Chapter 1

Introduction

Mixed notes from the book Convex Optimization [1]. Some parts will be supplemented by the book of Dattoro [2]. The book of Dattoro is extremely useful for concretely illustrating most of the concepts.

Quotes

- "We study convex geometry because it is the easiest of geometries. For that reason, much of a practitioners energy is expended seeking invertible transformation of problematic sets to convex ones". Dattoro [2].

Chapter 2

Convex Sets

2.1 Affine and convex sets and cones.

- **Affine set.** A set $C \subseteq \mathbf{R}^n$ is *affine* if the line through any two points in C lines in C, *i.e.* if $x_1, x_2 \in C$ and $\theta \in \mathbf{R}$ implies $\theta x_1 + (1-\theta)x_2 \in C$. More generally, an affine set is a set that contains all *affine combination* (see def. below) of two or more of its points.
  - * Any affine set is convex [2].
    * The intersection of an arbitrary collection of affine sets remains affine [2].
    * Any affine set is open in the sense that it contains no boundary, *e.g.* the empty set $\emptyset$, point, line, plane, hyperplane, subspace etc [2]. Converse not necessarily true (*e.g.* see point just below about subspace.)
    * If C is an affine set and $x_0 \in C$, then the set $V = C - x_o = \{x - x_0 | x \in C\}$ is a subspace.
  - *Affine combination.* A combination of points $\sum_{i=1}^k \theta_i x_i$ where $\sum_{i=1}^k \theta_i = 1$ is an affine combination.
  - *Ambient space.* The space where a given set lives in, *e.g.* a plane can live in $\mathbf{R}^2, \mathbf{R}^3$. The choice of ambient space has implications on, for example, the interior of a set ([2], p34)
  - *Affine hull*, denoted $\mathbf{aff}\, C$ is the smallest set that makes C affine.
  - *Affine dimension* of a set C is the dimension of $\mathbf{aff}\, C$. In fact dimension of a set is synonymous with affine dimension [2].
  - *Relative interior.* The interior of, for example, a plane in $\mathbf{R}^3$ is empty. To "fix" this issue, we define the relative interior of C as: $\mathbf{relint}\, C = \{x \in C | B(x,r) \cup \mathbf{aff}\, C \subseteq C \text{ for some } r > 0\}$
- **Convex sets**. A set C is convex if the line segment between any two points in C lies in C, *i.e.* $\theta x_1 + (1-\theta)x_2 \in C$ for any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$.
  - *Convex combination.* A combination of points $\sum_{i=1}^k \theta_i x_i$ where $\sum_{i=1}^k \theta_i = 1$ and $\theta_i \geq 0$ is a convex combination.
  - *Convex hull* $\mathbf{conv}\, C$ of a set C is the smallest set that makes C convex.
- **Cones**. A set C is called a cone if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. Cones can have very unintuitive shapes, see Fig. 35-41 in [2].
  - *Convex cone* is a set that is cone and also convex, *i.e.* $\theta_1 x_1 + \theta_2 x_2 \in C$ for any $x_1, x_2 \in C$ and for $\theta_1, \theta_2 \geq 0$. Some differences between a convex set and a convex cone: (i) A convex set doesn't have to include the origin, a convex cone does; (ii) a convex set can be bounded but a convex cone cannot.

Some important examples and notes ([1] p27):

- Any subspace is affine and a convex cone
- A line segment is convex but not affine
- A ray (*i.e.* $\{\theta v + x_0 : x \geq 0\}$) is convex but not affine. It is convex cone if its base x_0 is 0.
- Any line is affine.
- The empty set, any single point and the whole space are affine (hence convex) subsets of $\mathbf{R}^n$
- Halfspaces (see below) are convex but not affine.

2.2 Hyperplanes, halfspaces, balls and polyhedra

- **Hyperplane** is a set of the form
$$\{x|a^Tx = b\}$$
This set has several intuitive interpretations.
  1. It is the hyperplane with a normal vector a and an offset b from the origin.
  2. Let b be $a^Tx = b$. Then, $\{x|a^Tx = b\} = \{x|a^T(x - x_0)\} = x_0 + a^\perp$ where $a^\perp$ is the orthogonal complement of a.
  3. More interpretations on p27-28.
- **Halfspace**. Each hyperplane divides $\mathbf{R}^n$ into two halfspaces. A (closed) halfspace is of the form
$$\{x|a^Tx \leq b\},$$
where $a \neq 0$.
- **Norm ball**. A norm ball is the set of the form $B(x_c, r) = \{x : ||x_c - x|| \leq r, x \in \mathbf{R}^n\}$, where $||\cdot||$ is a given norm. Another common representation of the ball is $B(x_c, r) = \{x_c + ru : ||u|| \leq 1, u \in \mathbf{R}^n\}$.

  Norm ball is convex (p30).
- **Norm cone** is the set $C(x, t) = \{(x, t) : ||x|| \leq x \in \mathbf{R}^{n+1}, t \in \mathbf{R}\} \subseteq \mathbf{R}^{n+1}$. It is a convex cone.
- **Proper cone** is a cone $K \subseteq \mathbf{R}^n$ that satisfies the following:
  - is convex
  - is closed
  - is solid, *i.e.* it has nonempty interior
  - is pointed, which means that it contains no line (or, equivalently, $x \in K$, $-x \in K \implies x = 0$)

  The concept of Proper Cone will be central in defining *generalized inequalities.*
- **Polyhedra**. A polyhedron is the solution set of a finite number of equalities and inequalities:
$$\mathcal{P} = \{x : a_j^Tx \leq b_j, j = 1, \dots, m, c_i^Tx = d_i, i = 1, \dots, m\}.$$
A simpler notation is $\mathcal{P} = \{x|Ax \preceq b, Cx = d\}$, where the symbol $\preceq$ is *vector* or *componentwise* inequality (p32).
- **Simplexes.** Simplexes are a family of polyhedra; they are also a generalization of the triangle (and its interior); *i.e.* a 1D simplex is a line segment, a 2D simplex is the triangle and its interior, a 3D simplex is tetrahedron.
  - *Affine independence* means that for $v_0, \dots, v_k \in \mathbf{R}^n$, the points $v_1 - v_0, \dots, v_k - v_0$ are linearly independent.
  - A simplex can be defined in terms affinely independent points: $C = \mathbf{conv}\{v_0, \dots, v_k\}$.

2.3 Operations that preserve convexity

- **Intersection**: Convexity is preserved under intersection; the intersection of even infinite convex sets is convex.
- **Affine functions.** Let f be an affine function, *i.e.* $f(x) = Ax + b$. Then the image of S under f,
$$f(S) = \{f(x) \mid x \in S\}$$
and the inverse image of S under f,
$$f^{-1}(S) = \{x|f(x) \in S\}$$
are both convex if S is convex.
  - *Cartesian product.* $S = S_1 \times S_2$ for two convex sets S_1, S_2, $S = \{x_1 + x_2 : x_1 \in S_1, S_2 \in x_2\}$ is convex.
  - *Sum* The sum S of two convex sets S_1, S_2, $S = \{x_1 + x_2 : x_1 \in S_1, S_2 \in x_2\}$ is convex.
- **Linear-fractional and perspective functions.**
  - *Perspective function* is the $R^{n+1} \to \mathbf{R}^n$ function $P(x, t) = x/t$ with domain $\mathbf{dom}\, P = \mathbf{R}^{n+1} \times \mathbf{R}_{++}$. That is, the perspective function normalizes the input vector so the last element is one, and then drops this last element. If a set $C \subseteq \mathbf{dom}\, P$ is convex, then its image under P is also convex.
  - *Linear-fractional function* is the composition $P \circ g$ of a perspective function P with an affine function g.
    It is easy to show that linear-fraction functions preserve convexity: If S is convex, then its image $g(S)$ under g will be convex, then its image under perspective will also be convex.

2.4 Generalized inequalities, minimum and minimal elements

- **Generalized inequality**. A proper cone K (see above) can be used to define a *generalized inequality* as follows:

$$x \preceq_K y \iff y - x \in K$$

A strict generalized inequality $x \prec y$ is defined as

$$x \prec_K y \iff y - x \in \textbf{int}\, K$$

Example. The nonnegative orthant $\mathbf{R}_+^n$ is a proper cone and for $K = \mathbf{R}_+^n$ the associated inequality $\preceq_K$ corresponds to componentwise inequality $x \prec y$.

Properties of generalized inequalities:

1. $\preceq_K$ is preserved under addition: If $x \preceq_K y$ and $u \preceq_K y$, then $x + u \preceq_K y + v$
2. $\preceq_K$ is transitive: If $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
3. $\preceq_K$ is preserved under nonnegative scaling: If $x \preceq_K y$ and $\alpha > 0$, then $\alpha x \preceq_K \alpha y$.
4. $\preceq_K$ is reflexive: If $x \preceq_K x$.
5. $\preceq_K$ is antisymmetric: If $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
6. $\preceq_K$ is preserved under limits: If $x_i \preceq_K y_i$ for $i = 1, 2, \ldots$ and $x_i \to x$ and $y_i \to i$, then $x \preceq_K y$.

Properties 1,2,3 are shared by the strict inequality $x \prec_K y$ too, property 4 is strictly *not* shared by it. Also, and additional property for strict generalized inequalities (probably shared by non-strict too):

- if $x \prec_K y$, then for small enough u, v we have $x + u \prec_K y + v$.

- **Minimum and minimal elements**. An essential difference between a regular inequality and a generalized one is that *not all points are comparable*; that is, one of the two inequalities $x \leq y$ or $x \leq y$ has to hold. This is not the case for generalized inequality.

Example. Consider the proper cone $K = \mathbf{R}_+^n$, and points $x = (3, 3)$, $y = (5, 5)$ and $z = (4, 2)$. Clearly, x and y are comparable and $x \preceq_K y$. Similarly, y and z are comparable and $z \preceq_K y$. However, x and z are not comparable.

- *Minimum element.* We say that $x \in S$ is the minimum element of S (w.r.t. $\preceq_K$) if for every $y \in S$ we have $x \preceq_K y$, which happens if and only if

$$S \subseteq x + K$$

where $x + K$ is the set of all the points that are (i) comparable to x and (ii) greater than or equal to x (confer Fig. 2.17 of [1] or Fig. 43 of [2]).

There can be at most *one* minimum point.

- *Minimal element.* First of all, a minimum point is also a minimal point. But a minimal point can exist even if there is no minimum. There can be more than one minimal points.

We say that $x \in S$ is the minimal point of S (w.r.t. $\preceq_K$) if for any $y \in S$, $y \preceq_K$ holds only if $y = x$. Or, equivalently,

$$(x - K) \cap S = \{x\}$$

where $x - K$ denotes the set of all points that are comparable to x and are less then or equal to x w.r.t. $\preceq_K$. Confer Fig. 2.17 of [1] or Fig. 43 of [2]. Note that in Fig. 43b it's impossible to draw the cone K (centered on any of the minimal points) that would contain the entire $\mathcal{C}_2$, therefore $\mathcal{C}_2$ has no minimum.

2.5 Separating and supporting hyperplanes

- **Separating hyperplane** is a hyperplane that separates two convex sets.

Separating hyperplane theorem. Let C and D be two convex sets that do not intersect. Then, there exists $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.

- *Strict separation* is defined similarly when $\geq$ and $\leq$ are replaced by $>$ and $<$.

Converse of separating hyperplane theorem is not in general true, but one can obtain it by adding additional constraints. One variant of converse theorem would be (see p50): Any two convex sets at least one of which is open are disjoint if and only if there exists a separating hyperplane.

- **Supporting hyperplane**. Suppose that $C \subseteq \mathbf{R}^n$ and x_0 is a point in its boundary, *i.e.* $x_o \in \mathbf{bd}\, C = \mathbf{cl}\, C \setminus \mathbf{int}\, C$. If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$ (*i.e.* the entire set C lies on one side of the hyperplane), then the hyperplane $\{x : a^T x = a^T x_0\}$ is called a *supporting hyperplane.*

  Convexity and supporting hyperplanes are intimately connected:

  – *Supporting hyperplane theorem*: If C is convex, then there exists a supporting hyperplane for any $x_0 \in \mathbf{bd}\, C$.

  – Partial converse of the theorem: If a set is closed, has nonempty interior and has a supporting hyperplane at every point on its boundary, then it is convex.

2.6 Dual cones and generalized inequalities

- **Dual cone**. Let K be a cone. Then, the set $K^* = \{y : x^T y \geq 0 \text{ for all } x \in K\}$ is called the dual cone of K. Some properties of dual cone K^* (p51 and p53):

  – A dual cone K^* is a ... cone.

  – K^* is always *convex.*

  – K^* is always closed.

  – $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$.

  – If K has nonempty interior, then K^* is pointed.

  – If the closure of K is pointed, than K^* has nonempty interior.

  – K^{**} is the closure of the convex hull of K (hence, if K is convex and closed, $K^{**} = K$).

  – If K is a proper cone, then K^* is also a proper cone.

  An intuitive way of defining a dual cone is shown in Fig. ******************. *Examples to dual cones.*

  – The dual cone of $K = \mathbf{R}^n_+$ is itself.

  – The dual cone of a line in space is its orthogonal complement.

  – More generally, the dual cone of a subspace $V \subseteq \mathbf{R}^n$ is its orthogonal complement $\{y : y^T v = 0 \text{ for all } v \in V\}$

- **Dual generalized inequalities**. Like any proper cone, the dual of a proper cone, K^*, induces a generalized inequality $\preceq_{K^*}$. The following relationships relating the generalized inequality of proper cone and its dual seem to be fundamental:

  – $x \preceq_K y$ iff $\lambda^T x \leq \lambda^T y$ for all $\lambda \succeq_{K^*} 0$.

  – $x \prec_K y$ iff $\lambda^T x < \lambda^T y$ for all $\lambda \succ_{K^*} 0$, $\lambda \neq 0$.

  Those are central properties that allow us to characterize minimum and minimal points w.r.t. a cone K in terms of its dual generalized inequalities, $\preceq_{K^*}$.

  – *Minimum point.* $x \in S$ is the minimum point in S w.r.t. generalized inequality $\preceq_K$ iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over $z \in S$. Geometrically, this means that for any $\lambda \succ_{K^*}$, the hyperplane

  $$\{z : \lambda^T (z - x) = 0\}$$

  is a strict supporting hyperplane (strict means that the hyperplane intersects S only at one point, *e.g.* it's not a trivial hyperplane such a line being supporting hyperplane of a line).

  – *Minimal point.* There is a gap between necessary and sufficient conditions. A note to illustrate why: $x \in S$ may be minimal of S but there may be no λ for which x minimizes $\lambda^T z$ over $z \in S$ (see Figure 2.25, 2.26, p57).

Bibliography

[1] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.

[2] J. Dattorro, *Convex optimization & Euclidean distance geometry.* Lulu. com, 2010.

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