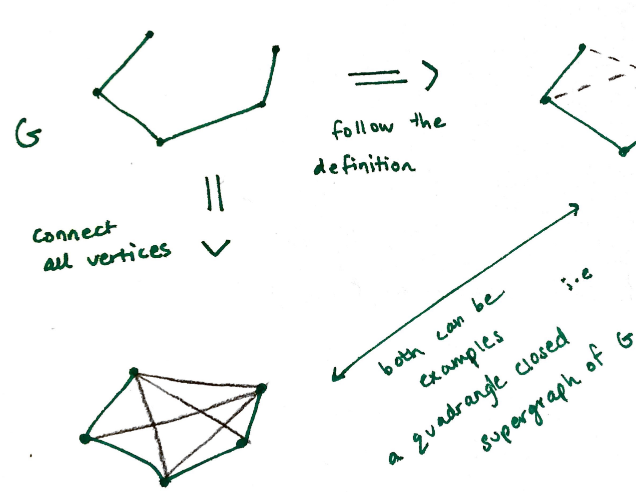
CS 202, PSET 9

9.1.1

Let A be the set of all quadrangle closed supergraphs of G. We want to find the smallest graph in this set and show it's a unique closure.

The first thing we do is show that set A is non-empty:
A simple proof of this is that, given any graph, we can use this method: connect all the vertices with edges. Such a method ensures the graph becomes quadrangle closed. For example:

Fig 9.1.1



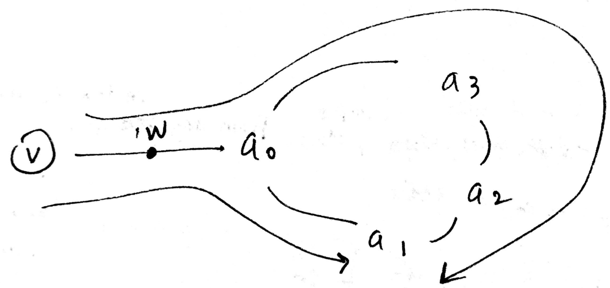

And also note that if the graph does not fulfill the criteria for having a quad. closed supergraph (i.e. does not have $a_0a_1a_2a_3$), then it is not considered. Hence we have proved that the set A is not empty.

Now we find the smallest element of A. Let $x_i \in A$. Consider the intersection of all the graphs in A:
$H = x_1 \cap x_2 \cap x_3 ... \cap x_n$
$H$ is a supergraph of G because it is an intersection of supergraphs.
$H$ is quadrangle closed:
$H$ contains the path $a_0a_1a_2a_3$, as do all the elements of $A$
So $a_0a_3$ is also in $H$
And $H \subseteq x \in A$

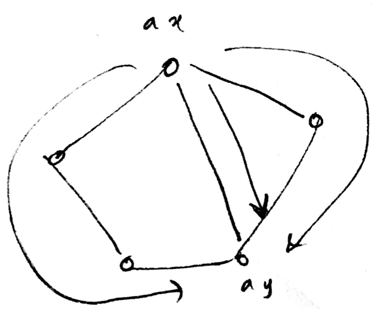
Now we show that $H$ is unique:
Say another quad. closed supergraph of $G$ exists, called $H'$
So $H' \subseteq x \in A$
This implies $H' \subseteq H$
But because $H$ is quad. closed, $H' \in A$
This implies: $H' \subseteq H$
Therefore $H = H'$ and is unique.

9.1.2
Want to show if $G$ is bipartite, then the quadrangle closure of $G$ is also bipartite.
Proof by induction:
Let $n$ represent the next edge added to graph $G$
Base case: the starting graph $G^0 = G$ is bipartite
Inductive hypothesis: if $G^n$ is bipartite $\to G^{n+1}$ is bipartite
Let $a_0a_1a_2a_3$ be a path in $G^n$
Without loss of generality, we can start with saying $a_0 \in S$. To maintain a bipartite property, $a_1 \in T$, $a_2 \in S$ and $a_3 \in T$
Now the edge $a_0a_3$ maintains the bipartite property because the two vertices are separated into $S$ and $T$
$\to G^{n+1}$

9.2
There $\exists$ cycle $C$ such that $C \subseteq G$:
We know that there are exactly two distinct simple paths that don't share edges, between any 2 vertices
Let's say $\exists u, v \in E_G$, there must be another simple path connecting them, which means there is a cycle.
Now we show that $C \supseteq G$
Let $V_c$ be the vertices of $C = \{a_0, a_1, a_2, ...a_n\}$ where $a_0a_n$ and $a_ia_{i+1}$ are edges
Proof:
1) First we show that any vertex in $G$ must be a vertex in $C$
Suppose $\exists v : v \in V_G \wedge v \notin V_C$
Then if there exists a simple path from $v$ to $a_i$, then there must exist another simple path, which contains the same edge. This leads to a contradiction.




2) Now we show every edge in $G$ is an edge in $C$
Suppose $\exists$ edge $a_x a_y \in E_G \wedge \notin E_C$
But any simple path from $a_x$ to $a_y$ would also imply the existence of other simple paths, which leads to a contradiction




So now we have 3 paths and a contradiction

$a_x, a_{x+1} \dots a_y$ is a path

$a_x, a_{x-1} \dots a_{y+1}\, a_y$ is a path

As a result of these two cases, $G = C$ and is a cycle.

9.3