Q3.

$f(n)$ is in $O(n)$: $\exists\, c_f$ st. $f(n) \leq c_f \cdot n$ for $n \geq N_f$

$g(n)$ is in $O(n)$: $\exists\, c_g$ st. $g(n) \leq c_g \cdot n$ for $n \geq N_g$

$f(g(n))$ is in $O(n)$: $\exists\, c_t$ st. $f(g(n)) \leq c_t \cdot n$ for $n \geq N_t$

$c_t \leq c_f \cdot c_g$

$N_t = \max\left\{N_g, \frac{b}{c_t}\right\}$

① if $g(n) > N_f$

then $f(g(n)) \leq c_f \cdot g(n)$

$n > N_g$

then $g(n) \leq c_g \cdot n$

$f(g(n)) \leq c_f \cdot g(n)$

$\leq \underbrace{c_f \cdot c_g}_{c_t}\, n$

② if $g(n) \leq N_f$

$g(n) \in \{0, 1, 2, \cdots, N_f\}$

$f(g(n)) \in \{f(0), f(1), \ldots, f(N_f)\}$

Let $b = \max\{f(0), f(1), \cdots, f(N_f)\}$

$f(g(n)) \leq b$

want $f(g(n)) \leq c_t \cdot n$

if $b \leq c_t \cdot n \Rightarrow f(g(n)) \leq b \leq c_t \cdot n$

$n \geq \frac{b}{c_t}$

③

$c_t = c_f \cdot c_g$

$N_t = \max\left\{N_g, \frac{b}{c_t}\right\}$

for $n \geq N_t$

if $g(n) > N_f$

then $f(g(n)) \leq c_f g(n) \leq c_f c_g n = c_t \cdot n$

if $g(n) \leq N_f$

$f(g(n)) \leq b = \frac{b}{c_t} \cdot c_t \leq c_t \cdot n$