≥ 0.95 $\Rightarrow 1 - (1-p)^3 \geq 0.95$

$\Rightarrow (1-p)^3 \leq 0.05 \Rightarrow 1-p \leq (0.05)^{1/3} \Rightarrow -p \leq (0.05)^{1/3} - 1 \Rightarrow p \geq -(0.05)^{1/3} + 1$

$\Rightarrow p \geq 1 - (0.05)^{1/3} \approx \underline{0.63}$

9. $\alpha = \frac{T}{5}$, PMF of B (no of buses arriving in T mins) • $P_B(b) = \begin{cases} \frac{\left(\frac{T}{5}\right)^b e^{-(T/5)}}{b!} & b = 0,1,\ldots \\ 0 & \text{otherwise} \end{cases}$

b) T = 2 mins. Prob that 3 buses arrive in T = 2 mins, $\alpha = 2/5$

$$P_B(3) = \frac{\left(\frac{2}{5}\right)^3 e^{-\left(\frac{2}{5}\right)}}{3!} = \frac{8}{125}\left(\frac{e^{-\frac{2}{5}}}{6}\right) = \frac{4}{375} e^{-2/5} \approx 0.0072$$

c) T = 10 mins. Prob that 0 buses arrive in T = 10 min, $\alpha = 10/5 \Rightarrow \alpha = 2$

$$P_B(0) = \frac{(2)^0 e^{-2}}{0!} = e^{-2} \approx 0.135$$

d) Prob that at least 1 bus arrives, $P[B \geq 1] = 1 - P[B=0] = 1 - e^{-T/5} \geq 0.99$

$\Rightarrow -e^{-T/5} \geq -0.01 \Rightarrow e^{-T/5} \leq 0.01 \Rightarrow \frac{-T}{5} \leq \ln(0.01)$

$\Rightarrow \frac{T}{5} \geq \ln(100) \Rightarrow T \geq 5 \ln 100 \Rightarrow T \geq 5(2)\ln 10 \Rightarrow T \geq 10(2.3)$

≈ 23 mins

10. Pkts can be retransmitted d times.

T → no of times pkt is transmitted. For t < d, pkt transmitted t times

(⇒ t−1 failed attempts & t^{th} attempt successful)

* $P_T(t) = \begin{cases} p(1-p)^{t-1} & t = 1, 2, \dots, d-1 \\ (1-p)^{d-1} & t = d \\ 0 & \text{otherwise} \end{cases}$

11. a) let S_n: event that Sixers win the playoffs in n games.
& C_n: " " Celtics " " " n "

Now, the Sixers can win the playoffs in 3 cases =

① If they win 3 consecutive games
② If they win 2 out of the first 3 games, lose in the 3rd game & win again in 4th game
③ If they win 2 out of the first 4 games, lose the 3rd & 4th " & win in the last (5th) game.

① $P[S_3] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

③ $P[S_5] = {}^4C_2 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) = 6\left(\frac{1}{32}\right) = \frac{3}{16}$

② $P[S_4] = {}^3C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) = 3\left(\frac{1}{16}\right) = \frac{3}{16}$

$\because$ each team is equally likely to win any game, $P[C_n] = P[S_n]$.

$N \to$ the no. of games played in the playoffs

$\Rightarrow P[N = n] = P[S_n] + P[C_n] = 2P[S_n]$

PMF $P_N(n) = \begin{cases} 2\left(\frac{1}{2}\right)^3 = \frac{1}{4} & n = 3 \\ 2\left({}^3C_1 \left(\frac{1}{2}\right)^4\right) = \frac{3}{8} & n = 4 \\ 2\left({}^4C_2 \left(\frac{1}{2}\right)^5\right) = \frac{3}{8} & n = 5 \\ 0 & \text{otherwise} \end{cases}$

b) Celtics win playoffs if they win $(3 + w)$ games. (where $w \to$ no. of wins)
Also, they win 3 games if they win the playoffs in 3, 4 or 5 games.

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$$P_W(w) = \begin{cases} P[C_3] & w=0 \\ P[C_4] & w=1 \\ P[C_5] & w=2 \\ P[C_3]+P[C_4]+P[C_5] & w=3 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow P_W(w) = \begin{cases} 1/8 & , w=0 \\ 3/16 & , w=1 \\ 3/16 & , w=2 \\ 1/8 + 3/16 + 3/16 = \frac{1}{2} & , w=3 \\ 0 & \text{otherwise} \end{cases}$$

9. no of celtics' losses L_c = no. of sixers' wins W_s.

$\Rightarrow P_{L_c}(l) = P_{W_s}(l)$ & $P_{W_s}(w) = P_{W_c}(w)$

$\Rightarrow P_{L_c}(l) = P_{W_s}(l) = P_{W_c}(l)$

where L_c → celtics' losses, W_c → celtics' wins, W_s → sixers' wins.

$$P_{L_c}(l) = P_{W_c}(l) = \begin{cases} 1/8 & l=0 \\ 3/16 & l=1 \\ 3/16 & l=2 \\ 1/2 & l=3 \\ 0 & \text{otherwise} \end{cases}$$

12. $$P_N(n) = \begin{cases} \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right) & n=0,1,\dots,99 \\ \left(\frac{1}{3}\right)^n & n=100 \\ 0 & \text{otherwise} \end{cases}$$

for $n < 100$, CDF of N, $F_N(n) = \sum_{i=0}^{n} P_N(i) = \sum_{i=0}^{n} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right) = \frac{2}{3}\left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \dots \left(\frac{1}{3}\right)^n\right]$ $\quad r = \frac{1}{3}$ $\left(S = \frac{1-r^{n+1}}{1-r}\right)$

$= \frac{2}{3}\left[\frac{1-\left(\frac{1}{3}\right)^{n+1}}{\left(1-\frac{1}{3}\right)}\right]$

$F_N(n) = 1 - \left(\frac{1}{3}\right)^{n+1}$.

$$F_N(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{1}{3}\right)^{\lfloor x \rfloor + 1} & 0 \leq x < 100 \\ 1 & x \geq 100 \end{cases}$$

*
$\left[\begin{array}{l} F_X(x) = 0 \quad x \leq x_{min} \\ F_X(x) = 1 \quad x \geq x_{max} \end{array}\right]$