Outline of Foundations of Signal Processing

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Chapter 3 – Sequences and Discrete-time Systems

Discrete(-time) Fourier Transform, Z-Transform, Stochastic Processes

- Discrete-time (DT) Systems *Operators* having sequences as their inputs and outputs
  Some types #195: Linear, Memoryless, Causal, Shift-invariant, Stable (BIBO). **We focus on Linear&Shift-Invariant**
- Basic systems Shift, modulator, accumulator, averaging operator, maximum
- Table 3.3 (#203) !important! illustrates types of systems and their properties!
- Difference equations (DE): many systems can be described via DEs.
  DE systems are NOT linear or shift-invariant UNLESS the initial conditions are zero (Appendix 3.A.2)
- Linear Shift-invariant Systems (LSISs)
  - Impulse response h: the of an LSI sys for input δ
  - Convolution $(Hx)_n = (h * x)_n = \sum_{k\in\mathbb{Z}} x_k h_{n-k} = \sum_{k\in\mathbb{Z}} h_k x_{n-k}$
    Connection to inner product: $(h * x)_n = \sum_{x\in\mathbb{Z}} x_k h_{n-k} = \langle x_k, h^*_{n-k}\rangle$
    Commutativity, Associativity
    Deterministic autocorrelation (AC): $a_n = x_n *_n x^*_{-n}$
    Shifting: $x_n *_n \delta_{n-k} = x_{n-k}$
    IMPORTANT! Most properties above depend on the sums – must be abs. converging.
  - Filter: Impulse response of a system; Filtering: convolving a seq with impulse response.
  - THM 3.8: (BIBO) Stability An LSIS is BIBO-stable iff its impulse response is abs. summable. Limiting in $\ell^2(\mathbb{Z})$ avoids convergence issues #209
  - Circular conv: applies to finite-length (N) seqs, extended circularly
    $(h \circledast x)_n = \sum_{k=0}^{N-1} h_k x(n-k) \mod N$
  - THM 3.10 - Equiv of circular and linear conv:
    If length of x is M and of h is L, linear and circular conv is equiv if $N \geq L + M - 1$
- Discrete-time Fourier Transform (DTFT) is a cont fn of ω; $X(e^{j\omega}) = \sum_{n\in\mathbb{Z}} x_n \upsilon_n^{-1}$ where $\upsilon_n = e^{j\omega n}$
  Inverse DTFT: $x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega n}) e^{j\omega n} d\omega$
  IMPORTANT! Why FT?: "Simply b/c they are based on eigensequences of LSIS":
  $(H\upsilon)_n = \sum_{k\in\mathbb{Z}} h_k e^{j\omega(n-k)} = \sum_{k\in\mathbb{Z}} h_k e^{-j\omega k} e^{j\omega n} = \lambda_\omega e^{j\omega n} = \lambda_\omega \upsilon_n$ — an eigendecomposition
  - Convergence in the mean square
    if sequence not abs. convergent, we can pursue convergence in mean square $(\mathcal{L}^2)$. There may be points that don't converge, but we may remove them through Dirac delta fn #220.
  - Properties of DTFT: (Table 3.4 #222)
    Linearity, shift in time, shift in freq, scaling in time, time reversal, differentiation, moment computation, conv $(x * y = XY)$, circular convolution in freq, deterministic AC $(a_n = |X(e^{j\omega})|^2)$, deterministic crosscorelation (CC) $(c_{x,y,n} = X(e^{j\omega})Y^*(e^{j\omega}))$

- Parseval equality #226 (energy conservation): $||X||^2 = 2\pi||x||^2$; $X = Fx$, $F : \ell^2(\mathbb{Z}) \to \mathcal{L}^2([-\pi,\pi])$
- Freq Resp of filter: The DTFT of a filter (*i.e.* impulse resp of an LSIS) h: $H(e^{j\omega}) = \sum_{n\in\mathbb{Z}} h_n e^{-j\omega n}$

  We often write magnitude and phase resp separately: $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\arg(H(e^{j\omega}))}$

  Zero-phase Filter: Filter that has real freq resp

  Generalised Linear-Phase Filter: $H(e^{j\omega}) = |H(e^{j\omega})|e^{\alpha j+\beta}$

  Linear-Phase Filter: $H(e^{j\omega}) = |H(e^{j\omega})|e^{\alpha j}$
- Ideal Filter: Unrealizable filter, needs infinite support but able to pass requested frequencies exactly

  Ideal lowpass filt: $\sqrt{\frac{\omega_0}{2\pi}}\text{sinc}(\frac{1}{2}\omega_0 n) \to \begin{cases}\sqrt{2\pi/\omega_0} & |\omega| \le \frac{1}{2}\omega_0 \\ 0 & other\end{cases}$

  Ideal Nth-band filter: $\frac{1}{\sqrt{N}}\text{sinc}(\frac{\pi n}{N}) \to \begin{cases}\sqrt{N} & |\omega| \le \frac{\pi}{N} \\ 0 & other\end{cases}$
- Finite Impulse Response (FIR) Filters: realizable filters
- Linear-Phase Filters: Real-valued FIR filters that are symmetric or antisymmetric
- Allpass filters:

  i) Energy conservation: $||y||^2 = ||x||^2$ #231 ii) Ortonormal set: Shifts of h, $\{h_{n-k}\}_{k\in\mathbb{Z}}$, form an orthonormal set

  iii) Orthonormal basis: $\{\phi_k\}_{k\in\mathbb{Z}}$ where $\phi_{k,n} = h_{n-k}$ form an orthonormal basis for $\ell^2(\mathbb{Z})$

- z-transform DTFT cool but assumes convergence. z-transforms relaxes this through region of convergence (ROC) concept; $X(z)|_{z=e^{j\omega}} = X(e^{j\omega}) \implies$ DTFT is a special case of z-transform.
  - Defn: $X(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n}$ and ROC $= \{z : |X(z)| < \infty\}$ where $\upsilon_n = z^n = r^n e^{j\omega n}$
  - Properties similar to DTFT; Table 3.6 #243
  - Rational z-transforms important class, $H(z) = \frac{B(z)}{A(z)}$
  - THM 3.13: Rational AC A rational fn A(z) is the deterministic AC of a stable real sequence x, iff ... #245
  - Corollary 3.14: Spectral Factorization #247 A rational z-trans $A(z)$ is the deterministic AC of a stable real sequence x iff it can be factored as $A(z) = X(z)X(z^{-1})$
  - THM 3.15 (BIBO stability with rational fns) A causal LSIS is BIBO-stble iff the poles of its Transfer Function (TF) are inside the unit circle.
  - !important! ROC can't contain any poles #238, #246
- Discrete Fourier Trans (DFT) While DTFT is $F_{\text{DTFT}} : \ell^2(\mathbb{Z}) \to \mathcal{L}^2([-\pi,\pi])$, DFT is $F_{\text{DFT}} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

  DFT: $X_k = (Fx)_k = \sum_{n=0}^{N-1} x_n W_N^{kn}$;

  $W_N^{nk} = e^{-jkn2\pi/N}$; $\upsilon_n = e^{j\omega n}$

  Inverse DFT: $x_n = \frac{1}{N}(F^*X)_n = \frac{1}{N}\sum_{n=0}^{N_1} X_k W_N^{-kn}$
  - Again, is eigenseq #252: $(H\upsilon)_n = (h \circledast \upsilon)_n = \sum_{i=0}^{N-1} \upsilon_{(n-i) \mod N} h_i = \sum_{i=0}^{N-1} W_N^{ki)} h_i W_N^{-kn} = \sum_{i=0}^{N-1} \lambda_k \upsilon_n$
  - k: Discrete Frequency
  - Relation to DFT #255: $X(e^{j\omega})|_{\omega=\omega_k} = X_k$; $\omega_k = \frac{k2\pi}{N}$
  - Properties similar to DTFT, Table 3.7 #256
  - Modulation: A shift k_0 in frequency #257
  - Frequency response of filters: DFT of a N-dim filter (impulse resp of an LSIS) h is frequency response:

    $H_k = \sum_{n=0}^{N-1} h_n W_N^{kn}$

    Again, we separate as magnitude and phase response $H_k = |H_k|e^{j\arg(H_k)}$

- !Beware! DFT analysis of infinite-long sequences can be misleading!
  Misleading allpass behaviour #260
  Misleading linear-phase
- !Beware! On periodic seqs, windowing becomes crucial to avoid SPURIOUS high frequencies

- ... Multirate seqs and systems (Sec 3.7, #264–#285) ...
  - Upsampling and downsampling not commutative in general; only when upsampling and downsampling rates M, N are relatively prime (they have no common factors)
  - Hermitian transposition equals time reversal (#277)
  - Quadrature mirror formula (3.217 #277) or *power complementarity*
    This is central in the design of orthonormal filter banks
- DT Stochastic Processes (SP)
  A countably infinite collection of jointly distributed RVs $\{\ldots, x_0, x_1, \ldots\}$
  We study systems that act deterministically on Random signals.
  - 2nd order statistics: mean, var, std, AC, CC
  - For iid, $\mu_{x,n} = \mu_x, \sigma_{x,n} = \sigma_x$ etc. #286
  - Stationarity #287 Generalizes the iid prop by allowing dependence bw RVs
    An SP is stationary if the joint distros of $(x_{n_0}, x_{n_1}, \ldots, x_{n_L})$ and $(x_{n_0+k}, \ldots, x_{n_L+k})$ are identical for finite $\{n_0, \ldots, n_L\}$
  - Wide-sense SP (WSS) More relaxed than stationary processes
    When $\mu_{x,n} = \mu_x$, AC $a_{x,n,k} = a_{x,k}$. Two SPs x, y are jointly WSS when $c_{x,y,n,k} = c_{x,y,k}$
  - White Noise: An SP such that $\mu_{x,n} = 0$ and $\sigma_{x,n} = \sigma_x$, $a_{x,n,k} = \sigma_x^2 \delta_k$
  - Orthogonality (Tab 3.10 #301) $c_{x,y,k} = \mathrm{E}[x_n y^*_{n-k}] = 0$; in frequency: $C_{x,y}(e^{j\omega}) = 0$
  - Whitening (decorrelation) Processing that results in a white noise proc
  - System analysis Props of sys output y when SP x is input
    $\mu_{y,n} = \mu_y$, $a_{y,n,k} = a_{y,k}$, $c_{x,y,n,k} = c_{x,y,k}$
  - Autoregressive Moving-average (ARMA) Process Output of a BIBO-stable causal LSIS when input is white noise x.
    One generative model is: $y_n = \sum_{k=0}^{M} b_k x_{n-k} + \sum_{k=1}^{N} a_k y_{n-k}$
  - Moving Average (MA) Process when $a_i = 0 \forall i$ in the eq above
  - Autoregressive (AR) Process when $b_i = 0$!for $i > 0$!

The definitions of power and energy can be very confusing for discrete vs continuous or deterministic seqs vs SPs. Tab 3.8 (copied below) #293 clarifies:

Deterministic sequences	**WSS discrete-time stochastic processes**
Energy spectral density $A(e^{j\omega}) = \lvert X(e^{j\omega})\rvert^2$	Power spectral density $A(e^{j\omega}) = \sum_{k\in\mathbb{Z}} \mathrm{E}[x_n x^*_{n-k} e^{-j\omega k}]$
Energy $E = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{j\omega}) d\omega$ $E = a_0 = \sum_{n\in\mathbb{Z}} \lvert x_n\rvert^2$	Power $P = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{j\omega}) d\omega$ $P = a_0 = \mathrm{E}[\lvert x_n\rvert^2]$