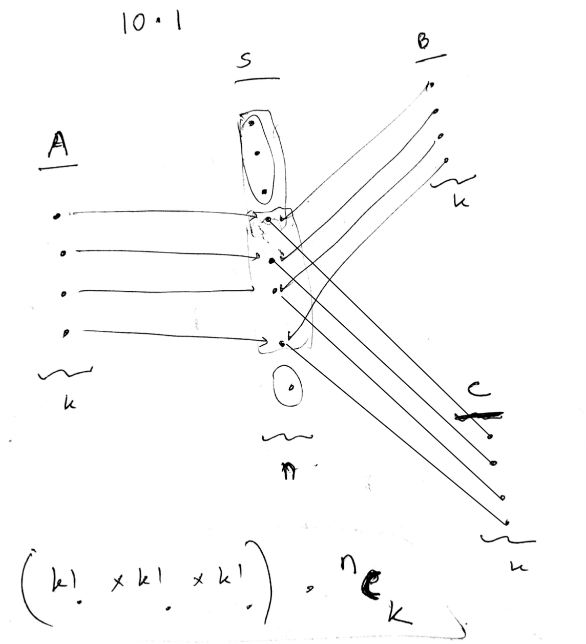
CS 202, PSET 10

10.1
We have three sets of k-elements, A, B and C that map to an n-element set S. Let's collect all the possible injective $f : A \to S$ inside F, all $g : B \to S$ inside G, and all $h : C \to S$ inside H. Since these functions are injective, the size of the subset of S that they map to is k, and they share a co-domain.
How many different functions from k elements can map to a set of k elements?
$k!$ functions
We also know that $f(A) = g(B) = h(C)$, so there are $k! * k! * k! = (k!)^3$ possibilities.
And there are nCk many subsets of S that these functions can map to.
In total we have: $nCk * (k!)^3$



10.2
Claim: $\binom{n}{m} * \binom{m}{k} = \binom{n}{k} * \binom{n-k}{m-k}$
I'm using these definitions:
$\binom{n}{k} = \frac{(n)_k}{k!}$
$(n)_k = \frac{n!}{(n-k)!}$
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
So now, expanding the claim:
$\frac{(n)_m}{m!} * \frac{(m)_k}{k!} = \frac{(n)_k}{k!} * \frac{(n-k)_{m-k}}{(m-k)!}$
$\frac{n!}{m!(n-m)!} * \frac{m!}{k!(m-k)!} = \frac{n!}{k!(n-k)!} * \frac{(n-k)!}{(m-k)!(n-k-(m-k))!}$

$\frac{n!}{(n-m)!k!(m-k)!} = \frac{n!}{k!(m-k)!(n-k-m+k)!}$
$\frac{n!}{(n-m)!k!(m-k)!} = \frac{n!}{(n-m)!k!(m-k)!}$
These two sides are equal, so we have shown the claim.

10.3
Letters: $\{a, b, c\}$
Numbers: $\{0, 1, 2, 3\}$
Restriction: Numbers after letters, no mixing
A generating function to generate letter strings of length k:
$f(x) = 3^0x^0 + 3^1x^1 + 3^2x^2 + ...3^kx^k$
Reasoning: empty string for length $k = 0$, 3 choices for $k = 1$, 3^2 choices for $k = 2$ and so on.
A generating function to generate number strings of length p:
$g(x) = 4^0x^0 + 4^1x^1 + 4^2x^2 + ...4^px^p$
Reasoning: same as for letters
A generating function to generate letter$\rightarrow$number strings of length n, where $n = k + p$:
$f(x)g(x) = ...$
Tangent: test this function for n = 2. We need to collect all terms that give x^2:
$(1 + 3x + 9x^2...)(1 + 4x + 16x^2...)$
$1 + 3x + 4x + 9x^2 + 12x^2 + 16x^2 + 36x^3 + 48x^3 + ...$
And the x^2 term has a combined coefficient of 37, which checks out.
So $f(x)g(x)$ works, and we can convert to sums:
$f(x) = \frac{1}{1-3x}$
Reasoning: Common ratio is $3x$, starting number is 1
$g(x) = \frac{1}{1-4x}$
Reasoning: same as for $f(x)$
$f(x)g(x) = \frac{1}{(1-3x)(1-4x)}$
Partial fraction decomposition:
$\frac{a}{1-3x} + \frac{b}{1-4x}$
$\frac{a(1-4x)+b(1-3x)}{(1-3x)(1-4x)}$
Find out unknowns a and b:
$a(1 - 4x) + b(1 - 3x) = 1$
$a - 4ax + b - 3bx = 1 + 0x$
Equation 1: $a + b = 1$
Equation 2: $-4a - 3b = 0$
Solve simultaneously:
$a = \frac{-3b}{4}$
$\frac{-3b}{4} + b = 1$
$b = 4$
$a = -3$
So now our partial fractions are:
$f(x)g(x) = \frac{-3}{1-3x} + \frac{4}{1-4x}$
Rewrite as a long-form geometric sum:
$f(x)g(x) = [-3(3x)^0 + -3(3x)^1 + ... - 3(3x)^n] + [4(4x)^0 + 4(4x)^1 + ...4(4x)^n]$

The n'th term is: $-3(3x)^n + 4(4x)^n$
Expand: $(-3)(3)^n x^n + (4)(4)^n x^n$
$(4^{n+1} - 3^{n+1})x^n$
Answer: $4^{n+1} - 3^{n+1}$
Phew. Test if this works, e.g. $n = 2$
$4^3 - 3^3 = 37$ which works