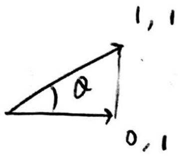
The following pages have the mathematical derivation and intuition.

I have also attached a Jupyter Notebook and image, which shows the use of the derived matrix.

The desired x-axis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The current x-axis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



rotate /change of basis

$$\tan \theta = 1$$
$$\theta = 45^\circ.$$

1) Rotate desired to current axis by $-\theta$

$$R_{gen} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

2) Scale in the new x-axis direction

$$S_{gen} = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$$

we only want x-axis scaling, so $\sigma_y == 1$

$$S = \begin{bmatrix} \sigma_{diag} & 0 \\ 0 & 1 \end{bmatrix}$$

3) Rotate back to original axis

$$\underbrace{R^{-1} = R^{T}}_{\text{Special property of rotation matrices}} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

4) Compose all 3 matrices together

$$\text{transformed } \vec{v} = \underbrace{R^{-1} \cdot S \cdot R}_{C} \cdot \vec{v}$$

$$C = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} \sigma_{diag} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0.707\sigma_{diag} & 0.707\sigma_{diag} \\ -0.707 & 0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\sigma_{diag} + 0.5 & 0.5\sigma_{diag} - 0.5 \\ 0.5\sigma_{diag} - 0.5 & 0.5\sigma_{diag} + 0.5 \end{bmatrix}$$