

Outline of Foundations of Signal Processing

Evangelos Sariyanidi

January 26, 2015

Chapter 3

Discrete(-time) Fourier Transform, Z-Transform, Stochastic Processes

- **Discrete-time (DT) Systems** *Operators* having sequences as their inputs and outputs
Some types #195: Linear, Memoryless, Causal, Shift-invariant, Stable (BIBO). **We focus on Linear&Shift-Invariant**
- **Basic systems** Shift, modulator, accumulator, averaging operator, maximum
- **Table 3.3 (#203)** !important! illustrates types of systems and their properties!
- **Difference equations (DE):** many systems can be described via DEs.
DE systems are NOT linear or shift-invariant UNLESS the initial conditions are zero (Appendix 3.A.2)
- **Linear Shift-invariant Systems (LSISs)**
 - **Impulse response** h : the of an LSI sys for input δ
 - **Convolution** $(Hx)_n = (h * x)_n = \sum_{k \in \mathbb{Z}} x_k h_{n-k} = \sum_{k \in \mathbb{Z}} h_k x_{n-k}$
Connection to inner product: $(h * x)_n = \sum_{x \in \mathbb{Z}} x_k h_{n-k} = \langle x_k, h_{n-k}^* \rangle$
Commutativity, Associativity
Deterministic autocorrelation (AC): $a_n = x_n * x_n^*$
Shifting: $x_n * \delta_{n-k} = x_{n-k}$
IMPORTANT! Most properties above depend on the sums – must be abs. converging.
 - **Filter:** Impulse response of a system; **Filtering:** convolving a seq with impulse response.
 - **THM 3.8: (BIBO) Stability** An LSIS is BIBO-stable iff its impulse response is abs. summable.
Limiting in $\ell^2(\mathbb{Z})$ avoids convergence issues #209
 - **Circular conv:** applies to finite-length (N) seqs, extended circularly
 $(h \otimes x)_n = \sum_{k=0}^{N-1} h_k x_{(n-k) \bmod N}$
 - **THM 3.10 - Equiv of circular and linear conv:**
If length of x is M and of h is L , linear and circular conv is equiv if $N \geq L + M - 1$
- **Discrete-time Fourier Transform (DTFT)** is a cont fn of ω ; $X(e^{j\omega}) = \sum_{n \in \mathbb{Z}} x_n v_n^{-1}$ where $v_n = e^{j\omega n}$
Inverse DTFT: $x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega n}) e^{j\omega n} d\omega$
IMPORTANT! Why FT?: "Simply b/c they are based on eigensequences of LSIS":
 $(Hv)_n = \sum_{k \in \mathbb{Z}} h_k e^{j\omega(n-k)} = \sum_{k \in \mathbb{Z}} h_k e^{-j\omega k} e^{j\omega n} = \lambda_\omega e^{j\omega n} = \lambda_\omega v_n$ — an eigendecomposition
 - **Convergence in the mean square**
if sequence not abs. convergent, we can pursue convergence in mean square (\mathcal{L}^2). There may be points that don't converge, but we may remove them through Dirac delta fn #220.
 - **Properties of DTFT: (Table 3.4 #222)**
Linearity, shift in time, shift in freq, scaling in time, time reversal, differentiation, moment computation, conv ($x * y = XY$), circular convolution in freq, deterministic AC ($a_n = |X(e^{j\omega})|^2$), deterministic crosscorrelation (CC) ($c_{x,y,n} = X(e^{j\omega}) Y^*(e^{j\omega})$)

- Parseval equality #226 (energy conservation): $\|X\|^2 = 2\pi\|x\|^2$; $X = Fx$, $F: \ell^2(\mathbb{Z}) \rightarrow \mathcal{L}^2([-\pi, \pi])$
- Freq Resp of filter: The DTFT of a filter (*i.e.* impulse resp of an LSIS) h : $H(e^{j\omega}) = \sum_{n \in \mathbb{Z}} h_n e^{-j\omega n}$
We often write magnitude and phase resp separately: $H(e^{j\omega}) = |H(e^{j\omega})|e^{j \arg(H(e^{j\omega}))}$
Zero-phase Filter: Filter that has real freq resp
Generalised Linear-Phase Filter: $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\alpha\omega}$
Linear-Phase Filter: $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\alpha\omega}$
- Ideal Filter: Unrealizable filter, needs infinite support but able to pass requested frequencies exactly
Ideal lowpass filter: $\sqrt{\frac{\omega_0}{2\pi}} \text{sinc}(\frac{1}{2}\omega_0 n) \rightarrow \begin{cases} \sqrt{2\pi/\omega_0} & |\omega| \leq \frac{1}{2}\omega_0 \\ 0 & \text{other} \end{cases}$
Ideal N th-band filter: $\frac{1}{\sqrt{N}} \text{sinc}(\frac{\pi n}{N}) \rightarrow \begin{cases} \sqrt{N} & |\omega| \leq \frac{\pi}{N} \\ 0 & \text{other} \end{cases}$
- Finite Impulse Response (FIR) Filters: realizable filters
- Linear-Phase Filters: Real-valued FIR filters that are symmetric or antisymmetric
- Allpass filters:
 - i) Energy conservation: $\|y\|^2 = \|x\|^2$ #231
 - ii) Orthonormal set: Shifts of h , $\{h_{n-k}\}_{k \in \mathbb{Z}}$, form an orthonormal set
 - iii) Orthonormal basis: $\{\phi_k\}_{k \in \mathbb{Z}}$ where $\phi_{k,n} = h_{n-k}$ form an orthonormal basis for $\ell^2(\mathbb{Z})$
- z -transform DTFT cool but assumes convergence. z -transforms relaxes this through region of convergence (ROC) concept; $X(z)|_{z=e^{j\omega}} = X(e^{j\omega}) \implies$ DTFT is a special case of z -transform.
 - Defn: $X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n}$ and $\text{ROC} = \{z : |X(z)| < \infty\}$ where $v_n = z^n = r^n e^{j\omega n}$
 - Properties similar to DTFT; Table 3.6 #243
 - Rational z -transforms important class, $H(z) = \frac{B(z)}{A(z)}$
 - THM 3.13: Rational AC A rational fn $A(z)$ is the deterministic AC of a stable real sequence x , iff ... #245
 - Corollary 3.14: Spectral Factorization #247 A rational z -trans $A(z)$ is the deterministic AC of a stable real sequence x iff it can be factored as $A(z) = X(z)X(z^{-1})$
 - THM 3.15 (BIBO stability with rational fns) A causal LSIS is BIBO-stable iff the poles of its Transfer Function (TF) are inside the unit circle.
 - !important! ROC can't contain any poles #238, #246
- Discrete Fourier Trans (DFT) While DTFT is $F_{\text{DTFT}}: \ell^2(\mathbb{Z}) \rightarrow \mathcal{L}^2([-\pi, \pi])$, DFT is $F_{\text{DFT}}: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$
DFT: $X_k = (Fx)_k = \sum_{n=0}^{N-1} x_n W_N^{kn}$;
 $W_N^{nk} = e^{-jkn2\pi/N}$; $v_n = e^{j\omega n}$
Inverse DFT: $x_n = \frac{1}{N}(F^* X)_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k W_N^{-kn}$
 - Again, is eigenseq #252: $(Hv)_n = (h \otimes v)_n = \sum_{i=0}^{N-1} v_{(n-i) \bmod N} h_i = \sum_{i=0}^{N-1} W_N^{ki} h_i W_N^{-kn} = \sum_{i=0}^{N-1} \lambda_k v_n$
 - k : Discrete Frequency
 - Relation to DFT #255: $X(e^{j\omega})|_{\omega=\omega_k} = X_k$; $\omega_k = \frac{k2\pi}{N}$
 - Properties similar to DTFT, Table 3.7 #256
 - Modulation: A shift k_0 in frequency #257
 - Frequency response of filters: DFT of a N -dim filter (impulse resp of an LSIS) h is frequency response:
 $H_k = \sum_{n=0}^{N-1} h_n W_N^{kn}$
Again, we separate as magnitude and phase response $H_k = |H_k|e^{j \arg(H_k)}$

- !Beware! DFT analysis of infinite-long sequences can be misleading!
 - Misleading allpass behaviour #260
 - Misleading linear-phase
- !Beware! On periodic seqs, windowing becomes crucial to avoid SPURIOUS high frequencies
- ... Multirate seqs and systems (Sec 3.7, #264–#285) ...
 - Upsampling and downsampling not commutative in general; only when upsampling and downsampling rates M, N are relatively prime (they have no common factors)
 - Hermitian transposition equals time reversal (#277)
 - Quadrature mirror formula (3.217 #277) or *power complementarity*
 - This is central in the design of orthonormal filter banks
- DT Stochastic Processes (SP)

A countably infinite collection of jointly distributed RVs $\{\dots, x_0, x_1, \dots\}$
 We study systems that act deterministically on Random signals.

 - 2nd order statistics: mean, var, std, AC, CC
 - For iid, $\mu_{x,n} = \mu_x, \sigma_{x,n} = \sigma_x$ etc. #286
 - Stationarity #287 Generalizes the iid prop by allowing dependence bw RVs
 - An SP is stationary if the joint distros of $(x_{n_0}, x_{n_1}, \dots, x_{n_L})$ and $(x_{n_0+k}, \dots, x_{n_L+k})$ are identical for finite $\{n_0, \dots, n_L\}$
 - Wide-sense SP (WSS) More relaxed than stationary processes
 - When $\mu_{x,n} = \mu_x$, AC $a_{x,n,k} = a_{x,k}$. Two SPs x, y are jointly WSS when $c_{x,y,n,k} = c_{x,y,k}$
 - White Noise: An SP such that $\mu_{x,n} = 0$ and $\sigma_{x,n} = \sigma_x$, $a_{x,n,k} = \sigma_x^2 \delta_k$
 - Orthogonality (Tab 3.10 #301) $c_{x,y,k} = E[x_n y_{n-k}^*] = 0$; in frequency: $C_{x,y}(e^{j\omega}) = 0$
 - Whitening (decorrelation) Processing that results in a white noise proc
 - System analysis Props of sys output y when SP x is input
 - $\mu_{y,n} = \mu_y$, $a_{y,n,k} = a_{y,k}$, $c_{x,y,n,k} = c_{x,y,k}$
 - Autoregressive Moving-average (ARMA) Process Output of a BIBO-stable causal LSIS when input is white noise x .
 - One generative model is: $y_n = \sum_{k=0}^M b_k x_{n-k} + \sum_{k=1}^N a_k y_{n-k}$
 - Moving Average (MA) Process when $a_i = 0 \forall i$ in the eq above
 - Autoregressive (AR) Process when $b_i = 0$!for $i > 0$!

The definitions of power and energy can be very confusing for discrete vs continuous or deterministic seqs vs SPs. Tab 3.8 (copied below) #293 clarifies:

Deterministic sequences	WSS discrete-time stochastic processes
Energy spectral density $A(e^{j\omega}) = X(e^{j\omega}) ^2$	Power spectral density $A(e^{j\omega}) = \sum_{k \in \mathbb{Z}} E[x_n x_{n-k}^* e^{-j\omega k}]$
Energy $E = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega}) d\omega$ $E = a_0 = \sum_{n \in \mathbb{Z}} x_n ^2$	Power $P = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega}) d\omega$ $P = a_0 = E[x_n ^2]$