Introduction

These are my notes from the book of Harville [?]. This are probably very esoteric notes, I noted whatever was important for me. That is, however basic, anything that I didn't know found a place here. It is also a summary of what I gained from this book. My coverage of chapters will probably be insanely unbalanced.

Chapter 1

Matrices

Preliminaries and basics of matrix algebra are presented here. But one has to go through to become familiar with the notation that will follow (*e.g.* $\mathbf{J}_{mn}$ is an $m \times n$ matrix of ones). Plus going through the entire text and solving the exercises helped me to start and become familiar with *proving* linear algebra. Exercise 8 is particularly interesting.

Some interesting notes:

$$\mathbf{J}_{mn}\mathbf{J}_{np} = n\mathbf{J}_{mp}m$$

1.0.1 Redo in future

- Exercise 4.a–c
- Exercise 8

Chapter 2

Submatrices and Partitioned Matrices

A basic topic that I wasn't sufficiently familiar is covered in this chapter (see title).

Notes

- A partitioning of a matrix is obtained by putting horizontal/vertical lines between its various columns or rows. The only requirement is that the lines shouldn't be *staggered* (see #15).
- The fact that multiplication of partitioned matrices is similar to that of the regular matrices (see 2.7) is very intuitive. Proves useful when proving thing in future chapters.

Interesting identities/lemmas/theorems

- (2.9): $\mathbf{AB} = \sum_{k=1}^{c} \mathbf{A}_k \mathbf{B}_k$, where $\mathbf{A}, \mathbf{B}$ are appropriately defined partitionings (see #19), i.e. $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_c)$, $B = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \ldots \\ \mathbf{B}_c \end{bmatrix}$. Particularly interesting when $\mathbf{A}_i$ and $\mathbf{B}_j$ are cols and rows of A and B respectively, i.e. $\mathbf{C} = \mathbf{AB} = \sum_{i=1}^{c} \mathbf{a}_i' \mathbf{b}_i$.

Important concepts

- Block-diagonal matrix
- Block-triangular matrix

2.0.1 Redo in future

- Exercise 4

Chapter 3

Linear Dependence and Independence

Very short chapter, but its lemmas are very useful, they can be used extensively in proofs that come later in the chapters. Lemma 3.2.1 and 3.2.2 are fundamental and they should be practically memorized. Lemma 3.2.4 is an elegant one, it seems like it can be the key to some complicated proofs.

3.1 Redo in future

- Lemma 3.2.1
- Lemma 3.2.2
- Lemma 3.2.4
- Exercise 2

Chapter 4

Linear Space: Row and Column Spaces

Very useful chapter, it's title doesn't do justice. Fundamental concepts of rank, dimension, basis, linear space, subspace, column/row space are introduced and they are made concrete with many lemmas and theorems.

4.1 Important Lemmas/Theorems/Equations

- Lemma 4.2.2 is easily the most important result in the chapter. Many results in the chapter can be proved directly through this lemma.
- Corollary 4.2.3 also very useful.
- Theorem 4.3.2
- Corollary 4.3.3
- Theorem 4.3.4
- Lemma 4.3.5
- Theorem 4.3.7
- Theorem 4.4.2
- The restatements of column/row space results in terms of rank results in Section 4.4.b
- Theorem 4.4.8
- Lemma 4.5.1
- Lemma 4.5.4
- Lemma 4.5.7
- Corollary 4.5.9
- Theorem 4.4.10

4.2 Redo

- All listed above
- Theorem 4.4.4
- Theorem 4.4.9
- Theorem 4.4.10
- Lemma 4.5.11
- Theorem 4.4.1
- Exercise 4
- Exercise 5

- Exercise 6
- Exercise 7
- Exercise 9
- Exercise 10
- Exercise 11a

4.3 Important concepts

- Linear Space
- Row/Column space
- Basis
- Linear Space
- Rank
- Spanning set
- Dimension

Chapter 5

Trace of a (Square) Matrix

The chapter gives some basic identities about traces, and then a very important equivalence for finding whether a matrix is null, $\mathbf{0}$.

Notes

- Basic trace identities and some properties (Section 5.2 and 5.3)
  - $\text{tr}(k\mathbf{A}) = k\text{tr}(\mathbf{A})$
  - $\text{tr}(\mathbf{A}+\mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$
  - $\text{tr}(\mathbf{A}'\mathbf{A}) \geq 0$
  - $\text{tr}(\mathbf{A}') = \text{tr}(\mathbf{A})$
  - $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{B}'\mathbf{A}') = \text{tr}(\mathbf{BA}) = \text{tr}(\mathbf{A}'\mathbf{B}')$
  - The last leads to allowing any cyclical rotations: $\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{CAB}) = \text{tr}(\mathbf{BCA})$
    (It is easy to remember that only cyclicals are allowed—the matrices have to be multipliable)
  - **L5.3.1**: $\mathbf{A} = \mathbf{0} \iff \text{tr}(\mathbf{A}'\mathbf{A}) = 0$
    C5.3.2: $\mathbf{A} = \mathbf{0} \iff \mathbf{A}'\mathbf{A} = \mathbf{0}$
    C5.3.3: $\mathbf{AB} = \mathbf{AC} \iff \mathbf{A}'\mathbf{AB} = \mathbf{A}'\mathbf{AC}$ and (transposed) $\mathbf{BA}' = \mathbf{CA}' \iff \mathbf{BA}'\mathbf{A} = \mathbf{A}'\mathbf{AC}$

Redo

- Prove Corollary 5.3.2
- Prove Corollary 5.3.3
- Exercise 3

Chapter 6

Geometrical Considerations

Geometry is very important because it allows for an intuitive understanding of many linear algebra concepts. There are two opinions about those who teach and write about linear statistical models. Those who advocate the "geometrical approach" emphasize its elegance and intuitiveness, and those who advocate a more "algebraic approach" highlight its rigour and the fact that it's more suggestive of computational approaches.

Important concepts

- Inner (or dot) Product (the "usual" one): $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}'\mathbf{y}$ for vecs $\mathbf{A}' \cdot \mathbf{B} = \text{tr}(\mathbf{AB})$ for mats
- Norm: $||\mathbf{x}|| = (\mathbf{x}' \cdot \mathbf{x})^{1/2}$
- Angle: $\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$ (similar for matrices)
- Distance: $||\mathbf{A} - \mathbf{B}||$
- Orthogonality

Notes

- Schwarz Inequality: $|\mathbf{A} \cdot \mathbf{B}| \leq ||\mathbf{A}||\,||\mathbf{B}||$
- Gram-Schmidt Orthogonalization: $x_{ij} = \frac{\mathbf{A}_j \cdot \mathbf{B}_i}{\mathbf{B}_i \cdot \mathbf{B}_i}$
- Any linear space has an orthonormal basis (T6.4.3): Follows from T4.3.9 and the fact that we can apply Gram-Schmidt Orthogonalization to any set of bases
- QR decomposition for (full-column matrix): $\mathbf{A} = \mathbf{BX} = (\mathbf{BD})(\mathbf{EX}) = \mathbf{QR}$, where cols of $\mathbf{Q}$ are orthonormal and $\mathbf{R}$ is upper triangular; cols of $\mathbf{B}$ are only orthogonal and $\mathbf{D}, \mathbf{E}$ are diagonal matrices such that (clearly) $\mathbf{D}^{-1} = \mathbf{E}$ (See Exercise 5 for generalization of QR decomposition to column rank deficient matrices.)

Redo

- Schwarz Inequality
- Exercise 1
- Exercise 5

Chapter 7

Consistency and Compatibility

This chapter is about the two important concepts in the title that relate to whether a linear system has solution(s). Finding the solution(s) is the topic of Chapter 11.

Important Concepts

- Linear system (in $\mathbf{x}$) is: $\mathbf{Ax} = \mathbf{B}$. $\mathbf{A}$ is the *coefficient matrix*, B is the *right side* and any x that satisfies the equality is a solution.
- A linear system is called *homogeneous* if $\mathbf{B} = \mathbf{0}$ and *non-homogeneous* otherwise.
- A system is called *consistent* if it has one or more solutions.

Notes

- **T7.2.1** gives four necessary and sufficient conditions for consistency. All four can be derived from the first condition: $\mathcal{C}(\mathbf{B}) \subset \mathcal{C}(\mathbf{A})$ (and then using L4.2.2, L4.5.1 and C4.5.2).
- **T7.2.2**: A system is *consistent* if $\mathbf{A}$ has full row rank.
- A linear system is *compatible* if $\mathbf{k}'\mathbf{B} = \mathbf{0}$ for every $\mathbf{k}'$ such that $\mathbf{k}'\mathbf{A} = \mathbf{0}$
- **T7.3.1** is the the chapter's main theorem that connects the two main concepts: A linear system is compatible iff it is consistent
- **T7.4.1** connects the chapter with the Trace chapter; specifically with C5.3.3. This leads to important equivalences between $\mathbf{A}'\mathbf{A}$ and $\mathbf{A}$ in terms of column/row spaces and ranks (see Section 7.4.b).

Redo

- Prove T7.2.1 and T7.2.2
- Prove T7.3.1
- Prove T7.4.1
- Prove all T's/C's in Section 7.4.b: this will help with consolidating parts of Chapter 4.
- Prove T7.4.8
- Exercise 1b

Chapter 8

Inverse Matrices

The chapter introduces the (left, right) inverses and their basic properties. Orthogonal matrices are also introduced. There is a detailed treatment on the use of partitioned matrices for computing (i) the rank and (ii) the inverse. Specifically, if one of the diagonal blocks of a 2×2 partitioning is non-singular, then computing the rank and inverse becomes easy; the larger the non-singular block the easier to compute the rank/inverse. We can use permutation matrices to make the non-singular block as large as possible and thus facilitate computations (see example in #103).

Basic identities

- The following assume that $\mathbf{A}, \mathbf{B}$ are non-singular $n \times n$ matrices
  - $(k\mathbf{A})^{-1} = (1/k)\mathbf{A}^{-1}$
  - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
  - $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$
  - $\text{rank}(\mathbf{AB}) = n$
- The following assume that $\mathbf{Q}, \mathbf{Q}_i$ are *orthogonal*
  - $\mathbf{QQ}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$
  - $\mathbf{Q}^{-1} = \mathbf{Q}'$
  - $\mathbf{Q}_1\mathbf{Q}_2 \dots \mathbf{Q}_k$ is orthogonal

Important Concepts

- Left inverse (LI), right inverse (RI), inverse (*a.k.a.* left *and* right inverse)
- Orthogonal matrix: A matrix is orthogonal if its columns form an *ortho**normal*** set.
- Permutation matrices
- Schur complement of a paritioned matrix $\begin{bmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{U} & \mathbf{T} \end{bmatrix}$ where $\mathbf{T}$ is non-singular is $\mathbf{Q} = \mathbf{W} - \mathbf{V}\mathbf{T}^{-1}\mathbf{U}$. It is used, among others, when computing inverses (T8.5.11)

Notes

- **L8.1.1** is used very widely: (i) $\mathbf{A}$ has RI $\iff$ $\mathbf{A}$ has full row rank, and (ii) $\mathbf{A}$ has LI $\iff$ $\mathbf{A}$ has full col rank. (Remember: **R**ight inverse if full **r**ow rank.)
- **C8.3.3** If $\mathbf{A}$ non-singular, then $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{B})$ (i.e. rank limited by $\mathbf{B}$).
- Helmert matrix computes an orthogonal matrix from one vector whose none element is 0.
- Permutation matrices can become very convenient for computing rank (see L8.5.1 and S8.5.f)
- Block-triangular matrices are convenient for computing inverses (see L8.5.4, T8.5.11)
  - The easiest is with block-diagonal matrices $\begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{-1} \end{bmatrix}$

– Then with unit block triangular matrices, L8.5.2, $\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{V} & \mathbf{I}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{V} & \mathbf{I}\end{bmatrix}$ (sim. for upper up. matrix)
– Next, L8.5.4, $\begin{bmatrix}\mathbf{W} & \mathbf{V}\\ \mathbf{0} & \mathbf{T}\end{bmatrix} = \begin{bmatrix}\mathbf{W}^{-1} & -\mathbf{W}^{-1}\mathbf{V}\mathbf{T}^{-1}\\ \mathbf{0} & \mathbf{T}^{-1}\end{bmatrix}$ (sim. for upper low. matrix)
– Generalization (S8.5.d): The lemmas above lead to generalizations (through induction) and recursive algorithms for computing the inverses of matrices partitioned to larger than 2×2 submatrices

- **T8.5.10**: If $\mathbf{T}$ is $m \times m$ and nonsingular, then rank $\begin{pmatrix}\mathbf{T} & \mathbf{U}\\ \mathbf{V} & \mathbf{W}\end{pmatrix} = m + \text{rank}(\mathbf{W} - \mathbf{V}\mathbf{T}^{-1}\mathbf{U})$

- **T8.5.11**: Inverse of non-block-triangular part. m. $\begin{bmatrix}\mathbf{W} & \mathbf{V}\\ \mathbf{U} & \mathbf{T}\end{bmatrix}$, given that $\mathbf{T}$ is non-singular, is computed using the results for block-triangular matrices and the Schur complement, $\mathbf{Q}$ (see def of $\mathbf{Q}$ above):

$$\begin{bmatrix}\mathbf{T} & \mathbf{U}\\ \mathbf{V} & \mathbf{W}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{T}^{-1} + \mathbf{T}^{-1}\mathbf{U}\mathbf{Q}^{-1}\mathbf{V}\mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{U}\mathbf{Q}^{-1}\\ -\mathbf{Q}^{-1}\mathbf{V}\mathbf{T}^{-1} & \mathbf{Q}^{-1}\end{bmatrix} \tag{8.1}$$

- The same theorem is used to establish the non-singularity of the part. m.

- Summary on how to compute (i) the rank or (ii) the inverse of a part. m. $\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}$, given that either $\mathbf{A}_{11}$ or $\mathbf{A}_{22}$ is non-singular (see #101).
  – Rank: (1) Invert $\mathbf{A}_{11}$ or $\mathbf{A}_{22}$, (2) form the Schur complement, (3) find the rank of the Schur complement
  – Inverse: (1) Compute $\mathbf{A}_{11}^{-1}$ or $\mathbf{A}_{22}^{-1}$, (2) compute the Schur complement, (3) use formula (8.5.16) or (8.5.17).

Redo

- Prove L8.1.1
- Prove L8.4.1
- Prove L8.5.3
- Prove L8.5.4
- Prove T8.5.5

Chapter 9

Generalized Inverses

This chapter is about the finding inverses for matrices that are not square or nonsingular. Those so-called generalized inverses apparently become ubiquitous in statistical analysis and possibly other disciplines. The results here are used heavily in Chapter 11 where solutions to linear systems are discussed.

Important Concepts

- A *generalized inverse* (GI) of an $m \times n$ matrix $\mathbf{A}$ is any $n \times m$ matrix $\mathbf{G}$ such that $\mathbf{AGA} = \mathbf{A}$. A GI of $\mathbf{A}$ is typically denoted as $\mathbf{A}^-$

Basic Identities and Properties

- $(1/k)\mathbf{A}^-$ is a GI of $k\mathbf{A}$ (L9.3.1)
- $(\mathbf{A}^-)'$ is a GI of $\mathbf{A}'$ (L.9.3.3)
- If $\mathbf{A}$ is symmetric, then $(\mathbf{A}^-)'$ is a GI of $\mathbf{A}$ (a weaker prop compared to nonsingular $(\mathbf{A}^{-1})' = \mathbf{A}^{-1}$)
- $\text{rank}(\mathbf{A}^-\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^-) = \text{rank}(\mathbf{A})$ (C9.3.8)
- **L9.3.5**: $\mathcal{C}(\mathbf{B}) \subset \mathcal{C}(\mathbf{A}) \iff \mathbf{B} = \mathbf{AA}^-\mathbf{B}$ and $\mathcal{R}(\mathbf{C}) \subset \mathcal{R}(\mathbf{A}) \iff \mathbf{C} = \mathbf{CA}^-\mathbf{A}$
(A weaker prop compared to nonsingular $\mathbf{AA^{-1}} = \mathbf{I}$ or, equivalently, $\mathbf{AA^{-1}} - \mathbf{I} = \mathbf{0}$)
  - This lemma immediately relates to the consistency of a linear system:
  - **L9.5.1**: A linear system $\mathbf{AX} = \mathbf{B}$ is consistent iff $\mathbf{AA}^-\mathbf{B} = \mathbf{B}$

Notes

- Existence of GI:
  - **L9.1.4**: Every matrix has at least one GI
  - **L9.1.1**: A nonsingular matrix has exactly one GI (*i.e.* its inverse)
  - A matrix that is not nonsingular has infinite numbers of matrices (see last para of #114)
- On computing a GI of a matrix: The theorems in Sec 9.2.a give a way to compute a GI through partitioning
  - **T9.2.1**: Let $\mathbf{A}$ be $m \times n$ with rank r and partition it, $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$. Suppose $\mathbf{A}_{11}$ is nonsingular. Then, $\mathbf{G}$ is a GI of $\mathbf{A}$ iff $\mathbf{G} = \begin{bmatrix} \mathbf{A}_{11}^{-1} - \mathbf{X}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} - \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{Y} - \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{Z}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix}$ for some $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ (of appropriate dimensions)
  - The theorem above becomes particularly powerful when combined with T9.2.3
  - **T9.2.3**: Let $\mathbf{A}$ be $m \times n$ with rank r. Let $\mathbf{B}$ be obtained from $\mathbf{A}$ by permuting its rows/columns, *i.e.* $\mathbf{B} = \mathbf{PAQ}$, so that $\mathbf{B}_{11}$, the principal leading $r \times r$ submatrix of $\mathbf{B}$, is of rank r. Then, $\mathbf{G}$ is GI of $\mathbf{B}$ iff $\mathbf{G} = \mathbf{Q} \begin{bmatrix} \mathbf{B}_{11}^{-1} - \mathbf{X}\mathbf{B}_{21}\mathbf{B}_{11}^{-1} - \mathbf{B}_{11}^{-1}\mathbf{B}_{12}\mathbf{Y} - \mathbf{B}_{11}^{-1}\mathbf{B}_{12}\mathbf{Z}\mathbf{B}_{21}\mathbf{B}_{11}^{-1} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \mathbf{P}$

- The theorems above can be used to define a 5-step procedure to compute a GI of $\mathbf{A}$: 1) find the rank of $\mathbf{A}$, r; 2) locate the r independent rows and columns of $\mathbf{A}$; 3) Form a submatrix of $\mathbf{A}$, $\mathbf{B}_{11}$, by striking out all except those r rows and columns; 4) Compute the inverse of $\mathbf{B}_{11}$; 5) Use T9.2.1 by setting $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ to zero.

- Obtaining *every* GI in terms of a particular GI: There us an elegant theorem that provides two formulae for this purpose
  - **T9.2.7**: Let $\mathbf{A}$ be $m \times n$ and $\mathbf{G}$ a particular inverse. Then, a matrix $\mathbf{G}^*$ is a GI of $\mathbf{A}$ iff $\mathbf{G}^* = \mathbf{G} + \mathbf{Z} - \mathbf{GAZAG}$ for some $\mathbf{Z}$. Also, $\mathbf{G}^*$ is a GI iff $\mathbf{G}^* = \mathbf{G} + (\mathbf{I} - \mathbf{GA})\mathbf{T} + \mathbf{S}(\mathbf{I} - \mathbf{AG})$ for some matrices $\mathbf{T}$ and $\mathbf{S}$.
  - By letting $\mathbf{Z}$ or $\mathbf{T}, \mathbf{S}$ vary over all matrices, one can compute all GI's from one $\mathbf{G}$.

- Obtaining a matrix invariant to the choice of GI
  - **T9.4.1**: Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be matrices such that $\mathcal{R}(\mathbf{B}) \subset \mathcal{R}(\mathbf{A})$ and $\mathcal{C}(\mathbf{C}) \subset \mathcal{C}(\mathbf{B})$, then $\mathbf{BA}^-\mathbf{C}$ is invariant to the choice of $\mathbf{A}^-$. (Nearly-converse holds too.)

- The Schur treatment for GIs on Partitioned Matrices
  - A treatment similar to that of computing the inverse (see Section 8.5) applies to GIs
  - The Schur complement is now defined as $\mathbf{Q} = \mathbf{W} - \mathbf{VT^{-1}U}$.
  - **T9.6.1**: Computing the rank (v similar to T8.5.10). Suppose $\mathcal{C}(\mathbf{U}) \subset \mathcal{C}(\mathbf{T})$ and $\mathcal{R}(\mathbf{V}) \subset \mathcal{C}(\mathbf{T})$. Then, $\text{rank}\begin{pmatrix}\mathbf{T} & \mathbf{U} \\ \mathbf{V} & \mathbf{W}\end{pmatrix} = \text{rank}(\mathbf{T}) + \text{rank}(\mathbf{Q})$. Moreover, the GI of the partitioned matrix can be computed in terms of the GI of $\mathbf{T}^-, \mathbf{Q}^-, \mathbf{V}, \mathbf{U}$ (#121). The bottom right of the GI is $\mathbf{Q}^-$
  - **T9.6.4**: The bottom right of *any* GI of the above partitioned matrix is a GI of $\mathbf{Q}$.

- Comparison of the 5-step algorithm (above) and the Schur way: 5-step requires finding a basis that spans $\mathbf{A}$, whereas Schur-way requires $\mathcal{C}(\mathbf{U}) \subset \mathcal{C}(\mathbf{T})$ and $\mathcal{R}(\mathbf{V}) \subset \mathcal{C}(\mathbf{T})$.

Important Lemmas and Theorems

- **L9.3.5**: See about $\mathbf{B} = \mathbf{AA}^-\mathbf{B}$ – see definition above in Basic Identities and Properties
- **T9.2.7**: On computing all GIs from a particular GI (above)
- **T9.4.1**: Obtaining an matrix invariant to the choice of GI

Redo

- Prove T9.1.2
- Prove T9.2.7
- Prove T9.4.1
- Prove T9.6.4
- Solve Exercise 7
- Solve Exercise 12a

Chapter 10

Idempotent Matrices

Apparently the class of idempotent matrices is very important in statistics and other disciplines. They do have some very interesting and non-obvious properties (*e.g.* L10.1.1, L10.1.2 or T10.2.1)

Definition and Some Properties

- $\mathbf{A}$ is said to be idempotent if $\mathbf{A}^2 = A$
- The only $n \times n$ idempotent matrix is $\mathbf{I}_n$ (L10.1.1)
- $\mathbf{A}'$ is idempotent $\iff$ $\mathbf{A}$ is idempotent (L10.1.2)
- $\mathbf{A}$ is idempotent $\iff$ $\mathbf{I} - \mathbf{A}$ is idempotent (L10.1.2)

Notes

- One of the most non-intuitive relations is through the following theorem (and its corollary):
  - **T10.2.1**: For any square matrix $\mathbf{A}$ such that $\mathbf{A}^2 = k\mathbf{A}$, $\mathrm{tr}(\mathbf{A}) = k\,\mathrm{rank}(\mathbf{A})$
  - **C10.2.2**: For any idempotent matrix $\mathbf{A}$, $\mathrm{rank}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ (I guess I find this non-intuitive because the rank –an integer– is computed with tr which does not in general yield an integer!)
  - **T10.2.7**: $\mathbf{B}$ is GI of $\mathbf{A}$ iff $\mathbf{BA}$ is idempotent and $\mathrm{rank}(\mathbf{BA}) = \mathrm{rank}(\mathbf{A})$ (This also is non-intuitive b/c idempotent matrices seem like they are rare but yet so easy to obtain from a matrix and its inverse!)

Redo

- Prove T10.2.1
- Prove T10.2.7
- Solve Exercise 3

Chapter 11

Linear Systems: Solutions

Notes

- Relationships between $\mathcal{N}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{X})$
  - $\mathcal{N}(\mathbf{A}) \subset \mathcal{C}(\mathbf{I}-\mathbf{A})$
  - $\mathcal{N}(\mathbf{I}-\mathbf{A}) \subset \mathcal{C}(\mathbf{A})$ (p140)
  - **C11.2.2**: $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{I}-\mathbf{A}^{-}\mathbf{A})$
  - (1) If $\mathbf{AX} = \mathbf{0}$, then $\mathcal{C}(\mathbf{X}) \subset \mathcal{N}(\mathbf{A})$. (2) If also $\text{rank}(\mathbf{X}) = n - \text{rank}(\mathbf{A})$, then $\mathcal{C}(\mathbf{X}) = \mathcal{N}(\mathbf{A})$
  - **T11.7.1** Let $\mathbf{A}$ be $n \times n$. Then, $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{I}-\mathbf{A}) \iff \mathbf{A}$ is idempotent.
- General forms of solutions (useful for finding/checking alternative solutions etc.)
  - **T11.2.1** (Homogeneous): $\mathbf{X}^*$ is solution to homogeneous LS $\mathbf{AX} = \mathbf{0} \iff \mathbf{X}^* = (\mathbf{I}-\mathbf{A}^{-}\mathbf{A})\mathbf{Y}$ for some $\mathbf{Y}$
  - **T11.2.3** (Nonhom.): $\mathbf{X}_0$ is solution to $\mathbf{AX} = \mathbf{B}$. $\mathbf{X}^*$ is sol. $\iff \mathbf{X}^* = \mathbf{X}_0 + \mathbf{Z}^*$ for some $\mathbf{Z}^*$ sol. to $\mathbf{AZ} = \mathbf{0}$
  - **T11.2.4** (Nonhom.): $\mathbf{X}^*$ is solution to $\mathbf{AX} = \mathbf{B} \iff \mathbf{X}^* = \mathbf{A}^{-}\mathbf{B} + (\mathbf{I}-\mathbf{A}^{-}\mathbf{A})\mathbf{Y}$ for some $\mathbf{Y}$
  - **T11.5.1**: Let $\text{rank}(\mathbf{A}) = n$, $\text{rank}(\mathbf{B}) = p$. Then, $\mathbf{X}^*$ is solution to $\mathbf{AX} = \mathbf{B} \iff \mathbf{X}^* = \mathbf{A}^{-}\mathbf{B}$ for some GI $\mathbf{A}^{-}$
- Number of solutions
  - **L11.3.1** (homogeneous, vec.): $\mathbf{A}$ is $m \times n$. Then, $\dim[\mathcal{N}(\mathbf{A})] = n - \text{rank}(\mathbf{A})$. Note: system is $\mathbf{Ax} = \mathbf{0}$
  - **L11.3.2** (homogeneous, mat.) The dimension of solution space for $\mathbf{AX} = \mathbf{0}$ (in $n \times p$ matrix $\mathbf{X}$) is $p[n - \text{rank}(\mathbf{A})]$
  - Summary based on p142-143:
    - (homog.) If $\mathbf{A}$ has full col. rank, $\mathbf{AX} = \mathbf{0}$ has *one* solution (*i.e.* the null mat. $\mathbf{0}$)
    - (homog.) If $\mathbf{A}$ is col. rank deficient, then $\mathbf{AX} = 0$ has infinite solutions
    - (nonhom.) Since there is one-to-one correspondence between solutions to $\mathbf{AX} = \mathbf{0}$ and sol.s to $\mathbf{AX} = \mathbf{B}$, the two above apply to nonhomogeneous case too given that the nonhom. system is consistent
- Equivalence of linear systems: When are the solution sets of two linear systems the same?
  - **L11.6.1.** If $\mathbf{C}$ is of full col. rank, then the systems $\mathbf{AX} = \mathbf{B}$ and $\mathbf{CAX} = \mathbf{B}$ are equivalent
  - **L11.6.2.** For any $\mathbf{A}$, the linear systems $\mathbf{A}'\mathbf{AX} = \mathbf{A}'\mathbf{AF}$ and $\mathbf{AX} = \mathbf{AF}$ are equivalent
  - **Exercise 6.** If $\text{rank}(\mathbf{CD}) = \text{rank}(\mathbf{D})$, then system $\mathbf{CDAX} = \mathbf{CDB}$ is equivalent to the system $\mathbf{DAX} = \mathbf{DB}$
  - **Exercise 7.** If $\text{rank}[\mathbf{C}(\mathbf{A}, \mathbf{B})] = \text{rank}(\mathbf{A}, \mathbf{B})$, then the systems $\mathbf{AX} = \mathbf{B}$ and $\mathbf{CAX} = \mathbf{B}$ are equivalent
- Can we derive a vector/matrix $\mathbf{K}'\mathbf{X}$ invariant to the choice of solution $\mathbf{X}$?
  - **T11.10.1**. Let $\mathbf{K}$ be $n \times q$. Then, $\mathbf{K}'\mathbf{X}$ is invariant to solution $\mathbf{X} \iff \mathcal{R}(\mathbf{K}') \subset \mathcal{R}(\mathbf{A})$
  - **T11.10.3**. Let $\mathbf{X}, \mathbf{Y}$ be solutions to $\mathbf{AX} = \mathbf{B}$ and $\mathbf{A}'\mathbf{Y} = \mathbf{K}$. Then, $\mathbf{K}'\mathbf{X}_0 = \mathbf{Y}'_0\mathbf{B}$ for any solutions $\mathbf{X}_0, \mathbf{Y}_0$.
- Approaches to solve linear systems
  - If $\mathbf{A}$ is upper block-triangular such that the diagonal blocks $\mathbf{A}_{ii}$ are non-singular, then back-substitution (Section 11.8)
  - If $\mathbf{A}$ is lower block-triangular such that the diagonal blocks $\mathbf{A}_{ii}$ are non-singular, then forward elimination
  - Find a decomposition $\mathbf{A} = \mathbf{KT}$ such that solving becomes easier. More on Ch. 14 and 21 (p.149)
  - **T11.11.1**. Absorption: see Section p.152 and 4-step summary in p. 153

Chapter 12

Projections and Projection Matrices

Definitions and some properties

- *Orthogonality*: If matrix $\mathbf{Y}$ is orthogonal to every matrix in a subspace $\mathcal{U}$, then $\mathbf{Y}$ is orthogonal to $\mathcal{U}$: $\mathbf{Y} \perp \mathcal{U}$.
- *Projection*: Let $\mathbf{Y} \in \mathcal{V}$, $\mathbf{Z} \in \mathcal{V}$ and $\mathcal{U} \subset \mathcal{V}$. $\mathbf{Z}$ is the projection of $\mathbf{Y}$ on $\mathcal{U}$ if $(\mathbf{Y} - \mathbf{Z}) \perp \mathcal{U}$
  For any $\mathbf{Y} \in \mathcal{U}$, $\mathbf{Y}$ itself is the projection of $\mathbf{Y}$ on $\mathcal{U}$
- *Projection matrix*: $\mathbf{P_X} := \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$. That is, $\mathbf{P_X y}$ is the projection of any $\mathbf{y} \in \mathcal{R}^n$ on $\mathcal{C}(\mathbf{X})$
  - $\mathbf{P_X X} = \mathbf{X}$ (T12.3.4)
  - $\mathbf{P}'_\mathbf{X} = \mathbf{P_X}$ (*i.e.* symmetric, T12.3.4)
  - $\mathbf{P}^2_\mathbf{X} = \mathbf{P_X}$ (*i.e.* idempotent, T12.3.4)
  - $(\mathbf{I} - \mathbf{P_X})$ is also idempotent and symmetric
- The *orthogonal complement* $\mathcal{U}^\perp$ of $\mathcal{U} \subset \mathcal{V}$ is the set of all matrices in $\mathcal{V}$ that are orthogonal to $\mathcal{U}$
  - $\mathcal{U} = (\mathcal{U}^\perp)^\perp$ (T12.5.4)
  - $\mathcal{W} \subset \mathcal{U} \iff \mathcal{U}^\perp \subset \mathcal{W}^\perp$ (C12.5.6)
  - $\dim(\mathcal{V}) = \dim(\mathcal{U}) + \dim(\mathcal{U}^\perp)$ for $\mathcal{U} \subset \mathcal{U}^\perp$ (T12.5.12)

Notes

- Orthogonality conditions
  - **L12.1.1**. $\mathbf{Y} \perp \mathcal{U} = \mathrm{sp}(\mathbf{X}_1, \ldots, \mathbf{X}_k) \iff \mathbf{Y} \cdot \mathbf{X}_i$ for $i = 1, ..., k$. $\mathcal{U} \perp \mathcal{V} = \mathrm{sp}(\mathbf{Z}_1, \ldots, \mathbf{Z}_t) \iff \mathbf{X}_i \perp \mathbf{Z}_j \ \forall i, j$.
  - **C12.1.2**. $\mathbf{y}\ (\in \mathcal{R}^m) \perp \mathcal{C}(\mathbf{X}) \iff \mathbf{X}'\mathbf{y} = 0$
  - **T12.1.3**. Let $\mathbf{Y} \in \mathcal{V}, \mathcal{U} \subset \mathcal{V}$ and $\dim(\mathcal{U}) = r$. Then, there is *unique* $\mathbf{Z} \in \mathcal{U}$ s.t. $(\mathbf{Y} - \mathbf{Z}) \perp \mathcal{U}$. Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_r\}$ be an orthonormal basis for $\mathcal{U}$. Then, $\mathbf{Z} = c_1\mathbf{X}_1 + \ldots + c_r\mathbf{X}_r$ where $c_i = \mathbf{X} \cdot \mathbf{Y}_i$. Moreover, $\mathbf{Z} = \mathbf{Y} \iff \mathbf{Y} \in \mathcal{U}$.
- Projection of column vector and Projection Matrix
  - **T12.2.1**. $\mathbf{z}$ is the projection of $\mathbf{y} \in \mathcal{R}^n$ on $\mathcal{U} \subset \mathcal{R}^n \implies \mathbf{z} = \mathbf{X}\mathbf{b}^*$ for any solution $\mathbf{b}^*$ to $\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}$
    Also (C12.2.2), $\mathbf{z} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}$
  - **T12.3.1** (uniqueness). Let $\mathcal{U} \subset \mathcal{R}^n$. There is *unique* $n \times n$ matrix $\mathbf{A}$ s.t. $\mathbf{Ay}$ is the projection of any $\mathbf{y} \in \mathcal{R}^n$ on $\mathcal{U}$. For any $\mathbf{X}$ s.t. $\mathcal{C}(\mathbf{X}) = \mathcal{U}$, $\mathbf{A} = \mathbf{P_X}$. (see def. of $\mathbf{P_X}$ above)
  - **T12.3.4** (other outcomes of this theorem are properties above). $\mathbf{P_X} = \mathbf{X}\mathbf{B}^*$ for any solution to $\mathbf{X}'\mathbf{X}\mathbf{B} = \mathbf{X}'$
  - **T12.5.8** (orth. compl). Let $\mathbf{Y} \in \mathcal{V}, \mathcal{U} \subset \mathcal{V}$. Projection of $\mathbf{Y}$ on $\mathcal{U}^\perp$ is $\mathbf{Y} - \mathbf{Z}$ where $\mathbf{Z}$ is proj. of $\mathbf{Y}$ on $\mathcal{U}$
  - **12.5.11**. Let $\mathbf{Y} \in \mathcal{V}, \mathcal{U} \subset \mathcal{V}$. $\exists$ *unique* $\mathbf{Z} \in \mathcal{U}$, $\mathbf{W} \in \mathcal{U}^\perp$ s.t. $\mathbf{Y} = \mathbf{Z} + \mathbf{W}$. Also $\mathbf{Z}$ ($\mathbf{W}$) is proj. of $\mathbf{Y}$ on $\mathcal{U}$ ($\mathcal{U}^\perp$).
- Least squares connection
  - **T12.4.1**. Let $\mathbf{Y} \in \mathcal{V}$ and $\mathcal{U} \subset V$. Then, for $\mathbf{W} \in \mathcal{U}$, $||\mathbf{Y} - \mathbf{W}||$ is minimized by $\mathbf{W} = \mathbf{Z}$ where $\mathbf{Z}$ is $\mathbf{Y}$'s projection on $\mathcal{U}$. Moreover, $||\mathbf{Y} - \mathbf{Z}||^2 = \mathbf{Y} \cdot (\mathbf{Y} - \mathbf{Z})$.
  - **T12.4.2** (Residue). For a column vector projection $\mathbf{z}$, the residue $||\mathbf{y} - \mathbf{z}||^2$ is $\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}$

Chapter 13

Determinant

13.1 Definitions and some properties

- *Determinant* of $\mathbf{A}$: $\det(\mathbf{A}) = |\mathbf{A}| = \sum_{\mathbf{j}\in\mathcal{J}} (-1)^{\phi_n(j_1,\dots,j_n)} a_{1j_1} \dots a_{nj_n}$ where $\mathbf{j} = (j_1, \dots, j_n)$ and $\mathcal{J}$ is the set that contains all permutations of first n positive integers, and $\phi(\cdot)$ is, *e.g.* $\phi_5(3, 7, 2, 1, 4) = 2 + 3 + 1 + 0 = 6$ (p178)
  - $|\mathbf{A}'| = |\mathbf{A}|$ (L13.2.1)
  - If $\mathbf{B}$ is same with $\mathbf{A}$ except that one of its rows/columns is multiplied with k, then $|\mathbf{B}| = k|\mathbf{A}|$ (L13.2.1)
  - $|k\mathbf{A}| = k^n|\mathbf{A}|$ (C13.2.4)
  - $|-\mathbf{A}| = (-1)^n|\mathbf{A}|$
  - If $\mathbf{B}$ is formed from $\mathbf{A}$ by interchanging 2 of its columns or rows, then $|\mathbf{B}| = -|\mathbf{A}|$ (T13.2.6)
  - If $\mathbf{B}$ is $n \times p$ and $\mathbf{C}$ is $n \times q$, then $|\mathbf{B}, \mathbf{C}| = (-1)^{pq}|\mathbf{B}, \mathbf{C}|$ (T13.2.7)
  - $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$, and more generally $|\mathbf{A}_{11}\mathbf{A}_{22} \dots \mathbf{A}_{rr}| = |\mathbf{A}_{11}||\mathbf{A}_{22}| \dots |\mathbf{A}_{rr}|$ (T13.3.4)
  - $|\mathbf{A}^k| = |\mathbf{A}|^k$
  - $|\mathbf{A}'\mathbf{A}| = |\mathbf{A}|^2$
  - $\mathbf{A}$ is nonsingular $\iff$ $|\mathbf{A}| \neq 0$ (T13.3.7)
  - If $\mathbf{A}$ is nonsingular, then $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$
- Cofactor, minor and adjoint
  - *Minor*. Let $\mathbf{A}_{ij}$ be the submatrix of $\mathbf{A}$ obtained by striking out its ith row and jth column. Then, $|\mathbf{A}_{ij}|$ is called the minor of the element a_{ij} of $\mathbf{A}$.
  - *Cofactor*, α_{ij} is the signed minor: $\alpha_{ij} = (-1)^{i+j}|\mathbf{A}_{ij}|$
  - The *adjoint* of $\mathbf{A}$ is the (transposed) matrix of cofactors: $\mathrm{adj}(\mathbf{A}) = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \dots & \alpha_{nn} \end{bmatrix}$
- Important properties of adjoint
  - $\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = |\mathbf{A}|\mathbf{I}_n$ (T13.5.3)
  - If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1} = (1/|\mathbf{A}|)\,\mathrm{adj}(\mathbf{A})$
  - $\mathrm{adj}(\mathbf{AB}) = \mathrm{adj}(\mathbf{B})\mathrm{adj}(\mathbf{A})$ (Exercise 14)

13.2 Notes

- Computation in special cases
  - If $\mathbf{A}$ is (upper or lower) triangular, then $|\mathbf{A}| = a_{11}a_{22} \dots a_{nn}$
  - **T13.1.1.** $\begin{vmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{V} & \mathbf{W} \end{vmatrix} = |\mathbf{T}||\mathbf{W}|$. Repeated application $\implies$ For any block triangular $\mathbf{A}$, $|\mathbf{A}| = |\mathbf{A}_{11}||\mathbf{A}_{22}| \dots |\mathbf{A}_{rr}|$.
  - **C13.3.2.** $\begin{vmatrix} \mathbf{T} & \mathbf{T} \\ \mathbf{W} & \mathbf{V} \end{vmatrix} = \begin{vmatrix} \mathbf{V} & \mathbf{W} \\ \mathbf{T} & \mathbf{0} \end{vmatrix} = (-1)^{mn}|\mathbf{T}||\mathbf{W}|$

- If $\mathbf{T}$ is nonsingular, then $\begin{vmatrix} \mathbf{T} & \mathbf{U} \\ \mathbf{V} & \mathbf{W} \end{vmatrix} = |\mathbf{T}||\mathbf{W} - \mathbf{V}\mathbf{T}^{-1}\mathbf{U}|$

- Some cases where determinants are the same
  - **T13.2.10**. If $\mathbf{B}$ is formed from $\mathbf{A}$ by adding to any one row or column of $\mathbf{A}$, scalar multiples of one or more other rows, then $|\mathbf{B}| = |\mathbf{A}|$.
  - **T13.2.11**. For any upper/lower *unit* triangular $\mathbf{T}$, $|\mathbf{AT}| = |\mathbf{TA}| = |\mathbf{A}|$

- Some cases where determinant is 0 or ± 1
  - **L13.2.9**. If a row or column of $\mathbf{A}$ is multiple of another row or column, then $|\mathbf{A}| = 0$
  - **T13.3.7**. $|\mathbf{A}| = 0 \iff \mathbf{A}$ is nonsingular (see also properties above)
  - **C13.3.6** For any orthogonal matrix $\mathbf{P}$, $|\mathbf{P}| = \pm 1$

Chapter 14

Linear, Bilinear and Quadratic Forms

14.1 Definitions and some properties

- Linear, bilinear, quadratic forms
  - Linear form: $\mathbf{ax}$, linear function: $f(\mathbf{x})$. $\mathbf{a}$: coefficient vector of linear form
  - Bilinear form: $\mathbf{x}'\mathbf{Ay}$, bilinear function: $f(\mathbf{x}, \mathbf{y})$. $\mathbf{A}$: matrix of the linear form
  - Symmetric bilinear form is when $\mathbf{x}'\mathbf{Ay} = \mathbf{y}'\mathbf{Ax}$ for all $\mathbf{x}, \mathbf{y}$.
  - Quadratic form: $\mathbf{x}'\mathbf{Ax}$. $\mathbf{A}$: matrix of the quadratic form.
- Nonnegative definite quadratic forms
  - A quadratic form $\mathbf{x}'\mathbf{Ax}$ (or its matrix $\mathbf{A}$) is called *nonnegative definite* if $\mathbf{x}'\mathbf{Ax} \geq 0$ for all $\mathbf{x} \in \mathcal{R}^n$
  - A quadratic form $\mathbf{x}'\mathbf{Ax}$ (or its matrix $\mathbf{A}$) is called *positive definite* if $\mathbf{x}'\mathbf{Ax} > 0$ for all $\mathbf{x} \in \mathcal{R}^n$
  - A nonnegative quadratic form $\mathbf{x}'\mathbf{Ax}$ (or its matrix $\mathbf{A}$) is called *positive semidefinite* if there exist nonnull $\mathbf{x} \in \mathcal{R}^n$ such that $\mathbf{x}'\mathbf{Ax} = 0$.
  - A quadratic form $\mathbf{x}'\mathbf{Ax}$ (or its matrix $\mathbf{A}$) is called *nonpositive definite*, *negative definite* or *negative semidefinite* if $-\mathbf{x}'\mathbf{Ax}$ is nonnegative definite, positive definite or positive semidefinite.
  - A quadratic form that is neither nonnegative nor nonpositive definite is called *indefinite*.
- Some properties of nonnegative matrices/forms
  - If $\mathbf{A}$ is nonnegative and k is a scalar such that $k > 0$, then $k\mathbf{A}$ is also nonnegative (L14.2.3).
  - If $\mathbf{A}, \mathbf{B}$ are nonnegative, then $\mathbf{A} + \mathbf{B}$ is nonnegative. If one of them is positive, then $\mathbf{A} + \mathbf{B}$ is positive. (L14.2.4)
  - Any positive definite matrix is nonsingular (L14.2.8).
  - The inverse of a positive definite (semidefinite nonsingular) matrix is positive definite (semidefinite). (C13.2.11)
  - Every symmetric idempotent matrix is nonnegative definite (L14.2.7).
  - A symmetric nonnegative definite $\mathbf{A}$ is positive definite $\iff$ it $\mathbf{A}$ is nonsingular (C14.3.12).

14.2 Notes

- Equivalence/symmetry of bilinear/quadratic forms
  - A bilinear form $\mathbf{x}'\mathbf{Ay}$ is symmetric $\iff$ $\mathbf{A}$ is a symmetric matrix (p209).
  - **L14.1.1**. Two quadratic forms $\mathbf{x}'\mathbf{Ax}$ $\mathbf{x}'\mathbf{Bx}$ are identically equal $\iff$ $\mathbf{A} + \mathbf{A}' = \mathbf{B} + \mathbf{B}'$
  - **C14.1.2** For any quadratic form $\mathbf{x}'\mathbf{Ax}$, there is a *unique* symmetric matrix $\mathbf{B}$ such as $\mathbf{x}'\mathbf{Ax} = \mathbf{x}'\mathbf{Bx}$, namely $\mathbf{B} = (1/2)(\mathbf{A} + \mathbf{A}')$
- On nonnegative and positive definiteness
  - **T14.2.9**. Let $\mathbf{A}$ be $n \times n$ and $\mathbf{P}$ any $n \times m$. (1) If $\mathbf{A}$ is nonnegative, then $\mathbf{P}'\mathbf{AP}$ is nonnegative. (2) If $\mathbf{A}$ nonnegative and $\operatorname{rank}(\mathbf{P}) < m$, then $\mathbf{P}'\mathbf{AP}$ is positive semidefinite. (3) If $\mathbf{A}$ is positive, then $\mathbf{P}'\mathbf{AP}$ is positive.
  - **C14.2.10**. If $\mathbf{P}$ is nonsingular and $\mathbf{A}$ pos. definite (semidefinite), then $\mathbf{P}'\mathbf{AP}$ is pos. definite (semidefinite).
  - **C14.2.12**. Any principal submatrix of a positive definite (semidefinite) matrix is positive (nonnegative) definite.

- **C14.2.13**. The diagonal elements of a positive (nonnegative) definite matrix are positive (nonnegative).
- **C13.2.14**. Let $\mathbf{P}$ be any $n \times m$. If rank($\mathbf{P}$) $= m$, $\mathbf{P}'\mathbf{P}$ is positive definite, otherwise, it's positive semidefinite.
- **C14.2.14**. Let, $\mathbf{P}$ be nonsingular and $\mathbf{D}$ diagonal. Then, (1) $\mathbf{P}'\mathbf{D}\mathbf{P}$ is nonnegative (positive) $\iff$ $\mathbf{D}$'s elements are nonnegative (positive). (2) $\mathbf{P}'\mathbf{D}\mathbf{P}$ is positive definite if $\mathbf{D}$'s elements are nonnegative and include some zeros.

- Decomposition of symmetric matrices
  - **T14.3.4**. For any $n \times n$ symmetric $\mathbf{A}$, $\exists$ nonsingular $\mathbf{Q}$ such that $\mathbf{Q}'\mathbf{A}\mathbf{Q}$ is a diagonal matrix.
  - **T14.3.4**. For any $n \times n$ symmetric $\mathbf{A}$, $\exists$ nonsingular $\mathbf{P}$ and diagonal $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}'\mathbf{D}\mathbf{P}$.
  - **C14.3.6**. For any $n \times n$ $\mathbf{A}$, $\exists$ nonsingular $\mathbf{P}$ and scalars $d_1, \ldots, d_n$ s.t. the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is expressible as a linear combination $\sum_{i=1}^{n} d_i y_i^2$ of the squares of the elements $y_1, \ldots, y_n$ of the transformed vector $\mathbf{y} = \mathbf{P}\mathbf{x}$

- Decomposition of symmetric nonnegative matrices
  - **T14.3.7**. A non-null $n \times n$ $\mathbf{A}$ is symmetric nonnegative of rank r $\iff$ $\exists$ $r \times n$ $\mathbf{A}$ s.t. $\mathbf{A} = \mathbf{P}'\mathbf{P}$.
  - **C14.3.13**. An $n \times n$ symmetric $\mathbf{A}$ is positive definite $\iff$ $\exists$ nonsingular $\mathbf{P}$ s.t. $\mathbf{A} = \mathbf{P}'\mathbf{P}$

14.3 Redo:

Chapter 4

- Theorem 4.4.10

Chapter 11

- Exercise 3

Chapter 13

- Exercise 12.a