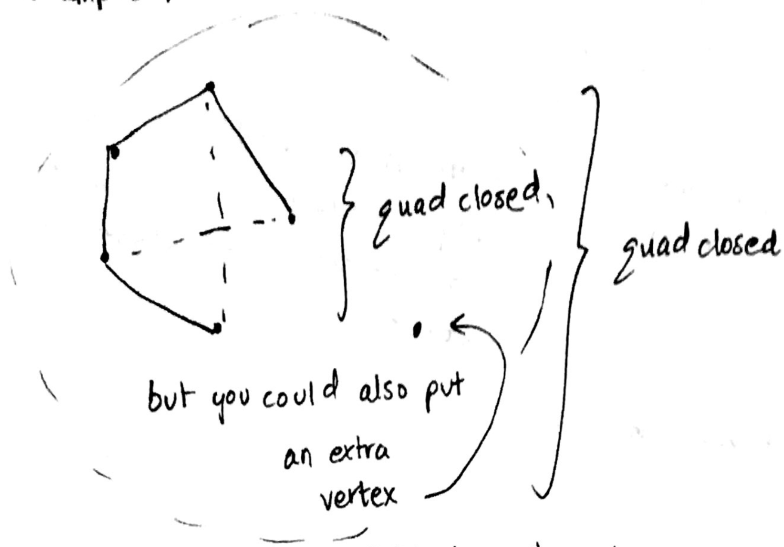
Q1.1

an example :



and all of this is a ~~subgraph~~ supergraph

Supergraphs :

set A such that $\forall x \in A$, x is a quad. supergraph of G.

Now we want to find the smallest graph in this set A and show its a unique closure.

1. show not empty
2. find the "smallest" element of A
   ↳ show that its a subgraph of all graphs in A.

Take the intersection of all the graphs in A and then show it is also quadangle closed ~~supergraph of G~~. i.e is in set A

e.g: $A \cap B$ gives a graph C with all its vertices and edges in A and B.

~~eg~~. $A = \{V_1, E_1\}$ $\quad B = \{V_2, E_2\}$

$$A \cap B = \{V_1 \cap V_2, E_1 \cap E_2\}$$

1. show not empty:

in order to show something exists, come up with one example that works with any graph.

given a G, the quad supergraph of G :

For any $a_0 a_1 a_2$, if they are connected, then $a_0 a_3$ also exists.

- $\boxed{\text{you could connect every vertex to each other}}$ ← the algorithm
  to create ~~quad supergraph~~ a quad supergraph
  (i.e adding vertices & edges)

- So ~~this~~ the set ~~is not empty~~ A of all quadrangle supergraphs is not empty.

even if you had two vertices, they wouldn't be relevant (b/c no $a_0 a_1 a_2 a_3$) to the set so the algorithm still works

2. find the smallest element of A.

consider intersection of all graphs in A.

$x_i \in A$

$H = x_1 \cap x_2 \cap \dots \cap x_n$

we ~~know~~ this a supergraph of G b/c it is an intersection of supergraphs. $G \subseteq H$

Now we show it is quadrangle closed:

Let's say H has some simple paths:

Let $a_0 a_1 a_2 a_3$ be a simple path in H. This simple path must exist in all other graphs x in A (b/c H is an intersection)

want to show: $a_0 a_3$ is also in H.

We know that set A has elements, each of which is quadrangle closed

which is why

so

we know $a_0 a_1 a_2 a_3$ is in every $x \in A$ because of how H was formed

So $a_0 a_3$ in H

and we know $H \subseteq X \subseteq A$.

Uniqueness: proof by contradiction

Say there is another quad closure of G called H'.

So $H' \subseteq X \in A$

This implies $H' \subseteq H$

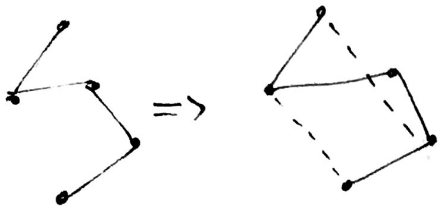
But b/c H' is quad closed, $H' \in A$. This means $H \underset{\text{subgraph}}{\subseteq} H'$. So $H = H'$ which proves uniqueness

Q1.2

We can take any graph G and make it quad closed.

in Q1.1 our algorithm was connecting all the points/vertices.

But there is another algorithm/method:



with this worst case scenario

for any simple path $a_0 a_1 a_2 a_3$, make $a_0 a_3$ if it does not exist already. This is a finite no. of moves.

want to show: if $\underbrace{G \text{ is bipartite}}_{\text{assume this}}$, the quadrangle closure of G is also bipartite

then show that every time you add another edge to it (to make a quad closure), its still bipartite

$$\underbrace{G \to G' \to G'' \to \cdots}_{\text{a sequence}} G^{n'}$$

So proof by induction:

$$G^n \to G^{n+1}$$

if G^n is bipartite then so is G^{n+1}

Let $a_0 a_1 a_2 a_3$ be a simple path in G^n

where $a_0 a_3 \notin E$ ← edges of

without loss of generality a_0 must be in either S or T.

if a_0 is in S, a_1 must be in T, $a_2 \in S$, $a_3 \in T$.

So $a_0 a_3$ will not change anything since both points are separated

$a_0 \in S$ $a_3 \in T$

so connecting them does not affect anything.

so the base case is just the existing G, which is bipartite, and we keep adding edges to it.

also note that with each iteration

$$G^{n} \subseteq x \in A$$

this is a subgraph of everything/element in set A

we already know every $x \in A$ is a supergraph of G.

easy inductive proof:

if $G^{n} \subseteq x \in A \longrightarrow G^{n+1} \subseteq x \in A$

actually ...

Look at any G^{i}. It contains $a_0 a_1 a_2 a_3$.

~~any~~ if it exists in G^{i}, it must also exist in G.

So the simple

And every element of A is closed

proof by induction:

$$G^{i} \subseteq x \in A \longrightarrow G^{i+1} \subseteq x \in A.$$

G^{i} contains some simple edge $a_0 a_1 a_2 a_3$ without $a_0 a_3$, which is why we are making G^{i+1}.

Because $G^{i} \subseteq x \in A$, ~~this edge exists for every x~~ $\in A$.

Every x is subgraph quad closed in A. ($\forall x \in A$, x contains a_0, a_3)

all graphs in A contain all of the edges in G^{i} i.e $a_0 a_1 a_2 a_3$.

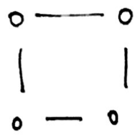
But b/c they are all quad closed, they must also contain $a_0 a_3$.

So now if we add $a_0 a_3$ to get G^{i+1}, this won't change anything so

$G^{i+1} \subseteq x \in A$ and the proof is complete.

Q2. simple

A cycle is a graph where every vertex & edge is distinct.



is a simple cycle

is a regular cycle b/c we can go from all vertices to another without a repeated edge.

First step: show G is a cycle.

Can we find a cycle in G ?

1. $\exists$ a cycle $\subseteq G$

2. Show that this cycle $= G$

If we know $A \subseteq B$, to show $A = B$,

we need to show $A \supseteq B$.

so we want to show cycle $\supseteq G$

- it contains all its edges & vertices

1. To show $\exists$ a cycle :

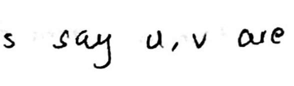
~~Let's look at~~ There is EXACTLY two distinct simple paths from any vertex to another / between any 2 vertices

between u, v are 2 distinct simple path

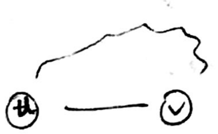
they don't have the same edges.

Let's say u, v are neighbors : $\exists\, u,v \in E_G$

edges of G.



so there must be another simple path connecting them, which means there is a cycle



} a cycle C

2. Show its a supergraph of G.

Let the vertices of C be $a_0 a_1 \text{---} a_n$ where $a_i a_{i+1}$ is an edge and $a_n a_0$ is also an edge

i.e. $V_c = \{a_0, a_1, \dots a_n\}$

$a_i a_{i+1} \in E$

$a_n a_0 \in E$

...rtex in C

Ca

Now we want to show any vertex in G must be a vertex in cycle :

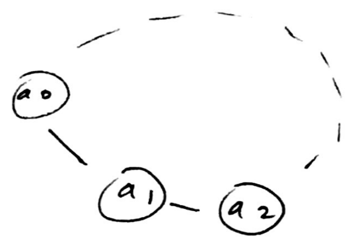
Contradiction :

~~Assume $\exists V \neq a_0 a_1 \dots a_n$.~~

assume $\exists V$ that is not one of the vertices $a_0, a_1, \dots a_n$.
exists a vertex

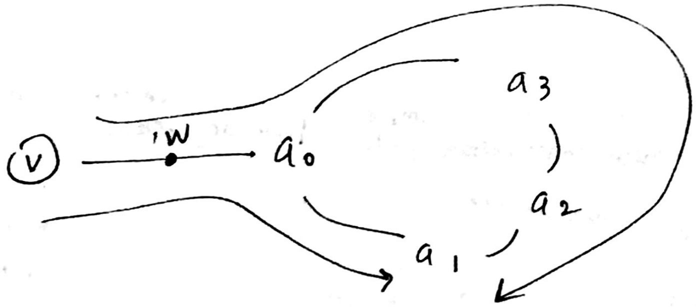
i.e $\exists V \notin V_c$ ← the set of

case 1)



Lets say there is a path from a (V) to (a_1)

you could have a path going directly to (a_1) or other wise will go into the cycle. within the cycle it can go both ways.



but this is a contradiction b/c ~~the path into the cycle is repeated twice~~ there are two simple paths which contain the same edge

So V ~~is~~ cannot exist outside the cycle

so the cycle must contain all its edges and vertices.

$\exists V \in V_G$ but $\notin V_c$

Again : $\exists$ simple path from V to a_i

Let w be the last vertex in the path outside of the cycle

Let a_i be the first vertex in the cycle of the graph

so w and a_i must be connected and $w, a_i \in E_G$

So $w, a_i, a_{i-1} \dots a_1$ is one simple path
$w, a_i, a_{i+1} \dots a_n a_0 a_i$ is another simple path } contradiction

So far we have proven every vertex in G is a vertex in C

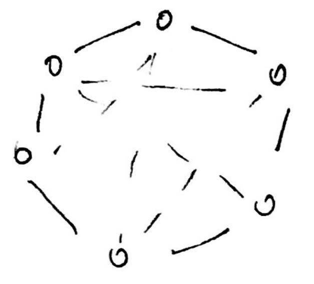
Case 2) now we need to show every edge in G is an edge in C.

Now we can rename all the vertices in G

$a_0, a_1, \ldots, a_n$.

$a_i, a_{i+1} \in E_C$

$a_n, a_0 \in E_C$



we want to prove the dotted edges do not exist.

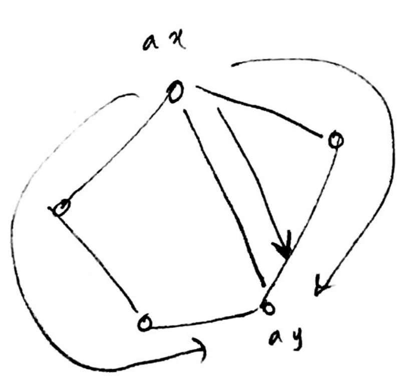
Again, proof by contradiction:

Assume $\exists$ edge $a_x a_y \in E_G$

such that $a_x a_y \notin E_C$ (they are not neighbors)

So if $a_x a_y$ is an edge, it is a simple path from a_x to a_y.

But there are two other simple paths because of this



So now we have 3 paths and a contradiction

$a_x, a_{x+1} \ldots a_y$ is a path

$a_x, a_{x-1} \ldots a_{y+1}\, a_y$ is a path

END of question?