Outline of Foundations of Signal Processing

February 14, 2015

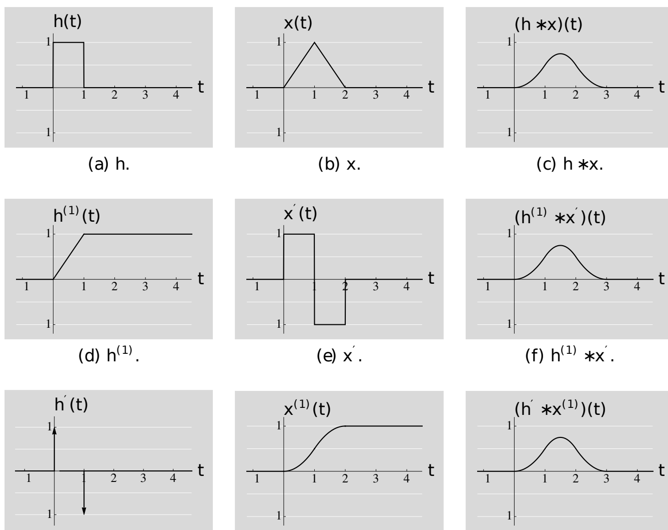
Chapter 4

Main discussion around Continuous-Time Fourier Transform (CTFT) and Fourier Series (FS), subjects like Laplace Transform and Cont. Stochastic Processes are touched very briefly.
Particularly useful is the usage of CTFT to determine the smoothness or differentiability of a function x.

- Smooth functions Global *smoothness* of a function described via its continuity and continuity of its derivs
- Systems: Continuous-Time (CT) systems are operators with functions as their input and output.
  Linear, Memoryless, Causal, Shift-invariant, (BIBO) stable
- Differential equations can define most system types. They are typically solved via Fourier Laplace Transforms
- CTFT (or simply FT) is an operator $F : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$
  - Fourier Transform so popular because complex exponentials are eigenfunctions of LSI systems: $(h * v)(t) = \int v(t-\tau)h(\tau)d\tau = \int e^{j\omega(t-\tau)}h(\tau)d\tau = \int h(\tau)e^{j\omega\tau}d\tau e^{j\omega t}$, i.e. $e^{j\omega t}$ comes out as eigenfunction
    Defn, Fourier Transform(FT): $X(\omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
    Inverse FT (IFT): $x(t) = \frac{1}{2\pi}\int\limits_{\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
  - Fourier operator $F : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ is a unitary operator up to scaling factor 2π!
    (defn of unitary $U^*U = I$, alternatively U is surjective AND $\langle Ux, Uy\rangle = \langle x, y\rangle$)
    See more here: `https://people.math.osu.edu/gerlach.1/math/BVtypset/node32.html`
  - Parseval equality, $||x||^2 = \frac{1}{2\pi}||X||^2$ and $\langle x, y\rangle = \langle X, Y\rangle$ (equivalent with $F/\sqrt{2\pi}$ being unitary operator, see above)
  - Properties of FT in Table 4.1 #366.
  - Common transform pairs also in Table 4.1 (e.g. FT of box fn, Heaviside fn, Gaussian fn).
  - When the FT of a fn can't be evaluated directly, we can evaluate through IFT (e.g. FT of constant function, see Ex 4.6 #365)
  - !important! Differentation and Convolution via FT – see the relationship in Fig. 1below.
  - !IMPORTANT! FT facilitates the characterization of boundedness and continuousness (and differentiability)
    For some positive γ, ϵ:
    (i) $|X(\omega)| \leq \frac{\gamma}{1+|\omega|^{1+\epsilon}} \implies x$ is bounded and continuous (*i.e.* $x \in C^0$)
    (ii) $|X(\omega)| \leq \frac{\gamma}{1+|\omega|^{1+\epsilon+q}} \implies x \in C^q$ (has q cont derivs)
    (iii) $x \in C^q$ and x is bounded (but not necess. cont.) $\implies |X(\omega)| \leq \frac{\gamma}{1+|\omega|^{1+q}}$
    (to understand these better see example 4.8 #376)
  - Lipschitz Regularity generalizes the above to differentiablity of fractional order:

- !critical! Lipschitz regularity can describe whether a fn is continuous ALMOST everywhere (*i.e.* being of Lipschitz order of $1-\epsilon$ for any $\epsilon > 0$ means being almost of order 1, see #378) Let $\alpha \in [0, 1)$.
  Then x is *pointwise Lipschitz* of order α at t_0 when $|x(t) - x(t_0)| \leq c|t - t_0|^\alpha$.
  In FT context; a fn x is bounded and *uniformly* Lipschitz over α when: $\int\limits_{-\infty}^{\infty} |X(\omega)|(1 + |\omega|^\alpha) d\omega < \infty$

- Laplace Transform: Extends FT with more general complex exponentials.
- Fourier Series (FS) is an operator $F : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z})$
  Defined for periodic signals over a single period T:
  $X_k = \frac{1}{T} \int\limits_{-T/2}^{T/2} x(t) e^{-j(2\pi/T)k} dt,\ k \in \mathbb{Z}.$
  Reconstruction via FS: $x(t) = \sum\limits_{k \in \mathbb{Z}} X_k e^{j(2\pi T)kt}, t \in [-\frac{T}{2}, \frac{T}{2})$
  - Properties at Table 4.3 (#386)
  - FS is a **countable** orthonormal basis
    The set $\{\phi_k\}_{k \in \mathbb{Z}}$ with $\phi_k(t) = \frac{1}{\sqrt{T}} e^{j(2\pi/T)kt}$ forms an orthonormal basis for $\mathcal{L}^2([-\frac{T}{2}, \frac{T}{2}))$
    To prove this, show that i) $\langle \phi_k, \phi_l \rangle = \delta_{k-l}$ (see page #384) and ii) any $x \in \mathcal{L}^2([-\frac{T}{2}, \frac{T}{2}))$ is in $\overline{\text{span}}(\{\phi_k\}_{k \in \mathbb{Z}})$ (book not clear, see Supp1, Section 7.1)
  - Thm 4.15 (#384) Fundamental 3-fold Theorem:
    Let $\hat{x}$ be the FS reconstruction of x from $X = Fx$. Then:
    i) $\mathcal{L}^2$ inversion: $||x - \hat{x}|| = 0$
    ii) (Parseval) $||x||^2 = T \sum\limits_{k \in \mathbb{Z}} |X_k|^2$ and $\langle x, y \rangle = T \sum_{k \in \mathbb{Z}} X_k Y_k^*$
    iii) Least-squares approx: The fn $\hat{x}_N = X_k e^{j(2\pi/T)kt}$ is the *least-squares approx* of x on subspace $\overline{\text{span}}(\{\phi_k(t)\}_{k=-N}^{N})$
  - Defn - Dirac Comb: $s_T(t) = \sum\limits_{n \in \mathbb{Z}} \delta()$
  - Relation bw FT and FS: $X(\omega) = 2\pi \sum\limits_{k \in \mathbb{Z}} X_k \delta(\omega - \frac{2\pi}{T} k)$
  - Furthermore: when the FT of $\tilde{x} = 1_{[-T/2, T/2)} x$ exists, the FS of x exists too.
  - Thm 4.16 (#392) Poisson Sum Formula (!critical! for sampling theorem):
    Let x be a fn with sufficient decay for the periodization $(s_T * x)(t) = \sum_{n \in \mathbb{Z}} x(t - nT)$ to converge absolutely for all t.
    Then (simplified version): $\sum_{n \in \mathbb{Z}} x(n) = \sum_{k \in \mathbb{Z}} X(2\pi k)$ — see #392
  - Regularity and spectral decay — like FT, FS can too be used to characterize the boundedness and continuity of fns: For some positive γ, ϵ:
    (i) $|X_k| \leq \frac{\gamma}{1 + |k|^{1+\epsilon}} \implies x$ is bounded and continuous (*i.e.* $x \in C^0$)
    (ii) $|X_k| \leq \frac{\gamma}{1 + |k|^{1+\epsilon}} \implies x \in C^q$ (has q cont derivatives)
    (iii) $x \in C^q$ and x is bounded (but not necess. cont.) $\implies |X_k| \leq \frac{\gamma}{1 + |k|^{1+q}}$



(a) h. (b) x. (c) $h * x$.

(d) $h^{(1)}$. (e) x'. (f) $h^{(1)} * x'$.

(g) h'. (h) $x^{(1)}$. (i) $h' * x^{(1)}$.

Figure 1: The relation bw convolution and differentiation