Outline of Foundations of Signal Processing

January 18, 2015

Chapter 2

Introduce the basic concepts for signal representation.

- Vector spaces (def'n and properties in #18)
  - Subspace, Span, Linear independence, Dimension (or rank?)
  - Inner product
    i) distributivity $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
    ii) linearity in 1st argument $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
    iii) Hermitian symmetry $\langle x, y\rangle^* = \langle y, x\rangle$
    iv) positive definiteness $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = \mathbf{0}$
    v) linearity in 2nd argument (by ii and iii): $\langle x, \alpha y\rangle = \alpha^*\langle x, y\rangle$
  - Orthogonality (#25)
    can be defined for vector vs vector, vector vs set of vectors, vector vs space, space vs space etc.
  - Norm see theorems
    i) positive def: $||x|| \geq 0$, and $||x|| = 0$ iff $x = \mathbf{0}$
    ii) positive scalability: $||\alpha x|| = |a|\,||x||$
    iii) triangle inequality: $||x+y|| \leq ||x|| + ||y||$, with equality if $y = \alpha x$
  - Metric
- Standard Spaces
  - Standard inner product spaces – inner product must be finite for space to be inner product sp.
    * $\mathbb{C}^N, \ell^2(\mathbb{Z}), \mathcal{L}^2(\mathbb{R})$, ... (#30)
  - Standard normed vector spaces – space must have a finite norm
    * $\mathbb{C}^N, \ell^p(\mathbb{Z}), \mathcal{L}^p(\mathbb{R})$, ... (# 33)
- Hilbert Spaces
  - Convergence
  - Closed Subspace (#37 and #135)
  - Cauchy Sequence – like convergence, but limit value doesn't have to be in defined metric space
  - Complete Space – A normed vector sp. V where every Cauchy seq. converges to a vector in V
  - Banach Space – A normed vector sp. V where each Cauchy Sequence converges to a v in V
  - Hilbert Space – A complete inner product space
  - Linear Operator, $A : H_0 \rightarrow H_1$ – i) $A(x+y) = Ax + Ay$, ii) $A(\alpha x) = \alpha(Ax)$
  - Bounded Linear Operator
  - Operator Norm $||A|| = \sup_{||x=1||} ||Ax||$
  - Inverse

- Adjoint operator (see #46 for properties)

  $$\langle Ax, y\rangle_{H_0} = \langle x, A^*y\rangle, \text{ for every } x \in H_0 \text{ and } y \in H_1. \tag{1}$$

- Unitary Operator: A bounded linear operator (BLO) that is invertible and preserves inner products: $\langle Ax, Ay\rangle_{H_1} = \langle x, y\rangle_{H_0}\ \ \forall x, y \in H_0$.
  Theorem: A BLO is unitary iff $A^{-1} = A*$
- Eigenvalues/vectors: $A : H \to H; v$ is eigvector if $Av = lv$ for some $\lambda \in \mathbb{C}$.
- Definite Linear Operator – a self-adjoint ($A = A^*$) operator such that:
  positive semi-definite: $\langle Ax, x\rangle \geq 0$, positive definite: $\langle Ax, x\rangle > 0$
  negative semi-definite: $\langle Ax, x\rangle \leq 0$, negative definite: $\langle Ax, x\rangle < 0$

- Approximations
  - Best approximation: orthogonal projection
  - Projection Theorem (#51): Let S be a closed subspace of a Hilbert sp. H, and let $x \in H$
    i) Existence: There exists $\hat{x}$ such that $||x - \hat{x}|| \leq ||x - s||$ for all $s \in S$.
    ii) Orthogonality: $x - \hat{x} \perp S$ is necessary and sufficient for determining $\hat{x}$.
    iii) Uniqueness: $\hat{x}$ is unique
    iv) Linearity: $\hat{x} = Px$, where P is a linear operator that depends on S and not on x
    v) Idempotency: $P(Px) = Px\ \ \forall x \in S$
    vi) Self-adjointess: $P = P^*$
  - Projection Operator (not necessarily orthogonal, #55)
    i) A projection operator is a BLO that is idempotent ($P^2 = P$)
    ii) Orthogonal projection operator is self-adjoint, Oblique projection operator is not self-adjoint
  - Theorem: Orthogonal Projection Operator: $\langle x - Px, Py\rangle = 0\ \ \forall x, y \in H$
  - Pseudoinverse (see here for theorem).
  - Direct Sum:
    A vec. sp. V is a *direct sum* of subspaces S and T, denoted $V = S \oplus T$, if any non-zero x can be written uniquely as: $x = x_S + x_T$, where $x_S \in S, x_T \in T$
  - Decomposition: S and T form a *decomposition* of V, and x_S, x_T the *decomposition* of x.
  - Orthogonal Random Vectors: RVs x, y are *orthogonal* when $\mathbb{E}[xy^*] = \mathbf{0}$. (not inner product).

- Bases and Frames
  - Basis:
    $\Phi = \{\phi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is finite or countably infinite (FOCI). Φ is a basis for normed V when:
    i) it is *complete* in V: $\forall x \in V$, there is sequence $\alpha \in \mathbb{C}^{\mathcal{K}}$ s.t. $x = \sum_{k\in\mathcal{K}} \alpha_k\phi_k$
    ii) for any $x \in V$, α that satisfies above is unique.
  - Riesz Basis (#72):
    A Basis Φ with *stability constraints* $\lambda_{\min}, \lambda_{\max}$ s.t. $\lambda_{\min}||x||^2 \leq \sum_{k\in\mathcal{K}} |\alpha_k|^2 \leq \lambda_{\max}||x||^2$
    The largest $\lambda_{\min}$ and smallest $\lambda_{\max}$ are *optimal stability constants.*
    !!! The stability constants are very useful – the farther they are from optimum, the more a matrix becomes vulnerable to numerical ill-conditioning
  - Basis Synthesis Operator Φ and Basis Analysis Operator Φ^* (defined for Riesz Bases #75)
    $\Phi : \ell^2(\mathcal{K}) \to H$ with $\Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k\phi_k$
    $\Phi^* : H \to \ell^2(\mathcal{K})$ with $\alpha_k = (\Phi^*x)_k = \langle x, \phi_k\rangle$ (the k^{th} analysis coefficient).
  - Orthonormal Basis: A basis $\Phi = \{\phi_k\}$ s.t. $\langle\phi_i, \phi_k\rangle = \delta_{i-k}$
    Are unitary $\Phi\Phi^* = I$
    The above implies $\Phi^{-1} = \Phi^*$
  - Gram-Schmidt Orthogonalization (recursive algorithm to derive orthonormal basis #84).

- Biorthogonal Pairs of Bases: Bases Φ and $\tilde{\Phi}$ that are *biorthogonal*, *i.e.* $\langle \phi_i, \tilde{\phi}_k \rangle \delta_{i-k}$
  $\alpha = \tilde{\Phi}^* x$
  $x = \alpha\Phi = \Phi\tilde{\Phi}^* x$ (these two due to Theorem 2.44 #88)
  $\tilde{\Phi}^*\Phi = I$ on $\ell^2(\mathcal{K})$
  $\tilde{\Phi}^* = \Phi^{-1}$
  $\tilde{\lambda}_{\min} = 1/\lambda_{\max}$
  $\tilde{\lambda}_{\max} = 1/\lambda_{\min}$
- Gram Matrix (enables computations w.r.t only 1 basis):
  $G = \Phi^*\Phi$, each value $G_{ik} = \langle \phi_k, \phi_i \rangle$
  $\langle x, y \rangle = \beta^* G\alpha$
- Dual basis properties (Theorem 2.46 #94):
  Let $A = (\Phi^*\Phi)^{-1}$ (inverse Gram matrix).
  Dual basis vectors ϕ_k can be computed as $\tilde{\phi}_k = \sum_{\ell\in\mathcal{K}} a_{\ell,k}\phi_\ell$.
  Synthesis operator: $\tilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1}$
- Successive Approximation (algo to compute *canonical* dual basis)
- Frame: A vector set $\{\phi_k\}_{k\in\mathcal{J}} \subset H$ that spans H but is overcomplete (like having more vectors than the rank).
  $\lambda_{\min}||x||^2 \leq \sum_{k\in\mathcal{J}} |\langle x, \phi_k \rangle|^2 \leq \lambda_{\max}||x||^2$
- Tight Frame or λ-tight Frame: frame such that $\lambda_{\min} = \lambda_{\max}$. $\Phi\Phi^* = I$
  Acc. to Theorem 2.51 (#105), the analysis and synthesis operations of 1-tight frames is analogous to orthonormal basis situation, but expansion is not unique anymore.

- Matrix Representations of Linear Operators:
  - Change of bases
    Consider two bases Φ, Ψ such that $x = \Phi\alpha$ and $x = \Psi\beta$
    A change of bases operator $C_{\Phi,\Psi}\alpha = (\Psi^{-1}\Phi)\alpha = \Psi^{-1}(\Phi\alpha) = \Psi^{-1}x = \beta$.
    We don't want to apply the latter operation from right to left, because we'll move back to original space H which can be complicated. Instead, we want to have a matrix operator $C_{\Phi,\Psi} = (\Psi^{-1}\Phi)$ that will allow the operations to stay within $\ell^2(\mathcal{K})$ See #113.

1 Norm- and Inner Product-related Theorems

The opposite triangle inequality $||v - w|| \geq \big|||v|| - ||w||\big|$
Pythagorean theorem $x \perp y$ implies $||x + y||^2 = ||x||^2 + ||y||^2$. Generalize: $\{x_k\}_{k\in\mathcal{K}}$ implies ... (# 29)
Parallelogram law $||x + y||^2 + ||x - y|| = 2(||x||^2 + ||y||^2)$
Hölder's inequality Let $p, q \in [1, \infty]$ satisfy $1/p + 1/q = 1$; then, $||xy||_1 \leq ||x||_p||y||_q$ with equal. iff $|x|^p$ and $|y|^q$ are scalar multiples of each other.
Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq ||x||\,||y||$ with equality iff $x = \alpha y$. (it's a special case of Hölder's ineq.) Using this, we can compute the angle bw any two vectors x,y as $\cos\theta = \langle x, y \rangle/||x||\,||y||$
Minwosky's inequality for any $p \in [1, \inf)$ (there are equivalents for integrals instead of sums as well, see #139):

$$\left(\sum_{k\in\mathbb{Z}} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k\in\mathbb{Z}} |x_k|^p\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}} |y_k|^p\right)^{1/p} \tag{2}$$

2 Projection-related Theorems

Orthogonal proj. via pseudoinverse Let $A : H_0 \to H_1$ be a BLO.
i) if AA^* invertible, then $B = A^*(AA^*)^{-1}$ is the *pseudoinverse* of A, and $BA = A^*(AA^*)^{-1}A$ is the orthogonal projection operator onto the range of A^*
ii) If A^*A is invertible, then $B = (A^*A)^{-1}A^*$ is the *pseudoinverse* of A and $AB = A(A^*A)^{-1}A^*$ is the orthogonal projection operator onto the range of A.

Diect-sum decomposition from Projection Operator (#61):
i) Let P be a projection op on H. P generates a direct-sum decomposition $H = \mathcal{R}(P) \oplus \mathcal{N}(P)$
ii) Conversely, if $H = S \oplus T$, then there is P on H s.t. $S = \mathcal{R}(P)$ and $T = \mathcal{N}(P)$
Orthonormal Basis Expansions Let $\Phi = \{\phi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for H. The unique *expansion coefficients* α_k can be obtained:

$$\alpha_k = \langle x, \phi_k\rangle \tag{3}$$
$$\alpha = \Phi^* x \tag{4}$$

Synthesis:

$$x = \textstyle\sum_{k\in\mathcal{K}} \langle x, \phi_k\rangle \phi_k \tag{5}$$
$$= \Phi\alpha = \Phi\Phi^* x \tag{6}$$

Parseval Equalities
For orthonormal bases (#77) and 1-tight frames (#105): $||x||^2 = ||\alpha||^2$ (more generally $\langle x, y\rangle = \langle \alpha, \beta\rangle$)
For biorthogonal pair of bases (#89): $||x||^2 = \langle \tilde{\alpha}, \alpha\rangle$ (more generally $\langle x, y\rangle = \langle \tilde{\alpha}, \beta\rangle$)
Orthogonal projection onto a*subspace* Let $\{\phi_k\}_{k\in\mathcal{I}}$. Then, $P_{\mathcal{I}} = \Phi_{\mathcal{I}}\Phi_{\mathcal{I}}^* x$ is the orthogonal projection of x onto the (subspace) $S_{\mathcal{I}} = \overline{\text{span}}(\{\phi_k\}_{k\in\mathcal{I}})$
Orthogonal projection onto a*subspace* Analogous to above, only that $P_{\mathcal{I}} = \Phi_{\mathcal{I}}\tilde{\Phi}_{\mathcal{I}}^* x$
Bessel's Inequality $||x||^2 \geq ||\Phi_{\mathcal{I}}^* x||^2$ (definitions above).

Quiz

1. **What is a Unitary Operator?**
2. **What is a Direct Sum?**
3. **What is a finite-energy function?**
4. **Order the following spaces from largest to smallest: Banach, Hilbert, Inner Product, Normed**
5. **Which matrices are called Hermitian?**
6. **What is the infinity norm or uniform norm or supremum norm or Chebyshev norm?**
7. **What is a tight frame? [options?]**
8. **What is the difference between a frame and a basis?**
9. **Group the synonyms among these four terms: cross product, inner product, scalar product, dot product.**