

# Chapter 1

## Introduction

Mixed notes from the book Convex Optimization [1]. Some parts will be supplemented by the book of Dattoro [2]. The book of Dattoro is extremely useful for concretely illustrating most of the concepts.

### Quotes

- “We study convex geometry because it is the easiest of geometries. For that reason, much of a practitioners energy is expended seeking invertible transformation of problematic sets to convex ones”. Dattoro [2].

# Chapter 2

## Convex Sets

### 2.1 Affine and convex sets and cones.

- **Affine set.** A set  $C \subseteq \mathbf{R}^n$  is *affine* if the line through any two points in  $C$  lies in  $C$ , i.e. if  $x_1, x_2 \in C$  and  $\theta \in \mathbf{R}$  implies  $\theta x_1 + (1 - \theta)x_2 \in C$ . More generally, an affine set is a set that contains all *affine combination* (see def. below) of two or more of its points.
  - \* Any affine set is convex [2].
  - \* The intersection of an arbitrary collection of affine sets remains affine [2].
  - \* Any affine set is open in the sense that it contains no boundary, e.g. the empty set  $\emptyset$ , point, line, plane, hyperplane, subspace etc [2]. Converse not necessarily true (e.g. see point just below about subspace.)
  - \* If  $C$  is an affine set and  $x_0 \in C$ , then the set  $V = C - x_0 = \{x - x_0 | x \in C\}$  is a subspace.
  - *Affine combination.* A combination of points  $\sum_{i=1}^k \theta_i x_i$  where  $\sum_{i=1}^k \theta_i = 1$  is an affine combination.
  - *Ambient space.* The space where a given set lives in, e.g. a plane can live in  $\mathbf{R}^2, \mathbf{R}^3$ . The choice of ambient space has implications on, for example, the interior of a set ([2], p34)
  - *Affine hull*, denoted  $\mathbf{aff} C$  is the smallest set that makes  $C$  affine.
  - *Affine dimension* of a set  $C$  is the dimension of  $\mathbf{aff} C$ . In fact dimension of a set is synonymous with affine dimension [2].
  - *Relative interior.* The interior of, for example, a plane in  $\mathbf{R}^3$  is empty. To “fix” this issue, we define the relative interior of  $C$  as:  $\mathbf{relint} C = \{x \in C | B(x, r) \cap \mathbf{aff} C \subseteq C \text{ for some } r > 0\}$
- **Convex sets.** A set  $C$  is convex if the line segment between any two points in  $C$  lies in  $C$ , i.e.  $\theta x_1 + (1 - \theta)x_2 \in C$  for any  $x_1, x_2 \in C$  and  $0 \leq \theta \leq 1$ .
  - *Convex combination.* A combination of points  $\sum_{i=1}^k \theta_i x_i$  where  $\sum_{i=1}^k \theta_i = 1$  and  $\theta_i \geq 0$  is a convex combination.
  - *Convex hull*  $\mathbf{conv} C$  of a set  $C$  is the smallest set that makes  $C$  convex.
- **Cones.** A set  $C$  is called a cone if for every  $x \in C$  and  $\theta \geq 0$  we have  $\theta x \in C$ . Cones can have very unintuitive shapes, see Fig. 35-41 in [2].
  - *Convex cone* is a set that is cone and also convex, i.e.  $\theta_1 x_1 + \theta_2 x_2 \in C$  for any  $x_1, x_2 \in C$  and for  $\theta_1, \theta_2 \geq 0$ .  
Some differences between a convex set and a convex cone: (i) A convex set doesn't have to include the origin, a convex cone does; (ii) a convex set can be bounded but a convex cone cannot.

Some important examples and notes ([1] p27):

- Any subspace is affine and a convex cone
- A line segment is convex but not affine
- A ray (i.e.  $\{\theta v + x_0 : \theta \geq 0\}$ ) is convex but not affine. It is convex cone if its base  $x_0$  is 0.
- Any line is affine.
- The empty set, any single point and the whole space are affine (hence convex) subsets of  $\mathbf{R}^n$
- Halfspaces (see below) are convex but not affine.

## 2.2 Hyperplanes, halfspaces, balls and polyhedra

- **Hyperplane** is a set of the form

$$\{x | a^T x = b\}$$

This set has several intuitive interpretations.

1. It is the hyperplane with a normal vector  $a$  and an offset  $b$  from the origin.
2. Let  $b$  be  $a^T x = b$ . Then,  $\{x | a^T x = b\} = \{x | a^T (x - x_0)\} = x_0 + a^\perp$  where  $a^\perp$  is the orthogonal complement of  $a$ .
3. More interpretations on p27-28.

- **Halfspace.** Each hyperplane divides  $\mathbf{R}^n$  into two halfspaces. A (closed) halfspace is of the form

$$\{x | a^T x \leq b\},$$

where  $a \neq 0$ .

- **Norm ball.** A norm ball is the set of the form  $B(x_c, r) = \{x : \|x_c - x\| \leq r, x \in \mathbf{R}^n\}$ , where  $\|\cdot\|$  is a given norm. Another common representation of the ball is  $B(x_c, r) = \{x_c + ru : \|u\| \leq 1, u \in \mathbf{R}^n\}$ .

Norm ball is convex (p30).

- **Norm cone** is the set  $C(x, t) = \{(x, t) : \|x\| \leq t, t \in \mathbf{R}\} \subseteq \mathbf{R}^{n+1}$ . It is a convex cone.
- **Proper cone** is a cone  $K \subseteq \mathbf{R}^n$  that satisfies the following:
  - is convex
  - is closed
  - is solid, *i.e.* it has nonempty interior
  - is pointed, which means that it contains no line (or, equivalently,  $x \in K, -x \in K \implies x = 0$ )

The concept of Proper Cone will be central in defining *generalized inequalities*.

- **Polyhedra.** A polyhedron is the solution set of a finite number of equalities and inequalities:

$$\mathcal{P} = \{x : a_j^T x \leq b_j, j = 1, \dots, m, c_i^T x = d_i, i = 1, \dots, m\}.$$

A simpler notation is  $\mathcal{P} = \{x | Ax \preceq b, Cx = d\}$ , where the symbol  $\preceq$  is *vector* or *componentwise* inequality (p32).

- **Simplexes.** Simplexes are a family of polyhedra; they are also a generalization of the triangle (and its interior); *i.e.* a 1D simplex is a line segment, a 2D simplex is the triangle and its interior, a 3D simplex is tetrahedron.
  - *Affine independence* means that for  $v_0, \dots, v_k \in \mathbf{R}^n$ , the points  $v_1 - v_0, \dots, v_k - v_0$  are linearly independent.
  - A simplex can be defined in terms affinely independent points:  $C = \mathbf{conv}\{v_0, \dots, v_k\}$ .

## 2.3 Operations that preserve convexity

- **Intersection:** Convexity is preserved under intersection; the intersection of even infinite convex sets is convex.
- **Affine functions.** Let  $f$  be an affine function, *i.e.*  $f(x) = Ax + b$ . Then the image of  $S$  under  $f$ ,

$$f(S) = \{f(x) | x \in S\}$$

and the inverse image of  $S$  under  $f$ ,

$$f^{-1}(S) = \{x | f(x) \in S\}$$

are both convex if  $S$  is convex.

- *Cartesian product.*  $S = S_1 \times S_2$  for two convex sets  $S_1, S_2$ ,  $S = \{x_1 + x_2 : x_1 \in S_1, S_2 \in x_2\}$  is convex.
- *Sum* The sum  $S$  of two convex sets  $S_1, S_2$ ,  $S = \{x_1 + x_2 : x_1 \in S_1, S_2 \in x_2\}$  is convex.

- **Linear-fractional and perspective functions.**

- *Perspective function* is the  $\mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$  function  $P(x, t) = x/t$  with domain  $\mathbf{dom} P = \mathbf{R}^{n+1} \times \mathbf{R}_{++}$ . That is, the perspective function normalizes the input vector so the last element is one, and then drops this last element. If a set  $C \subseteq \mathbf{dom} P$  is convex, then its image under  $P$  is also convex.
- *Linear-fractional function* is the composition  $P \circ g$  of a perspective function  $P$  with an affine function  $g$ . It is easy to show that linear-fraction functions preserve convexity: If  $S$  is convex, then its image  $g(S)$  under  $g$  will be convex, then its image under perspective will also be convex.

## 2.4 Generalized inequalities, minimum and minimal elements

- **Generalized inequality.** A proper cone  $K$  (see above) can be used to define a *generalized inequality* as follows:

$$x \preceq_K y \iff y - x \in K$$

A strict generalized inequality  $x \prec y$  is defined as

$$x \prec_K y \iff y - x \in \text{int } K$$

*Example.* The nonnegative orthant  $\mathbf{R}_+^n$  is a proper cone and for  $K = \mathbf{R}_+^n$  the associated inequality  $\preceq_K$  corresponds to componentwise inequality  $x \prec y$ .

Properties of generalized inequalities:

1.  $\preceq_K$  is preserved under addition: If  $x \preceq_K y$  and  $u \preceq_K v$ , then  $x + u \preceq_K y + v$
2.  $\preceq_K$  is transitive: If  $x \preceq_K y$  and  $y \preceq_K z$ , then  $x \preceq_K z$ .
3.  $\preceq_K$  is preserved under nonnegative scaling: If  $x \preceq_K y$  and  $\alpha > 0$ , then  $\alpha x \preceq_K \alpha y$ .
4.  $\preceq_K$  is reflexive: If  $x \preceq_K x$ .
5.  $\preceq_K$  is antisymmetric: If  $x \preceq_K y$  and  $y \preceq_K x$ , then  $x = y$ .
6.  $\preceq_K$  is preserved under limits: If  $x_i \preceq_K y_i$  for  $i = 1, 2, \dots$  and  $x_i \rightarrow x$  and  $y_i \rightarrow y$ , then  $x \preceq_K y$ .

Properties 1,2,3 are shared by the strict inequality  $x \prec_K y$  too, property 4 is strictly *not* shared by it. Also, and additional property for strict generalized inequalities (probably shared by non-strict too):

- if  $x \prec_K y$ , then for small enough  $u, v$  we have  $x + u \prec_K y + v$ .

- **Minimum and minimal elements.** An essential difference between a regular inequality and a generalized one is that *not all points are comparable*; that is, one of the two inequalities  $x \leq y$  or  $x \geq y$  has to hold. This is not the case for generalized inequality.

*Example.* Consider the proper cone  $K = \mathbf{R}_+^n$ , and points  $x = (3, 3)$ ,  $y = (5, 5)$  and  $z = (4, 2)$ . Clearly,  $x$  and  $y$  are comparable and  $x \preceq_K y$ . Similarly,  $y$  and  $z$  are comparable and  $z \preceq_K y$ . However,  $x$  and  $z$  are not comparable.

- *Minimum element.* We say that  $x \in S$  is the minimum element of  $S$  (w.r.t.  $\preceq_K$ ) if for every  $y \in S$  we have  $x \preceq_K y$ , which happens if and only if

$$S \subseteq x + K$$

where  $x + K$  is the set of all the points that are (i) comparable to  $x$  and (ii) greater than or equal to  $x$  (confer Fig. 2.17 of [1] or Fig. 43 of [2]).

There can be at most *one* minimum point.

- *Minimal element.* First of all, a minimum point is also a minimal point. But a minimal point can exist even if there is no minimum. There can be more than one minimal points.

We say that  $x \in S$  is the minimal point of  $S$  (w.r.t.  $\preceq_K$ ) if for any  $y \in S$ ,  $y \preceq_K x$  holds only if  $y = x$ . Or, equivalently,

$$(x - K) \cap S = \{x\}$$

where  $x - K$  denotes the set of all points that are comparable to  $x$  and are less than or equal to  $x$  w.r.t.  $\preceq_K$ .

Confer Fig. 2.17 of [1] or Fig. 43 of [2]. Note that in Fig. 43b it's impossible to draw the cone  $K$  (centered on any of the minimal points) that would contain the entire  $\mathcal{C}_2$ , therefore  $\mathcal{C}_2$  has no minimum.

## 2.5 Separating and supporting hyperplanes

- **Separating hyperplane** is a hyperplane that separates two convex sets.

*Separating hyperplane theorem.* Let  $C$  and  $D$  be two convex sets that do not intersect. Then, there exists  $a \neq 0$  and  $b$  such that  $a^T x \leq b$  for all  $x \in C$  and  $a^T x \geq b$  for all  $x \in D$ .

- *Strict separation* is defined similarly when  $\geq$  and  $\leq$  are replaced by  $>$  and  $<$ .

*Converse of separating hyperplane theorem* is not in general true, but one can obtain it by adding additional constraints. One variant of converse theorem would be (see p50): Any two convex sets at least one of which is open are disjoint if and only if there exists a separating hyperplane.

- **Supporting hyperplane.** Suppose that  $C \subseteq \mathbf{R}^n$  and  $x_0$  is a point in its boundary, *i.e.*  $x_0 \in \mathbf{bd} C = \mathbf{cl} C \setminus \mathbf{int} C$ . If  $a \neq 0$  satisfies  $a^T x \leq a^T x_0$  for all  $x \in C$  (*i.e.* the entire set  $C$  lies on one side of the hyperplane), then the hyperplane  $\{x : a^T x = a^T x_0\}$  is called a *supporting hyperplane*.

Convexity and supporting hyperplanes are intimately connected:

- *Supporting hyperplane theorem:* If  $C$  is convex, then there exists a supporting hyperplane for any  $x_0 \in \mathbf{bd} C$ .
- Partial converse of the theorem: If a set is closed, has nonempty interior and has a supporting hyperplane at every point on its boundary, then it is convex.

## 2.6 Dual cones and generalized inequalities

- **Dual cone.** Let  $K$  be a cone. Then, the set  $K^* = \{y : x^T y \geq 0 \text{ for all } x \in K\}$  is called the dual cone of  $K$ . Some properties of dual cone  $K^*$  (p51 and p53):
  - A dual cone  $K^*$  is a ... cone.
  - $K^*$  is always *convex*.
  - $K^*$  is always closed.
  - $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$ .
  - If  $K$  has nonempty interior, then  $K^*$  is pointed.
  - If the closure of  $K$  is pointed, then  $K^*$  has nonempty interior.
  - $K^{**}$  is the closure of the convex hull of  $K$  (hence, if  $K$  is convex and closed,  $K^{**} = K$ ).
  - If  $K$  is a proper cone, then  $K^*$  is also a proper cone.

An intuitive way of defining a dual cone is shown in Fig. \*\*\*\*\* *Examples to dual cones.*

- The dual cone of  $K = \mathbf{R}_+^n$  is itself.
- The dual cone of a line in space is its orthogonal complement.
- More generally, the dual cone of a subspace  $V \subseteq \mathbf{R}^n$  is its orthogonal complement  $\{y : y^T v = 0 \text{ for all } v \in V\}$
- **Dual generalized inequalities.** Like any proper cone, the dual of a proper cone,  $K^*$ , induces a generalized inequality  $\preceq_{K^*}$ . The following relationships relating the generalized inequality of proper cone and its dual seem to be fundamental:

- $x \preceq_K y$  iff  $\lambda^T x \leq \lambda^T y$  for all  $\lambda \succeq_{K^*} 0$ .
- $x \prec_K y$  iff  $\lambda^T x < \lambda^T y$  for all  $\lambda \succ_{K^*} 0, \lambda \neq 0$ .

Those are central properties that allow us to characterize minimum and minimal points w.r.t. a cone  $K$  in terms of its dual generalized inequalities,  $\preceq_{K^*}$ .

- *Minimum point.*  $x \in S$  is the minimum point in  $S$  w.r.t. generalized inequality  $\preceq_K$  iff for all  $\lambda \succ_{K^*} 0$ ,  $x$  is the unique minimizer of  $\lambda^T z$  over  $z \in S$ . Geometrically, this means that for any  $\lambda \succ_{K^*}$ , the hyperplane

$$\{z : \lambda^T(z - x) = 0\}$$

is a strict supporting hyperplane (strict means that the hyperplane intersects  $S$  only at one point, *e.g.* it's not a trivial hyperplane such a line being supporting hyperplane of a line).

- *Minimal point.* There is a gap between necessary and sufficient conditions. A note to illustrate why:  $x \in S$  may be minimal of  $S$  but there may be no  $\lambda$  for which  $x$  minimizes  $\lambda^T z$  over  $z \in S$  (see Figure 2.25, 2.26, p57).

# Bibliography

- [1] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.
- [2] J. Dattorro, *Convex optimization & Euclidean distance geometry*, 2010.