# Chapter 1

# Introduction

Mixed notes from the book Convex Optimization [1]. Some parts will be supplemented by the book of Dattoro [2]. The book of Dattoro is extremely useful for concretely illustrating most of the concepts.

## Quotes

- "We study convex geometry because it is the easiest of geometries. For that reason, much of a practitioners energy is expended seeking invertible transformation of problematic sets to convex ones". Dattoro [2].

# Chapter 2

# Convex Sets

## 2.1 Affine and convex sets and cones.

- **Affine set.** A set $C \subseteq \mathbf{R}^n$ is *affine* if the line through any two points in $C$ lines in $C$, *i.e.* if $x_1, x_2 \in C$ and $\theta \in \mathbf{R}$ implies $\theta x_1 + (1-\theta)x_2 \in C$. More generally, an affine set is a set that contains all *affine combination* (see def. below) of two or more of its points.
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    * Any affine set is convex [2].
    * The intersection of an arbitrary collection of affine sets remains affine [2].
    * Any affine set is open in the sense that it contains no boundary, *e.g.* the empty set $\emptyset$, point, line, plane, hyperplane, subspace etc [2]. Converse not necessarily true (*e.g.* see point just below about subspace.)
    * If $C$ is an affine set and $x_0 \in C$, then the set $V = C - x_o = \{x - x_0 | x \in C\}$ is a subspace.
  - *Affine combination.* A combination of points $\sum_{i=1}^k \theta_i x_i$ where $\sum_{i=1}^k \theta_i = 1$ is an affine combination.
  - *Ambient space.* The space where a given set lives in, *e.g.* a plane can live in $\mathbf{R}^2, \mathbf{R}^3$. The choice of ambient space has implications on, for example, the interior of a set ([2], p34)
  - *Affine hull*, denoted $\mathbf{aff}\, C$ is the smallest set that makes $C$ affine.
  - *Affine dimension* of a set $C$ is the dimension of $\mathbf{aff}\, C$. In fact dimension of a set is synonymous with affine dimension [2].
  - *Relative interior.* The interior of, for example, a plane in $\mathbf{R}^3$ is empty. To "fix" this issue, we define the relative interior of $C$ as: $\mathbf{relint}\, C = \{x \in C | B(x,r) \cup \mathbf{aff}\, C \subseteq C \text{ for some } r > 0\}$
- **Convex sets**. A set $C$ is convex if the line segment between any two points in $C$ lies in $C$, *i.e.* $\theta x_1 + (1-\theta)x_2 \in C$ for any $x_1, x_2 \in C$ and $0 \le \theta \le 1$.
  - *Convex combination.* A combination of points $\sum_{i=1}^k \theta_i x_i$ where $\sum_{i=1}^k \theta_i = 1$ and $\theta_i \ge 0$ is a convex combination.
  - *Convex hull* $\mathbf{conv}\, C$ of a set $C$ is the smallest set that makes $C$ convex.
- **Cones**. A set $C$ is called a cone if for every $x \in C$ and $\theta \ge 0$ we have $\theta x \in C$.
  - *Convex cone* is a set that is cone and also convex, *i.e.* $\theta_1 x_1 + \theta_2 x_2 \in C$ for any $x_1, x_2 \in C$ and for $\theta_1, \theta_2 \ge 0$. Some differences between a convex set and a convex cone: (i) A convex set doesn't have to include the origin, a convex cone does; (ii) a convex set can be bounded but a convex cone cannot.

Some important examples and notes ([1] p27):

- Any subspace is affine and a convex cone
- A line segment is convex but not affine
- A ray (*i.e.* $\{\theta v + x_0 : x \ge 0\}$) is convex but not affine. It is convex cone if its base $x_0$ is 0.
- Any line is affine.
- The empty set, any single point and the whole space are affine (hence convex) subsets of $\mathbf{R}^n$
- Halfspaces (see below) are convex but not affine.

## 2.2 Hyperplanes and halfspaces

- **Hyperplane** is a set of the form
$$\{x|a^T x = b\}$$
This set has several intuitive interpretations.
  1. It is the hyperplane with a normal vector $a$ and an offset $b$ from the origin.
  2. Let $b$ be $a^T x = b$. Then, $\{x|a^T x = b\} = \{x|a^T (x - x_0)\} = x_0 + a^{\perp}$ where $a^{\perp}$ is the orthogonal complement of $a$.
  3. More interpretations on p27-28.

- **Halfspace**. Each hyperplane divides $\mathbf{R}^n$ into two halfspaces. A (closed) halfspace is of the form
$$\{x|a^T x \leq b\},$$
where $a \neq 0$.

# Bibliography

[1] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.

[2] J. Dattorro, *Convex optimization & Euclidean distance geometry.* Lulu. com, 2010.