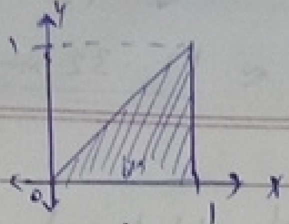
Q1. HW6

a) $f_{X,Y}(x,y) = \begin{cases} cy & 0 \le y \le x \le 1 \\ 0 & \text{else} \end{cases}$

region of non zero prob. =

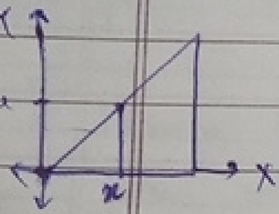


b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \left(\int_0^x cy\,dy\right)dx = \int_0^1 \left(\frac{cx^2}{2}\right)dx = \frac{cx^3}{6}\Big]_0^1 = \frac{c}{6} = 1$

$\rightarrow \boxed{c = 6}$

c) $F_X(x) = 0$ for $x < 0$ & for $x > 1$, $F_X(x) = 1$

for $x \in [0,1]$, $F_X(x) = \int_0^x \left(\int_0^u 6v\,dv\right)du = \int_0^x 3u^2\,du = x^3$

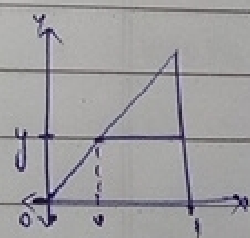


$F_X(x) = \begin{cases} 0 & , x < 0 \\ x^3 & , x \in [0,1] \\ 1 & , x > 1 \end{cases}$

d) $F_Y(y) = 0$ for $y < 0$ & $F_Y(y) = 1$ for $y > 1$

$F_Y(y) = \int_0^y \int_v^1 6v\,du\,dv = \int_0^y 6v \left(\int_v^1 du\right)dv$

$= \int_0^y 6v(1-v)\,dv$



$= \int_0^y (6v - 6v^2)\,dv = 3v^2 - 2v^3\Big]_0^y = 3y^2 - 2y^3$

$F_Y(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & y \in [0,1] \\ 1 & y > 1 \end{cases}$

c) $P[Y \leq \frac{X}{2}] = \int_0^1 \int_0^{x/2} 6y \, dy \, dx = \int_0^1 3y^2 \Big]_0^{x/2} dx = \int_0^1 \frac{3x^2}{4} dx = \frac{3}{4} \frac{x^3}{3}\Big|_0^1 = \frac{x^3}{4}\Big]_0^1 = \frac{1}{4}$

Page No.
Date : / /

Q2. Hint = is it possible to observe $W < 1$?

Either Y will be greater than or equal to X ($Y \geq X$) or
X " " " " or " " Y ($X \geq Y$)

$\rightarrow$ either $\left(\frac{Y}{X}\right) \geq 1$ or $\left(\frac{X}{Y}\right) \geq 1 \rightarrow \therefore \underline{W \geq 1}$

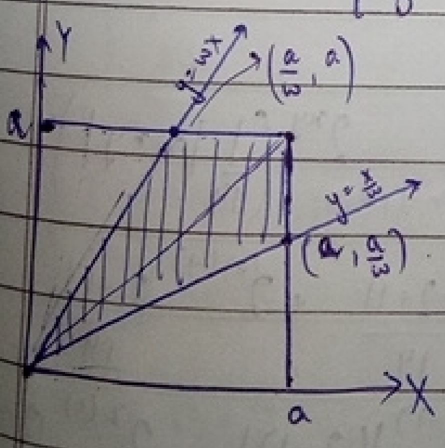
We know, $F_W(w) = 0$ for $w < 1$. $W = \max\left[\left(\frac{X}{Y}\right), \left(\frac{Y}{X}\right)\right]$

For $w \geq 1$, $F_W(w) = P[W \leq w] = P\left[\max\left[\left(\frac{X}{Y}\right), \left(\frac{Y}{X}\right)\right] \leq w\right]$

$= P\left[\left(\frac{X}{Y}\right) \leq w, \left(\frac{Y}{X}\right) \leq w\right] = P\left[Y \geq \frac{X}{w}, Y \leq wX\right]$

$= P\left[\frac{X}{w} \leq Y \leq wX\right]$

$f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & , x \in [0,a], y \in [0,a] \\ 0 & , \text{else} \end{cases}$



▨ $\rightarrow$ Shaded region has prob.

$\left(1 - P\left[\frac{X}{w} \leq Y \leq wX\right]\right)$

$1 - \left(1 - \left[\left(\frac{a^2}{2w}\right) 2\right] \left(\frac{1}{a^2}\right)\right)$

$= \cancel{1} - \left(\cancel{1} - \frac{1}{w}\right) = \frac{1}{w}$

$\therefore$ unshaded region has prob. $P\left[\frac{X}{w} \le Y \le wX\right] = 1 - \left(\frac{1}{w}\right)$

Page No.
Date : / /

$$F_W(w) = \begin{cases} 0 & , w < 1 \\ 1 - \frac{1}{w} & , w \ge 1 \end{cases}$$
(CDF)

$$f_W(w) = \frac{d F_W(w)}{dw} = \begin{cases} 0 & , w < 1 \\ 1/w^2 & , w \ge 1 \end{cases}$$
(PDF)

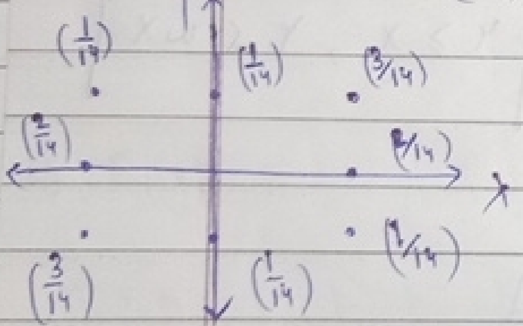
Q3. $\Rightarrow \boxed{\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x,y) = 1} \Rightarrow \sum_{x \in S_X} \sum_{y \in S_Y} c|x+y|$

$$= \sum_{x=-2,0,2} \sum_{y=-1,0,1} c|x+y| = \sum_{x=-2,0,2} \left(c|x-1| + c|x| + c|x+1|\right)$$

$$= \left(c|-2-1| + c|-1| + c|-2+1|\right) + \left(c|-2| + c|0| + c|2|\right) + \left(c|2+1| + c|0+1| + c|-2+1|\right)$$

$$= (3c + c + c) + (2c + 2c) + (3c + c + c)$$

$$= (5c) \times 2 + 4c \Rightarrow \underline{\underline{14c = 1}} \Rightarrow \boxed{c = 1/14}$$



a) $E[g(X,Y)] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) P_{X,Y}(x,y)$

$[W = g(X,Y) = 2^{XY}] \Rightarrow E[2^{XY}] = \sum_{x=-2,0,2} \sum_{y=-1,0,1} 2^{xy} \left(\frac{1}{14}|x+y|\right)$

$$= 2^{(-2)(-1)} \frac{|-2-1|}{14} + 2^{(-2)(0)} \frac{|-2|}{14} + 2^{(-2)(1)} \frac{|-2+1|}{14} + 2^{(0)(-1)} \frac{|0-1|}{14}$$

$$+ 2^{(0)(0)} \cancel{\frac{(0)}{14}}^{\,0} + 2^{(0)(1)} \frac{(1)}{14} + 2^{2(-1)} \frac{|(2-1)|}{14} + 2^{2(0)} \left(\frac{2}{14}\right)$$

$$+ 2^{2(1)} \frac{|(2+1)|}{14} = 2\left(2^2\left(\frac{3}{14}\right) + \left(\frac{2}{14}\right) + 2^{-1}\left(\frac{1}{14}\right) + \left(\frac{1}{14}\right)\right) = \frac{61}{28}$$

b) correlation $E[XY]$ or $r_{X,Y}$ $\left[E[g(x,y)] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y)\, P_{X,Y}(x,y)\right]$

$$r_{X,Y} = \sum_{x=-2,0,2} \sum_{y=-1,0,1} xy\, P_{X,Y}(x,y)$$

Page No.

Date : / /

$$= \frac{(-2)(-1)|-2-1|}{14} + \frac{(-2)(0)|-2|}{14} + \frac{(-2)(1)|-2+1|}{14}$$

$$+ \cancel{\frac{(0)(0)(0)}{14}} + \frac{2(-1)|2-1|}{14} + \frac{2(0)|2+0|}{14} + \frac{(2)(1)(|2+1|)}{14}$$

$$= \frac{4}{7}.$$

c) $Cov[X,Y] = E[XY] - E[X]\,E[Y]$

(To find - $E[X] = ?$ $E[Y] = ?$)

$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y)$ & $P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y)$ [marginal PMFs]

$$\Rightarrow P_X(x) = \sum_{y=-1,0,1} \frac{1}{14}|x+y| = \frac{1}{14}\left(|x-1| + |x| + |x+1|\right)$$

$$\Rightarrow P_X(x) = \begin{cases} \frac{1}{14}(|-2-1| + |-2| + |-2+1|), & x=-2 \\ \frac{1}{14}(|-1| + 0 + |1|), & x=0 \\ \frac{1}{14}(|2-1| + |2| + |2+1|), & x=2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow P_X(x) = \begin{cases} \frac{6}{14}, & x=-2 \\ \frac{2}{14}, & x=0 \\ \frac{6}{14}, & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \frac{1}{14}(|y-2| + |y| + |y+2|)$$

$$\Rightarrow P_Y(y) = \begin{cases} \frac{1}{14}(|-1-2| + |-1| + |-1+2|), & y=-1 \\ \frac{1}{14}(|-2| + |0| + |2|), & y=0 \\ \frac{1}{14}(|1-2| + |1| + |1+2|), & y=1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow P_Y(y) = \begin{cases} \frac{5}{14}, & y=-1 \\ \frac{4}{14}, & y=0 \\ \frac{5}{14}, & y=1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x \in S_X} x P_X(x) = \sum_{x=-2,0,2} x P_X(x) = (-2)\cancel{\left(\frac{6}{14}\right)} + 0\cancel{\left(\frac{2}{14}\right)} + 2\cancel{\left(\frac{6}{14}\right)} = \underline{0}$$

Page No.

Date : / /

$$E[Y] = \sum_{y \in S_Y} y P_Y(y) = \sum_{y=-1,0,1} y P_Y(y) = (-1)\cancel{\left(\frac{5}{14}\right)} + (0)\cancel{\left(\frac{4}{14}\right)} + 1\cancel{\left(\frac{5}{14}\right)} = \underline{0}$$

$\therefore \; E[X] = E[Y] = 0$

$$\rightarrow \left(Cov[X,Y] = E[X]E[Y] = \frac{4}{7} \right)$$

d) correlation coeff $\rho_{X,Y} = \dfrac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}}$

$$E[X^2] = \sum_{x=-2,0,2} x^2 P_X(x) = (-2)^2\left(\frac{6}{14}\right) + \cancel{0^2\left(\frac{2}{14}\right)}^{0} + (2)^2\left(\frac{6}{14}\right) = \left[4\left(\frac{6}{14}\right)\right]2 = \frac{4\times 6}{7} = \left(\frac{24}{7}\right)$$

$$E[Y^2] = \sum_{y=-1,0,1} y^2 P_Y(y) = (-1)^2\left(\frac{5}{14}\right) + \cancel{0^2\left(\frac{4}{14}\right)}^{0} + (1)^2\left(\frac{5}{14}\right) = \frac{10}{14} = \left(\frac{5}{7}\right)$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{24}{7} - 0 = \frac{24}{7}$$

$$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{5}{7} - 0 = \frac{5}{7}$$

$$\rho_{X,Y} = \frac{\frac{4}{7}}{\sqrt{\left(\frac{24}{7}\right)\left(\frac{5}{7}\right)}} = \frac{4}{\cancel{7}}\left(\frac{\cancel{7}}{\sqrt{(24)(5)}}\right) = \frac{4}{\sqrt{120}} = \frac{2}{\sqrt{30}}$$

e) $Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$

$$= \frac{24}{7} + \frac{5}{7} + 2\left(\frac{4}{7}\right) = \frac{37}{7}$$

Page No.
Date : / /

Q4. $P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y) = \sum_{y=0}^{x} \frac{1}{21} = \frac{1}{21}(x+1)$

$P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y) = \sum_{x=y}^{5} \left(\frac{1}{21}\right) = \frac{1}{21}(6-y)$

$$P_X(x) = \begin{cases} \frac{x+1}{21} & x = 0, \dots, 5 \\ 0 & \text{otherwise} \end{cases} \qquad P_Y(y) = \begin{cases} \frac{6-y}{21} & y = 0, \dots, 5 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x \in S_X} x P_X(x) = \sum_{x=0}^{5} x\left(\frac{x+1}{21}\right) = \frac{1}{21}\left(\sum_{x=0}^{5} x^2 + \sum_{x=0}^{5} x\right)$$

$$= \left(\frac{1}{21}\right)\left[\frac{5(6)(11)}{6} + \frac{5(6)}{2}\right] = \frac{1}{21}\left[55 + 15\right] = \left(\frac{70}{21}\right) = \frac{10}{3}$$

$$E[Y] = \sum_{y \in S_Y} y P_Y(y) = \sum_{y=0}^{5} y\left(\frac{6-y}{21}\right) = \frac{1}{21}\left(\sum_{y=0}^{5} 6y - \sum_{y=0}^{5} y^2\right)$$

$$= \frac{1}{21}\left(6\left[\frac{(5)(6)}{2}\right] - \frac{5(6)(11)}{6}\right) = \frac{1}{21}(90 - 55) = \left(\frac{35}{21}\right) = \frac{5}{3}$$

$$E[XY] = \sum_{x=0}^{5}\sum_{y=0}^{x} \frac{(xy)}{21} = \left(\frac{1}{21}\right)\sum_{x=1} x \sum_{y=1}^{x} y = \left(\frac{1}{21}\right)\sum_{x=1}^{5} x^2 \frac{(x+1)}{2}$$

$$= \left(\frac{1}{42}\right)\sum_{x=1}^{5}(x^3 + x^2) = \frac{1}{42}\left[\sum_{x=1}^{5} x^3 + \sum_{x=1}^{5} x^2\right]$$

$$= \left(\frac{1}{42}\right)\left[\frac{5^2(5+1)^2}{4} + \frac{5(5+1)(10+1)}{6}\right]$$

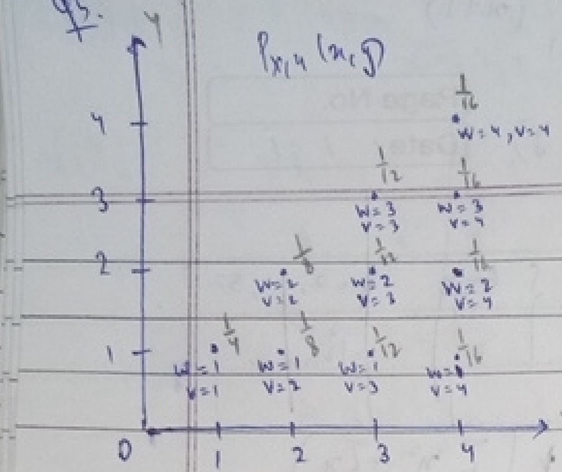
$$= \frac{1}{42}\left[(25)\frac{(36)}{4} + 5(11)\right] = \frac{1}{42}\left((25)(9) + 55\right) = \frac{(225 + 55)}{42}$$

$$= \frac{280}{42} = \frac{20}{3}$$

$$\Rightarrow \operatorname{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{20}{3} - \frac{50}{9} = \frac{10}{9}$$

Q5. $W = \min(X,Y)$, $V = \max(X,Y)$

Page No.
Date : / /



Acc. to graph (after writing vals of W & V for each pair (x,y))

① $W = \min(X,Y) = Y$ &

② $V = \max(X,Y) = X$

a) $E[W] = E[Y] = \sum_{y \in S_Y} y\, P_Y(y)$; $E[V] = E[X] = \sum_{x \in S_X} x\, P_X(x)$

$$P_X(x) = \begin{cases} \frac{1}{4} & x = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}, \quad P_Y(y) = \begin{cases} \frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16} & y=1 \\ \frac{1}{8}+\frac{1}{12}+\frac{1}{16} & y=2 \\ \frac{1}{12}+\frac{1}{16} & y=3 \\ \frac{1}{16} & y=4 \\ 0 & \text{otherwise} \end{cases} \Rightarrow P_Y(y) = \begin{cases} \frac{25}{48}, & y=1 \\ \frac{13}{48}, & y=2 \\ \frac{7}{48}, & y=3 \\ \frac{1}{16}, & y=4 \\ 0 & \text{else} \end{cases}$$

$$E[W] = \sum_{y=1}^{4} y\, P_Y(y) = \left(\frac{25}{48}\right)(1) + \left(\frac{13}{48}\right)(2) + \left(\frac{7}{48}\right)(3) + \left(\frac{1}{16}\right)(4) = \frac{7}{4}$$

$$E[X] = \sum_{x=1}^{4} x\, P_X(x) = \frac{1}{4}(1+2+3+4) = \frac{5}{2} = E[V]$$

b) $$E[X^2] = \sum_{x=1}^{4} x^2 P_X(x) = \frac{1}{4}(1^2+2^2+3^2+4^2) = \frac{15}{2}$$

$$E[Y^2] = \sum_{y=1}^{4} y^2 P_Y(y) = 1^2\left(\frac{25}{48}\right) + 2^2\left(\frac{13}{48}\right) + 3^2\left(\frac{7}{48}\right) + 4^2\left(\frac{1}{16}\right) = \frac{47}{12}$$

$$\text{var}[Y] = E[Y^2] - (E[Y])^2 = \frac{47}{12} - \left(\frac{7}{4}\right)^2 = \frac{41}{48} \Rightarrow \text{var}[W] = \frac{41}{48}$$

$$\text{var}[X] = E[X^2] - (E[X])^2 = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{5}{4} \Rightarrow \text{var}[V] = \frac{5}{4}$$

Page No.
Date : / /

c) $E[WV] = E[YX] = \sum_{y=1}^{x} \sum_{x=1}^{4} (xy)\, P_{x,y}(x,y)$

$= \frac{1}{4} + \frac{2}{8} + \frac{3}{12} + \frac{4}{16} + \frac{4}{8} + \frac{6}{12} + \frac{8}{16} + \frac{9}{12} + \frac{12}{16} + \frac{16}{16}$

$= 5 = E[WV]$

d) $Cov[W,V] = Cov[Y,X]$ or $Cov[X,Y]$

$Cov[X,Y] = E[XY] - E[X]E[Y] = 5 - \left(\frac{7}{4}\right)\left(\frac{5}{2}\right) = \frac{10}{16}$

e) correlation coeff $\rho_{W,V} = \rho_{Y,X} = \dfrac{Cov[W,V]}{\sqrt{var[W]\, var[W]}} = \dfrac{\frac{10}{16}}{\sqrt{\left(\frac{41}{48}\right)\left(\frac{5}{4}\right)}}$

$= \frac{10}{16} \frac{\sqrt{(12)(4)(4)}}{\sqrt{41}\sqrt{5}} = \frac{10}{4} \frac{\sqrt{3\times 4}}{\sqrt{41}\sqrt{5}} = 5\sqrt{\frac{3}{(41)(5)}}$

$= \sqrt{\left(\frac{15}{41}\right)} \approx \underline{0.605}$.

Q6. $f_{X,Y}(x,y) = \begin{cases} (4x + 2y)/3 & x \in [0,1],\ y \in [0,1] \\ 0 & \text{else} \end{cases}$ $\quad A = \{Y \leq \frac{1}{2}\}$

i) $P[A] = \iint_{y \leq \left(\frac{1}{2}\right)} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^{1/2} \left(\frac{4x + 2y}{3}\right) dy\, dx$

$= \frac{1}{3} \int_0^1 \int_0^{1/2} (4x\, dy + 2y\, dy)\, dx = \frac{1}{3} \int_0^1 \left(4xy + y^2 \Big]_0^{y = \frac{1}{2}}\right) dx$

$= \frac{1}{3} \int_0^1 \left(4x\left(\frac{1}{2}\right) + \frac{1}{4}\right) dx = \frac{1}{3} \int_0^1 \left(2x + \frac{1}{4}\right) dx$

$= \frac{1}{3}\left(x^2 + \frac{x}{4}\right)\Big]_0^1 = \frac{1}{3}\left(1 + \frac{1}{4}\right) = \frac{5}{4}\left(\frac{1}{3}\right) = \frac{5}{12}$

Page No.
Date : / /

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{(4x+2y)}{3} \dfrac{1}{\left(\frac{5}{12}\right)} & (x,y) \in A \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{8(2x+y)}{5} & 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

(cond^n marginal PDF)

For $x \in [0,1]$, $f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dy = \frac{8}{5}\int_0^{1/2} (2x+y)\,dy$

$$= \frac{8}{5}\left(2xy + \frac{y^2}{2}\right)\Big|_0^{1/2} = \frac{8}{5}\left(2x\left(\frac{1}{2}\right) + \frac{1}{8}\right) = \frac{8}{5}\left(x + \frac{1}{8}\right)$$

$$f_{X|A}(x) = \begin{cases} \dfrac{8x+1}{5} & x \in [0,1] \\ 0 & \text{else} \end{cases}$$

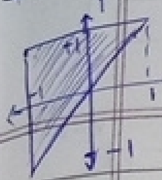
For $y \in [0, \frac{1}{2}]$, $f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dx = \frac{8}{5}\int_0^1 (2x+y)\,dx$

$$= \frac{8x^2 + 8xy}{5}\Big]_{x=0}^{x=1} = \frac{8(y+1)}{5}$$

$$f_{Y|A}(y) = \begin{cases} \dfrac{8(y+1)}{5} & y \in [0, \frac{1}{2}] \\ 0 & \text{else} \end{cases}$$

Page No.
Date : / /

Q7 $f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & -1 \leqslant x \leqslant y \leqslant 1 \\ 0 & \text{else} \end{cases}$



a) for $y \in [-1, 1]$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_{-1}^{y} \left(\frac{1}{2}\right) dx = \left(\frac{y+1}{2}\right).$$

$$f_Y(y) = \begin{cases} \frac{(y+1)}{2} & y \in [1,1] \\ 0 & \text{else} \end{cases}$$

b) $f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{\left(\frac{1}{2}\right)}{\left(\frac{y+1}{2}\right)} & -1 \leq x \leq y \\ 0 & \text{else} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \left(\frac{1}{1+y}\right), & -1 \leq x \leq y \\ 0 & \text{, else} \end{cases}$$

c) $E[X|Y=y]$ is uniform cond^n RV where $x \in [-1, y]$

&

$$E[X|Y=y] = \frac{y+(-1)}{2} = \frac{y-1}{2}.$$

Q8. $P_{M|N}(m|n)$ → cond^n prob. that there ~~hv been~~ are m calls up to and including the 1st voice call (M) given that n calls hv been observed upto & including 2nd voice call (N).

for $n > m$, $P[M=m, N=n] = \underbrace{(1-p)^{m-1} \cdot p}_{\text{for } (m-1) \text{ calls}} \times \underbrace{(1-p)^{n-m-1} \cdot p}_{\text{for } (n-m-1) \text{ calls}}$

$$= (p^2)\ (1-p)^{\cancel{m}-1+n-\cancel{m}-1} = \underline{\underline{p^2 (1-p)^{n-2}}}$$

$$P_{M,N}(m,n) = P[M=m, N=n] = \begin{cases} p^2 (1-p)^{n-2} & m \in [1, n-1];\ n = m+1, m+2 \ldots \\ 0 & \text{else} \end{cases}$$

$$P_N(n) = \sum_{m=1}^{n-1} \underbrace{(1-p)^{n-2}p^2}_{\text{const.}} \qquad \left[\because\ p_N(n) = \sum_{n \in S_N} P_{M,N}(m,n) \right.$$
$$\left. \text{(marginal PMF of N)} \right]$$

Page No.
Date: / /

$$\Rightarrow P_N(n) = (1-p)^{n-2} p^2 \left(\sum_{m=1}^{n-1} (1) \right) = \underline{(n-1)\, p^2\, (1-p)^{n-2}}$$

$$\therefore P_N(n) = \begin{cases} (n-1)\, p^2\, (1-p)^{n-2} & n=2,3,\dots \\ 0 & \text{else} \end{cases}$$

$$P_M(m) = \sum_{m \in S_M} P_{M,N}(m,n) \rightarrow P_M(m) = \sum_{n=m+1}^{\infty} (1-p)^{n-2} \underbrace{p^2}_{\text{const}}$$

$$= p^2 \left(\sum_{n=m+1}^{\infty} (1-p)^{n-2} \right) \qquad \left(S = \frac{a}{r-1}\right)$$

$$= p^2 \left[\underbrace{(1-p)^{m-1} + (1-p)^m + \dots + (1-p)^{\infty}}_{\infty\ GP} \right] = p^2 \frac{(1-p)^{m-1}}{(1-(1-p))}$$

$$= \frac{p^2 (1-p)^{m-1}}{p} = \underline{p(1-p)^{m-1}}$$

$$\therefore P_M(m) = \begin{cases} p(1-p)^{m-1} & m=1,2,\dots \\ 0 & \text{else} \end{cases}$$

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-2} p^2 \left(\frac{1}{p(1-p)^{m-1}} \right) & n=m+1, m+2, \dots \\ 0 & \text{else} \end{cases}$$

$$P_{N|M}(n|m) = \begin{cases} (1-p)^{n-m-1}\, p & n=m+1, m+2, \dots \\ 0 & \text{else} \end{cases}$$

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} \frac{\cancel{(1-p)^{n-2}p^2}}{(n-1)\cancel{(1-p)^{n-2}p^2}} & m=1,\dots,n-1 \\ 0 & \text{else} \end{cases}$$

Page No.
Date : / /

$$P_{M|N}(m|n) = \begin{cases} \frac{1}{n-1} & m=1,\dots,n-1 \\ 0 & \text{else} \end{cases}$$

Q. X_1 & X_2 are iid RVs w/ PDF $f_X(x) = \begin{cases} x/2 & x \in [0,2] \\ 0 & \text{else} \end{cases}$

a) X_1 & X_2 will hv same CDF as X. ($\because$ they are iid)

$$F_X(x) = \int_0^x f_X(u)\,du = \begin{cases} 0 & , x < 0 \\ x^2/4 & , x \in [0,2] \\ 1 & , x > 2 \end{cases}$$

b) $\because X_1$ & X_2 are independent. $\left(\begin{array}{l} P[X_1 \le x_1, X_2 \le x_2] \\ = P[X_1 \le x_1] \cdot P[X_2 \le x_2] \end{array} \right)$

$$\Rightarrow P[X_1 \le 1] \cdot P[X_2 \le 1] = F_{X_1}(1) F_{X_2}(1) = [F_X(1)]^2 = \frac{1}{16}$$

c) $W = \max(X_1, X_2) \Rightarrow F_W(1) = P[\max(X_1, X_2) \le 1]$

$$= P[X_1 \le 1, X_2 \le 1] = \frac{1}{16}$$ obtained in b)

d) $F_W(w) = P[W \le w] = P[\max(X_1, X_2) \le w] = P[X_1 \le w, X_2 \le w]$

$$F_W(w) = P[X_1 \le w] \cdot P[X_2 \le w] = F_X(w)^2 = \begin{cases} 0 & , w < 0 \\ w^4/16 & , w \in [0,2] \\ 1 & , w > 2 \end{cases}$$

(X_1, X_2 r independent)

Q10. $f_X(x) = \begin{cases} 2x & , 0 \le x \le 1 \\ 0 & , \text{else} \end{cases}$ $\quad f_Y(y) = \begin{cases} 3y^2 & , 0 \le y \le 1 \\ 0 & , \text{else} \end{cases}$

Page No.

Date : / /

$A = \{X > Y\}$ (X & Y r independent RVs)

E[Y|A] =

a) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = \frac{2x^3}{3}\Big]_0^1 = \frac{2}{3}$

$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 3y^3\, dy = \frac{3y^4}{4}\Big]_0^1 = \frac{3}{4}$

b) $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ ($\because$ X & Y r independent)

$\Rightarrow f_{X,Y}(x,y) = \begin{cases} 6xy^2 & x \in [0,1], y \in [0,1] \\ 0 & \text{else} \end{cases}$

$P[A] = \iint_{x>y} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^x (6xy^2)\, dy\, dx = \int_0^1 2x^4\, dx = \frac{2x^5}{5}\Big]_0^1$

$A = \{X > Y\}$

$= \frac{2}{5}$

$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{else} \end{cases}$

$= \begin{cases} 6xy^2 \left(\frac{5}{2}\right) & (x,y) \in A \\ 0 & \text{else} \end{cases} = \begin{cases} 15xy^2 & 0 \le y \le x \le 1 \\ 0 & \text{else} \end{cases}$

$E[X|A] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y|A}(x,y)\, dy\, dx$

$= 15 \int_0^1 x^2 \left(\int_0^x y^2\, dy \right) dx = 15 \int_0^1 x^2 \left(\frac{x^3}{3} \right) dx = 5 \int_0^1 x^5\, dx$

$= \frac{5x^6}{6}\Big]_0^1 = \frac{5}{6}$

Page No.

Date : / /

$$E[Y|A] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y|A}(x,y)\, dy\, dx$$

$$= 15\int_0^1 x\int_0^x y^3\, dy\, dx = \frac{15}{4}\int_0^1 x^5\, dx$$

$$= \frac{15}{4}\left(\frac{x^6}{6}\right)\Big]_0^1$$

$$= \frac{15}{4}\left(\frac{1}{6}\right) = \frac{5}{8}$$