Outline of Foundations of Signal Processing

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March 2, 2015

Chapter 5 – Sampling and Interpolation

The chapter considers how to move from discrete to continuous spaces (and vice versa), which were studied in earlier chapters. This is essential for many studies including wavelet analysis.

Throughout the chapter two things are placed to focus: Interpolation followed by sampling (IFS) and sampling followed by interpolation (SFI). These two are discussed for three types of inputs: i) Continuous functions, ii) sequences and iii) periodic functions. For each type of input the discussion is repeated for a) orthogonal vectors and b) nonorthogonal vectors (often biorthogonal pairs of bases).

Finite-dimensional Sequences

- Orthogonal vectors
  - Thm 5.1 – Recovery from orthogonal vecs (#423)
    Let $\Phi^* : \mathbb{C}^M \to \mathbb{C}^N$ be sampling operator and $\Phi : \mathbb{C}^N \to \mathbb{C}^M$ interpolation operator.
    Approximated fn is: $\hat{x} = Px = \Phi\Phi^* x$, the best approximation of x in S.
    If rows of Φ^* orthonormal, error is orthogonal to $S = \mathcal{R}(\Phi)$, *i.e.* $x - \hat{x} \perp S$
- Non-orthogonal vectors
  - We have sampling op $\tilde{\Phi}^*$ and interpolation op Φ.
    Define $\tilde{S} = \mathcal{N}(\tilde{\Phi}^*)^\perp = \text{span}(\{\tilde{\phi_k}\}_{k=0}^{N-1})$ and
    $S = \mathcal{R}(\Phi)$
  - Def - Consistency (#428): When $\tilde{\Phi}^*\Phi = I$, ops $\tilde{\Phi}^*, \Phi$ are called *consistent*
  - Def - Ideally Matched (#427): When $S = \tilde{S}$.
  - Thm 5.2 - Recovery from non-orthogonal vectors (#428)
    Similar to Thm 5.1 but we need consistency to satisfy $x - \hat{x} \perp \tilde{S}$.
    If ideal matching also exists, approximation $\hat{x} = Px = \Phi\tilde{\Phi}^* x$ is best approximation, and also $S = \tilde{S}$

Sequences and Functions

Now we consider (infinite-dimensional) sequences (Sec 5.3) and functions (Sec 5.4). Of particular importance will be the *bandlimited* sequences, which we'll be able to represent with a finite number of coefficients. The abstract space S above will be typically replaced by the space of bandlimited sequences $\text{BL}[\omega_1, \omega_2]$.

- Sequences
  - Def - Shift-invariant Subspace of $\ell^2(\mathbb{Z})$
    $S \subset \ell^2(\mathbb{Z})$ is shift-inv. subspace *with respect to a shift* $L \in \mathbb{Z}^+$ when $x_n \in S$ implies $xn - kL \in S$ for every $k \in \mathbb{Z}$.
  - Def - Generator of S
    $s \in \ell^2(\mathbb{Z})$ is called a *generator* of S when $S = \overline{\text{span}}(\{s_{n-kL}\}_{k\in\mathbb{Z}})$
  - Thm 5.4 - Recovery from sequences (very similar to Thm 5.1 above)

- Def - Bandlimited sequence (#437) (otherwise called full-band sequence)
  A seq x is called *bandlimited* when there is $\omega_0 \in [0, 2\pi)$ s.t. its DTFT X satisfies $X(e^{j\omega} = 0$ for all $|\omega| \in (\frac{1}{2}\omega_0, \pi]$
- Def - Bandwidth the smallest ω_0 for bandlimited sequence.
- Def - Subspace of Bandlimited seqs (#438)
  Set of sequences in $\ell^2(\mathbb{Z})$ with bandwidth at most ω_0 is *a closed subspace* denoted $\mathrm{BL}[-\frac{1}{2}\omega_0, \frac{1}{2}\omega_0]$.
- Thm 5.7 (#439) Projection to BL spaces
  The best approximation of x on $\mathrm{BL}[\pi/N, \pi/N]$ involves ideal LP filters (*i.e.* trunctation of spectrum x to $[-\pi/n, \pi/N]$):
  $\hat{x}_n = \frac{1}{\sqrt{N}} \sum\limits_{k\in\mathbb{Z}} y_k \mathrm{sinc}(\frac{\pi}{N}(n-kN))$, $n \in \mathbb{Z}$ where
  $y_k = \frac{1}{\sqrt{N}} \sum\limits_{n\in\mathbb{Z}} x_n \mathrm{sinc}(\frac{\pi}{N}(n-kN))$
- Thm 5.8 - Sampling theorem for seqs (#440)
  If x in $\mathrm{BL}[pi/N, \pi/N]$:
  $x_n = \sum\limits_{k\in\mathbb{Z}} x_{kN} \mathrm{sinc}(\frac{\pi}{N}(n-kN))$
- Def - Aliasing (#440): When the component of x at freq ω affects a component $\tilde{\omega} \neq \omega$ (i.e. assumes -aliases- the role of the component)
- Thm 5.9 - Recovery for sequences, non-orthogonal
  Counterpart of Thm 5.2 for infinite-dim sequences.

• Functions

- Very similar definitions for bandwidth (#452), subspace of bandlimited functions (#453), aliasing (#460) and theorem of projection to bandlimited subspace (#454).
- Thm 5.15 – Sampling Theorem (cornerstone theorem)
  If $x \in BL[-pi/T, pi/T]$, then $x(t) = \sum\limits_{n\in\mathbb{Z}} x(nT) sinc(\frac{\pi}{T}(t-nT))$
- Thm 5.16: Continuous convolution via discrete operators (#462)
  Let $x \in BL[\frac{-\pi/T}{\pi/T}]$, and $\tilde{h}_n = \langle h(t), sinc(\frac{\pi}{T}(n-T))_t \rangle$. Let sampled signal be $\tilde{x}_n = \sqrt{T}x(nT)$ and discrete conv output be $\tilde{y} = \tilde{x} * \tilde{x}$. Then:
  $y(t) = \sqrt{T} \sum_{n\in\mathbb{Z}} \tilde{y}_n sinc(\frac{\pi}{T}(t-nT))$

• Periodic Functions

- Similar definitions for bandlimited and bandwidth (k_0 s.t. $X_k = 0$ for $k > k_0$, see #481), subspace of bandlimited fns (#482), projection to bandlimited subspace (Thm 5.25, #484), and sampling thm (Thm 5.26 #489).
- Dirichlet Kernel (#482-#483): The counterpart of the ideal LP filter (with sinc)for periodic functions.
  $d(t) = \sum\limits_{k=-K}^{K} e^{j(2\pi/T)} kt$
  * Acts like a LP filter and forms a basis for $\mathrm{BL}[\cdot,\cdot]$ space. (#483)
  * Plays the role of interpolating filter sinc in sampling Thm 5.26 (#489)

• Projection onto convex sets via Papoulis-Gerhberg algorithm.
A nice algorithm for signal reconstruction.
Assumption: x is bandlimited perioduc function in $\mathcal{L}^2([0,1])$ where a part for $t \in (\alpha, \beta)$ is unobserved/missing.
Then, we know x belongs to two sets: $S_1 = BL\{\ldots\}$ and $S_2 = \mathcal{L}^2([0,1])$.
We find the full signal by exploiting these two set memberships in an alternating and iterative manner (#492)
Discrete version from in #494