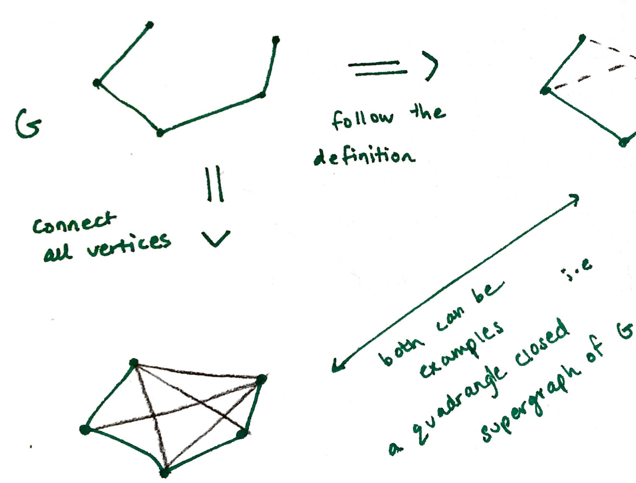
CS 202, PSET 9

9.1.1

Let A be the set of all quadrangle closed supergraphs of G. We want to find the smallest graph in this set and show it's a unique closure.

The first thing we do is show that set A is non-empty:
A simple proof of this is that, given any graph, we can use this method: connect all the vertices with edges. Such a method ensures the graph becomes quadrangle closed. For example:

Fig 9.1.1



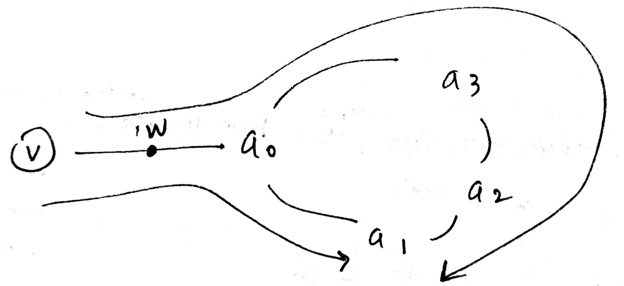
And also note that if the graph does not fulfill the criteria for having a quad. closed supergraph (i.e. does not have $a_0a_1a_2a_3$), then it is not considered. Hence we have proved that the set A is not empty.

Now we find the smallest element of A. Let $x_i \in A$. Consider the intersection of all the graphs in A:
$H = x_1 \cap x_2 \cap x_3 ... \cap x_n$
H is a supergraph of G because it is an intersection of supergraphs.
H is quadrangle closed:
H contains the path $a_0a_1a_2a_3$, as do all the elements of A
So a_0a_3 is also in H
And $H \subseteq x \in A$

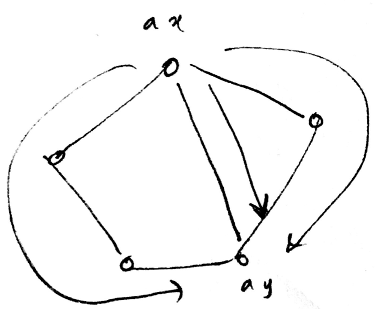
Now we show that H is unique:
Say another quad. closed supergraph of G exists, called H'
So $H' \subseteq x \in A$
This implies $H' \subseteq H$
But because H is quad. closed, $H' \in A$
This implies: $H' \subseteq H$
Therefore $H = H'$ and is unique.

9.1.2
Want to show if G is bipartite, then the quadrangle closure of G is also bipartite.
Proof by induction:
Let n represent the next edge added to graph G
Base case: the starting graph $G^0 = G$ is bipartite
Inductive hypothesis: if G^n is bipartite $\rightarrow G^{n+1}$ is bipartite
Let $a_0a_1a_2a_3$ be a path in G^n
Without loss of generality, we can start with saying $a_0 \in S$. To maintain a bipartite property, $a_1 \in T$, $a_2 \in S$ and $a_3 \in T$
Now the edge a_0a_3 maintains the bipartite property because the two vertices are separated into S and T
$\rightarrow G^{n+1}$

9.2
There $\exists$ cycle C such that $C \subseteq G$:
We know that, for any 2 vertices, there are exactly two distinct simple paths that don't share edges between them
Let's say $\exists u, v \in E_G$, there must be another simple path connecting them, which means there is a cycle.
Now we show that $C \supseteq G$
Let V_c be the vertices of $C = \{a_0, a_1, a_2, ...a_n\}$ where a_0a_n and a_ia_{i+1} are edges
Proof:
1) First we show that any vertex in G must be a vertex in C
Suppose $\exists v : v \in V_G \wedge v \notin V_C$
Then if there exists a simple path from v to a_i, then there must exist another simple path, which contains the same edge. This leads to a contradiction.



2) Now we show every edge in G is an edge in C
Suppose $\exists$ edge $a_x a_y \in E_G \wedge \notin E_C$
But any simple path from a_x to a_y would also imply the existence of other simple paths, which leads to a contradiction



So now we have 3 paths and a contradiction

$a_x, a_{x+1} \dots a_y$ is a path

$a_x, a_{x-1} \dots a_{y+1}\, a_y$ is a path

As a result of these two cases, $G = C$ and is a cycle.

9.3
To show we can delete the graph if and only if it is acyclic, we consider two sides of the iff
1) We cannot delete a graph if it is cyclic:
A cyclic graph either is a cycle or contains a cycle
If it contains a cycle, we can remove all vertices not in the cycle by the described process to be left with the cycle
In the cycle: $\forall v \in V_c : d(v) \geq 2$ as per the definition of a cycle, so we cannot remove them.
2) If the graph is acyclic, we can delete it:
Showing by induction on $|V|$
Base case:

$|V| = 0$ is the empty graph
$|V| = 1$ is a single vertex, $d(v) = 1$ and can be removed.
Inductive step:
Assume $|V| = n$, we can completely delete this graph
Because if the graph with $|V| = n + 1$ is acyclic, $\exists v : d(v) \leq 1$ and so we can delete at least one vertex and get back to our original graph
As per the notes, acyclic graphs also have the property that $|E| \leq |V| - 1$, so if we were to think that $d(v) \geq 2$, then by the handshake lemma we would arrive at a contradiction to this property:
$2|E| = \sum_{v \in V} d(v) \geq \sum_{v \in V} 2 = 2|V|$
$2|E| \geq 2|V|$
$E \geq |V|$
Having shown both cases, a graph can be reduced to the empty graph iff it is acyclic.