TUTORIAL 3: VECTOR TRANSFORMATIONS AND MATRICES

Overview

- Matrices
- Vector2D transformations
- Vector3D and 3x3 Matrices
- Object Layout

References

- Stanley B. Lippman, Josée Lajoie, and Barbara E. Moo: C++ Primer. 5th Edition. Addison-Wesley (2013)
- Eric Lengyel: Mathematics for 3D Game Programming and Computer Graphics, Course Technology (2012)

NXM MATRIX

- An n x m matrix **M** is an array of numbers having n rows and m columns. If n=m, the we say that the matrix **M** is a square matrix.

$$\mathbf{M}_{3x4} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \end{bmatrix}$$

$$row(\mathbf{M}_{3x4}, 2) = \begin{bmatrix} M_{21} & M_{22} & M_{23} & M_{24} \end{bmatrix} \qquad column(\mathbf{M}_{3x4}, 3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \end{bmatrix}$$

MATRIX SCALAR MULTIPLICATION

- Given a scalar **α** and an n x m matrix **M**, the product **α**M = M**α** is given by

$$\alpha\mathbf{M} = \mathbf{M}\alpha = \begin{bmatrix} \alpha M_{11} & \alpha M_{12} & \cdots & \alpha M_{1m} \\ \alpha M_{21} & \alpha M_{22} & \cdots & \alpha M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha M_{n1} & \alpha M_{n2} & \cdots & \alpha M_{nm} \end{bmatrix}$$

MATRIX ADDITION

- Matrices add element wise. Given two nxm matrices **F** and **G**, the sum **F**+**G** is given by

$$\mathbf{F} + \mathbf{G} = \begin{bmatrix} F_{11} + G_{11} & F_{12} + G_{12} & \cdots & F_{1m} + G_{1m} \\ F_{21} + G_{21} & F_{22} + G_{22} & \cdots & F_{2m} + G_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1} + G_{n1} & F_{n2} + G_{n2} & \cdots & F_{nm} + G_{nm} \end{bmatrix}$$

MATRIX MULTIPLICATION

- Two matrices **F** and **G** can be multiplied, provided that the number of columns in **F** is equal to the number of rows in **G**. If **F** is n x m matrix and **G** is an m x p matrix, then the product **FG** is an n x p matrix whose (i, j) entry is given by

$$(\mathbf{FG})_{ij} = \sum_{k=1}^{m} F_{ik} G_{kj}$$

- Actually, an (i, j) entry in **FG** is the dot project of the vectors row(**F**, i) and column(**G**, j).

- An n-dimensional vector can be thought of as n x 1 matrix. Hence, multiplying an m x n matrix **M** with an n-dimensional vector **V**, written **MV**, yields an m-dimensional vector **V'**.

SCALING A 2D VECTOR

- To scale a vector **V** by a factor of **α**, we simply calculate **V**' = α**V**.

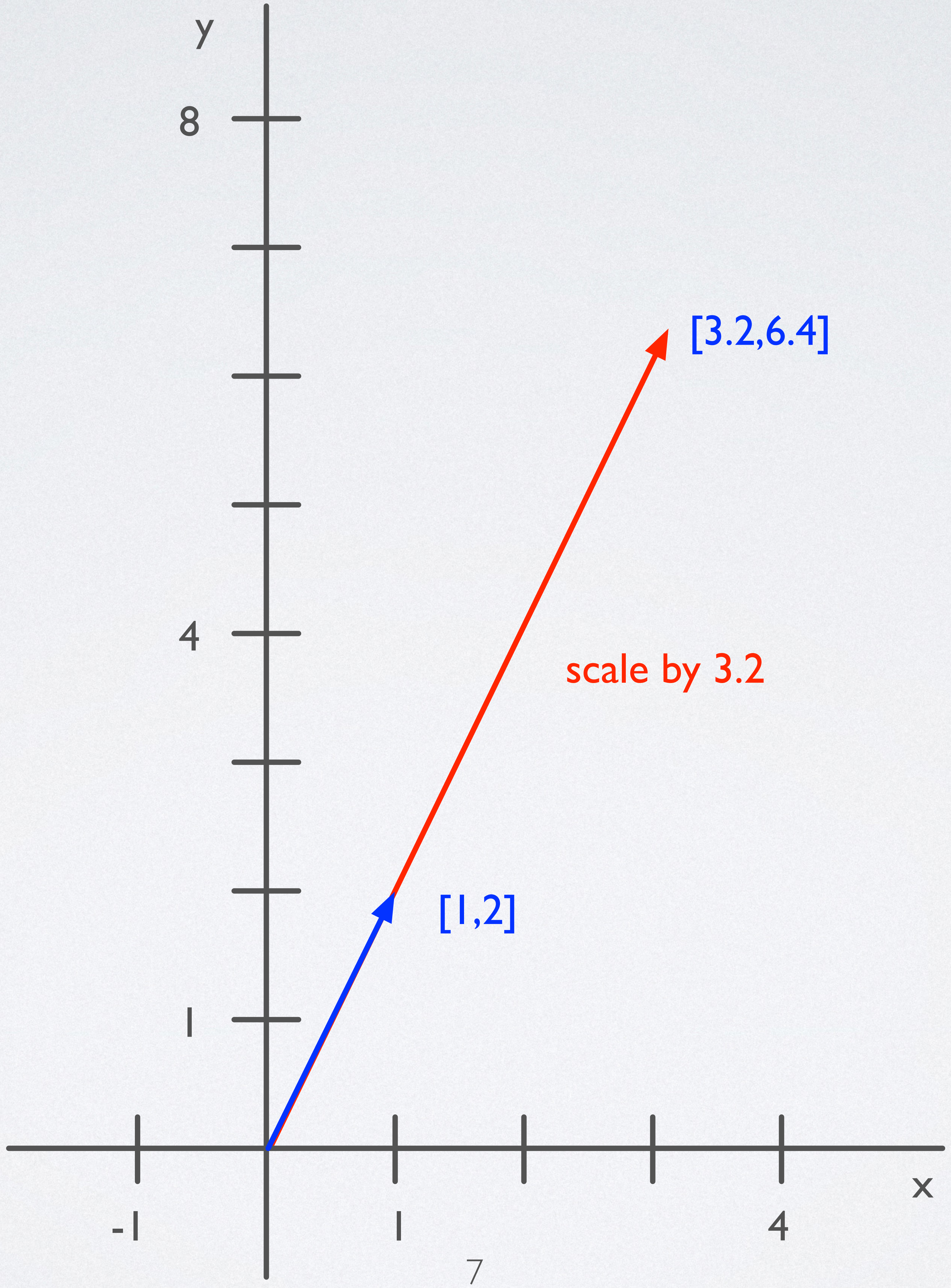
- This operation can also be expressed as the (2-dimensional) matrix product

$$\mathbf{V}' = \begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

- For example, the vector [1.0 2.0] scaled by 3.2 yields a vector [3.2, 6.4]:

$$\begin{bmatrix} 3.2 \\ 6.4 \end{bmatrix} = \begin{bmatrix} 3.2 & 0 \\ 0 & 3.2 \end{bmatrix} \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix}$$

SCALING BY 3.2



ROTATION OF A 2D VECTOR

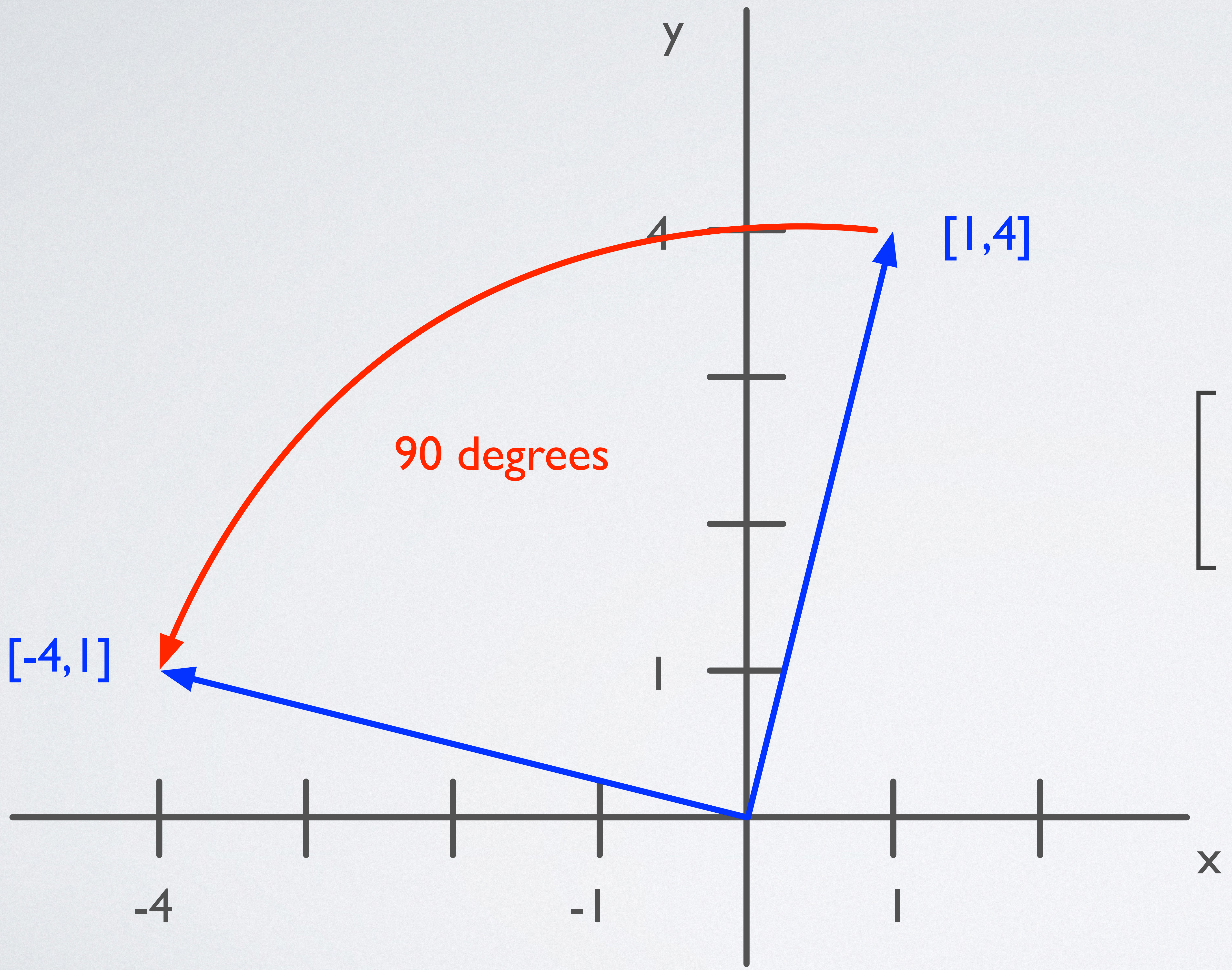
- In 2 dimensions, we can rotate a point in the plane by the following matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- For example, the vector [1.0 4.0] rotated by 90 degrees yields a vector [-4.0, 1.0]:

$$\begin{bmatrix} -4.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 4.0 \end{bmatrix}$$

ROTATION BY 90 DEGREES



$$\begin{bmatrix} -4.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 4.0 \end{bmatrix}$$

TRANSLATION OF A 2D VECTOR

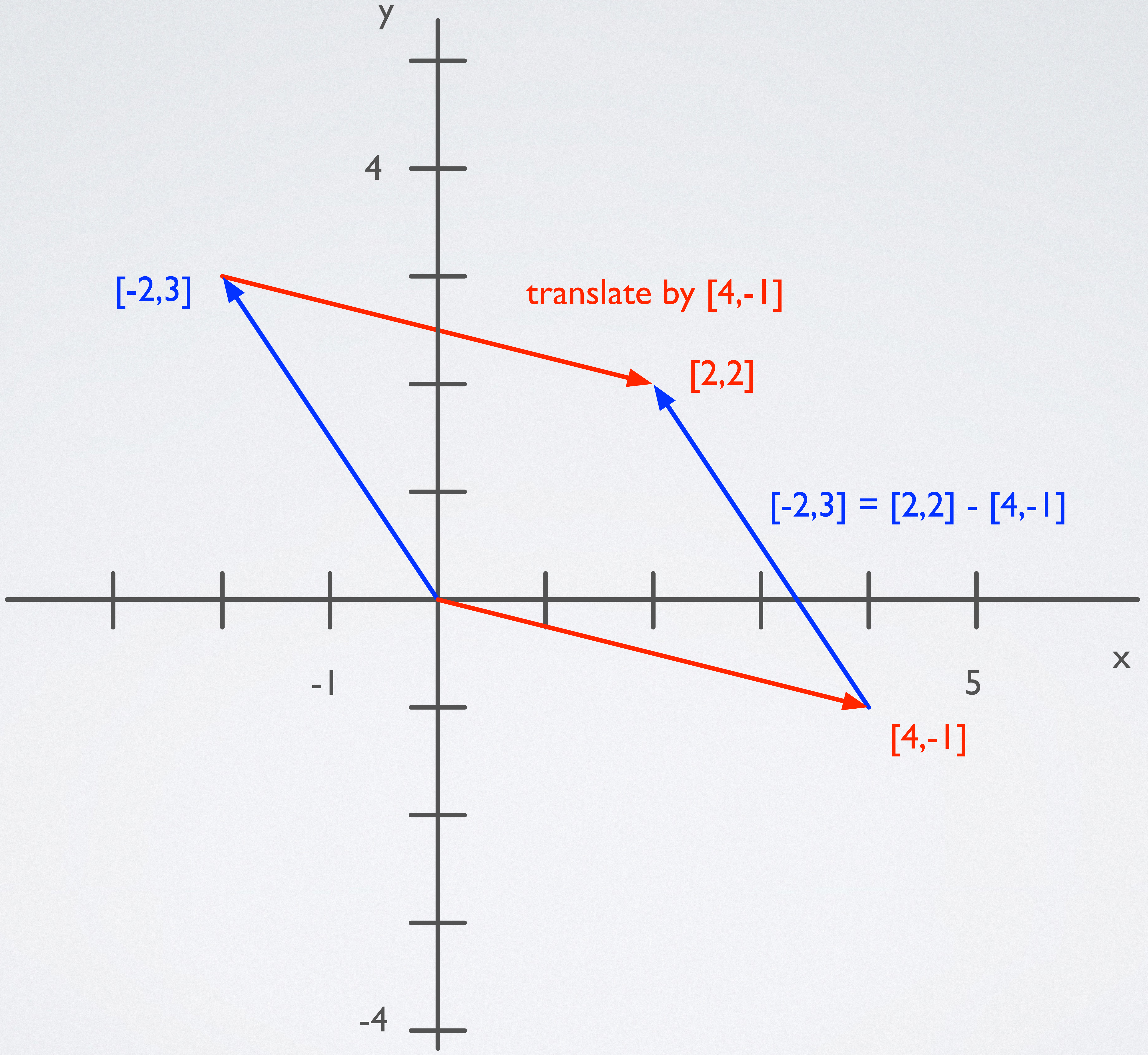
- Translation of a vector can be achieved by simply adding an offset to it:

$$\mathbf{V}' = \begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \end{bmatrix} + \begin{bmatrix} \Delta_x \\ \Delta_y \end{bmatrix}$$

- Translation does not affect the orientation or scale of the axes.

- Unfortunately, this operation cannot be expressed in terms of a multiplication by a 2x2 matrix. We are left with matrix addition, which is computationally undesirable.

TRANSLATION BY [4,-1]



HOMOGENEOUS COORDINATES

- There exists a compact and elegant way to represent all transformations, including translations, within a single mathematical entity.

- We extend 2D vectors to 3D homogeneous coordinates (i.e, 3D vectors) and uses 3x3 matrices to transform them:

$$\mathbf{V}_3 = \begin{bmatrix} V_x \\ V_y \\ 1 \end{bmatrix} \qquad \mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{T} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} M_{11} & M_{12} & T_x \\ M_{21} & M_{22} & T_y \\ \hline 0 & 0 & 1 \end{array}\right]$$

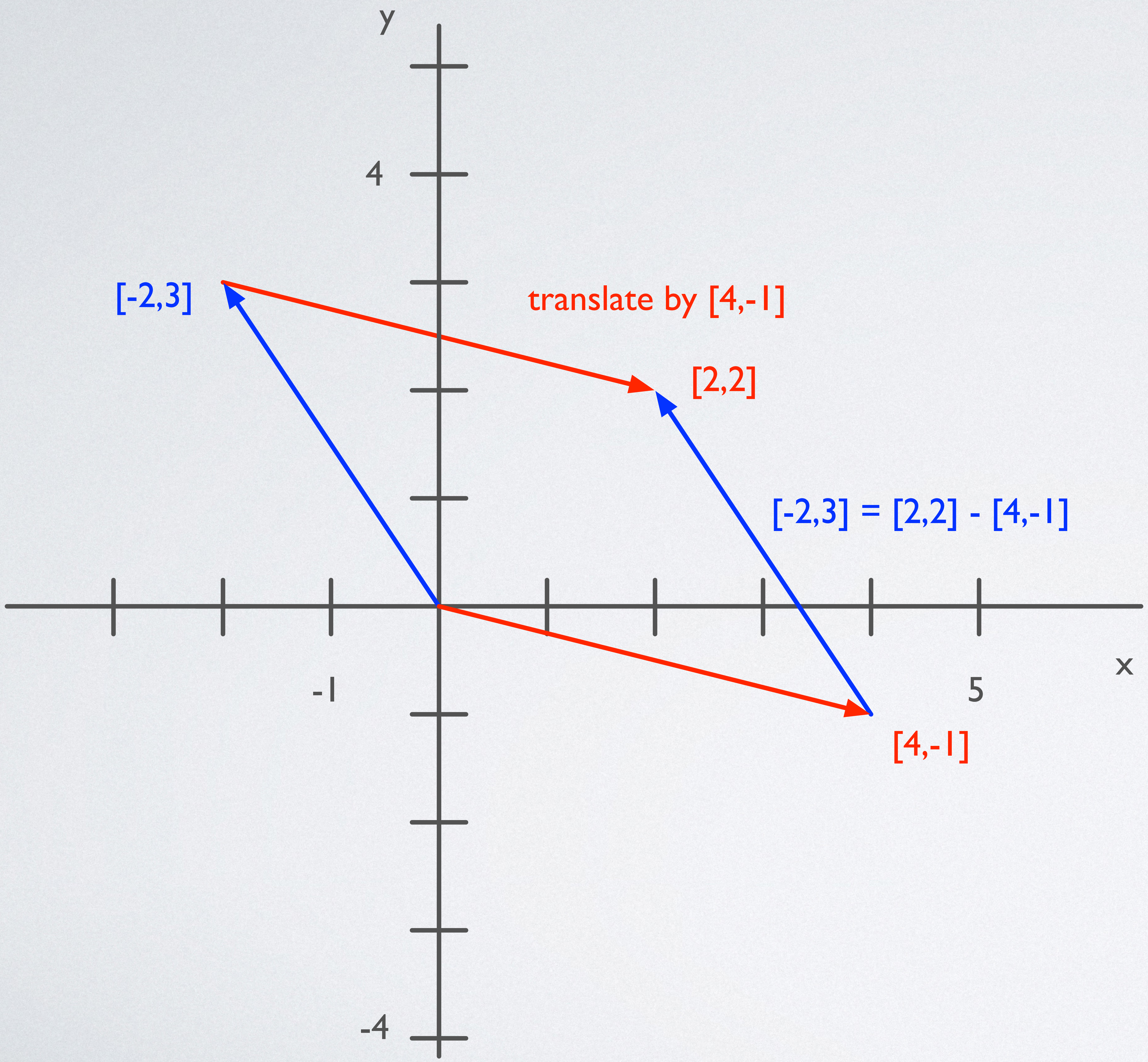
SCALE, ROTATION, AND TRANSLATION MATRIX

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Note: In the translation matrix **T**, the 2x2 matrix **M** is the identity matrix. That is, under translation, scale and rotation remain unchanged.

TRANSLATION BY [4,-1]



Translate [-2,3]:

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

Translate origin:

$$\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

TO AND FROM HOMOGENEOUS COORDINATES

In order to obtain 3-dimensional homogeneous coordinates, we add a w-coordinate with the value 1:

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix} \Rightarrow \begin{bmatrix} V_x \\ V_y \\ V_w \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ 1 \end{bmatrix}$$

To return to 2D, we divide all coordinates by V_w and drop the w-coordinate:

$$\begin{bmatrix} V'_x/V_w \\ V'_y/V_w \\ V_w/V_w \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

CLASS VECTOR3D

```
#include "Vector2D.h"

class Vector3D
{
private:
    Vector2D fBaseVector;
    float fW;

public:
    Vector3D( float aX = 1.0f, float aY = 0.0f, float aW = 1.0f ) noexcept;
    Vector3D( const Vector2D aVector ) noexcept;                  // implicit type conversion form Vector2D to Vector3D

    float x() const noexcept { return fX; }
    float y() const noexcept { return fY; }
    float w() const noexcept { return fW; }

    float operator[]( size_t aIndex ) const;                      // index-based coordinate access

    explicit operator Vector2D() const noexcept;                  // explicit conversion operator to Vector2D

    Vector3D operator*( const float aScalar ) const noexcept;     // 3D scalar multiplication
    Vector3D operator+( const Vector3D& aOther ) const noexcept;  // 3D vector addition
    float dot( const Vector3D& aOther ) const noexcept;           // 3D dot product

    friend std::ostream& operator<<( std::ostream& aOStream, const Vector3D& aVector );
};
```

CLASS MATRIX3X3

```
#include "Vector3D.h"

class Matrix3x3
{
private:
    Vector3D fRows[3];

public:
    Matrix3x3() noexcept;
    Matrix3x3( const Vector3D& aRow1, const Vector3D& aRow2, const Vector3D& aRow3 ) noexcept;

    Matrix3x3 operator*( const float aScalar ) const noexcept;           // multiplication with scalar
    Matrix3x3 operator+( const Matrix3x3& aOther ) const noexcept;       // matrix addition

    Vector3D operator*( const Vector3D& aVector ) const noexcept;        // multiplication with a vector

    static Matrix3x3 scale( const float aX = 1.0f, const float aY = 1.0f );        // build scaling matrix
    static Matrix3x3 translate( const float aX = 0.0f, const float aY = 0.0f );    // build translation matrix
    static Matrix3x3 rotate( const float aAngleInDegree = 0.0f );                  // build rotation matrix

    const Vector3D row( size_t aRowIndex ) const;                        // return read-only row vector
    const Vector3D column( size_t aColumnIndex ) const;                  // return read-only column vector
};
```

OBJECT LAYOUT VECTOR2D & VECTOR3D

```
   Vector2D:
0:     float fX;
1:     float fY;
```

```
   Vector3D:
       Vector2D fBaseVector:
0:         float fX;
1:         float fY;
2:     float fW;
```

- Typically, the memory layout of classes in C++ is left to the implementation. However, the layout usually follows some standard principles:
  - Non-static member variables with the same access level are places adjacent to each other with the layout preserving the order of the declaration of member variables. Technically, the C++ object layout yields a record structure in which the fields are the member variables.
  - If there are no virtual members, the object record just contains the member variables. In case of Vector2D and Vector3D, the member variables fX, fY, and fW can be viewed as elements of a packed array of **float** values.