# CATEGORICAL TORELLI THEOREMS FOR GUSHEL–MUKAI THREEFOLDS

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Abstract. We show that a general ordinary Gushel–Mukai (GM) threefold $X$ can be reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ together with an extra piece of data coming from tautological subbundle of the Grassmannian $\mathrm{Gr}(2,5)$. We also prove that $\mathcal{K}u(X)$ determines the birational isomorphism class of $X$, while $\mathcal{K}u(X')$ determines the isomorphism class of a special GM threefold $X'$ if it is general. As an application, we prove a conjecture of Kuznetsov–Perry in dimension three under a mild assumption. Finally, we use $\mathcal{K}u(X)$ to restate a conjecture of Debarre–Iliev–Manivel regarding fibers of the period map for ordinary GM threefolds.

## 1. Introduction

In recent times, derived categories have played an important role in algebraic geometry; in many cases, much of the geometric information of a variety/scheme $X$ is encoded by its bounded derived category of coherent sheaves $\mathrm{D}^b(X)$. In this setting, one of the most fundamental questions that can be asked is whether $\mathrm{D}^b(X)$ recovers $X$ up to isomorphism, in other words, whether a *derived Torelli theorem* holds for $X$. For varieties with ample or anti-ample canonical bundle (which include Fano varieties and varieties of general type), this question was answered affirmatively by Bondal–Orlov in [BO01].

### 1.1. Kuznetsov components and categorical Torelli theorems.

Therefore, for the class of varieties above, it is natural to ask whether they are also determined up to isomorphism by *less* information than the whole derived category $\mathrm{D}^b(X)$. A natural candidate for this is a subcategory $\mathcal{K}u(X)$ of $\mathrm{D}^b(X)$ called the *Kuznetsov component*. This subcategory has been studied extensively by Kuznetsov and others (e.g. [Kuz03, Kuz09, KP18b]) for many Fano varieties, including Gushel–Mukai (GM) varieties.

In fact, the question of whether $\mathcal{K}u(X)$ determines $X$ up to isomorphism has been studied for certain cases in the setting of Fano threefolds. In [BMMS12], the authors show that the Kuznetsov component completely determines cubic threefolds up to isomorphism, in other words, a *categorical Torelli theorem* holds for cubic threefolds $Y$. The same result was also verified in [PY20]. On the other hand, for many Fano varieties, the Kuznetsov component $\mathcal{K}u(X)$ does not determine the isomorphism class, but only the birational isomorphism class of $X$. This is known as a *birational categorical Torelli theorem*. For instance, Kuznetsov components determine birational isomorphism class of every index 1 prime Fano threefolds of even genus $g \geq 8$. For GM threefolds – the focus of our paper – by [KP19] it is known that there are birational GM threefolds with equivalent Kuznetsov components. So there are two natural questions to ask in this setting:

**Question 1.1.**

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(1) Does $\mathcal{K}u(X)$ determine the birational equivalence class of $X$?
(2) What extra data along with $\mathcal{K}u(X)$ do we need to identify a particular GM threefold $X$ from its birational equivalence class?

1.2. **Main Results.**

1.2.1. *(Refined) categorical Torelli for Gushel–Mukai threefolds.* In the present paper, we deal with the case of index 1 prime Fano threefolds of degree 10, also known as Gushel–Mukai threefolds (GM threefolds for short), which are split into two types: ordinary GM threefolds which arise as a quadric sections of a linear sections of the Grassmannian $\mathrm{Gr}(2,5)$, and special GM threefolds which arise as double covers of a codimension three linear sections of $\mathrm{Gr}(2,5)$, branched over a degree ten K3 surface. Our first main theorem is concerned with ordinary GM threefolds, and gives an answer to Question 1.1 (2):

**Theorem 1.2** (Theorem 10.2)**.** *Let $X$ be a general ordinary GM threefold, and set $\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E}\rangle \subset \mathrm{D}^b(X)$ where $\mathcal{E}$ is the restriction of the tautological bundle on $\mathrm{Gr}(2,5)$ to $X$. Let $\pi : \mathcal{D} \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $\mathcal{K}u(X) \subset \mathcal{D}$. Then the data of $\mathcal{K}u(X)$ along with the object $\pi(\mathcal{E})$ is enough to recover $X$ up to isomorphism.*

We prove the above theorem by considering the moduli space of Bridgeland stable objects in the alternative Kuznetsov component $\mathcal{A}_X$ with respect to $-x$, one of the two $(-1)$-classes in the numerical Grothendieck group of $\mathcal{A}_X$, i.e. $\chi(x,x) = -1$ where $\chi$ is the Euler form. The alternative Kuznetsov component is defined by the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee\rangle$ and there is an equivalence $\Xi : \mathcal{K}u(X) \simeq \mathcal{A}_X$. In particular, we show that this moduli space is isomorphic to the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics (Theorem 7.13). Indeed, first we show that the unique exceptional curve contracted in $\mathcal{C}(X)$ is the rational curve of conics whose ideal sheaf $I_C$ is not in $\mathcal{A}_X$ and that the image is the smooth point represented by $\pi(\mathcal{E})$ (Proposition 7.1), so $\mathcal{C}_m(X)$ forms an irreducible component of the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects in $\mathcal{A}_X$ with respect to $-x$. Then we show this component actually occupies the whole moduli space, which is the most difficult and technical part of the article; we briefly sketch the argument. We start with a stable object $F \in \mathcal{A}_X$. It suffices to show that $F$ is isomorphic to the projection of ideal sheaf $I_C$ of some conic $C \subset X$. First we assume that $F$ is semistable in the double tilted heart $\mathrm{Coh}^0_{\alpha,\beta}(X)$ (cf. Section 4.5). Then by a standard wall-crossing argument as in [PY20] and [APR19], we prove that $F[-1]$ is a slope-semistable sheaf of rank one. Since its class is $[F] = -[I_C]$, we get $F \cong I_C[1]$. Then we assume that $F$ is not semistable in the double tilted heart $\mathrm{Coh}^0_{\alpha,\beta}(X)$. Our main tools are inequalities in [PR20], [PY20, Proposition 4.1], Lemma 4.7 and Lemma 4.8, which allow us to bound the rank and first two Chern characters $\mathrm{ch}_1, \mathrm{ch}_2$ of the destabilizing objects and their cohomology objects. Since $F \in \mathcal{A}_X$, by using the Euler characteristics $\chi(\mathcal{O}_X, -)$ and $\chi(\mathcal{E}^\vee, -)$ we can obtain a bound on $\mathrm{ch}_3$. Then we deduce that the Harder–Narasimhan factors of $F$ are the expected ones. As a result, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$. Thus the data $(\mathcal{K}u(X), \pi(\mathcal{E}))$ recovers $\mathcal{C}(X)$. A classical result of Logachev [Log12] states that $X$ can be recovered up to isomorphism from $\mathcal{C}(X)$. Thus the theorem is proved. On the other hand, for special GM threefolds which are general ("general special" for short), we show that a categorical Torelli theorem holds:

**Theorem 1.3** (Theorem 10.9)**.** *Let $X$ and $X'$ be general special GM threefolds, and assume that there is an equivalence of categories $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X$ and $X'$ are isomorphic.*

We prove this theorem by considering the equivariant Kuznetsov components $\mathcal{K}u(X)^{\mu_2}$, first discussed in [KP18a], and exploiting the fact that $X$ is the double cover of a degree 5 index 2

Picard rank 1 Fano threefold $Y$, branched over a quadric hypersurface $\mathcal{B} \subset Y$. In this case, the equivariant Kuznetsov component is equivalent to $\mathrm{D}^b(\mathcal{B})$ where $\mathcal{B}$ is a K3 surface. Therefore, a number of results concerning the Fourier–Mukai partners of K3 surfaces can be used to deduce that $\mathcal{K}u(X)^{\mu_2} \simeq \mathcal{K}u(X')^{\mu_2}$ implies $\mathcal{B} \cong \mathcal{B}'$. Then the fact that the moduli space of Fano threefolds of such $Y$ is a point can be used to deduce that indeed, $X \cong X'$.

1.2.2. *Birational categorical Torelli for Gushel–Mukai threefolds.* Next, returning to the setting of ordinary GM threefolds, we show that a birational categorical Torelli theorem holds for general ordinary GM threefolds, which answers Question 1.1 (2).

**Theorem 1.4** (Theorem 10.3)**.** *Let $X$ and $X'$ be general ordinary GM threefolds, and suppose that there is an equivalence of categories $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X$ is birationally equivalent to $X'$.*

To prove this result, first we study the Bridgeland moduli space of stable objects $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ in $\mathcal{A}_X$ with respect to $y-2x$, the other $(-1)$-class in the numerical Grothendieck group of $\mathcal{A}_X$. We identify the moduli space $M_G^X(2,1,5)$ of Gieseker semistable sheaves of rank 2, $c_1 = 1$ and $c_2 = 5$ on $X$ with the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ (Theorem 8.5). We then invoke a few more results from [DIM12]. More precisely, an equivalence of categories $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ identifies the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ with either $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ or $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$. The former case gives an isomorphism of minimal surfaces $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$. Blowing $\mathcal{C}_m(X)$ up at the smooth point associated to $\pi(\mathcal{E})$ gives $\mathcal{C}(X)$, and blowing up $\mathcal{C}_m(X')$ at the image of $\pi(\mathcal{E})$ under $\Phi$ gives $\mathcal{C}(X'_c)$, where $X'_c$ is certain birational transformation of $X'$, associated with a conic $c \subset X'$. Then by Logachev's Reconstruction Theorem for $\mathcal{C}(X)$, $X$ is isomorphic to $X'_c$ which is birational to $X'$. For the latter case, we start with the isomorphism $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. In fact, $M_G^{X'}(2,1,5)$ is birational to $\mathcal{C}(X'_L)$, where $X'_L$ is another birational transformation of $X'$, associated with a line $L \subset X'$. Since $\mathcal{C}(X'_L)$ is a surface of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$. Then by the same argument as in the previous case, $X$ is isomorphic to some birational transformation of $X'$.

1.2.3. *The Kuznetsov–Perry Conjecture.* In [KP19], the authors studied GM varieties of arbitrary dimension and proved the Duality Conjecture [KP18b, Conjecture 3.7] for them, i.e. they showed that the period partner or period dual of a GM variety $X$ shares the same Kuznetsov component $\mathcal{A}_X$ as $X$. Combining earlier results [DK15, Theorem 4.20] on the birational equivalence of these varieties, this gives a strong evidence for the following conjecture:

**Conjecture 1.5** ([KP19, Conjecture 1.7])**.** *If $X$ and $X'$ are GM varieties of the same dimension such that there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X$ and $X'$ are birationally equivalent.*

By a careful study of Bridgeland moduli spaces of stable objects in the Kuznetsov components $\mathcal{A}_X$ for not only smooth ordinary GM threefolds, but also special GM threefolds $X$, we can prove Conjecture 1.5 in dimension three with assumptions on the genericity of the GM threefolds.

**Theorem 1.6** (Theorem 10.7 and Corollary 10.8)**.** *If $X$ and $X'$ are smooth* general *GM threefolds such that there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X$ and $X'$ are birationally equivalent.*

The difference between Theorems 1.4 and 1.6 is that, in 1.6 GM threefolds can be either ordinary or special, while 1.4 is stated only for ordinary ones. The proof is similar to that of Theorem 1.4. Firstly, we identify the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ on a special GM threefold $X'$ with $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ respectively (Theorem 7.13 and Theorem 8.5), where $\mathcal{C}_m(X')$ is the contraction of the Fano surface $\mathcal{C}(X')$ of conics on $X'$ along one of the components to a singular point. Then if $X$ is ordinary, the equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify those moduli spaces on a general ordinary GM threefold $X$ with those on a special GM threefold $X'$; we show that this is impossible by analyzing their singularities. Then Theorem 1.6 reduces to Theorem 1.4 and Theorem 1.3.

1.2.4. *The Debarre–Iliev–Manivel Conjecture.* In [DIM12], the authors conjecture that the general fiber of the classical period map from the moduli space of ordinary GM threefolds to the moduli space of 10 dimensional principally polarised abelian varieties is birational to the disjoint union of $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$, both quotiented by involutions, which we call the *Debarre–Iliev–Manivel* Conjecture 11.1. Within the moduli space of smooth GM threefolds, we define the fiber of the "categorical period map" through $[X]$ as the isomorphism classes of all ordinary GM threefolds $X'$ whose Kuznetsov components satisfy $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$. Then the following categorical analogue of the *Debarre–Iliev–Manivel conjecture* follows from Theorem 1.6, Theorem 1.4 and results on Bridgeland moduli spaces with respect to the two $(-1)$-classes in the numerical Grothendieck group of $\mathcal{A}_X$.

**Theorem 1.7** (Theorem 11.3)**.** *A general fiber of the "categorical period map" through an ordinary GM threefold $X$ is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$ where $\iota, \iota'$ are geometrically meaningful involutions.*

As an application, the *Debarre–Iliev–Manivel* Conjecture 11.1 can be restated in an equivalent form as follows:

**Conjecture 1.8.** *Let $X$ be a general ordinary GM threefold. The intermediate Jacobian $J(X)$ determines the Kuznetsov component $\mathcal{K}u(X)$.*

**Remark 1.9.** In [DIM12], the authors actually conjecture that a general fiber of the period map is *birational* to the disjoint union of two surfaces, parametrizing conic transforms and conic transforms of a line transform of $X$, which is birational to the disjoint union of $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$, both quotiented by involutions. In Corollary 10.5 we show that this birational map is indeed an *isomorphism*.

1.2.5. *Uniqueness of Serre-invariant stability conditions.* A stability condition $\sigma$ on the Kuznetsov component $\mathcal{K}u(X)$ of a prime Fano threefold $X$ is *Serre-invariant* if $S_{\mathcal{K}u(X)} \cdot \sigma = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Serre-invariance is one of the fundamental tools in studying relationship of classical Gieseker moduli spaces and Bridgeland moduli spaces for Kuznetsov components(cf.[Zha20], [LZ21], [PY20], [FP21]). A natural question is whether any two Serre-invariant stability conditions are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit. In the present paper, we answer this question affirmatively.

**Theorem 1.10** (Theorem 4.20)**.** *Let $X$ be a prime Fano threefold of index 1 of genus $g \geq 6$, or a del Pezzo threefold of degree $d \geq 2$. Then all Serre-invariant stability conditions on $\mathcal{K}u(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit.*

Theorem 1.10 allows us to prove the stability of some objects with respect to every Serre-invariant stability condition. This is one of the key steps when we identify Bridgeland moduli spaces via an equivalence of Kuznetsov components in the proofs of Theorems 1.2 and 1.4.

## 1.3. **Related Work.**

1.3.1. *Categorical Torelli theorems.* There is a very nice survey article [PS22] on recent results and remaining open questions on this topic. In [BMMS12] and [PY20], the authors prove categorical Torelli theorems for cubic threefolds. In [APR19] and [BT16], the authors prove categorical Torelli theorems for general quartic double solids. In [LNSZ21] and [LSZ22], the authors prove a refined categorical Torelli theorem for Enriques surfaces. In [JLZ21], the authors generalize Theorem 10.2 to all prime Fano threefolds of genus $g \geq 6$. In [GLZ22], the authors prove a birational categorical Torelli theorem for general non-Hodge-special Gushel–Mukai fourfolds.

1.3.2. *Identifying classical moduli spaces as Bridgeland moduli spaces for Kuznetsov components.* In the present article, we realize the Fano surface of conics and a certain Gieseker moduli space of semistable sheaves as Bridgeland moduli spaces of stable objects in Kuznetsov components of GM threefolds. In [PY20], the authors realize the Fano surface of lines $\Sigma(Y_d)$ (for $d \geq 2$) as a Bridgeland moduli space of stable objects in the Kuznetsov component $\mathcal{K}u(Y_d)$. In [LZ21], the authors realize the moduli space of rank two instanton sheaves on a del Pezzo threefold $Y_d$ (for $d \geq 3$) and the compactification of the moduli space of ACM sheaves on $X_{4d+2}$ (for $d \geq 3$) as Bridgeland moduli spaces of stable objects in $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$, respectively. In [FP21], the authors realize the moduli space of Ulrich bundles of arbitrary rank on a cubic threefold $Y_3$ as a Bridgeland moduli space of stable objects in $\mathcal{K}u(Y_3)$.

1.3.3. *Serre-invariant stability conditions.* In [PR21] and [PY20]y, the authors prove that stability conditions on Kuznetsov components of every del Pezzo threefold $Y_d$ of degree $d \geq 1$ and every index 1 prime Fano threefold of genus $g \geq 6$ are Serre-invariant. In [FP21], the authors prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of very natural assumptions, which include Kuznetsov components of a series of prime Fano threefolds.

1.4. **Notation and conventions.**

- We work over the complex numbers $k = \mathbb{C}$.
- We denote the bounded derived category of a smooth projective variety $X$ by $\mathrm{D}^b(X)$. The derived dual functor is denoted by $\mathbb{D} := \mathrm{R}\mathcal{H}om_X(-, \mathcal{O}_X)$.
- $\mathcal{K}u(X)$ is the ordinary Kuznetsov component of $X$, and $\mathcal{A}_X$ is the alternative Kuznetsov component of $X$ (see Definition 3.1).
- We call slope stability $\mu$-stability. The pairs $(\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$, $(\mathrm{Coh}^\mu_{\alpha,\beta}(X), Z^\mu_{\alpha,\beta})$, and $(\mathcal{A}(\alpha,\beta), Z(\alpha,\beta))$ denote once-tilted stability conditions, twice-tilted weak stability conditions, and Bridgeland stability conditions on the alternative Kuznetsov component $\mathcal{A}_X$, respectively.
- We denote the phase and slope with respect to a weak stability condition $\sigma$ by $\phi_\sigma$ and $\mu_\sigma$, respectively. The maximal and minimal slopes (phases) of the Harder–Narasimhan factors of a given object $F$ will be denoted by $\mu^+_\sigma(F)$ $(\phi^+_\sigma(F))$ and $\mu^-_\sigma(F)$ $(\phi^-_\sigma(F))$, respectively.
- $\mathcal{H}^i_{\mathcal{A}}$ means the $i$-th cohomology with respect to the heart $\mathcal{A}$. When the $\mathcal{A}$ subscript is dropped, we take the heart to be $\mathrm{Coh}(X)$.
- The symbol $\simeq$ denotes an equivalence of categories, and a birational equivalence of varieties. The symbol $\cong$ denotes an isomorphism of varieties.

1.5. **Organization of the paper.** In Section 2, we collect basic facts about semiorthogonal decompositions. In Section 3, we introduce Gushel–Mukai threefolds and their Kuznetsov components. In Section 4, we introduce the definition of weak stability conditions on $\mathrm{D}^b(X)$, and then apply [BLMS17, Theorem 5.1] to induce stability conditions on the alternative Kuznetsov components $\mathcal{A}_X$ of GM threefolds. We then introduce Serre-invariant stability conditions on Kuznetsov components and show that they are contained in one $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ orbit. In Section 5, we introduce a distinguished object $\pi(\mathcal{E}) \in \mathcal{K}u(X)$ and its alternative Kuznetsov component analogue $\Xi(\pi(\mathcal{E})) \in \mathcal{A}_X$ and prove its stability. In Section 6 we discuss the geometry of the Fano surface of conics of a GM threefold. In Section 7, we construct the Bridgeland moduli space of $\sigma$-stable objects with class $-x = -(1-2L)$ in $\mathcal{A}_X$. In Section 8, we construct the Bridgeland moduli space of $\sigma$-stable objects with respect to the other $(-1)$-class in $\mathcal{A}_X$. In Section 9, we prove irreducibility of the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ for GM threefolds. In Section 10, we prove several birational/refined categorical Torelli theorems (Theorems 1.2, 1.3 and 1.4) and Conjecture 1.5 in dimension three with mild assumptions. In Section 11, we describe the general fiber of the "categorical period map" for ordinary GM threefolds 1.7, and restate the *Debarre–Iliev–Manivel conjecture* in terms of Conjecture 11.7.

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## 2. Semiorthogonal decompositions

In this section, we collect some useful facts about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06], for example. From now on, let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on $X$, and for $E, F \in \mathrm{D}^b(X)$, define

$$\mathrm{RHom}^\bullet(E,F) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(E,F)[-i].$$

### 2.1. Exceptional collections and semiorthogonal decompositions.

**Definition 2.1.** Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$. We say that $E$ is an *exceptional object* if $\mathrm{RHom}^\bullet(E,E) = k$. Now let $(E_1, \dots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

**Definition 2.2.** Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y,X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

**Definition 2.3.** Let $\mathcal{D}$ be a triangulated category. We say a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is *admissible*, if the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ has left adjoint $i^*$ and right adjoint $i^!$.

**Definition 2.4.** Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ be a collection of full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{C}_j \subset \mathcal{C}_i^{\perp}$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ generate $\mathcal{D}$, i.e. the category resulting from taking all shifts and cones of objects in the categories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ is equivalent to $\mathcal{D}$.

Let $S_{\mathcal{D}}$ be the Serre functor of $\mathcal{D}$, then we have:

**Proposition 2.5.** *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} \simeq \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

2.2. **Mutations.** Let $\mathcal{C} \subset \mathcal{D}$ be an admissible triangulated subcategory. Then the *left mutation functor* $\mathbf{L}_{\mathcal{C}}$ through $\mathcal{C}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \to \mathrm{id} \to \mathbf{L}_{\mathcal{C}}$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{C}}$ through $\mathcal{C}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{C}} \to \mathrm{id} \to ii^*.$$

When $E \in \mathrm{D}^b(X)$ is an exceptional object, and $F \in \mathrm{D}^b(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{RHom}^\bullet(E, F) \to F \to \mathbf{L}_E F,$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \to F \to E \otimes \mathrm{RHom}^\bullet(F, E)^\vee.$$

**Proposition 2.6.** *Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then*

$$S_{\mathcal{B}} = \mathbf{R}_{\mathcal{A}} \circ S_{\mathcal{D}} \quad \textit{and} \quad S_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{B}} \circ S_{\mathcal{D}}^{-1}.$$

**Lemma 2.7** ([Kuz10, Lemma 2.7])**.** *Let $\mathcal{D} = \langle \mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n \rangle$ be a semiorthogonal decomposition with all components being admissible. Then for each $1 \leq k \leq n-1$, there is a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{C}_1, ..., \mathcal{C}_{k-1}, \mathbf{L}_{\mathcal{C}_k}\mathcal{C}_{k+1}, \mathcal{C}_k, \mathcal{C}_{k+2} ..., \mathcal{C}_n \rangle$$

*and for each $2 \leq k \leq n$ there is a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{C}_1, ..., \mathcal{C}_{k-2}, \mathcal{C}_k, \mathbf{L}_{\mathcal{C}_k}\mathcal{C}_{k-1}, \mathcal{C}_{k+1} ..., \mathcal{C}_n \rangle.$$

## 3. Gushel–Mukai threefolds and their derived categories

Let $X$ be a prime Fano threefold of index 1 and degree $H^3 = 10$. Then $X$ is either a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5)$, in which case it is called an ordinary Gushel–Mukai (GM) threefold, or $X$ is a double cover of a degree 5 and index 2 Fano threefold $Y_5$ ramified in a quadric hypersurface, in which case it is called a special GM threefold. In the latter case, it has a natural involution $\tau : X \to X$ induced by the double cover $\pi : X \to Y_5$. There exists a unique stable vector bundle $\mathcal{E}$ of rank 2 with $c_1(\mathcal{E}) = -H$ and $c_2(\mathcal{E}) = 4L$, where $L$ is the class of a line on $X$. In addition, $\mathcal{E}$ is exceptional and $H^\bullet(X, \mathcal{E}) = 0$. In fact, $\mathcal{E}$ is the pullback of the tautological bundle on the Grassmannian $\mathrm{Gr}(2,5)$. Furthermore, there is a standard short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to \mathcal{Q} \to 0 \tag{1}$$

where $\mathcal{Q}$ is the restriction of the tautological quotient bundle on $\mathrm{Gr}(2,5)$ to $X$.

**Definition 3.1.** Let $X$ be a GM threefold.

- The *Kuznetsov component* of $X$ is defined as $\mathcal{K}u(X) := \langle \mathcal{E}, \mathcal{O}_X \rangle^{\perp}$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$;
- The *alternative Kuznetsov component* of $X$ [KP18b, Proposition 2.3] is defined as $\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{E}^\vee \rangle^{\perp}$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$.

**Definition 3.2.** The left adjoint to the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$ is given by $\mathrm{pr} := \mathbf{L}_{\mathcal{O}_X} \mathbf{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$. We call this the *natural projection functor*.

The analogous natural projection functor can be defined for $\mathcal{K}u(X)$, and we denote it by $\mathrm{pr}' := \mathbf{L}_{\mathcal{E}} \mathbf{L}_{\mathcal{O}_X}$.

3.1. **Kuznetsov components.** Let $K_0(\mathcal{D})$ denote the Grothendieck group of a triangulated category $\mathcal{D}$. We have the bilinear Euler form

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{ext}^i(E, F)$$

for $E, F \in K_0(\mathcal{D})$. By the Hirzebruch–Riemann–Roch formula, it takes the following form on GM threefolds. We have [Kuz09, p. 5] $\chi(u, v) = \chi_0(u^* \cap v)$ where $u \mapsto u^*$ is an involution of $\oplus_{i=0}^3 H^i(X, \mathbb{Q})$ given by multiplication with $(-1)^i$ on $H^{2i}(X, \mathbb{Q})$, and $\chi_0$ is given by

$$\chi_0(x + yH + zL + wP) = x + \frac{17}{6} y + \frac{1}{2} z + w.$$

The *numerical Grothendieck group* of $\mathcal{D}$ is $\mathcal{N}(\mathcal{D}) = K_0(\mathcal{D}) / \ker \chi$.

**Lemma 3.3** ([Kuz09, p. 5])**.** *The numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X))$ of the Kuznetsov component is a rank 2 integral lattice generated by the basis elements $v = 1 - 3L + \frac{1}{2}P$ and $w = H - 6L + \frac{1}{6}P$. Using this basis, $\chi$ is given by the matrix*

$$\begin{pmatrix} -2 & -3 \\ -3 & -5 \end{pmatrix}.$$

3.2. **Alternative Kuznetsov components.** As in [Kuz09, Proposition 3.9], the following lemma follows from a straightforward computation.

**Lemma 3.4.** *The numerical Grothendieck group of $\mathcal{A}_X$ is a rank 2 integral lattice with basis vectors $x = 1 - 2L$ and $y = H - 4L - \frac{5}{6}P$, and the Euler form with respect to the basis is*

$$\begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix}.$$

**Remark 3.5.** It is straightforward to check that the $(-1)$-classes of $\mathcal{N}(\mathcal{A}_X)$ are $x = 1 - 2L$ and $2x - y = 2 - H + \frac{5}{6}P$, up to sign.

It is true that the Kuznetsov components from Subsection 3.1 and the alternative Kuznetsov components from this section are equivalent:

**Lemma 3.6.** *The original and alternative Kuznetsov components are equivalent. More precisely, there is an equivalence of categories $\Xi : \mathcal{K}u(X) \xrightarrow{\sim} \mathcal{A}_X$ given by $E \mapsto \mathbf{L}_{\mathcal{O}_X}(E \otimes \mathcal{O}_X(H))$, with inverse given by $F \mapsto (\mathbf{R}_{\mathcal{O}_X} F) \otimes \mathcal{O}_X(-H)$.*

*Proof.* Using Lemma 2.7 and noting that $\mathcal{E} \otimes \mathcal{O}_X(H) \cong \mathcal{E}^\vee$, we manipulate the semiorthogonal decomposition as follows:

$$\begin{aligned}
\mathrm{D}^b(X) &= \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle \\
&\simeq \langle \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee, \mathcal{O}_X(H) \rangle \\
&\simeq \langle \mathcal{O}_X, \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee \rangle \\
&\simeq \langle \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H)), \mathcal{O}_X, \mathcal{E}^\vee \rangle.
\end{aligned}$$

Now comparing with the definition of $\mathcal{A}_X$, we get $\mathcal{A}_X \simeq \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H))$ and the desired result follows. The reverse direction is similar. ◻

## 4. Bridgeland stability conditions

In this section, we recall (weak) Bridgeland stability conditions on $\mathrm{D}^b(X)$, and the notions of tilt stability, double-tilt stability, and stability conditions induced on Kuznetsov components from weak stability conditions on $\mathrm{D}^b(X)$. We follow [BLMS17, § 2].

4.1. **Weak stability conditions.** Let $\mathcal{D}$ be a triangulated category, and $K_0(\mathcal{D})$ its Grothendieck group. Fix a surjective morphism $v : K_0(\mathcal{D}) \to \Lambda$ to a finite rank lattice.

**Definition 4.1.** The *heart of a bounded t-structure* on $\mathcal{D}$ is an abelian subcategory $\mathcal{A} \subset \mathcal{D}$ such that the following conditions are satisfied:

(1) for any $E, F \in \mathcal{A}$ and $n < 0$, we have $\mathrm{Hom}(E, F[n]) = 0$;
(2) for any object $E \in \mathcal{D}$ there exist objects $E_i \in \mathcal{A}$ and maps

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

such that $\mathrm{cone}(\phi_i) = A_i[k_i]$ where $A_i \in \mathcal{A}$ and the $k_i$ are integers such that $k_1 > k_2 > \cdots > k_m$.

**Definition 4.2.** Let $\mathcal{A}$ be an abelian category and $Z \colon K_0(\mathcal{A}) \to k$ be a group homomorphism such that for any $E \in \mathcal{D}$ we have $\operatorname{Im} Z(E) \geq 0$ and if $\operatorname{Im} Z(E) = 0$ then $\operatorname{Re} Z(E) \leq 0$. Then we call $Z$ a *weak stability function* on $\mathcal{A}$. If furthermore we have for any $0 \neq E$ that $\operatorname{Im} Z(E) \geq 0$, and $\operatorname{Im} Z(E) = 0$ implies that $\operatorname{Re} Z(E) < 0$, then we call $Z$ a *stability function* on $\mathcal{A}$.

**Definition 4.3.** A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z \colon \Lambda \to \mathbb{C}$ is a group homomorphism such that the following conditions hold:

(1) The composition $Z \circ v \colon K_0(\mathcal{A}) \cong K_0(\mathcal{D}) \to \mathbb{C}$ is a weak stability function on $\mathcal{A}$. From now on, we write $Z(E)$ rather than $Z(v(E))$.

Much like the slope from classical $\mu$-stability, we can define a *slope* $\mu_\sigma$ for $\sigma$ using $Z$. For any $E \in \mathcal{A}$, set

$$\mu_\sigma(E) := \begin{cases} -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)}, & \operatorname{Im} Z(E) > 0 \\ +\infty, & \text{else.} \end{cases}$$

We say an object $0 \neq E \in \mathcal{A}$ is $\sigma$-(semi)stable if $\mu_\sigma(F) < \mu_\sigma(E)$ (respectively $\mu_\sigma(F) \leq \mu_\sigma(E)$) for all proper subobjects $F \subset E$.

(2) Any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of $\sigma$-semistability defined above.
(3) There exists a quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$ for all $\sigma$-semistable objects $E \in \mathcal{A}$. This is known as the *support property*.

If the composition $Z \circ v$ is a stability function, then $\sigma$ is a *stability condition* on $\mathcal{D}$.

For this paper, we let $\Lambda$ be the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ which is $K_0(\mathcal{D})$ modulo the kernel of the Euler form $\chi(E,F) = \sum_i (-1)^i \operatorname{ext}^i(E,F)$.

4.2. **Tilt stability.** Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on a triangulated category $\mathcal{D}$. Now consider the following subcategories[1] of $\mathcal{A}$:

$$\mathcal{T}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle$$
$$\mathcal{F}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leq \mu \rangle.$$

Then it is a result of [HRS96] that

**Proposition 4.4.** *The abelian category $\mathcal{A}_\sigma^\mu := \langle \mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu[1] \rangle$ is the heart of a bounded t-structure on $\mathcal{D}$.*

We call $\mathcal{A}_\sigma^\mu$ the *tilt* of $\mathcal{A}$ around the torsion pair $(\mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu)$. Let $X$ be an $n$-dimensional smooth projective complex variety. Tilting can be applied to the heart $\mathrm{Coh}(X) \subset \mathrm{D}^b(X)$ to form the once-tilted heart $\mathrm{Coh}^\beta(X)$. Define for $E \in \mathrm{Coh}^\beta(X)$

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^n \mathrm{ch}_0^\beta(E) - H^{n-2}\mathrm{ch}_2^\beta(E) + \mathfrak{i} H^{n-1}\mathrm{ch}_1^\beta(E).$$

**Proposition 4.5** ([BMT11, BMS16])**.** *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$. The quadratic form $Q$ is given by the discriminant*

$$\Delta_H(E) = (H^{n-1}\mathrm{ch}_1(E))^2 - 2H^n \mathrm{ch}_0(E) H^{n-2}\mathrm{ch}_2(E).$$

*The stability conditions $\sigma_{\alpha,\beta}$ vary continuously as $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies. Furthermore, for any $v \in \Lambda_H^2$ there is a locally-finite wall-and-chamber structure on $\mathbb{R}_{>0} \times \mathbb{R}$ which controls stability of objects with class $v$.*

We now state a useful lemma which relates Gieseker-stability and tilt stability.

**Lemma 4.6** ([BMS16, Lemma 2.7], [BBF+20, Proposition 4.8, 4.9])**.** *Let $E \in \mathrm{D}^b(X)$.*

*(1) Let $\beta < \mu(E)$. Then $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-(semi)stable for $\alpha \gg 0$ if and only if $E \in \mathrm{Coh}(X)$ and $E$ is 2-Gieseker-(semi)stable.*

*(2) If $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable for $\beta \geq \mu(E)$ $\alpha \gg 0$, then $\mathcal{H}^{-1}(E)$ is a torsion free $\mu$-semistable sheaf and $\mathcal{H}^0(E)$ is supported in dimension not greater than one. If $\beta > \mu(E)$ and $\alpha > 0$, then $\mathcal{H}^{-1}(E)$ is also reflexive.*

4.3. **Stronger BG inequalities.** In this subsection, we state stronger Bogomolov–Gieseker (BG) style inequalities, which hold for tilt-semistable objects. These will be useful later on for ruling out potential walls for tilt-stability of objects in $\mathrm{D}^b(X)$. The first is a stronger version of Proposition 4.5, which was proved by Chunyi Li in [Li16, Proposition 3.2] for Fano threefolds of Picard number one.

**Lemma 4.7** (Stronger BG I)**.** *Let $X$ be an index 1 prime Fano threefold with degree $d$, and $E \in \mathrm{D}^b(X)$ a $\sigma_{\alpha,\beta}$-stable object where $\alpha > 0$. Let $k := \lfloor \mu(E) \rfloor$. Then we have:*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leq \max\left\{ k\mu_H(E) - \frac{k^2}{2}, \frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, (k+1)\mu_H(E) - \frac{(k+1)^2}{2} \right\}.$$

*Moreover, if the equality holds, then $E$ has rank one or two.*

[1] The angle brackets here mean extension closure.

The second is due to Naoki Koseki and Chunyi Li. It is based on [Kos20, Lemma 4.2, Theorem 4.3], however for our purposes we quote a reformulation for Fano threefolds from the upcoming paper [JLZ21]. Chunyi Li also sent us a similar inequality from his upcoming paper [Li21].

**Lemma 4.8** (Stronger BG II)**.** *Let $X_{2g-2}$ be an index 1 Fano threefold of degree $d = 2g-2$, and $E \in \mathrm{Coh}^0(X)$ be a $\mu_{\alpha,0}$-semistable object for some $\alpha > 0$ with $|\mu_H(E)| \in [0,1]$ and $\mathrm{rk}(E) \geq 2$. Then*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leq \max\left\{\frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, \mu_H(E)^2 - \frac{1}{2}|\mu_H(E)|\right\}.$$

4.4. **Double-tilting.** Now pick a weak stability condition $\sigma_{\alpha,\beta}$ and tilt the once-tilted heart $\mathrm{Coh}^\beta(X)$ with respect to the tilt slope $\mu_{\alpha,\beta}$ and a second tilt parameter $\mu$. One gets a torsion pair $(\mathcal{T}^\mu_{\alpha,\beta}, \mathcal{F}^\mu_{\alpha,\beta})$ and another heart $\mathrm{Coh}^\mu_{\alpha,\beta}(X)$ of $\mathrm{D}^b(X)$. Now "rotate" the stability function $Z_{\alpha,\beta}$ by setting

$$Z^\mu_{\alpha,\beta} := \frac{1}{u} Z_{\alpha,\beta}$$

where $u \in \mathbb{C}$ such that $|u| = 1$ and $\mu = -\frac{\mathrm{Re}\, u}{\mathrm{Im}\, u}$. Then we have the following result:

**Proposition 4.9** ([BLMS17, Proposition 2.15])**.** *The pair $(\mathrm{Coh}^\mu_{\alpha,\beta}(X), Z^\mu_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$.*

4.5. **Stability conditions on the Kuznetsov component of a GM threefold.** Proposition 5.1 in [BLMS17] gives a criterion for checking when weak stability conditions on a triangulated category can be used to induce stability conditions on a subcategory. Each of the criteria of this proposition can be checked for $\mathcal{A}_X \subset \mathrm{D}^b(X)$ to give stability conditions on $\mathcal{A}_X$.

More precisely, let $\mathcal{A}(\alpha,\beta) = \mathrm{Coh}^0_{\alpha,\beta}(X) \cap \mathcal{A}_X$ and $Z(\alpha,\beta) = Z^0_{\alpha,\beta}|_{\mathcal{A}_X}$. Furthermore, if we take suitable $(\alpha,\beta)$, by [BLMS17, Theorem 6.9] and [PR21, Proposition 3.2] we have:

**Theorem 4.10.** *Let $X$ be a GM threefold. Then $\sigma(\alpha,\beta)$ is a stability condition on $\mathcal{A}_X$ for all $(\alpha,\beta) \in V$, where*

$$V := \{(\alpha,\beta) \colon -\frac{1}{10} < \beta < 0, 0 < \alpha < -\beta\}.$$

4.6. **Serre-invariant stability conditions on GM threefolds.**

**Definition 4.11.** Let $\sigma$ be a stability condition on a triangulated category $\mathcal{T}$. It is called *Serre-invariant* if $S_{\mathcal{T}} \cdot \sigma = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, where $S_{\mathcal{T}}$ is the Serre functor of $\mathcal{T}$.

We recall several properties of Serre-invariant stability conditions from [PY20, Zha20] below:

**Proposition 4.12.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{A}_X$. Then*

*(1) the homological dimension of the heart of $\sigma$ is 2.*
*(2) $\mathrm{ext}^1(A,A) \geq 2$ for every non-trivial object $A$ in the heart of $\sigma$.*

*Proof.* Denote by $\mathcal{A}'$ the heart of $\sigma$, and let $A, B \in \mathcal{A}'$. Then $\mathrm{Hom}(A, B[i]) = 0$ for $i < 0$. Note that the phases of the semistable factors of $\tau(A)$ are in the interval $(0,1)$, and the phases of the semistable factors of $B[i]$ are in $(i, i+1)$. Then $\mathrm{Hom}(A, B[i]) \cong \mathrm{Hom}(B[i], \tau(A)[2]) = 0$ if $i \geq 3$. This proves (1). For (2), note that $\chi(A,A) \geq -1$ for all non-zero objects $A \in \mathcal{A}_X$, so the result follows. ◻

We recall a very recent result proved in [PR21].

**Theorem 4.13.** *Let $X$ be an ordinary GM threefold and $\sigma$ (or $\sigma'$) be a stability condition on $\mathcal{K}u(X)$ (or $\mathcal{A}_X$) defined by [BLMS17]. Then $\sigma$ (or $\sigma'$) is Serre-invariant.*

**Proposition 4.14.** *Let $X$ be a GM threefold and $E$ an object in $\mathcal{A}_X$ such that* $\mathrm{ext}^1(E,E)=2$ *or* $3$ *and* $\chi(E,E)=-1$. *Then $E$ is $\sigma$-stable for every Serre-invariant stability condition $\sigma$ on $\mathcal{A}_X$.*

*Proof.* The proof is the same as in [Zha20, Lemma 9.12]. We omit the details. ◻

4.7. **Uniqueness of Serre-invariant stability conditions.** Let $Y_d$ be smooth index 2 degree $d\geq 2$ prime Fano threefold and $X_{4d+2}$ an index 1 degree $4d+2$ prime Fano threefold. In this section, we show that all Serre-invariant stability conditions on $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ (or $\mathcal{A}_{X_{4d+2}}$) are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit for each $d\geq 2$.

Recall that the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y_d))$ is a rank two lattice generated by two classes $v$ and $w$. The only $(-1)$-classes are $v=1-2L$ and $w-v=-1+H-3L-\frac{1}{2}P$, up to sign.

**Lemma 4.15.** *Let $\sigma'$ be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ where $d\geq 2$. Then the heart of $\sigma'$ has homological dimension at most 2.*

*Proof.* When $d=2$, this follows from the same argument as in Proposition 4.12. When $d=3$, this follows from [PY20, Lemma 5.10]. When $d=4$ and $5$, since $\mathcal{K}u(Y_4)\cong \mathrm{D}^b(C_2)$ and $\mathcal{K}u(Y_5)\cong \mathrm{D}^b(\mathrm{Rep}(K(3)))$ where $C_2$ is a genus 2 smooth curve and $\mathrm{Rep}(K(3))$ is the category of representations of the 3-Kronecker quiver ([KPS18a]), then in these two cases the heart has homological dimension 1. ◻

**Lemma 4.16.** *Let $\sigma'$ be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ where $d\geq 2$. If $E$ and $F$ are two $\sigma'$-semistable objects with phases $\phi'(E)<\phi'(F)$, then* $\mathrm{Hom}(E,F[2])=0$.

*Proof.* When $d=4$ and $5$, this follows from the fact that the heart of $\sigma'$ has homological dimension 1. When $d=2$ and $3$, this is by [PY20, Sec. 5, Sec. 6]. ◻

**Lemma 4.17.** (Weak Mukai Lemma) *Let $F\to E\to G$ be an exact triangle in $\mathcal{K}u(Y_d)$ such that* $\mathrm{Hom}(F,G)=\mathrm{Hom}(G,F[2])=0$. *Then we have*

$$\mathrm{ext}^1(F,F)+\mathrm{ext}^1(G,G)\leq \mathrm{ext}^1(E,E).$$

**Lemma 4.18.** *Let $\sigma'$ be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ where $d\geq 2$. Assume that there is a triangle $F\to E\to G$ of $E\in\mathcal{K}u(Y_d)$ such that the phases of all the $\sigma'$-semistable factors of $F$ are greater than the phases of the $\sigma'$-semistable factors of $G$. Then we have* $\mathrm{ext}^1(F,F)<\mathrm{ext}^1(E,E)$ *and* $\mathrm{ext}^1(G,G)<\mathrm{ext}^1(E,E)$.

*Proof.* Since $\phi'^-(F)>\phi'^+(G)$, by Lemma 4.16 we have $\mathrm{Hom}(F,G)=0$ and

$$\mathrm{Hom}(G,F[2])=\mathrm{Hom}(F[2],S_{\mathcal{K}u(Y_d)}(E))=0.$$

Thus the result follows from Lemma 4.17. ◻

Let $\sigma=\sigma(\alpha,-\frac{1}{2})$ and $Y:=Y_d$ where $d\geq 2$. As shown in [PY20, Section 4], the moduli spaces $\mathcal{M}_\sigma(\mathcal{K}u(Y),-v)$ and $\mathcal{M}_\sigma(\mathcal{K}u(Y),w-v)$ are non-empty. Let $A,B\in\mathcal{A}(\alpha,-\frac{1}{2})$ with $[A]=-v,[B]=w-v$ be $\sigma$-stable objects. We denote the phase with respect to $\sigma=\sigma(\alpha,-\frac{1}{2})$ by $\phi(-)$.

Now let $\sigma_1$ be any Serre-invariant stability condition on $\mathcal{K}u(Y)$. By [PY20, Remark 5.14], there is a $T=(t_{ij})_{1\leq i,j\leq 2}\in \mathrm{GL}^+(2,\mathbb{R})$ such that $Z_1=T\cdot Z(\alpha,-\frac{1}{2})$. Since $A$ is stable with respect to every Serre-invariant stability condition by [PY20, Lemma 5.16], we can assume $A[m]\in\mathcal{A}_1$. Let $\sigma_2=\sigma\cdot\tilde{g}$ for $\widetilde{g}:=(g,T)\in\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $A[m]\in\mathcal{A}_2$ and $Z_2=Z_1$. Then we have $\phi_1(A)=\phi_2(A)$ and $\mathcal{A}_2=\mathcal{P}(\alpha,-\frac{1}{2})((g(0),g(0)+1])$.

**Lemma 4.19.** *Fix the notation as above. Then $A$ and $B$ are $\sigma_1$-stable with phase $\phi_1(A) = \phi_2(A)$ and $\phi_1(B) = \phi_2(B)$.*

*Proof.* The stability of $A$ and $B$ is from [PY20, Lemma 5.13]. By definition of $\sigma_2$, we know $\phi_1(A) = \phi_2(A)$ and $\phi_2(B) < \phi_2(A) < \phi_2(B) + 1$. Also, from [PY20, Remark 4.8] we know $\phi_1(B) < \phi_1(A) = \phi_2(A) < \phi_1(B) + 1$. Thus $\phi_1(B) = \phi_2(B)$. ∎

**Theorem 4.20.** *All Serre-invariant stability conditions on $\mathcal{K}u(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit. Here $X := X_{4d+2}$ or $Y_d$ for $d \geq 2$.*

*Proof.* Fix the notation as above. We are going to show that $\sigma_1 = \sigma_2$. Since $\mathcal{K}u(X_{12})$, $\mathcal{K}u(X_{16})$ and $\mathcal{K}u(X_{18}) \simeq \mathcal{K}u(Y_4)$ are equivalent to the bounded derived categories of some smooth curves of positive genus, the results for these three cases follow from [Mac07, Theorem 2.7]. The results for $\mathcal{K}u(X_{14})$ and $\mathcal{K}u(X_{22})$ are from the results for $\mathcal{K}u(Y_3)$ and $\mathcal{K}u(Y_5)$ and the equivalences $\mathcal{K}u(Y_d) \simeq \mathcal{K}u(X_{4d+2})$, where $d \geq 3$ (see [KPS18a]). Thus we only need to prove this for $Y_d$ when $d \geq 2$ and $X := X_{10}$.

We first prove this for $Y_d$ when $d \geq 2$. Let $E \in \mathcal{A}(\alpha, -\frac{1}{2})$ be a $\sigma$-semistable object with $[E] = av + bw$. First we are going to show that if $E$ is $\sigma_1$-semistable, then $\phi_2(E) = \phi_1(E)$. Note that we have the following relations:

(1) $\chi(E, A) = a + (d-1)b$, $\chi(A, E) = a + b$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(A) \iff b < 0$
(2) $\chi(E, B) = -b$, $\chi(B, E) = -[(d-2)a + (d-1)b]$; and $\mu^0_{\alpha,-\frac{1}{2}}(E) > \mu^0_{\alpha,-\frac{1}{2}}(B) \iff a+b < 0$.

From the definition of $\sigma = \sigma(\alpha, -\frac{1}{2})$-stability we have $a \leq 0$. When $a = 0$, by the definition of a stability condition we have $b < 0$. Thus in the case $b > 0$ we always have $a < 0$. Note that by Lemma 4.19, we have $\phi_2(B) < \phi_2(A)$ and both of them lie in the interval $(g(0), g(0) + 1]$.

- Assume that $b > 0$ and $a + b > 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(E) < \mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(E) < \phi_2(B) < \phi_2(A)$. We also have $\chi(E, A) > 0$. Thus by Lemma 4.16 we know $\mathrm{Hom}(E, A[2]) = 0$. Thus $\chi(E, A) = \hom(E, A) - \hom(E, A[1]) > 0$ implies $\hom(E, A) > 0$, and therefore $\phi_1(E) < \phi_1(A)$. Also from $\chi(B, E) < 0$ and Lemma 4.15 we have $\phi_1(B) - 1 < \phi_1(E)$. Then we have $\phi_1(B) - 1 < \phi_1(E) < \phi_1(A)$. But by Lemma 4.19 we know $\phi_1(B) = \phi_2(B)$, $\phi_1(A) = \phi_2(A)$. Also, from the definition of $\sigma_2$ we have $|\phi_2(B) - \phi_2(A)| < 1$ and $|\phi_2(A) - \phi_2(E)| < 1$. Thus $\phi_2(E) - \phi_1(E) = 0$ or $1$. But if $\phi_2(E) = \phi_1(E) + 1$, then $\phi_2(B) - 1 = \phi_1(B) - 1 < \phi_2(E) < \phi_1(B) = \phi_2(B)$. This implies $1 = \phi_1(B) - \phi_1(B) + 1 > \phi_2(E) - \phi_1(B) + 1 = \phi_1(E) - \phi_1(B) + 2$, which is impossible since $\phi_1(B) - 1 < \phi_1(E)$. Thus we have $\phi_1(E) = \phi_2(E)$.
- Assume that $b > 0$ and $a + b < 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(E) < \mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(B) < \phi_2(E) < \phi_2(A)$. Since $\chi(A, E) < 0$ and $\chi(E, B) < 0$, from Lemma 4.15 we know $\hom(A, E[1]) > 0$ and $\hom(E, B[1]) > 0$, hence $\phi_1(A) - 1 < \phi_1(E) < \phi_1(B) + 1$. This means $|\phi_1(E) - \phi_2(E)| = 0$ or $1$. But $|\phi_1(E) - \phi_2(E)| = 1$ is impossible since $\phi_1(B) = \phi_2(B) < \phi_2(E) < \phi_2(A) = \phi_1(A)$. Therefore we have $\phi_1(E) = \phi_2(E)$.
- Assume that $b < 0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(A) < \mu^0_{\alpha,-\frac{1}{2}}(E)$ and hence $\phi_2(B) < \phi_2(A) < \phi_2(E)$. Since $\chi(E, A) < 0$, from Lemma 4.15 we have $\hom(E, A[1]) > 0$ and $\phi_1(E) < \phi_1(A) + 1$. By Lemma 4.16, $\mu^0_{\alpha,-\frac{1}{2}}(B) < \mu^0_{\alpha,-\frac{1}{2}}(E)$ and $\chi(B, E) > 0$, we know that $\hom(B, E) > 0$. Thus $\phi_1(B) < \phi_1(E) < \phi_1(A) + 1$. Hence $\phi_1(E) - \phi_2(E) = 0$ or $1$. But since $\mu^0_{\alpha,-\frac{1}{2}}(A) < \mu^0_{\alpha,-\frac{1}{2}}(E)$, we have $\phi_2(A) = \phi_1(A) < \phi_2(E)$. Thus $\phi_1(A) < \phi_2(E) < \phi_1(A) + 1$. Then $\phi_1(E) - \phi_2(E) = 1$ is impossible since $\phi_1(E) < \phi_1(A) + 1$. Therefore we have $\phi_1(E) = \phi_2(E)$.

- When $b=0$, we have $[E]=-a\cdot[A]$. Hence $\chi(E,A)=\chi(A,E)<0$ and we have $\phi_1(A)-1\leq\phi_1(E)\leq\phi_1(A)+1$. But $\mu_1(E)=\mu_1(A)$, so we know $\phi_1(E)-\phi_1(A)$ is an integer. Thus $\phi_1(E)=\phi_1(A)\pm1$. But from the definition of a stability function, we have $Z_1(E[\pm1])=-Z_1(A)$. Thus $\phi_1(E)=\phi_1(A)=\phi_2(E)$.
- When $a+b=0$, we have $[E]=-a\cdot[B]$. Hence $\chi(E,B)=\chi(B,E)<0$ and we have $\phi_1(B)-1\leq\phi_1(E)\leq\phi_1(B)+1$. But $\mu_1(E)=\mu_1(B)$, so we know $\phi_1(E)-\phi_1(B)$ is an integer. Thus $\phi_1(E)=\phi_1(B)\pm1$. But from the definition of a stability function, we have $Z_1(E[\pm1])=-Z_1(B)$. Thus $\phi_1(E)=\phi_1(B)=\phi_2(E)$.

Next we show that $E\in\mathcal{A}_2$ is $\sigma_2$-semistable if and only if $E\in\mathcal{A}_1$ is $\sigma_1$-semistable. We prove this by induction.

If $\mathrm{ext}^1(E,E)<2$, this is by [PY20, Sec. 5]. Now assume this is true for all $E\in\mathcal{A}_2$ $\sigma_2$-semistable such that $\mathrm{ext}^1(E,E)<N$.

When $E\in\mathcal{A}_2$ is $\sigma_2$-semistable and has $\mathrm{ext}^1(E,E)=N$, assume otherwise that $E$ is not $\sigma_1$-semistable. Let $A_0$ be the first HN-factor of $E$ with respect to $\sigma_1$ and $A_n$ be the last one. Then $\phi_1(A_0)>\phi_1(A_n)$. By Lemma 4.18, $\mathrm{ext}^1(A_0,A_0)<N$ and $\mathrm{ext}^1(A_n,A_n)<N$. Thus $A_0$ and $A_n$ are $\sigma_2$-semistable by the induction hypothesis and $\phi_2(A_0)>\phi_2(A_n)$ by the results above. Since $\mathrm{Hom}(A_0,E)$ and $\mathrm{Hom}(E,A_n)$ are both non-zero, we know that $\phi_2(A_0)\leq\phi_2(E)$ and $\phi_2(E)\leq\phi_2(A_n)$, which implies $\phi_2(A_0)\leq\phi_2(A_n)$ and gives a contradiction. Thus $E$ is $\sigma_1$-semistable. When $E\in\mathcal{A}_1$ is $\sigma_1$-semistable, the same argument shows that $E\in\mathcal{A}_2$ is also $\sigma_2$-semistable.

Since every object in the heart is the extension of semistable objects, we have $\mathcal{A}_1=\mathcal{A}_2$. And from $Z_1=Z_2$, we know that $\sigma_1=\sigma_2=\sigma\cdot\widetilde{g}$. Hence $\sigma_1$ is in the orbit of $\sigma=\sigma(\alpha,-\frac{1}{2})$.

For a GM threefold $X_{10}$, the result follows from Lemma 4.21 and a similar argument as the previous index 2 cases. ∎

**Lemma 4.21.** *Let $X$ be a smooth GM threefold and $A,B\in\mathcal{K}u(X)$ be two $\sigma$-stable objects of numerical class $[A]=-(3s-2t)$ and $[B]=s$, where $\sigma$ is a Serre-invariant stability condition. Then we have $\phi_\sigma(B)<\phi_\sigma(A)<\phi_\sigma(B)+1$.*

*Proof.* Let $X$ be a GM threefold. Its numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X))=\langle s,t\rangle$ is a rank two lattice generated by $s=1-3L+\frac{1}{2}P=[I_C]$ and $t=H-6L+\frac{1}{6}P$, where $C$ is a twisted cubic on $X$. Let $A,B$ and $B'$ be $(-2)$-class $\sigma$-stable objects in $\mathcal{K}u(X)$ with respect to a Serre-invariant stability condition $\sigma$. In particular, let $B=\mathrm{pr}'(I_C)$ where $I_C\notin\mathcal{K}u(X)$. Thus $\mathrm{pr}'(I_C)\cong\mathrm{pr}'(G)$, where $G$ is the twisted derived dual of a line $L$ such that $L\cup C=Z(s)$, and where $s$ is a section of $\mathcal{E}^\vee$. Note that $G$ is given by the triangle

$$\mathcal{O}_X(-H)[1]\to G\to\mathcal{O}_L(-2).$$

Let $A:=\mathrm{pr}'(I_L)$ and $B':=\mathrm{pr}'(I_D)=I_D$, where $D$ is a twisted cubic with an irreducible component $L$ and $I_D\in\mathcal{K}u(X)$. Note that $[B]=[B']=s$. Then the result follows Lemma 4.22, Lemma 4.23 and Lemma 4.24. ∎

**Lemma 4.22.** *We have* $\mathrm{Hom}(A,B[1])\neq0$.

*Proof.* By adjunction, we have $\mathrm{Hom}(\mathrm{pr}'(I_L),\mathrm{pr}'(G)[1])\cong\mathrm{Hom}(I_L,\mathrm{pr}'(G)[1])$. Note that $\mathrm{pr}'(G)$ fits into the exact triangle

$$G\to\mathrm{pr}'(G)\to\mathcal{E}$$

by [Zha20, Proposition 5.3]. Next apply $\mathrm{Hom}(I_L,-)$ to the above triangle to get the exact sequence

$$\cdots\to\mathrm{Hom}(I_L,\mathcal{E})\to\mathrm{Ext}^1(I_L,G)\to\mathrm{Ext}^1(I_L,\mathrm{pr}'(G))\to\mathrm{Ext}^1(I_L,\mathcal{E})\to\cdots.$$

It is clear that $\mathrm{Hom}(I_L,\mathcal{E})=0$ and $\mathrm{Ext}^1(I_L,\mathcal{E})\cong\mathrm{Ext}^2(\mathcal{E}^\vee,I_L)=0$ so we get $\mathrm{Ext}^1(I_L,\mathrm{pr}'(G))\cong\mathrm{Ext}^1(I_L,G)$. Applying $\mathrm{Hom}(I_L,-)$ to the above triangle defining $G$, we get a long exact sequence

$$\cdots\to\mathrm{Ext}^i(I_L,\mathcal{O}_X(-H)[1])\to\mathrm{Ext}^i(I_L,G)\to\mathrm{Ext}^i(I_L,\mathcal{O}_L(-2))\to\cdots.$$

By Serre duality, we have $\mathrm{Ext}^i(I_L,\mathcal{O}_X(-H))=\mathrm{Ext}^{3-i}(\mathcal{O}_X,I_L)=0$ for all $i$. Then we have $\mathrm{Ext}^1(I_L,G)\cong\mathrm{Ext}^1(I_L,\mathcal{O}_L(-2))$. By the adjunction associated to the embedding $j:L\to X$, we get $\mathrm{Ext}^1(I_L,\mathcal{O}_L(-2))\cong\mathrm{Ext}^1_{\mathcal{O}_L}(j^*I_L,\mathcal{O}_L(-2))\cong\mathrm{Ext}^1_{\mathcal{O}_L}(\mathcal{N}_{L|X},\mathcal{O}_L(-2))$. As the normal bundle of $L$ in $X$ is either $\mathcal{N}_{L|X}=\mathcal{O}_L\oplus\mathcal{O}_L(-1)$ or $\mathcal{O}_L(1)\oplus\mathcal{O}_L(-2)$, we get $\mathrm{Ext}^1(I_L,\mathcal{O}_L(-2))\cong\mathrm{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_L\oplus\mathcal{O}_L(1),\mathcal{O}_L(-2))=k^3$ or $\mathrm{Ext}^1(I_L,\mathcal{O}_L(-2))\cong\mathrm{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_L(-1)\oplus\mathcal{O}_L(2),\mathcal{O}_L(-2))=k^3$. □

**Lemma 4.23.** *We have* $\mathrm{Hom}(I_D,\mathrm{pr}'(I_L))\neq 0$.

*Proof.* Applying $\mathrm{Hom}(I_D,-)$ to the triangle $\mathcal{E}^{\oplus 2}\to I_L\to\mathrm{pr}'(I_L)$, we get an exact sequence

$$0\to\mathrm{Hom}(I_D,\mathcal{E}^{\oplus 2})\to\mathrm{Hom}(I_D,I_L)\to\mathrm{Hom}(I_D,\mathrm{pr}'(I_L))\to\cdots.$$

Note that $\mathrm{Hom}(I_D,\mathcal{E})=0$, thus $\hom(I_D,\mathrm{pr}'(I_L))\geq\hom(I_D,I_L)$. Since $L\subset D$ is an irreducible component of $D$, $\hom(I_D,I_L)=1$. Then the result follows. □

**Lemma 4.24.** *The twisted cubic $C, D$ and a line $L$ as in Lemma 4.21 exist.*

*Proof.*

(1) Let $X$ be a special GM threefold, $\pi:X\to Y_5$ be the double cover and $\mathcal{B}\subset Y$ the branch locus. Let $c=l_1\cup l_2\cup l_3$ be a twisted cubic on $Y_5$ such that each $l_i$ is tangent to $\mathcal{B}$. Note that $l_1$ is in the conic $l_1\cup l_2$; pulling back to $X$ via $\pi$, we get a twisted cubic $C$ such that $L\cup C=\pi^{-1}(l_1\cup l_2)$ and $\tau(L)\subset C$. On the other hand, $l_1$ is in $c=l_1\cup l_2\cup l_3$; it is a twisted cubic triple tangent to $\mathcal{B}$, and pulling back to $X$ we get a twisted cubic $D$ and $I_D\in\mathcal{K}u(X)$. Note that $L\subset D$.

(2) If $X$ is an ordinary GM threefold, the locus of irreducible twisted cubics has dimension $\leq 2$, and the locus of twisted cubics that are in $\mathcal{K}u(X)$ has dimension 3. Thus there exists a twisted cubic $D$ that contains a line $L$ and $I_D\in\mathcal{K}u(X)$. On the other hand, since $\mathrm{Hom}(\mathcal{E},I_L)=k^2$, the locus of twisted cubics $C$ such that $C\cup L$ is the zero locus of a section of $\mathcal{E}^\vee$ is parametrized by $\mathbb{P}^1$, where $I_C\notin\mathcal{K}u(X)$. Choose one such twisted cubic $C$.

□

**Remark 4.25.** The idea of the proof of Theorem 4.20 was first explained to us by Arend Bayer. In [Zha20, Proposition 4.21], one of the authors made an attempt to prove this statement but the argument is incomplete. Here, we fill the gaps and give a uniform argument for all $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ when $d\geq 2$. In an upcoming paper [FP21, Theorem 3.1], the authors prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of very natural assumptions; these categories include Kuznetsov components of a series prime Fano threefolds.

## 5. Projection of $\mathcal{E}$ into $\mathcal{K}u(X)$

In this section, we consider the object that results from projecting the vector bundle $\mathcal{E}$ into $\mathcal{K}u(X)$, and its stability in $\mathcal{K}u(X)$.

5.1. **The projection of $\mathcal{E}$ into $\mathcal{K}u(X)$.** The set-up is as follows. Let

$$\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E} \rangle = \langle \mathcal{O}_X \rangle^{\perp} \subset \mathrm{D}^b(X),$$

and let $\pi := i^! : \mathcal{D} \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $i : \mathcal{K}u(X) \hookrightarrow \mathcal{D}$. Here $\mathcal{K}u(X)$ is the original Kuznetsov component.

**Lemma 5.1.** *The projection object $\pi(\mathcal{E})$ is given by $\mathbf{L}_{\mathcal{E}}\mathcal{Q}(-H)[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(\pi(\mathcal{E})) = \begin{cases} \mathcal{Q}(-H), & i = -1 \\ \mathcal{E}, & i = 0 \\ 0, & i \neq -1, 0. \end{cases}$$

*Proof.* Indeed, by e.g. [Kuz10, p. 4] we have the exact triangle

$$i\pi(\mathcal{E}) \to \mathcal{E} \to \mathbf{L}_{\mathcal{K}u(X)}\mathcal{E} \to .$$

But note that $\langle \mathcal{K}u(X), \mathcal{E} \rangle \simeq \langle S_{\mathcal{D}}(\mathcal{E}), \mathcal{K}u(X) \rangle \simeq \langle \mathbf{L}_{\mathcal{K}u(X)}\mathcal{E}, \mathcal{K}u(X) \rangle$. Therefore the triangle above becomes $i\pi(\mathcal{E}) \to \mathcal{E} \to S_{\mathcal{D}}(\mathcal{E})$. To find $S_{\mathcal{D}}(\mathcal{E})$ explicitly, note that $S_{\mathcal{D}} \cong \mathbf{R}_{\mathcal{O}_X(-H)} \circ S_{\mathrm{D}^b(X)}$. Since $\mathbf{R}_{\mathcal{O}_X(-H)}\mathcal{E}(-H) \cong \mathcal{Q}(-H)[-1]$, we have $S_{\mathcal{D}}(\mathcal{E}) \cong \mathcal{Q}(-H)[2]$. So the triangle above becomes

$$i\pi(\mathcal{E}) \to \mathcal{E} \to \mathcal{Q}(-H)[2].$$

Applying $i^* = \mathbf{L}_{\mathcal{E}}$ to the triangle and using the fact that $i^*i \cong \mathrm{id}$ and $i^*\mathcal{E} = 0$ gives $\pi(\mathcal{E}) \cong \mathbf{L}_{\mathcal{E}}\mathcal{Q}(-H)[1]$, as required. Taking the long exact sequence with respect to $\mathcal{H}^*$ gives the cohomology objects. ◻

**Remark 5.2.** Since $\hom(\mathcal{E}, \mathcal{Q}(-H)[2]) = 1$, the object $\pi(\mathcal{E})$ is the unique object that lies in the non-trivial triangle

$$\mathcal{Q}(-H)[1] \to \pi(\mathcal{E}) \to \mathcal{E}. \tag{2}$$

**Lemma 5.3.** *Let $X$ be a GM threefold. Then we have*

*(1)* $\mathrm{RHom}^\bullet(\mathcal{Q}(-H), \mathcal{E}) = \mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{Q}^\vee) = k^2$ *when $X$ is ordinary.*
*(2)* $\mathrm{RHom}^\bullet(\mathcal{Q}(-H), \mathcal{E}) = \mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{Q}^\vee) = k^3 \oplus k[-1]$ *when $X$ is special.*
*(3)* $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{Q}(-H)) = k[-2]$.

*Proof.* When $X$ is ordinary, this follows from the Koszul resolution of $X \subset \mathrm{Gr}(2,5)$ and the Borel–Weil–Bott Theorem. When $X$ is special, note that $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$. Then the result follows from the projection formula and [San14, Lemma 2.14, Proposition 2.15]. ◻

**Lemma 5.4.** *Let $X$ be a GM threefold. Then we have*

- $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = k \oplus k^2[-1]$ *when $X$ is ordinary.*
- $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = k \oplus k^3[-1] \oplus k[-2]$ *when $X$ is special.*

*Hence $\pi(\mathcal{E})$ is stable with respect to every Serre-invariant stability condition on $\mathcal{K}u(X)$.*

*Proof.* The first statement follows from applying $\mathrm{Hom}(-, \mathcal{E})$ to triangle (2) and Lemma 5.3, and also the fact that $\mathrm{RHom}^\bullet(\pi(\mathcal{E}), \pi(\mathcal{E})) = \mathrm{RHom}^\bullet(\pi(\mathcal{E}), \mathcal{E})$ which is by adjunction. The last statement follows from Lemma 4.14. ◻

5.2. **The analogous projection object for $\mathcal{A}_X$.** In this subsection, we state and prove the analogous results as in Subsection 5.1, except for $\mathcal{A}_X$ instead of $\mathcal{K}u(X)$. First note that

$$\begin{aligned}\mathrm{D}^b(X) &= \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle \\ &\simeq \langle \mathcal{A}_X, \mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee, \mathcal{O}_X \rangle \\ &\simeq \langle \mathcal{A}_X, \mathcal{Q}^\vee, \mathcal{O}_X \rangle.\end{aligned}$$

because $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee \cong \mathcal{Q}^\vee[1]$. Now let $\mathcal{D}' := \langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle$ and let $\pi' := (i')^! : \mathcal{D}' \to \mathcal{A}_X$ be the right adjoint to the inclusion $i' : \mathcal{A}_X \hookrightarrow \mathcal{D}'$.

**Lemma 5.5.** *The projection object $\pi'(\mathcal{Q}^\vee)$ is given by $\mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(\pi'(\mathcal{Q}^\vee)) = \begin{cases} \mathcal{E}, & i = -1 \\ \mathcal{Q}^\vee, & i = 0 \\ 0, & i \neq -1, 0. \end{cases}$$

*Proof.* The proof is completely analogous to the proof of Lemma 5.1. As before, we have the exact triangle

$$i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathbf{L}_{\mathcal{A}_X}\mathcal{Q}^\vee \to .$$

But note that $\langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle \simeq \langle S_{\mathcal{D}'}(\mathcal{Q}^\vee), \mathcal{A}_X \rangle \simeq \langle \mathbf{L}_{\mathcal{A}_X}\mathcal{Q}^\vee, \mathcal{A}_X \rangle$. Therefore the triangle above becomes $i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to S_{\mathcal{D}'}(\mathcal{Q}^\vee)$. To find $S_{\mathcal{D}'}(\mathcal{Q}^\vee)$ note that $S_{\mathcal{D}'}(\mathcal{Q}^\vee) = \mathbf{R}_{\mathcal{O}_X}(-H)(\mathcal{Q}^\vee(-H))[3]$. One can check that $\mathbf{R}_{\mathcal{O}_X}\mathcal{Q}^\vee = \mathcal{E}^\vee[-1]$, so $S_{\mathcal{D}'}(\mathcal{Q}^\vee) \cong \mathcal{E}[2]$. Hence our triangle becomes $i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathcal{E}[2]$. Now applying $(i')^* = \mathbf{L}_{\mathcal{Q}^\vee}$ to the triangle, we get $\pi'(\mathcal{Q}^\vee) = \mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}[1]$, as required. Since $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee, \mathcal{E}) = k[-2]$, we have the triangle $\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}$. Taking the long exact sequence of this triangle with respect to $\mathcal{H}^*$ gives the required cohomology objects. ∎

**Remark 5.6.** Later in Section 7, we will see that we in fact have $\pi'(\mathcal{Q}^\vee) \cong \mathrm{pr}(I_C)[1]$ where $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ is the projection functor, and $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$.

**Lemma 5.7.** *Let $X$ be a GM threefold. Then $\pi'(\mathcal{Q}^\vee)$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$. Moreover, it is a smooth point in its Bridgeland moduli space if and only if $X$ is ordinary.*

*Proof.* It is not hard to check that $\Xi(\pi(\mathcal{E})) \cong \pi'(\mathcal{Q}^\vee)[1]$, where $\Xi$ is the equivalence $\mathcal{K}u(X) \simeq \mathcal{A}_X$ from Lemma 3.6. Then the result follows from Lemma 5.4. ∎

## 6. Conics on GM threefolds

In this section, we collect some useful results regarding the birational geometry of GM threefolds and their Hilbert schemes of conics. The results in this section are all from [DIM12], [Log12] and [Ili94].

A conic means a closed subscheme $C \subset X$ with Hilbert polynomial $p_C(t) = 1 + 2t$, and a line means a closed subscheme $L \subset X$ with Hilbert polynomial $p_L(t) = 1 + t$. Denote their Hilbert schemes by $\mathcal{C}(X)$ and $\Gamma(X)$, respectively.

6.1. **Conics on ordinary GM threefolds.** Let $X$ be an ordinary GM threefold. Recall that it is a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5) = \mathrm{Gr}(2, V_5)$, where $V_5$ is a 5-dimensional complex vector space. Let $V_i$ be an $i$-dimensional vector subspace of $V_5$. There are two types of 2-planes in $\mathrm{Gr}(2,5)$; *$\sigma$-planes* are given set-theoretically as $\{[V_2] \mid V_1 \subset V_2 \subset V_4\}$, and *$\rho$-planes* are given by $\{[V_2] \mid V_2 \subset V_3\}$.

**Definition 6.1** ([DIM12, p. 5]).

- A conic $C \subset X$ is called a *$\sigma$-conic* if the 2-plane $\langle C \rangle$ spanned by $C$ is an $\sigma$-plane, and if there is a unique hyperplane $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2, V_4)$ and the union of the corresponding lines in $\mathbb{P}(V_5)$ is a quadric cone in $\mathbb{P}(V_4)$.
- A conic $C \subset X$ is called a *$\rho$-conic* if the 2-plane $\langle C \rangle$ spanned by $C$ is a $\rho$-plane, and the union of corresponding lines in $\mathbb{P}(V_5)$ is this 2-plane.

The following lemma is very useful for computations:

**Lemma 6.2.** *Let $X$ be an ordinary GM threefold and $C$ be a conic on $X$.*

*(1) If $C$ is a $\tau$-conic, then we have $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.*
*(2) If $C$ is a $\rho$-conic, then we have $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = 0$.*
*(3) If $C$ is a $\sigma$-conic, then we have $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = k[-1] \oplus k[-2]$.*

*Proof.* Note that if $\mathrm{Hom}(\mathcal{E}, I_C) = k^a$, then $C \subset \mathrm{Gr}(2, 5-a) \cap X$. Since for any conic $C$, there is some $V_4$ such that $C \subset \mathrm{Gr}(2, V_4)$, then we have $\hom(\mathcal{E}, I_C) \geq 1$ for any conic $C$. Now if $\hom(\mathcal{E}, I_C) \geq 2$, we know that $C$ is contained in a $\rho$-plane $\mathrm{Gr}(2,3)$. Since $\langle C \rangle$ is not in $\mathrm{Gr}(2,5)$ for a $\tau$-conic $C$, and $\langle C \rangle$ is a $\sigma$-plane $\{V_2 | V_1 \subset V_2 \subset V_4\}$ for a $\sigma$-conic, for these two types of conics we have $\mathrm{Hom}(\mathcal{E}, I_C) = k$. Also, for a $\rho$-conic $C$, since $\langle C \rangle = \mathrm{Gr}(2,3)$, we have $\hom(\mathcal{E}, I_C) \geq 2$. But if $\hom(\mathcal{E}, I_C) \geq 3$, we know that $C \subset \mathrm{Gr}(2,2)$ which is impossible. Hence for a $\rho$-conic $C$ we have $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$. Now the result for Ext groups follows from applying $\mathrm{Hom}(\mathcal{E}, -)$ to the short exact sequence $0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and $\chi(\mathcal{E}, I_C) = 1$.

First by stability and Serre duality, we have $\mathrm{Hom}(\mathcal{E}^\vee, I_C) = \mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$. From $\chi(\mathcal{E}^\vee, I_C) = 0$, we only need to compute $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. Since $\mathrm{RHom}^\bullet(\mathcal{O}_X, I_C) = 0$, applying $\mathrm{Hom}(-, I_C)$ to the tautological sequence, we have $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = \mathrm{Ext}^1(\mathcal{E}^\vee, I_C)$. Note that if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = k^a$, then $C \subset \mathrm{Gr}(2-a, 5-a) \cap X$. Thus we have $\hom(\mathcal{Q}^\vee, I_C) \leq 1$ for any conic $C$. And since $\hom(\mathcal{Q}^\vee, I_C) = 1$ if and only if $C$ is contained in the zero locus of a global section of $\mathcal{Q}$, which is a $\sigma$-3-plane in $\mathrm{Gr}(2,5)$, we know that $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = 0$ for $C$ of type $\tau$ or $\rho$, and $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = k$ for a $\sigma$-conic. Then the result follows. $\square$

Now we recall some properties of the Fano surface of conics $\mathcal{C}(X)$.

**Theorem 6.3** ([Log12], [DIM12])**.** *Let $X$ be an ordinary GM threefold. Then $\mathcal{C}(X)$ is an irreducible projective surface. If $X$ is furthermore general, then $\mathcal{C}(X)$ is smooth.*

It is a fact that there is a unique $\rho$-conic on $X$; denote it $c_X$. Furthermore, we have the following result which is a corollary of Logachev's Tangent Bundle Theorem (Section 4 in *loc. cit.*). Let $L_\sigma \subset \mathcal{C}(X)$ be the curve of $\sigma$-conics as defined above.

**Lemma 6.4** ([DIM12, p. 16])**.** *The only rational curve in $\mathcal{C}(X)$ is $L_\sigma$. Furthermore, there exists a surface $\mathcal{C}_m(X)$ and a map $\mathcal{C}(X) \to \mathcal{C}_m(X)$ which contracts $L_\sigma$ to a point $[\pi]$. If $X$ is general, then $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

**Theorem 6.5** ([DIM12, Section 5.2])**.** *Let $X$ be a general ordinary GM threefold. Then there is a natural involution $\iota$ on $\mathcal{C}_m(X)$, switching the points $[c_X]$ and $[\pi]$.*

Another very useful result which we require is Logachev's Reconstruction Theorem. This was originally proved in [Log12, Theorem 7.7], and then reproved later in [DIM12, Theorem 9.1].

**Theorem 6.6** (Logachev's Reconstruction Theorem)**.** *Let $X$ and $X'$ be general ordinary GM threefolds. If $\mathcal{C}(X) \cong \mathcal{C}(X')$, then $X \cong X'$.*

6.2. **Conic and line transforms.** For this section we follow [DIM12, § 6.1]. Let $X$ be a general ordinary GM threefold, and let $c$ be a smooth conic such that $c \neq c_X$. Let $\pi_c : \mathbb{P}^7 \to \mathbb{P}^4$ be the projection away from the 2-plane $\langle c \rangle$, and let $\epsilon : \widetilde{X} \to X$ be the blow-up of $X$ in $c$ with exceptional divisor $E$. Note that the composition $\pi_c \circ \epsilon$ is the morphism associated to the linear system $|-K_{\widetilde{X}}|$. This morphism has a Stein factorisation

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "\phi_{|-K_{\widetilde{X}}|}"] \arrow[dr, "\phi"'] & & \mathbb{P}^4 \\
& \overline{X} \arrow[ur] &
\end{tikzcd}$$

Since the conditions in [Isk99, Theorem 1.4.15] are all satisfied, there exists a $(-E)$-*flop*

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "f"] \arrow[dr, "\phi"'] & & \widetilde{X}_c \arrow[dl, "\phi_c"] \\
& \overline{X} &
\end{tikzcd}$$

A study of the properties of $-K_{\widetilde{X}_c}$ shows that there is a contraction $\epsilon_{c'} : \widetilde{X}_c \to X_c$, where $X_c$ is an ordinary GM threefold, and $\epsilon_{c'} : \widetilde{X}_c \to X_c$ is the blow-up of $X_c$ in a smooth conic $c'$ with exceptional divisor $E' = -2K_{\widetilde{X}_c} - f(E)$. In summary, there exists a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "f"] \arrow[dr, "\phi"] \arrow[dd, "\epsilon"'] & & \widetilde{X}_c \arrow[dl, "\phi_c"'] \arrow[dd, "\epsilon_{c'}"] \\
& \overline{X} & \\
X \arrow[rr, "\psi_c"] \arrow[ur, "\pi_c"] & & X_c \arrow[ul, "\pi_{c'}"']
\end{tikzcd}$$

where $\psi_c : X \dashrightarrow X_c$ is the *elementary transformation of $X$ along the conic $c$*. Note that the elementary transformation of $X_c$ along the conic $c'$ is $\psi_c^{-1} : X_c \dashrightarrow X$.

**Remark 6.7.** A similar flopping procedure can be done to construct the *elementary transformation of $X$ along the line $L$*, which we denote as $\psi_L : X \dashrightarrow X_L$ (see [DIM12, § 6.2]).

Conic transforms can be defined for any conic $c \subset X$. Such an $X_c$ is called the *period partner* of $X$ in [DK15], and the line transforms are called the *period duals*. We now list some important results about conic and line transforms below.

**Theorem 6.8** ([DIM12, Theorem 6.4]). *Let $X$ be a general ordinary GM threefold, and let $c \subset X$ be any conic. Then one can define a general ordinary GM threefold $X_c \simeq X$, such that $\mathcal{C}(X_c)$ is isomorphic to $\mathcal{C}_m(X)$ blown up at the point $[c] \in \mathcal{C}_m(X)$, where $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

**Proposition 6.9** ([DIM12, Theorem 6.4, Remark 7.2]). *Let $X$ be a general ordinary GM threefold. Then the isomorphism classes of conic transforms of $X$ are parametrized by the surface $\mathcal{C}_m(X)/\iota$.*

**Theorem 6.10** ([KP19, Theorem 1.6]). *Let $X$ be a general ordinary GM threefold. Then the Kuznetsov components of all conic transforms and line transforms of $X$ are equivalent to $\mathcal{A}_X$.*

6.3. **Conic on special GM threefolds.** Let $X$ be a special GM threefold. Recall that $X$ is a double cover $X \to Y$ of a degree five del Pezzo threefold $Y$ with branch locus a quadric hypersurface $\mathcal{B} \subset Y$. When $X$ is general, $\mathcal{B}$ is a smooth K3 surface of Picard number 1 and degree 10. Recall that $Y$ is a codimension 3 linear section of $\mathrm{Gr}(2,5)$. Let $\mathcal{V}$ be the tautological quotient bundle on $Y$. We recall some properties of $\mathcal{C}(X)$ from [Ili94].

**Theorem 6.11** ([Ili94])**.** *Let $X$ be a special GM threefold. Then $\mathcal{C}(X)$ has two components $\mathcal{C}_1$ and $\mathcal{C}_2$. One of the components $\mathcal{C}_2 \cong \Sigma(Y) \cong \mathbb{P}^2$ parametrizes the preimage of lines on $Y$. Moreover, when $X$ is general, $\mathcal{C}(X)$ is smooth away from $\mathcal{C}_1 \cap \mathcal{C}_2$.*

The following lemma will be useful in computations; it is similar to Lemma 6.2.

**Lemma 6.12.** *Let $X$ be a special GM threefold and $C$ a conic on $X$. Then $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if $C$ is the preimage of a line on $Y$. In this case $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) = k[-1] \oplus k[-2]$, and such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbb{P}^2$ on $Y$.*

*Proof.* The proof is almost the same as the proof of Proposition 7.1. The same argument shows that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_C) \neq 0$ if and only if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The image of a non-trivial map $\mathcal{Q}^\vee \to I_C$ is the ideal sheaf of the zero locus of a section $s$ of $\mathcal{Q}$, which is the preimage of the zero locus of a section of $\mathcal{V}$. By [San14, Lemma 2.18], the zero locus of a section of $\mathcal{V}$ is either a line or a point. Thus the zero locus of a section of $\mathcal{Q}$ is either the preimage of a line on $Y$ which is a conic on $X$, or a zero-dimensional closed subscheme of length two. But this zero locus contains a conic $C \subset X$, so $C = Z(s)$ is the preimage of a line on $Y$ and the map $\mathcal{Q}^\vee \to I_C$ is surjective. In particular, such conics are exactly the preimages of lines on $Y$, and are parametrized by $\Sigma(Y) \cong \mathbb{P}^2$. ◻

## 7. Conics and Bridgeland moduli spaces

In this section, we construct the moduli space of $\sigma$-stable objects of the $(-1)$-class $-x$ in the alternative Kuznetsov component $\mathcal{A}_X$ of a GM threefold $X$.

**Proposition 7.1.** *Let $C \subset X$ be a conic on a GM threefold $X$. Then $I_C \notin \mathcal{A}_X$ if and only if*

*(1) $C$ is a $\sigma$-conic when $X$ is ordinary. In particular, such a family of conics is parametrized by the line $L_\sigma$.*
*(2) $C$ is the preimage of a line on $Y$. In particular, such a family of conics is parametrized by the Hilbert scheme of lines $\Sigma(Y) \cong \mathbb{P}^2$ on $Y$.*

*Moreover, we have an exact sequence*

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

*Proof.* When $X$ is ordinary, the result follows from Lemma 6.2. When $X$ is special, this is by Lemma 6.12. Note that since $I_C \notin \mathcal{A}_X$, we have $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. The non-trivial map $\mathcal{Q}^\vee \to I_C$ is surjective by the arguments in Lemma 6.2 and 6.12. Note that by the stability of $\mathcal{Q}^\vee$, the kernel of $\mathcal{Q}^\vee \twoheadrightarrow I_C$ is $\mu$-stable with the same Chern character as $\mathcal{E}$, hence we have $\ker(\mathcal{Q}^\vee \twoheadrightarrow I_C) \cong \mathcal{E}$ by [DIM12, Proposition 4.1]. ◻

**Proposition 7.2.** *Let $X$ be a GM threefold and $C \subset X$ a conic on $X$. If $I_C \notin \mathcal{A}_X$, then we have the exact triangle*

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$$

*where $\mathcal{Q}$ is the tautological quotient bundle.*

*Proof.* By Proposition 7.1, $I_C$ fits into the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Applying the projection functor to this short exact sequence, we get a triangle

$$\mathrm{pr}(\mathcal{E}) \to \mathrm{pr}(\mathcal{Q}^\vee) \to \mathrm{pr}(I_C),$$

where $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}$. Note that applying the functor pr to the exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$ gives $\mathrm{pr}(\mathcal{Q}^\vee) = 0$. Then $\mathrm{pr}(I_C) \cong \mathrm{pr}(\mathcal{E})[1]$. Now we compute the projection $\mathrm{pr}(\mathcal{E}) = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. We have the triangle

$$\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \otimes \mathcal{E}^\vee \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}.$$

Since $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \cong k[-3]$, we get $\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. Now applying $\mathbf{L}_{\mathcal{O}_X}$ to this triangle, we get $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E} = \mathrm{pr}(\mathcal{E})$, which is equivalently

$$\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathrm{pr}(\mathcal{E}).$$

Therefore we obtain the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(\mathcal{E})[1] \to \mathcal{Q}^\vee$$

and the desired result follows. □

By [KP18b, Proposition 2.6], there is a natural involutive autoequivalence functor of $\mathcal{A}_X$, denoted by $\tau_\mathcal{A}$. When $X$ is special, it is induced by the natural involution $\tau$ on $X$, which comes from the double cover $X \to Y$. In this case it is easy to see that $\tau_\mathcal{A}(\mathrm{pr}(I_C)) \cong \mathrm{pr}(I_{\tau(C)})$.

When $X$ is ordinary, the situation is more subtle. In the following, we describe the action of $\tau_\mathcal{A}$ on the projection into $\mathcal{A}_X$ of an ideal sheaf of a conic $\mathrm{pr}(I_C)$ in this case.

**Proposition 7.3.** *Let $X$ be an ordinary GM threefold and $C$ a conic on $X$.*

*(1) If $I_C \in \mathcal{A}_X$, then $\tau_\mathcal{A}(I_C)$ is either*
- *$I_{C'}$ such that $C \cup C' = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$, where $Z(s)$ is the zero locus of the section $s$;*
- *or $\pi'(\mathcal{Q}^\vee)$, where $\pi'(\mathcal{Q}^\vee)$ is given by the triangle*

$$\mathcal{E}[1] \to \pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee.$$

*(2) If $I_C \notin \mathcal{A}_X$, then $\tau_\mathcal{A}(\mathrm{pr}(I_C)) \cong I_{C''}$ for a conic $C'' \subset X$.*

**Remark 7.4.** Once we have proved Theorem 7.13, this will imply that the involution induced by $\tau_\mathcal{A}$ on $\mathcal{C}_m(X)$ is the same as $\iota$ in Theorem 6.5, described in [DIM12, Section 5.2].

We first state some technical lemmas which we require for the proof of the proposition above.

**Lemma 7.5.** *Let $X$ be a GM threefold and $E$ a $\mu$-semistable sheaf on $X$ with truncated Chern character $\mathrm{ch}_{\leq 2}(E) = (2, -H, aL)$. Here $a \geq 1$ and $c_3(E) \geq 0$. Then we have $E \cong \mathcal{E}$.*

*Proof.* By Lemma 4.7, we have $a \leq 1$ which means $a = 1$ by our assumption. Then $c_1(E) = -1$ and $c_2(E) = 4$. Since $c_3(E) \geq 0$, by [BF14, Proposition 3.5] we have $\chi(E) = 0$. This implies $c_3(E) = 0$. Moreover, $E$ is a globally generated bundle by [BF14, Proposition 3.5]. Thus $E \cong \mathcal{E}$ by [DIM12, Proposition 4.1]. □

**Lemma 7.6.** *Let $X$ be a GM threefold and $E$ a $\mu$-semistable sheaf on $X$ with $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{Q})$. Then we have $E \cong \mathcal{Q}$.*

*Proof.* First we show that $h^2(E) = 0$; then from $\chi(E) = 5$ we have $h^0(E) \geq 5$. Indeed, if $h^2(E) \neq 0$, then $\mathrm{Hom}(E, \mathcal{O}_X(-H)[1]) \neq 0$ by Serre duality. Therefore, we have a non-trivial extension

$$0 \to \mathcal{O}_X(-H) \to F \to E \to 0.$$

If $F$ is not $\mu$-semistable, then by the stability of $\mathcal{O}_X(-H)$ and $E$, the minimal destabilizing quotient sheaf $F'$ of $F$ has $\mathrm{ch}_{\leq 1}(F') = (1, -H)$. Thus $F'^{\vee\vee} \cong \mathcal{O}_X(-H)$. But if we apply $\mathrm{Hom}(-, \mathcal{O}_X(-H))$ to the exact sequence above, we obtain $\mathrm{Hom}(F, \mathcal{O}_X(-H)) = 0$ since this extension is non-trivial, which gives a contradiction. Then $F$ is $\mu$-semistable with $\mathrm{ch}_{\leq 2}(F) = (4, 0, 4L)$, which is impossible since $\Delta(F) < 0$.

Now we can take five linearly independent elements in $H^0(E)$, and obtain a natural map $t : \mathcal{O}_X^{\oplus 5} \to E$. From the stability of $\mathcal{O}_X$ and $E$, we have $\mu(\mathrm{Im}(t)) = 0$ or $\mu(\mathrm{Im}(t)) = \frac{1}{3}$. But the first case cannot happen, since then $\mathrm{Im}(t)$ is the direct sum of a number of copies of $\mathcal{O}_X$, and this contradicts the construction of $t$. Thus $\mu(\mathrm{Im}(t)) = \frac{1}{3}$ and $\mathrm{ch}_{\leq 1}(\mathrm{Im}(t)) = (3, H)$. Also $\mathrm{ch}_{\leq 2}(\ker(t)) = (2, -H, xL)$, where $x \geq 1$. Note that $\ker(t)$ is reflexive, thus we have $c_3(\ker(t)) \geq 0$ since $\ker(t)$ has rank two. Then by stability of $\mathcal{O}_X$ and $\mathrm{Hom}(\mathcal{O}_X, \ker(t)) = 0$, it is not hard to see that $\ker(t)$ is $\mu$-semistable. Thus by Lemma 7.5 we have $\ker(t) \cong \mathcal{E}$. Therefore $\mathrm{ch}(\mathrm{Im}(t)) = \mathrm{ch}(E)$ and thus $t$ is surjective.

Now applying $\mathrm{Hom}(\mathcal{Q}, -)$ to the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to E \to 0,$$

from $\mathrm{RHom}^\bullet(\mathcal{Q}, \mathcal{O}_X) = 0$ and $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{E}) = k$ we have $\mathrm{Hom}(\mathcal{Q}, E) = k$. Thus from the stability of $E$ and $\mathcal{Q}$, we have $E \cong \mathcal{Q}$ and the result follows. ◻

**Lemma 7.7.** *Let $X$ be an ordinary GM threefold and $C$ a $\rho$-conic on $X$. Then the natural morphism $s' : \mathcal{E}^{\oplus 2} \to I_C$ is surjective and there is a short exact sequence*

$$0 \to \mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2} \to I_C \to 0.$$

*Proof.* By Lemma 6.2, we have $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$. Thus, taking two linearly independent elements in $\mathrm{Hom}(\mathcal{E}, I_C)$, we have a natural map $s' : \mathcal{E}^{\oplus 2} \to I_C$. Moreover, since $\langle C \rangle = \mathrm{Gr}(2, 3)$ and $\langle C \rangle \cap X = C$, we know that $s'$ is surjective. Let $K := \ker(s')$. Then it is not hard to see that $\mathrm{ch}(K) = \mathrm{ch}(Q(-H))$. Note that $\mathrm{Hom}(\mathcal{E}, K) = 0$ and $K$ is reflexive.

We claim that $K$ is $\mu$-semistable. Indeed, suppose $K$ is not $\mu$-semistable and let $K'$ be its maximal destabilizing subsheaf. Then $K'$ is also reflexive. Since $\mathrm{Hom}(\mathcal{E}, K) = 0$, we have $K' \neq \mathcal{E}$. By the stability of $\mathcal{E}$ and the fact that $K \subset \mathcal{E}^{\oplus 2}$, we know that $\mu(K') = -\frac{1}{2}$. Since $\mathrm{Hom}(K', \mathcal{E}) \neq 0$, by the stability of $K'$ and $\mathcal{E}$ we have $K' \subset \mathcal{E}$. Thus from $\mathrm{ch}_{\leq 1}(K') = \mathrm{ch}_{\leq 1}(\mathcal{E})$ we know that $\mathcal{E}/K'$ is supported in codimension $\geq 2$, which gives a contradiction since $\mathcal{E}$ and $K'$ are both reflexive.

Now the result follows from Lemma 7.6, since $K(H)$ is $\mu$-semistable with $\mathrm{ch}(K(H)) = \mathrm{ch}(\mathcal{Q})$. ◻

**Lemma 7.8.** *Let $X$ be an ordinary GM threefold. Consider the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$. Let $C$ be a conic on $X$. Then*

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E}), & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1] \end{cases}$$

*where $C'$ is the involutive conic of $C$.*

*Proof.* By Lemma 6.2, we have that $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either $k$ or $k^2 \oplus k[-1]$. If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then we have the triangle

$$\mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Taking cohomology with respect to the standard heart we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \xrightarrow{s} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to 0.$$

The image of the map $s$ is the ideal sheaf of an elliptic quartic $D$, thus we have following two short exact sequences: $0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to I_D \to 0$ and $0 \to I_D \to I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C) \to 0$. Then $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C))$ is a torsion-free sheaf of rank 1 with the same Chern character as $\mathcal{O}_X(-H)$. It is easy to show that it must be $\mathcal{O}_X(-H)$. On the other hand $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C))$ is supported on the residual curve $C'$ of $C$ in $D$ and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{O}_{C'}(-H)$. Thus we have the triangle

$$\mathcal{O}_X(-H)[1] \to \mathbf{L}_{\mathcal{E}}(I_C) \to \mathcal{O}_{C'}(-H)$$

and we observe that $\mathbf{L}_{\mathcal{E}}(I_C)$ is exactly the twisted derived dual of the ideal sheaf $I_{C'}$ of a conic $C' \subset X$, i.e. $\mathbf{L}_{\mathcal{E}}(I_C) \cong \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1]$.

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then we have the triangle

$$\mathcal{E}^2 \oplus \mathcal{E}[-1] \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Taking the long exact sequence in cohomology with respect to the standard heart, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E}^2 \xrightarrow{s'} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to 0.$$

Now by Lemma 7.7, $s'$ is surjective and the cohomology objects are given by $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{Q}(-H)$ and $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{E}$, which implies that $\mathbf{L}_{\mathcal{E}}(I_C) \cong \pi(\mathcal{E})$. ◻

*Proof of Proposition 7.3.* Since $\tau_{\mathcal{A}} \circ \tau_{\mathcal{A}} \cong \mathrm{id}$, we have $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1}$. It is easy to see $\tau_{\mathcal{A}}^{-1} \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(- \otimes \mathcal{O}_X(H))[-1]$. Then

$$\begin{aligned} \tau_A(I_C) &\cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(I_C \otimes \mathcal{O}_X(H))[-1] \\ &\cong \mathbf{L}_{\mathcal{O}_X}(\mathbf{L}_{\mathcal{E}}(I_C) \otimes \mathcal{O}_X(H))[-1]. \end{aligned}$$

The left mutation $\mathbf{L}_{\mathcal{E}}(I_C)$ is given by

$$\mathrm{RHom}^\bullet(\mathcal{E}, I_C) \otimes \mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Note that $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either $k$ or $k^2 \oplus k[-1]$. Then by Lemma 7.8,

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E}), & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1] \end{cases}$$

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))$. We have the triangle

$$\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \otimes \mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})).$$

Note that $\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \cong \mathrm{RHom}^\bullet(I_{C'}, \mathcal{O}_X) = k \oplus k[-1]$. Then we have the triangle

$$\mathcal{O}_X \oplus \mathcal{O}_X[-1] \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})). \tag{3}$$

The derived dual $\mathbb{D}(I_{C'})$ is given by the triangle $\mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathcal{O}_{C'}[-1]$. Then taking cohomology with respect to the standard heart of triangle (3) we have the long exact sequence

$$\begin{aligned} 0 = \mathcal{H}^{-1}(\mathbb{D}(I_{C'})) &\to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_X \\ &\to \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_{C'} \to \mathcal{H}^1(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to 0. \end{aligned}$$

Thus we have $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$, $\mathcal{H}^1(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$ and $\mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \cong I_{C'}$. Hence $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})) \cong I_{C'}$.

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(I_C \otimes \mathcal{O}_X(H))[-1] \cong \mathbf{L}_{\mathcal{O}_X}(\pi(\mathcal{E}) \otimes \mathcal{O}_X(H)[-1])$ by Lemma 7.8. Then using the triangle 2, we have $\tau_{\mathcal{A}}(I_C) \cong \pi'(\mathcal{Q}^\vee)$. Then (2) follows from $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1}$. ◻

Now the following two results follow from Lemma 5.4 and Lemma 5.7.

**Lemma 7.9.** *Let $X$ be a GM threefold. If $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$, then*

- $\mathrm{RHom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k \oplus k^2[-1]$ *when $X$ is ordinary.*
- $\mathrm{RHom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k \oplus k^3[-1] \oplus k[-2]$ *when $X$ is special.*

**Lemma 7.10.** *Let $X$ be a GM threefold. If $I_C \notin \mathcal{A}_X$, the projection $\mathrm{pr}(I_C)[1]$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

When $I_C \in \mathcal{A}_X$, we cannot use Lemma 4.14 to prove the Bridgeland stability of $I_C$, since $\mathcal{C}(X)$ can be singular and $\mathrm{Ext}^1(I_C, I_C)$ may have large dimension. Instead, we use a wall-crossing argument and the uniqueness of Serre-invariant stability conditions (Theorem 4.20).

**Lemma 7.11.** *Let $X$ be a GM threefold. Let $F$ be an object with $\mathrm{ch}_{\leq 2}(F) = (1, 0, -2L)$. Then there are no walls for $F$ in the range $-\frac{1}{2} \leq \beta < 0$ and $\alpha > 0$.*

*Proof.* Recall that by [BBF+20, Theorem 4.13], $\beta = 0$ is the unique vertical wall of $F$. Any other wall is a semicircle centered along the $\beta$-axis, and its apex lies on the hyperbola $\mu_{\alpha,\beta}(F) = 0$. Moreover, no two walls intersect.

Note that when $\mu_{\alpha,\beta}(F) = 0$ holds, we have $\beta < -\sqrt{\frac{2}{5}} < -\frac{1}{2}$, thus we know that there is no semicircular wall centered in the interval $-\frac{1}{2} \leq \beta < 0$. Therefore, any semicircular wall in the range $-\frac{1}{2} \leq \beta < 0$ will intersect $\beta = -\frac{1}{2}$. To prove the statement, we only need to show that there are no walls when $\beta = -\frac{1}{2}$. This follows from the fact that $\mathrm{ch}_1^{-\frac{1}{2}}(F)$ is minimal. ◻

**Proposition 7.12.** *Let $C \subset X$ be a conic on a GM threefold $X$ such that $I_C \in \mathcal{A}_X$. Then $I_C[1] \in \mathcal{A}_X$ is stable with respect to every Serre-invariant stability condition on $\mathcal{A}_X$.*

*Proof.* By Lemma 4.6 and Lemma 7.11, we know that $I_C$ is $\sigma_{\alpha,\beta}$-semistable for every $(\alpha, \beta) \in V$. Since $I_C$ is torsion-free, we know that $I_C[1] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable. Thus $I_C[1] \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha, \beta)$-semistable. Then stability with respect to every Serre-invariant stability condition follows from Theorem 4.13 and Theorem 4.20. ◻

Now we are ready to realize the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ as a contraction of the Fano surface $\mathcal{C}(X)$:

**Theorem 7.13.** *Let $X$ be a GM threefold and $\sigma$ a Serre-invariant stability condition on $\mathcal{A}_X$. The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $p(\mathcal{C}(X)) = \mathcal{S} \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of $\sigma$-stable objects in $\mathcal{A}_X$, where $p : \mathcal{C}(X) \to \mathcal{S}$ is*

- *a blow-down morphism to a smooth point when $X$ is ordinary;*
- *a contraction of the component $\mathbb{P}^2$ to a singular point when $X$ is special.*

*In particular, when $X$ is general and ordinary, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is isomorphic to the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics on $X$. When $X$ is general and special, the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ has only one singular point.*

*Proof.* We assume that $X$ is ordinary, but the argument for special GM threefold is almost the same. By Proposition 7.1, it is known that the family of conics $C \subset X$ with the property that $I_C \notin \mathcal{A}_X$ is parametrized by the line $L_\sigma$. By Lemma 7.10, $\mathrm{pr}(I_C)$ is $\sigma$-stable when $I_C \notin \mathcal{A}_X$. The ideal sheaves $I_C$ for all the conics $[C]$ in the complement of $L_\sigma$ in Fano surface $\mathcal{C}(X)$ of conics are contained in $\mathcal{A}_X$. Then $\mathrm{pr}(I_C[1]) = I_C[1] \in \mathcal{A}_X$, and they are $\sigma$-stable by Proposition 7.12.

Using the universal family of conics on $X \times \mathcal{C}(X)$, the functor pr induces a morphism $p : \mathcal{C}(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ factoring through one of the irreducible components $\mathcal{S}$ of $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ as in [LZ21, Lemma 4.4]. The complement of $L_\sigma$ in $\mathcal{C}(X)$ is a dense open subset $U_1$ of $\mathcal{C}(X)$ since $\mathcal{C}(X)$ is irreducible. The morphism $p|_{U_1}$ is injective and étale, so $p(U_1) \subset \mathcal{S}$ is also a dense open subset of $\mathcal{S}$. But $p$ is a proper morphism, so $p(\mathcal{C}(X)) = \mathcal{S}$. Then $p$ is a birational surjective proper morphism from $\mathcal{C}(X)$ to $\mathcal{S}$. In particular, $L_\sigma$ is contracted by $p$ to a smooth point by Lemma 7.9. Now from Proposition 9.4, we know $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$. Thus the result follows.

When $X$ is general and ordinary, the Fano surface $\mathcal{C}(X)$ is smooth. Thus $\mathcal{S}$ is a smooth surface obtained by blowing down a smooth rational curve $L_\sigma$ on the smooth irreducible projective surface $\mathcal{C}(X)$. This implies that $\mathcal{S}$ is also a smooth projective surface. On the other hand, it is known that there is a unique rational curve $L_\sigma \subset \mathcal{C}(X)$ and it is the unique exceptional curve by Lemma 6.4. Thus $\mathcal{S}$ is the minimal model $\mathcal{C}_m(X)$ of Fano surface of conics on $X$.

When $X$ is general and special, the last statement follows from Theorem 6.11 and Lemma 7.9. ◻

## 8. The moduli space $M_G(2,1,5)$ for GM threefolds

In this section we investigate the moduli space of rank 2 Gieseker-semistable torsion-free sheaves on a GM threefold $X$ with Chern classes $c_1 = H$ and $c_2 = 5L$, denoted $M_G^X(2,1,5)$. We drop $X$ from the notation when it is clear from context on which threefold we work. Note that if $F \in M_G(2,1,5)$, then

$$\mathrm{ch}(F) = (2, H, 0, -\frac{5}{6}P).$$

Recall the following theorem [DIM12, Section 8]:

**Theorem 8.1.** *Let $X$ be a GM threefold and $F \in M_G^X(2,1,5)$. Then $F$ is either a*

*(1) globally generated bundle which fits into a short exact sequence*

$$0 \to \mathcal{O}_X \to F \to I_Z(H) \to 0$$

*where $Z$ is a projective normal smooth elliptic quintic curve;*

*(2) non-locally free sheaf with a short exact sequence*

$$0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$$

*where $L$ is a line on $X$. Moreover, $F$ is uniquely determined by $L$;*

*(3) non-globally generated vector bundle which fits into the exact sequence*

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

*Moreover, $F$ is uniquely determined by $L$.*

*Furthermore, in all cases we have* $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$ *and* $\mathrm{RHom}^\bullet(\mathcal{O}_X, F(-H)) = 0$.

*Proof.* The proofs for statements in this theorem can be found in [DIM12, Section 8]. The result also follows from [BF14, Proposition 3.5]. ◻

A natural question to ask is what Bridgeland moduli space we get after projecting an object from $M_G(2,1,5)$ into the Kuznetsov component. Since it is easier in this setting, we will work with the alternative Kuznetsov component $\mathcal{A}_X$ in this section. Our analysis of the projections of objects in $M_G(2,1,5)$ is based on the three cases listed in Theorem 8.1. We begin with a Hom-vanishing result.

**Proposition 8.2.** *Let $X$ be a GM threefold and $F \in M_G^X(2,1,5)$. Then we have* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$.

*Proof.* By Serre duality and the stability of $\mathcal{E}^\vee$ and $F$, we have $\mathrm{Hom}(\mathcal{E}^\vee, F) = \mathrm{Ext}^3(\mathcal{E}^\vee, F) = 0$. Since $\chi(\mathcal{E}^\vee, F) = 0$, we only need to show that $\mathrm{Ext}^1(\mathcal{E}^\vee, F) = 0$ or $\mathrm{Ext}^2(\mathcal{E}^\vee, F) = 0$.

Firstly, let $F$ be globally generated. Applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence in Theorem 8.1 (1), from $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$ we obtain $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_Z(H)) \cong \mathrm{RHom}^\bullet(\mathcal{E}^\vee, F)$. Now we turn to $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_Z(1)) \cong \mathrm{RHom}^\bullet(\mathcal{E}, I_Z)$. We have $\mathrm{Hom}(\mathcal{E}, I_Z) = \mathrm{Ext}^3(\mathcal{E}, I_Z) = 0$ from Serre duality and stability. Since $\chi(\mathcal{E}, I_Z) = 0$, we only need to show that $\mathrm{Ext}^2(\mathcal{E}, I_Z) = 0$. To this end, we apply $\mathrm{Hom}(\mathcal{E}, -)$ to the ideal sheaf sequence $0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. Since $\mathrm{Ext}^i(\mathcal{E}, \mathcal{O}_X) = 0$ for $i \neq 0$, we only need to show that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{O}_Z) = 0$. We claim that

$$\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_Z) = \mathrm{RHom}^\bullet(\mathcal{O}_Z, \mathcal{E}^\vee|_Z) = k^5.$$

Indeed, by Atiyah's classification of vector bundles on elliptic curves [Ati57] and the case described in e.g. [IM05, § 5.2], we have that $\mathcal{E}^\vee|_Z$ can only split as the direct sum of line bundles with degrees $(2,3)$ or $(0,5)$. The second case in [IM05, § 5.2] is not possible because $\mathcal{E}^\vee|_Z$ has odd degree. But as shown in *loc. cit.*, $\mathcal{E}^\vee|_Z$ cannot split as the sum of line bundles with degrees $(0,5)$, otherwise $Z$ would not be projectively normal as explained in [IM05, § 5.2], which is a contradiction. So $\mathcal{E}^\vee|_Z \cong \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)$ where $p \in Z$ is a point. Then a simple cohomology computation shows that $H^0(Z, \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}^\vee|_Z) = k^5$. Finally, an Euler characteristic computation shows that

$$\chi(\mathcal{E}, \mathcal{O}_Z) = 5 = \hom(\mathcal{E}, \mathcal{O}_Z) - \mathrm{ext}^1(\mathcal{E}, \mathcal{O}_Z),$$

as required for the claim. Hence it follows that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

Now let $F$ be non-locally free. Apply $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence from Theorem 8.1 (2). Because $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}^\vee) = k$ by exceptionality, and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L) = k$ by a cohomology calculation ($\mathcal{E}|_L$ splits as $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$), we get the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}^\vee, F) \to k \to k \to \mathrm{Ext}^1(\mathcal{E}^\vee, F) \to 0.$$

Hence by $\mathrm{Hom}(\mathcal{E}^\vee, F) = 0$, we obtain $\mathrm{Ext}^1(\mathcal{E}^\vee, F) = 0$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

Now let $F$ be a non-globally generated bundle. Recall from Theorem 8.1 (3) the exact sequence

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

Let $G := \mathrm{im}(H^0(X, F) \otimes \mathcal{O}_X \to F)$. Then the exact sequence above can be split up into the short exact sequences

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to G \to 0 \tag{4}$$

and

$$0 \to G \to F \to \mathcal{O}_L(-1) \to 0. \tag{5}$$

Applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to sequence (4), we have the long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_X^{\oplus m}) \to \mathrm{Hom}(\mathcal{E}^\vee, G) \to \mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{E}) \to \cdots$$

where $m := h^0(X, F)$. First we know that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$. Next we find $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E})$. By Serre duality, $\mathrm{Ext}^i(\mathcal{E}^\vee, \mathcal{E}) \cong \mathrm{Ext}^{3-i}(\mathcal{E}, \mathcal{E})$ which is $k$ for $i = 3$ and 0 else by exceptionality of $\mathcal{E}$. Hence $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, G) = k[-2]$.

Next we apply $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence (5). We get the long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}^\vee, G) \to \mathrm{Hom}(\mathcal{E}^\vee, F) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_L(-1)) \to \mathrm{Ext}^1(\mathcal{E}^\vee, G) \to \cdots.$$

Since $\mathcal{E}|_L(-1)$ splits as $\mathcal{O}_L(-1)\oplus\mathcal{O}_L(-2)$, cohomology computations show that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L(-1)) = k[-1]$, so the resulting long exact sequence and the paragraph above gives that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$. ◻

8.1. **Involutions on $M_G(2,1,5)$.** In this subsection, we briefly recall the involutions which exist on $M_G(2,1,5)$. Let $F$ be a globally generated vector bundle, and consider the short exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X,F)\otimes\mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0.$$

Note that $\ker(\mathrm{ev})$ is a rank 2 vector bundle with $c_1 = -H$ and $c_2 = 5L$ and no global sections, hence $\ker(\mathrm{ev})^\vee \in M_G(2,1,5)$. Define $\iota F := \ker(\mathrm{ev})^\vee$. This bundle $\iota F$ is globally generated, and we have $H^0(X,\iota F)\cong H^0(X,F)^\vee$ [DIM12, p. 29]. If $F$ is a non-locally free sheaf, then the same construction gives a bundle $\iota F = \ker(\mathrm{ev})^\vee$.

Note that for a special GM threefold, there is another involution on $M_G(2,1,5)$ induced by the involution $\tau$ on $X$,

$$\tau^* : M_G(2,1,5) \to M_G(2,1,5), F \mapsto \tau^* F,$$

which is different from the one we just defined, since if $F$ is not a bundle, then $\iota F$ is a bundle but $\tau^* F$ is not.

8.2. **An explicit description of $\mathrm{pr}(F)$.** We are now ready to give an explicit description of $\mathrm{pr}(F)$, for all objects $F \in M_G(2,1,5)$.

**Lemma 8.3.** *Let $X$ be a GM threefold and $F \in M_G(2,1,5)$. Then we have*

$$\mathrm{pr}(F) = \begin{cases} (\iota F)^\vee[1] \cong \ker(\mathrm{ev})[1], & F \textit{ globally generated} \\ & \textit{or non-locally free} \\ \mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1), & F \textit{ non-globally generated} \end{cases}$$

*where $\iota$ is the involution on $M_G(2,1,5)$.*

*Proof.* As a result of Proposition 8.2, $\mathbf{L}_{\mathcal{E}}F = F$, so $\mathrm{pr}(F) = \mathbf{L}_{\mathcal{O}_X}F$. By Theorem 8.1 we have $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$, and the triangle defining the left mutation is

$$\mathcal{O}_X^{\oplus 4} \xrightarrow{\mathrm{ev}} F \to \mathrm{pr}(F). \tag{6}$$

In the cases where $F$ is globally generated or non-locally free, the evaluation map ev is surjective, so $\mathrm{pr}(F) = \ker(\mathrm{ev})[1]$. Subsection 8.1 relates $\ker(\mathrm{ev})$ to $\iota F$ as required.

If $F$ is non-globally generated, ev is not surjective. So we take the long exact sequence in cohomology with respect to $\mathrm{Coh}(X)$ of triangle (6). This gives an exact sequence

$$0 \to \mathcal{H}^{-1}(\mathrm{pr}(F)) \to \mathcal{O}_X^{\oplus 4} \to F \to \mathcal{H}^0(\mathrm{pr}(F)) \to 0. \tag{7}$$

Comparing the sequence (7) with the sequence in (3) of Theorem 8.1 gives that

$$\mathcal{H}^i(\mathrm{pr}(F)) = \begin{cases} \mathcal{E}, & i=-1 \\ \mathcal{O}_L(-1), & i = 0 \\ 0, & \text{else.} \end{cases}$$

Thus $\mathrm{pr}(F)$ in this case fits in the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1)$$

as required. ◻

8.3. **Compatibility of categorical and classical involutions for ordinary GM threefolds.** Let $X$ be an ordinary GM threefold, $\tau_{\mathcal{A}}$ be the involution of $\mathcal{A}_X$, and $\iota$ be the geometric involution of $M_G(2,1,5)$ defined in Section 8.1. Then $\tau_{\mathcal{A}}$ induces involutions of the Bridgeland moduli spaces of $\sigma$-stable objects $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)$. In Proposition 7.3, we already showed that the action of $\tau_{\mathcal{A}}$ on $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ induces a geometric involution on $\mathcal{C}_m(X)$. In this section, we show that the involution induced by $\tau_{\mathcal{A}}$ is also compatible with $\iota$ on $M_G(2,1,5)$.

**Proposition 8.4.** *Let $X$ be an ordinary GM threefold and $F\in M_G^X(2,1,5)$. Then $\tau_{\mathcal{A}}\mathrm{pr}(F)\cong \mathrm{pr}(\iota(F))$.*

*Proof.*

(1) If $F$ is a non-globally generated vector bundle, then by Corollary 8.3 we have the triangle $\mathcal{E}[1]\to\mathrm{pr}(F)\to\mathcal{O}_L(-1)$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F))$ is given by a triangle

$$\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee)\to\tau_{\mathcal{A}}(\mathrm{pr}(F))\to\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1].$$

Note that $\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee)=0$, hence $\tau_{A}(\mathrm{pr}(F))\cong\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1]$. It is easy to see $\mathrm{RHom}^\bullet(\mathcal{E}^\vee,\mathcal{O}_L)=k$, therefore we have $\mathcal{E}^\vee\to\mathcal{O}_L\to\mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L$. Also, since $\mathcal{E}^\vee\to\mathcal{O}_L$ is surjective, we have $\mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L\cong\ker(\mathcal{E}^\vee\to\mathcal{O}_L)[1]$, where $F':=\ker(\mathcal{E}^\vee\to\mathcal{O}_L)$ is a non-locally free sheaf in $M_G(2,1,5)$ by Theorem 8.1. Thus $\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\ker(\mathrm{ev})[1]$, where ev is the evaluation morphism $\mathrm{Hom}(\mathcal{O}_X,F')\otimes\mathcal{O}_X\xrightarrow{\mathrm{ev}}F'$. But note that $\ker(\mathcal{E}^\vee\to\mathcal{O}_L)\cong\iota(F)$ since $F'$ and $F$ are associated with the same line $L$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\mathbf{L}_{\mathcal{O}_X}(\iota F)$. Note that $\iota F$ is already a non-locally free sheaf and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee,\iota F)=0$ by Proposition 8.2. Thus we have $\iota F\cong\mathbf{L}_{\mathcal{E}^\vee}\iota F$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\iota F\cong\mathrm{pr}(\iota F)$ as required.

(2) If $F$ is a non-locally free sheaf in $M_G(2,1,5)$, then $F\cong\iota E$ for some non-globally generated vector bundle $E$. Thus we only need to check $\tau_{\mathcal{A}}(\mathrm{pr}(\iota E))\cong\mathrm{pr}(\iota\circ\iota(E))\cong\mathrm{pr}(E)$, but this is true by part (1) of the proof.

(3) If $F$ is a globally generated vector bundle, consider the standard short exact exact sequence

$$0\to\ker(\mathrm{ev})\to H^0(X,F)\otimes\mathcal{O}_X\xrightarrow{\mathrm{ev}}F\to 0.$$

Dualizing the sequence and applying pr, we get the triangle

$$\mathrm{pr}(F^\vee)\to\mathrm{pr}(\mathcal{O}_X^{\oplus 4})\to\mathrm{pr}(\ker(\mathrm{ev})^\vee)\cong\mathrm{pr}(\iota F).$$

Note that $F^\vee\in\mathcal{A}_X$ and $\mathrm{pr}(\mathcal{O}_X)=0$, thus we get $\mathrm{pr}(\iota F)\cong F^\vee[1]$. Since $F\in M_G(2,1,5)$ is a globally generated vector bundle, we have $F\cong\iota E$ for some globally generated vector bundle $E$. Then $\mathrm{pr}(F)=\mathrm{pr}(\iota E)\cong E^\vee[1]\cong E\otimes\mathcal{O}_X(-H)[1]$, hence $\tau_{\mathcal{A}}(\mathrm{pr}(F))\cong\tau_{\mathcal{A}}(E\otimes\mathcal{O}_X(-H))[1]\cong\mathrm{pr}(E)\cong\mathrm{pr}(\iota F)$.

$\square$

8.4. **A Bridgeland moduli space interpretation of $M_G(2,1,5)$.** We arrive at the first of the main results of Section 8:

**Theorem 8.5.** *Let $X$ be a GM threefold and $\sigma$ be a Serre-invariant stability condition on $\mathcal{A}_X$. Then the projection functor $\mathrm{pr}:\mathrm{D}^b(X)\to\mathcal{A}_X$ induces an isomorphism $M_G(2,1,5)\cong\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)$.*

We split the proof of this theorem into a series of lemmas and propositions.

**Proposition 8.6.** *The functor $\mathrm{pr}:\mathrm{D}^b(X)\to\mathcal{A}_X$ is injective on all objects in $M_G(2,1,5)$, i.e. if $\mathrm{pr}(F_1)\cong\mathrm{pr}(F_2)$, then $F_1\cong F_2$.*

*Proof.* For the case of globally generated vector bundles or non-locally free sheaves, by Corollary 8.3, $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies that

$$(\iota F_1)^\vee \cong (\iota F_2)^\vee. \tag{8}$$

Note that $(\iota F_i)^\vee \cong \iota F_i \otimes \mathcal{O}_X(-H)$ for $i = 1, 2$. Then we get $\iota F_1 \cong \iota F_2$. Finally, we apply $\iota$ to both sides. Since it is an involution $\iota^2 = \mathrm{id}$, so $F_1 \cong F_2$ as required.

For the case of non-globally generated vector bundles $F$, recall that from Lemma 8.3 we have $\mathcal{H}^{-1}(\mathrm{pr}(F)) = \mathcal{E}$ and $\mathcal{H}^0(\mathrm{pr}(F)) = \mathcal{O}_L(-H)$. Since $F$ is uniquely determined by the line $L$, and $\mathrm{Hom}(\mathcal{O}_L(-H), \mathcal{E}[2]) = k$, the object $\mathrm{pr}(F)$ is also uniquely determined by the line $L$. Thus $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies $F_1 \cong F_2$, as required. ◻

**Proposition 8.7.** *The functor* $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ *induces isomorphisms of* $\mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ *and* $\mathrm{Ext}^k(F_1, F_2)$ *for all* $k$ *and for all* $F_1, F_2 \in M_G(2, 1, 5)$.

*Proof.* We apply $\mathrm{Hom}(F_1, -)$ to the exact triangle $\mathcal{O}_X^{\oplus 4} \to F_2 \to \mathrm{pr}(F_2)$. By adjunction of pr and the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$, we have $\mathrm{Ext}^k(F_1, \mathrm{pr}(F_2)) = \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2))$ for all $k$. Thus we get a long exact sequence

$$\cdots \to \mathrm{Ext}^k(F_1, \mathcal{O}_X)^{\oplus 4} \to \mathrm{Ext}^k(F_1, F_2) \to \mathrm{Ext}^k(\mathrm{pr}(F_1), \mathrm{pr}(F_2)) \to \mathrm{Ext}^{k+1}(F_1, \mathcal{O}_X)^{\oplus 4} \to \cdots.$$

Note that $\mathrm{Ext}^k(F_1, \mathcal{O}_X) = \mathrm{Ext}^{3-k}(\mathcal{O}_X, F_1(-H)) = 0$ for all $k$ by [BF14, Proposition 3.5]. Thus the desired result follows. ◻

In what follows, we show the stability of $\mathrm{pr}(F)$ in $\mathcal{A}_X$.

**Proposition 8.8.** *Let* $X$ *be a GM threefold and* $F \in M_G^X(2, 1, 5)$. *Then we have*

*(1)* $\mathrm{RHom}^\bullet(F, F) = k \oplus k^2[-1]$ *when* $X$ *is ordinary;*
*(2)* $\mathrm{RHom}^\bullet(F, F) = k \oplus k^2[-1]$ *or* $\mathrm{RHom}^\bullet(F, F) = k \oplus k^3[-1] \oplus k[-2]$ *when* $X$ *is special.*

*Proof.* Fist we assume that $X$ is ordinary. By [DIM12, Theorem 8.2], we have $\mathrm{ext}^1(F, F) = 2$. Now $\hom(F, F) = 1$ and $\mathrm{ext}^3(F, F) = 0$ by Serre duality and the stability of $F$. Note that $\chi(F, F) = -1$, so $\mathrm{ext}^2(F, F) = 0$.

Now we assume that $X$ is special. Then by Proposition 8.7 and Serre duality in $\mathcal{K}u(X)$, we have

$$\begin{aligned}
\mathrm{Ext}^2(F, F) &\cong \mathrm{Ext}^2(\mathrm{pr}(F), \mathrm{pr}(F)) \\
&\cong \mathrm{Hom}(\mathrm{pr}(F), \tau_{\mathcal{A}}(\mathrm{pr}(F))) \\
&\cong \mathrm{Hom}(\mathrm{pr}(F), \mathrm{pr}(\tau^* F)) \cong \mathrm{Hom}(F, \tau^* F),
\end{aligned}$$

where $\tau$ is the involution on $X$ induced by the double cover. Thus when $F \cong \tau^* F$, we have $\mathrm{Ext}^2(F, F) = k$, and $\mathrm{Ext}^2(F, F) = 0$ otherwise. Since $\mathrm{Ext}^3(F, F) = 0$ and $\mathrm{Hom}(F, F) = k$, the result follows from $\chi(F, F) = 1$. ◻

**Lemma 8.9.** *For every* $F \in M_G(2, 1, 5)$, *the object* $\mathrm{pr}(F)$ *is stable with respect to every Serre-invariant stability condition on* $\mathcal{A}_X$.

*Proof.* This follows from Proposition 8.7, Proposition 8.8 and Lemma 4.14. ◻

*Proof of Theorem 8.5.* First note that $M_G(2, 1, 5)$ is a fine moduli space. This is a consequence of [HL10, Theorem 4.6.5]. Using Lemma 8.9, by the same argument as in [Zha20, Theorem 8.9], the projection functor pr induces a morphism

$$p : M_G(2, 1, 5) \to \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$$

which is bijective on points by Theorem 9.1 and bijective on tangent spaces by Proposition 8.7. Hence it is an isomorphism. ◻

## 9. Irreducibility of Bridgeland moduli spaces

In this section, we will prove some technical results. The only results that will be used in other sections are Theorems 9.1 and 9.4 in the proof of Theorems 7.13 and 8.5, so there is no harm in skipping this whole section.

We first fix some notation. Let $\alpha > 0$ and $\beta < 0$. For an object $E \in \mathrm{D}^b(X)$, the limit central charge $Z^0_{0,0}(E)$ is defined as the limit of $Z^0_{\alpha,\beta}(E)$ when $(\alpha,\beta) \to (0,0)$. Note that $Z^0_{\alpha,\beta}(E)$ is given by $\mathbb{Q}$-linear combinations of $\alpha, \beta, \alpha^2, \beta^2$, thus such a limit $Z^0_{0,0}(E)$ always exists. For $Z^0_{0,0}(E) \neq 0$, we can also define the limit slope $\mu^0_{0,0}(E)$ as follows:

- If $\mathrm{Im}(Z^0_{0,0}(E)) \neq 0$, then we define $\mu^0_{0,0}(E) := -\frac{\mathrm{Re}(Z^0_{0,0}(E))}{\mathrm{Im}(Z^0_{0,0}(E))}$.
- If $\mathrm{Im}(Z^0_{0,0}(E)) = 0$ and $\mathrm{Re}(Z^0_{0,0}(E)) > 0$, then we define $\mu^0_{0,0}(E) := -\infty$.
- If $\mathrm{Im}(Z^0_{0,0}(E)) = 0$ and $\mathrm{Re}(Z^0_{0,0}(E)) < 0$, then we define $\mu^0_{0,0}(E) := +\infty$.

Note that $Z^0_{0,0}(E) = 0$ if and only if $\mathrm{ch}_{\leq 2}(E)$ is a multiple of $\mathrm{ch}_{\leq 2}(\mathcal{O}_X)$.

Let $E \in \mathrm{Coh}^0_{\alpha,\beta}(X)$. By continuity, we can find a nerborhood $U_E$ of the origin such that for any $(\alpha,\beta) \in U_E$, the slopes $\mu^0_{\alpha,\beta}(E)$ and $\mu^0_{0,0}(E)$ are both negative or positive. Let $F \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ be another object such that $E, F$ are both $\sigma^0_{\alpha,\beta}$-semistable in a neighborhood $U_{E,F}$ of the origin. If $\mu^0_{0,0}(E) > \mu^0_{0,0}(F)$, then by continuity, we can find a smaller neighborhood $U'_{E,F}$ such that $\mu^0_{\alpha,\beta}(E) > \mu^0_{\alpha,\beta}(F)$ holds for every $(\alpha,\beta) \in U'_{E,F}$. Thus we have $\mathrm{Hom}(E,F) = 0$. We will use these two elementary facts repeatedly.

9.1. **The moduli space of class $y - 2x$.** In this subsection, we show that $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$.

Recall that in 4.10 we defined

$$V := \{(\alpha,\beta)\colon -\frac{1}{10} < \beta < 0, 0 < \alpha < -\beta\}.$$

**Theorem 9.1.** *Let $F \in \mathcal{A}(\alpha,\beta)$ be a $\sigma(\alpha,\beta)$-stable object with numerical class $y-2x$ for every $(\alpha,\beta) \in V$. Then $F = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$.*

*Proof.* First we argue as in [PY20, Proposition 4.6]. When $(\alpha_0,\beta_0) = (0,0)$, we have $\mu^0_{\alpha_0,\beta_0}(F) = -\infty$. Since there are no walls intersecting with $\beta = 0$ as in [PY20, Proposition 4.6], we know that $F$ is $\sigma^0_{\alpha,0}$-semistable for all $\alpha > 0$. By the definition of the double tilted heart, we have a triangle

$$A[1] \to F \to B$$

such that $A$ (respectively $B$) is in $\mathrm{Coh}^0(X)$ with its $\sigma_{\alpha,0}$-semistable factors having slope $\mu_{\alpha,0} \leq 0$ (respectively $\mu_{\alpha,0} > 0$). Since $F$ is $\sigma^0_{\alpha,0}$-semistable and $\mu^0_{\alpha,0}(F) < 0$, we have that $A[1] = 0$ and $B \cong F$. Since $\mathrm{ch}^0_1(F)$ is minimal, there are no walls on $\beta = 0$, and we know that $F$ is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. Thus by Lemma 4.6, $\mathcal{H}^{-1}(F)$ is a $\mu$-semistable reflexive sheaf and $\mathcal{H}^0(F)$ is 0 or supported in dimension $\leq 1$.

If $\mathcal{H}^0(F)$ is supported in dimension 0, then $\mathrm{ch}(\mathcal{H}^0(F)) = bP$ for $b \geq 1$. But this is impossible since then $c_3(\mathcal{H}^{-1}(F)) > 0$ and by [BF14, Proposition 3.5] we have $\chi(\mathcal{H}^{-1}(F)) = 0$, which implies $b = 0$.

If $\mathcal{H}^0(F)$ supported in dimension 1, we can assume $\mathrm{ch}(\mathcal{H}^0(F)) = aL + \frac{b}{2}P$ where $a \geq 1$ and $b$ are integers. Thus $\mathrm{ch}(\mathcal{H}^{-1}(F)) = 2 - H + aL + (\frac{5}{6} + \frac{b}{2})P$. Now from Lemma 7.5, we know

$\mathcal{H}^{-1}(F) \cong \mathcal{E}$ and $\mathrm{ch}(\mathcal{H}^0(F)) = L - \frac{P}{2}$. Thus $\mathcal{H}^0(F) \cong \mathcal{O}_L(-1)$ for some line $L$ on $X$. Therefore we have a triangle

$$\mathcal{E}[1] \to F \to \mathcal{O}_L(-1).$$

In this case we have $\mathrm{Hom}(\mathcal{O}_L(-1), \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee(1), \mathcal{O}_L[1]) = H^1(L, \mathcal{E}(-1)|_L) = H^1(L, \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) = k$. Hence by Lemma 8.3, $F \cong \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that $E$ is locally free but not globally generated.

If $\mathcal{H}^0(F) = 0$, we have $F[-1] \cong \mathcal{H}^{-1}(F)$. Then $F[-1]$ is a $\mu$-semistable sheaf. Since $F[-1]$ is reflexive and $c_3(F[-1]) = 0$, $F[-1] \in M_G(2,-1,5)$ is a stable vector bundle. Thus by Lemma 8.3, we know $F[-1] = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that $E$ is a globally generated vector bundle or non-locally free sheaf. ∎

9.2. **The moduli space of class $-x$.** In this subsection, we show that $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$.

**Lemma 9.2.** *If $F \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha,\beta)$-stable such that $[F] = -x$ and $F$ is $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha, \beta) \in V$, then $F \cong I_C[1]$ for some conic $C$ on $X$.*

*Proof.* Since $F$ is $\sigma^0_{\alpha,\beta}$-semistable and $\mu^0_{\alpha,\beta}(F) > 0$, as in [PY20, Proposition 4.6] there is a triangle

$$F_1[1] \to F \to F_2$$

where $F_1 \in \mathrm{Coh}^\beta(X)$ with $\mu^+_{\alpha,\beta}(F_1) < 0$ and $F_2$ is supported on points. Thus $\mathrm{ch}(F_1) = (1, 0, -2L, mP)$, where $m$ is the length of $F_2$. By Lemmas 7.11 and 4.6, $F_1$ is a rank one torsion free sheaf, hence it is the ideal sheaf of a closed subscheme. Thus by [San14, Corollary 1.38], we have $m \leq 0$, which means $F_2 = 0$ and $F_1 \cong F[-1]$. Thus by Lemma 7.11 again, $F[-1]$ is a $\mu$-semistable torsion free sheaf, which is of the form $F[-1] \cong I_C$ for some conic $C$ on $X$ since $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. ∎

When $F$ is not $\sigma^0_{\alpha,\beta}$-semistable for $(\alpha, \beta) \in V$, the argument is slightly more complicated. Our main tools are the inequalities in [PR20], [PY20, Proposition 4.1], Lemma 4.7 and Lemma 4.8, which allow us to bound the rank and first two Chern characters $\mathrm{ch}_1, \mathrm{ch}_2$ of the destabilizing objects and their cohomology objects. Since $F \in \mathcal{A}_X$, by using the Euler characteristics $\chi(\mathcal{O}_X, -)$ and $\chi(\mathcal{E}^\vee, -)$ we can obtain a bound on $\mathrm{ch}_3$. Finally, via a similar argument as in Lemma 7.5 we deduce that the Harder–Narasimhan factors of $F$ are the ones we expect.

**Lemma 9.3.** *If $F \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha,\beta)$-stable such that $[F] = -x$ and $F$ is not $\sigma^0_{\alpha,\beta}$-semistable for every $(\alpha, \beta) \in V$, then $F$ fits into a triangle*

$$\mathcal{E}[2] \to F \to \mathcal{Q}^\vee[1].$$

*Proof.* Since there are no walls for $F$ tangent to the wall $\beta = 0$, by the local finiteness of walls and [BMT11, Proposition 2.2.2] we can find an open neighborhood $U'$ of the origin such that the Harder–Narasimhan filtration with respect to $\sigma^0_{\alpha,\beta}$ is constant for every $(\alpha, \beta) \in U := U' \cap V$. In the following we will only consider $\sigma^0_{\alpha,\beta}$ for $(\alpha, \beta) \in U$.

Let $B$ be the minimal destabilizing quotient object of $F$ and $0 \to A \to F \to B \to 0$ be the destabilizing short exact sequence of $F$ in $\mathrm{Coh}^0_{\alpha,\beta}(X)$. Hence we know that $A, B \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and $B$ is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^{0,-}_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$ for all $(\alpha, \beta) \in U$. By [BLMS17, Remark 5.12], we have $\mu^0_{\alpha,\beta}(B) \geq \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$. Hence the following relations hold for all $(\alpha, \beta) \in U$:

(a) $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(F) > \mu^0_{\alpha,\beta}(B)$,
(b) $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$, $\mathrm{Im}(Z^0_{\alpha,\beta}(B)) > 0$,
(c) $\mu^0_{\alpha,\beta}(B) \geq \min\{\mu^0_{\alpha,\beta}(F), \mu^0_{\alpha,\beta}(\mathcal{O}_X), \mu^0_{\alpha,\beta}(\mathcal{E}^\vee)\}$,

(d) $\Delta(B) \geq 0$.

By continuity we have:

(1) $\mu^0_{0,0}(A) \geq \mu^0_{0,0}(F) = 0 \geq \mu^0_{0,0}(B)$,
(2) $\mathrm{Im}(Z^0_{0,0}(A)) \geq 0$, $\mathrm{Im}(Z^0_{0,0}(B)) \geq 0$,
(3) $\mu^0_{0,0}(B) \geq \min\{\mu^0_{0,0}(F), \mu^0_{0,0}(\mathcal{O}_X), \mu^0_{0,0}(\mathcal{E}^\vee)\}$,
(4) $\Delta(B) \geq 0$.

Assume $[A] = a[\mathcal{O}_X]+b[\mathcal{O}_H]+c[\mathcal{O}_L]+d[\mathcal{O}_P]$. Then $[B] = (-1-a)[\mathcal{O}_X]-b[\mathcal{O}_H]+(2-c)[\mathcal{O}_L]-(1+d)[\mathcal{O}_P]$. Then $\mathrm{ch}(A) = (a, bH, \frac{c-5b}{10}H^2, \frac{\frac{5}{3}b+\frac{c}{2}+d}{10}H^3)$ and $Z^0_{0,0}(A) = bH^3 + (\frac{c-5b}{10}H^3)\cdot i$, $Z^0_{0,0}(B) = -bH^3 + (\frac{2-c+5b}{10}H^3)\cdot i$ and $\mu^0_{0,0}(A) = \frac{10b}{5b-c}, \mu^0_{0,0}(B) = \frac{-10b}{c-5b-2}$. Note that $[F] = -[\mathcal{O}_X]+2[\mathcal{O}_L]-[\mathcal{O}_P]$. From (2) we know $c-5b = 0$, 1 or 2. But when $c-5b=2$, it is not hard to see that $(c)$ fails near the origin. Thus $c-5b = 0$ or 1.

We begin with two claims.

**Claim 1:** $\mathrm{RHom}^\bullet(\mathcal{O}_X, B) = \mathrm{Hom}(\mathcal{O}_X, B)$ and $\mathrm{RHom}^\bullet(\mathcal{O}_X, A) = \mathrm{Ext}^1(\mathcal{O}_X, A)[-1]$.

Since $F \in \mathcal{A}_X$, we only need to prove that $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$. Indeed, since $\mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ and $F \in \mathcal{A}_X$, we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for all $i \leq 0$. Also, by Serre duality we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = \mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i])$. Thus from $\mathcal{O}_X(-H) \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we obtain $\mathrm{Hom}(A, \mathcal{O}_X(-H)[3-i]) = 0$ for $i \geq 2$. Therefore we have $\mathrm{Ext}^i(\mathcal{O}_X, A) = 0$ for $i \neq 1$.

**Claim 2:** $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, B) = \mathrm{Hom}(\mathcal{E}^\vee, B)$ and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, A) = \mathrm{Ext}^1(\mathcal{E}^\vee, A)[-1]$.

Since $\mathcal{E}^\vee$ and $\mathcal{E}[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, the argument is the same as Claim 1.

Now we deal with the cases $c-5b=0$ and $c-5b=1$ separately.

**Case 1 ($c-5b=0$):**

First we assume that $c-5b=0$. By 9.2, we have:

(1) $-2 \leq b \leq 0$,
(2) $b^2 + \frac{2a+2}{5} \geq 0$.

**Case 1.1 ($b=0$):** If $b=0$, then $c=0$ and $a \geq -1$. In this case we have $\mathrm{ch}_{\leq 2}(B) = (-1-a, 0, 2L)$. If $a=-1$, then $\mathrm{ch}_{\leq 2}(B) = (0,0,2L)$, which is impossible since $B \in \mathrm{Coh}^0_{\alpha,\beta}(X)$. Thus $a \geq 0$, but then we have $\mu^0_{\alpha,\beta}(F) \geq \mu^0_{\alpha,\beta}(B)$ when $(\alpha,\beta) \in U$ is sufficiently close to the origin. This contradicts our assumption on $B$.

**Case 1.2 ($b=-1$):** If $b=-1$, we have $c=-5$. In this case $\mathrm{ch}_{\leq 2}(A) = (a, -H, 0)$. Since $A \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$ for every $(\alpha,\beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $\alpha < -\beta$, and we have $a \geq \frac{-2\beta}{\beta^2-\alpha^2}$. But note that when $\alpha = \frac{-\beta}{2}$ and $\beta \to -0$, we have $\frac{-2\beta}{\beta^2-\alpha^2} \to +\infty$, thus we get a contradiction since $a$ is a finite number.

**Case 1.3 ($b=-2$):** If $b=-2$, we have $c=-10$. In this case we have $\mathrm{ch}_{\leq 2}(A) = (a, -2H, 0)$. Similarly to case 1.2, we have $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) \geq 0$ for every $(\alpha,\beta) \in U$. Note that $\mathrm{Im}(Z^0_{\alpha,\beta}(A)) = (2\beta + \frac{a(\beta^2-\alpha^2)}{2})H^3$ and $\alpha < -\beta$, and we have $a \geq \frac{-4\beta}{\beta^2-\alpha^2}$. Then as in Case 1.2, we get a contradiction.

**Case 2 ($c-5b=1$):** Now we assume that $c-5b=1$. Then by 9.2, we have:

(1) $-1 \leq b \leq 0$,
(2) $b^2 + \frac{a+1}{5} \geq 0$.

**Case 2.1 ($b=0$):** If $b=0$, then $c=1$. Therefore $-1 \leq a$. If $a=-1$, since $B$ is $\sigma^0_{\alpha,\beta}$-semistable, we know $\mathcal{H}^0_{\mathrm{Coh}^\beta(X)}(B)$ is either 0 or supported on points. Thus $\mathrm{ch}_{\leq 2}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B)) = (0,0,-L)$. But $\mathrm{Re}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) > 0$ which is impossible since $\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B) \in \mathrm{Coh}^\beta(X)$ with $\mathrm{Im}(Z_{\alpha,\beta}(\mathcal{H}^{-1}_{\mathrm{Coh}^\beta(X)}(B))) = 0$.

Therefore we have $a \geq 0$. Hence $\mathrm{ch}_{\leq 2}(B) = -(a+1, 0, -L)$, where $a+1 \geq 1$. This is also impossible since when $(\alpha,\beta) \in U$ is sufficiently close to the origin, we have $\mu^0_{\alpha,\beta}(B) > \mu^0_{\alpha,\beta}(F)$.

**Case 2.2 ($b = -1$):** We have $b = -1$ and $c = -4$. Hence $-6 \leq a$. In this case $\mathrm{ch}_{\leq 2}(B) = (-1-a, H, L)$ and we have $\mu^0_{\alpha,\beta}(B) < 0$ for when $(\alpha,\beta) \in U$ is sufficiently close to the origin. Therefore, $B \in \mathrm{Coh}^{\beta}(X)$ is $\sigma_{\alpha,\beta}$-semistable. Applying Lemma 4.7 to $B$, we have $a \geq -3$.

We first prove a claim.

**Claim 3:** In the situation of Case 2.2, we have $A$ is $\sigma^0_{\alpha,\beta}$-semistable. Hence $\mathrm{RHom}^\bullet(\mathcal{O}_X, A) = 0$, $\mathrm{ch}(A) = (a, -H, L, (\frac{7}{3} - a)P)$ and $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

Assume $A$ is not $\sigma^0_{\alpha,\beta}$-semistable for some $(\alpha,\beta) \in U$. Then we can take a neighborhood $U'_A$ of the origin such that $A$ has constant Harder–Narasimhan factors for any $(\alpha,\beta) \in U_A := U \cap U'_A \cap V$. Let $C$ be the minimal destabilizing quotient object of $A$ with respect to $\sigma^0_{\alpha,\beta}$ for $(\alpha,\beta) \in U_A$. In this case we have $\mathrm{ch}_{\leq 2}(A) = (a, -H, L)$. Since $\mathrm{Im}(Z^0_{0,0}(A)) = \frac{1}{10}H^3$, we know that $\mathrm{Im}(Z^0_{0,0}(C)) = 0$ or $\frac{1}{10}H^3$. If $\mathrm{Im}(Z^0_{0,0}(C)) = 0$, then $\mu^0_{0,0}(C) = +\infty$ or $-\infty$. But the previous case contradicts $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ and the latter case contradicts $\mu^0_{\alpha,\beta}(C) > \mu^0_{\alpha,\beta}(F)$. Therefore we have $\mathrm{Im}(Z^0_{0,0}(C)) = \frac{1}{10}H^3$ and we can assume that $\mathrm{ch}_{\leq 2}(C) = (e, fH, L)$ where $e, f \in \mathbb{Z}$. Since $\mu^0_{0,0}(A) \geq \mu^0_{0,0}(C) \geq \mu^0_{0,0}(F) = 0$, we have $10 \geq -10f \geq 0$. If $f = 0$, then $\mathrm{ch}_{\leq 2}(C) = (e, 0, L)$ and $\mathrm{ch}_{\leq 2}(D) = (a-e, -H, 0)$, where $D = \mathrm{cone}(A \to C)[-1]$. Then $\mu^{0-}_{\alpha,\beta}(D) > \mu^0_{\alpha,\beta}(A)$ for any $(\alpha,\beta) \in U_A$. Hence $\mu^0_{\alpha,\beta}(D) = \frac{1+(a-e)\beta}{\beta + \frac{a-e}{2}(\beta^2 - \alpha^2)}$. But note that if we take $\alpha = -\frac{\beta}{2}$ and $|\beta| < |\frac{1}{a-e}|$, when $(\alpha,\beta) \in U_A$ and $|\beta|$ is sufficiently small we get $1 + (a-e)\beta > 0$ and $\beta + \frac{a-e}{2}(\beta^2 - \alpha^2) < 0$. This implies $\mu^0_{\alpha,\beta}(D) < 0$ for such $(\alpha,\beta)$, which gives a contradiction since $\mu^0_{\alpha,\beta}(D) > \mu^0_{\alpha_0,\beta_0}(F)$ holds for any $(\alpha,\beta) \in U_A$.

Therefore the only possible case is $f = -1$, and hence $\mu^0_{0,0}(C) = 10$. Since $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(C)$ for $(\alpha,\beta) \in U_A$, we have $\mathrm{rk}\, C > a$. But this is impossible since $D, \mathcal{O}_X \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ but $\mathrm{ch}_{\leq 2}(D) = (s, 0, 0) = s \cdot \mathrm{ch}_{\leq 2}(\mathcal{O}_X)$ where $s = a - \mathrm{rk}\, C < 0$. Now for the last statement, note that $\mathcal{O}_X(-H)[2] \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^0_{0,0}(\mathcal{O}_X(-H)[2]) = 2$, hence we have $\mathrm{Hom}(A, \mathcal{O}_X(-H)[2]) = \mathrm{Hom}(\mathcal{O}_X, A[1]) = 0$. Now combining with Claim 1, this proves our claim.

Now we deal with the three cases $a = -3$, $-2 \leq a \leq 1$ and $a \geq 2$ separately.

When $a = -3$, we have $\mathrm{ch}_{\leq 2}(B) = \mathrm{ch}_{\leq 2}(\mathcal{E}^\vee)$. Then since $\mathrm{ch}_{\leq 2}(B)$ is on the boundary of Lemma 4.7, by a standard argument we know that $B$ is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$ and $\beta < 0$, as explained in [PR21, Proposition 3.2]. Thus by Lemma 4.6, $B$ is a $\mu$-semistable sheaf. From Claim 3 we have $\chi(\mathcal{O}_X, B) = 0$, hence $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{E}^\vee)$ and by Lemma 7.5 we have $B \cong \mathcal{E}^\vee$. But this implies $\mathrm{Hom}(\mathcal{O}_X, A[1]) = k^5$ since $F \in \mathcal{A}_X$, which contradicts Claim 3.

When $-2 \leq a \leq 1$, we have $\mu^0_{\alpha,\beta}(A) > \mu^0_{\alpha,\beta}(\mathcal{E}[2])$. Since $A$ is $\sigma^0_{\alpha,\beta}$-semistable, we have $\mathrm{Hom}(A, \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee, A[1]) = 0$. Thus $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, A) = 0$ by Claim 2. But this contradicts Claim 3 since $\chi(\mathcal{E}^\vee, A) = 3 - 2a$.

When $a \geq 2$, applying Lemma 4.8 to $B$, we have $a = 2$. Thus $\mathrm{ch}_{\leq 2}(B) = \mathrm{ch}_{\leq 2}(\mathcal{Q}^\vee[1])$. By Claim 3, we know that $\mathrm{RHom}^\bullet(\mathcal{O}_X, B) = 0$ and we get $\mathrm{ch}(B) = \mathrm{ch}(\mathcal{Q}^\vee[1])$. Thus $\chi(\mathcal{E}^\vee, B) = \hom(\mathcal{E}^\vee, B) > 0$. Therefore, if we apply $\mathrm{Hom}(-, B)$ to the exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$, we obtain $\hom(\mathcal{Q}^\vee[1], B) > 0$. Now by stability, we have $B \cong \mathcal{Q}^\vee[1]$. Now $\mathrm{ch}(A) = \mathrm{ch}(\mathcal{E}[2])$. By Claim 2 and 3, we have $\mathrm{ext}^1(\mathcal{E}^\vee, A) = \hom(A, \mathcal{E}[2]) = 1$. Since $A$ is $\sigma^0_{\alpha,\beta}$-semistable and $\mathcal{E}[2]$ is $\sigma^0_{\alpha,\beta}$-stable, we have $A \cong \mathcal{E}[2]$. ◻

**Theorem 9.4.** *Let $X$ be a GM threefold. Then the the irreducible component $\mathcal{S}$ in Theorem 7.13 is the whole moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$.*

*Proof.* Note that $\hom(\mathcal{Q}^\vee[1], \mathcal{E}[2]) = 1$. Then the result follows from Lemma 9.2 and Lemma 9.3. □

## 10. Refined and birational categorical Torelli theorems for GM threefolds

In this section, we will prove several refined/birational categorical Torelli theorems for GM threefolds, using results from the previous sections.

**10.1. The universal family for $\mathcal{C}_m(X)$.** In this subsection, we show that $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ admits a universal family, which thus gives a fine moduli space. Let $\mathcal{I}$ be the universal ideal sheaf of conics on $X \times \mathcal{C}(X)$ and $\mathcal{I}_{L_\sigma}$ be the universal ideal sheaf of conics restricted to $X \times L_\sigma$. Let $q : X \times \mathcal{C}(X) \to X$ and $\pi : X \times \mathcal{C}(X) \to \mathcal{C}(X)$ be the projection maps on the first and second factors, respectively. Let $\mathcal{G}' := \mathrm{pr}(\mathcal{I}_{L_\sigma})$ be the projected family in $\mathcal{A}_{X\times L_\sigma}$. Let $t \in L_\sigma \cong \mathbb{P}^1$ be any point. Then $j_t^*\mathrm{pr}(\mathcal{I}_{L_\sigma}) \cong A$, where $j_t : X_t \to X_t \times L_\sigma$ and $A \in \mathcal{A}_X$ is $A \cong \mathrm{pr}(I_C)$ for $I_C \notin \mathcal{A}_X$ by Proposition 7.2. Then $\mathcal{G}' \cong q^*(A) \otimes \pi^*\mathcal{O}_{L_\sigma}(k)$ for some $k \in \mathbb{Z}$. Now let $\mathcal{G} := \mathrm{pr}(\mathcal{I}) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)$, where $E \cong L_\sigma \cong \mathbb{P}^1$ is the unique exceptional curve on $\mathcal{C}(X)$.

**Proposition 10.1.** *The object $(p_X)_*\mathcal{G}$ is the universal family of $\mathcal{C}_m(X)$, where $p_X = \mathrm{id}_X \times p : X \times \mathcal{C}(X) \to X \times \mathcal{C}_m(X)$.*

*Proof.*

(1) If $s = [A] = \pi \in \mathcal{C}_m(X)$, $s$ is contracted from the unique rational curve $L_\sigma \cong \mathbb{P}^1 \subset \mathcal{C}(X)$. Note that in this case $p_X|_{L_\sigma} = q$. Then

$$\begin{aligned} i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathcal{G}' \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong i_s^*q_*(q^*(A) \otimes \pi^*\mathcal{O}_{L_\sigma}(k) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong i_s^*q_*(q^*(A) \otimes (\pi^*\mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(kE))) \\ &\cong i_s^*q_*(q^*(A) \otimes \pi^*(\mathcal{O}_{L_\sigma}(k) \otimes \mathcal{O}_{L_\sigma}(-k))) \\ &\cong i_s^*q_*(q^*(A)) \cong i_s^*(A) \cong A. \end{aligned}$$

(2) If $s = [I_C]$, then $\mathcal{C}_m(X)$ and $\mathcal{C}(X)$ are isomorphic outside $L_\sigma$. Note that $p$ restricts to id on $\mathcal{C}(X) \setminus L_\sigma$. Then

$$\begin{aligned} i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathrm{pr}(\mathcal{I}) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong j_s^*(\mathrm{pr}(\mathcal{I})) \otimes j_s^*\pi^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\ &\cong I_C \otimes (\pi \circ j_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\ &\cong I_C \otimes (i_s \circ \pi_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \cong I_C. \end{aligned}$$

See below for the commutative diagrams which summarise the maps in the proof:

$$\begin{tikzcd} X_s \arrow[r, "\cong"] \arrow[d, hook, "j_s"] & X_s \arrow[r, "\pi_s"] \arrow[d, hook, "i_s"] & \{s\} \arrow[d] \\ X \times \mathcal{C}(X) \arrow[r, "p_X"'] & X \times \mathcal{C}_m(X) \arrow[r] & \mathcal{C}_m(X) \end{tikzcd}$$

$$\begin{tikzcd} X_s \arrow[r, "j_s"] \arrow[d, "\pi_s"] & X \times \mathcal{C}(X) \arrow[d, "\pi"] \\ \{s\} \arrow[r, "i_s"'] & \mathcal{C}(X) \end{tikzcd}$$

□

10.2. **A refined categorical Torelli theorem for ordinary GM threefolds.** We now prove a refined categorical Torelli theorem for ordinary GM threefolds.

**Theorem 10.2.** *Let $X$ and $X'$ be general ordinary GM threefolds such that $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is an equivalence and $\Phi(\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$. Then $X \cong X'$.*

*Proof.* Note that $\Xi(\pi(\mathcal{E})) = \mathrm{pr}(I_C)[1]$, where $I_C \notin \mathcal{A}_X$. Then the equivalence $\Phi$ induces an equivalence $\Psi = \Xi^{-1} \circ \Phi \circ \Xi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that $\Psi(\pi) = \pi'$, where $\pi := \mathrm{pr}(I_C)[1] \in \mathcal{A}_X$ and $\pi' := \mathrm{pr}(I_{C'})[1] \in \mathcal{A}_{X'}$. The existence of the universal family on $\mathcal{C}_m(X)$ guarantees a projective dominant morphism from $\mathcal{C}_m(X)$ to $\mathcal{C}_m(X')$, denoted by $\psi$, which is induced by $\Psi$ (for more details on the construction of the morphism $\psi$, see [BMMS12, APR19]). Since $\Psi$ is an equivalence, $\psi$ is bijective on closed points by Theorem 7.13 and Theorem 4.20. It also identifies the tangent spaces of each point on $\mathcal{C}_m(X)$ and $\mathcal{C}_m(X')$, hence it is an isomorphism. On the other hand, we have $\psi(\pi) = \pi'$. Then $\psi$ induces an isomorphism $\phi : \mathcal{C}(X) \cong \mathcal{C}(X')$ by blowing up $\pi \in \mathcal{C}_m(X)$ and $\pi' \in \mathcal{C}_m(X')$, respectively. Then we have $X \cong X'$ by Logachev's Reconstruction Theorem 6.6. ∎

10.3. **Birational categorical Torelli theorem for ordinary GM threefolds.** In this subsection, we show a birational categorical Torelli theorem for ordinary GM threefolds, i.e. assuming the Kuznetsov components are equivalent leads to a birational equivalence of the ordinary GM threefolds.

**Theorem 10.3.** *Let $X$ and $X'$ be general ordinary GM threefolds such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then $X'$ is a conic transform, or a conic transform of a line transform of $X$. In particular, we have $X \simeq X'$.*

*Proof.* Assume that $\Phi : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$, and fix a $(-1)$-class $-x$ in $\mathcal{N}(\mathcal{A}_X)$. The equivalence $\Phi$ sends $-x$ to either itself or $y - 2x$ in $\mathcal{N}(\mathcal{A}_{X'})$ up to sign. By the same argument as in [BMMS12, APR19], we thus get two possible induced morphisms between Bridgeland moduli spaces

$$\begin{tikzcd} M_\sigma(\mathcal{A}_X, -x) \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & M_\sigma(\mathcal{A}_{X'}, -x) \\ & M_\sigma(\mathcal{A}_{X'}, y-2x) \end{tikzcd}$$

As we have seen in Theorems 7.13 and 8.5, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G(2,1,5)$. So we have two cases: either $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$ or $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$.

For the first case, blow up $\mathcal{C}_m(X)$ at the distinguished point $[\pi] := [\Xi(\pi(\mathcal{E}))]$, and blow up $\mathcal{C}_m(X')$ at the point $[c] := [\Phi(\pi)]$. We have $\mathcal{C}(X) \cong \mathrm{Bl}_{[c]}\mathcal{C}_m(X)$ and we have $\mathrm{Bl}_{[c]}\mathcal{C}_m(X') \cong \mathcal{C}(X'_c)$ by Theorem 6.8, so $\mathcal{C}(X) \cong \mathcal{C}(X'_c)$. Therefore by Logachev's Reconstruction Theorem 6.6 we have $X \cong X'_c$. But $X'_c$ is birational to $X'$, so $X$ and $X'$ are birational.

For the second case, we get $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$ but we have a birational equivalence $M_G^{X'}(2,1,5) \simeq \mathcal{C}(X'_L)$ of surfaces by [DIM12, Proposition 8.1]. Thus $\mathcal{C}_m(X)$ is birationally equivalent to $\mathcal{C}(X'_L)$. Let $\mathcal{C}_m(X'_L)$ be the minimal surface of $\mathcal{C}(X'_L)$. Note that the surfaces here are all smooth surfaces of general type. By the uniqueness of minimal models of surfaces of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$, which implies $X \cong (X'_L)_c \simeq X'$ as in the first case. ∎

**Remark 10.4.** Theorem 10.3 proves a conjecture [KP19, Conjecture 1.7] of Kuznetsov–Perry for *general* ordinary GM varieties of dimension 3.

As a corollary, we can obtain a stronger result proved in [DIM12], which claims that $\mathcal{C}_m(X_L)$ is birational to $M_G^X(2,1,5)$.

**Corollary 10.5.** *Let $X$ be a general ordinary GM threefold, and $X_L$ be a line transform of $X$. Then we have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Moreover, this isomorphism commutes with involutions $\iota$ and $\iota'$ on both sides, thus giving an isomorphism $\mathcal{C}_m(X_L)/\iota \cong M_G^X(2,1,5)/\iota'$.*

*Proof.* By the same argument as in the proof of Theorem 10.3, we have $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ or $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. Note that $\mathcal{C}_m(X_L) \cong \mathcal{C}_m(X)$ implies that $X_L \cong X_c$ for some conic $c \subset X$ as in Theorem 10.3. But this is impossible by [DIM12, Remark 7.3]. Thus we always have $\mathcal{C}_m(X_L) \cong M_G^X(2,1,5)$. The last statement follows from the fact that any equivalence between Kuznetsov components commutes with Serre functors, and the involutions on $\mathcal{C}_m(X_L)$ and $M_G^X(2,1,5)$ can be induced by Serre functors up to shift by Propositions 7.3 and 8.4. ◻

Since the intermediate Jacobian $J(X)$ is invariant under conic and line transforms, as a corollary we have

**Corollary 10.6.** *Let $X$ and $X'$ be general ordinary GM threefolds. If $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then we have $J(X) \cong J(X')$.*

In fact, we can relax the assumptions on $X$ by looking at the singularities of Bridgeland moduli spaces.

**Theorem 10.7.** *Let $X$ and $X'$ be general GM threefolds (they can be either ordinary or special) and suppose their Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ are equivalent. Then $X$ is birationally equivalent to $X'$.*

*Proof.* First we claim that if $X$ and $X'$ are general GM threefolds such that $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then both $X$ and $X'$ are ordinary or special simultaneously. Indeed, we may assume $X'$ is ordinary and $X$ is special. Then the equivalence would identify the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects of class $-x$ in $\mathcal{A}_X$ with either the moduli space $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, -x)$ or $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, y-2x)$. Then the surface $\mathcal{C}_m(X)$ for a special GM threefold $X$ would be identified with the minimal Fano surface $\mathcal{C}_m(X')$ or the moduli space $M_G^{X'}(2,1,5)$ for a general ordinary GM threefold $X'$. But $\mathcal{C}_m(X)$ has a unique singular point and both $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ are smooth for $X'$ general. This means that neither identification is possible, so the claim follows.

Now $X$ and $X'$ are both general ordinary or general special, hence the result follows from Theorem 10.3 and 10.4. ◻

**Corollary 10.8.** *Let $X$ and $X'$ be general GM threefolds such that one of them is ordinary and their Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ are equivalent. Then they are both general ordinary and $X$ is birationally equivalent to $X'$.*

10.4. **A categorical Torelli theorem for special GM threefolds.** In this subsection, we show that the Kuznetsov component of a general special GM threefold $X$ determines the isomorphism class of $X$.

Recall from Section 3 that every special GM threefold $X$ is a double cover of a degree 5 index 2 prime Fano threefold $Y$ branched over a quadric hypersurface $\mathcal{B}$ in $Y$. Since $X$ is smooth and general, $(\mathcal{B}, h)$ is a smooth degree $h^2 = 10$ K3 surface with Picard number 1. There is a natural geometric involution $\tau$ on $X$ induced by the double cover. The Serre functor on $\mathcal{K}u(X)$ is given by $S_{\mathcal{K}u(X)} = \tau \circ [2]$.

**Theorem 10.9.** *Let $X$ and $X'$ be smooth general special GM threefolds with $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X \cong X'$.*

*Proof.* By [KP18a, Theorem 1.1, Section 8.2], the equivariant triangulated category $\mathcal{K}u(X)^{\mu_2}$ is equivalent to $\mathrm{D}^b(\mathcal{B})$, where $\mu_2$ is the group of square roots of 1 generated by the involution $\tau$

acting on $\mathcal{K}u(X)$. Assume there is an equivalence $\Phi : \mathcal{K}u(X) \cong \mathcal{K}u(X')$. Note that $\Phi$ commutes with the involutions $\tau$ and $\tau'$ on $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$, respectively. Then we get an induced equivalence

$$\Psi : \mathcal{K}u(X)^{\mu_2} \cong \mathcal{K}u(X')^{\mu_2'}$$

where $\mu_2 = \langle \tau \rangle$, $\mu_2' = \langle \Phi \circ \tau \circ \Phi^{-1} = \tau' \rangle$ and $\mu_2 \cong \mu_2'$. Thus we have $\Psi : \mathrm{D}^b(\mathcal{B}) \cong \mathrm{D}^b(\mathcal{B}')$. We know that $\mathcal{B}$ and $\mathcal{B}'$ are smooth projective surfaces with polarizations $h$ and $h'$, respectively, so $\Psi$ is a Fourier–Mukai functor by Orlov's Representability Theorem [Orl97, Theorem 2.2]. Moreover, $(\mathcal{B}, h)$ and $(\mathcal{B}', h')$ are both Picard number 1 smooth projective K3 surfaces of degree $h^2 = h'^2 = 10 = 2 \times 5$. Then by [Ogu02, Theorem 1.10] and [HLOY02, Corollary 1.7], there is an isomorphism $\phi : \mathcal{B} \cong \mathcal{B}'$. Since they both have Picard number one, we obtain $\phi^*(h') = h$. On the other hand $Y_5$ is rigid [Kuz09, § 4.1], which implies $X \cong X'$ ◻

**Remark 10.10.** Theorem 10.9 can also be proved via Bridgeland moduli spaces with respect to the Kuznetsov component $\mathcal{A}_X$. The details are contained in an upcoming paper [JLZ21]; we only sketch the proof here. One can show that the Gieseker moduli space $M_G(2, 1, 5)$ for a general special GM threefold $X$ is also a smooth projective surface. On the other hand, the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is a surface with a unique singular point represented by $\pi := \Xi(\pi(\mathcal{E}))$ by Theorem 7.13. Assume there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then it will induce an equivalence $\Psi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that the gluing data $\pi$ is preserved by $\Psi$ (because $\pi$ is the unique singular point of the moduli space). Thus $\pi(\mathcal{E})$ is automatically preserved by $\Phi$. Then $X$ is reconstructed as the Brill–Noether locus of a Bridgeland moduli space of $\sigma$-stable objects in $\mathcal{K}u(X)$ with respect to $\pi(\mathcal{E})$.

## 11. The Debarre–Iliev–Manivel conjecture

In [DIM12, pp. 3-4], the authors make the following conjecture regarding the general fiber of the period map:

**Conjecture 11.1** ([DIM12, pp. 3-4])**.** *A general fiber $\mathcal{P}^{-1}([J(X)])$ of the period map $\mathcal{P} : \mathcal{X}_{10} \to \mathcal{A}_{10}$ through an ordinary GM threefold $X$ is the union of $\mathcal{C}_m(X)/\iota$ and a surface birationally equivalent to $M_G(2, 1, 5)/\iota'$, where $\iota, \iota'$ are geometrically meaningful involutions.*

**Remark 11.2.** Note that by Corollary 10.5, the surface birationally equivalent to $M_G(2, 1, 5)/\iota'$ in [DIM12], parametrizing conic transforms of a line transform of $X$, is actually isomorphic to $M_G(2, 1, 5)/\iota'$. Thus this conjecture predicts that a general fiber $\mathcal{P}^{-1}([J(X)])$ is actually the disjoint union of $\mathcal{C}_m(X)/\iota$ and $M_G(2, 1, 5)/\iota'$.

We will prove a categorical analogue of this conjecture. Consider the "categorical period map"

$$\mathcal{P}_{\mathrm{cat}} : \mathcal{X}_{10} \to \{\mathcal{A}_X\}/\simeq, \ X \mapsto \mathcal{A}_X$$

where $\mathcal{X}_{10}$ is the moduli space of isomorphism classes of GM threefolds and $\{\mathcal{A}_X\}/\simeq$ is the set of equivalence classes of Kuznetsov components of GM threefolds. Note that a global description of a "moduli of Kuznetsov components" $\{\mathcal{A}_X\}/\simeq$ is not known, however local deformations are controlled by the second Hochschild cohomology $\mathrm{HH}^2(\mathcal{A}_X)$. The fiber of the "categorical period map" $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for an ordinary GM threefold is defined as the isomorphism classes of all ordinary GM threefolds $X'$ such that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$.

**Theorem 11.3.** *The general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ of the categorical period map over the alternative Kuznetsov component of an ordinary GM threefold $X$ is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2, 1, 5)/\iota'$ where $\iota, \iota'$ are geometrically meaningful involutions.*

*Proof.* The general fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ of the categorical period map consists of GM threefolds $X'$ such that there is an equivalence of Kuznetsov components $\mathcal{A}_{X'} \simeq \mathcal{A}_X$. Then by Theorem 10.7, $X'$ is also a general ordinary GM threefold. Thus by Theorem 10.3 and Theorem 6.10, we know that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ if and only if $X'$ is a conic transform of $X$, or a conic transform of a line transform of $X$. Then the result follows from Proposition 6.9 and Corollary 10.5. ◻

**Remark 11.4.** The Kuznetsov components of prime Fano threefolds of index 1 and 2 are often regarded as categorical analogues of the intermediate Jacobians of these threefolds, and it is known that if there is a Fourier–Mukai type equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ (or $\mathcal{A}_X \simeq \mathcal{A}_{X'}$), then $J(X) \cong J(X')$ by [Per20]. As a corollary of Theorem 10.3, we know that for two general ordinary GM threefolds, $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ implies $J(X) \cong J(X')$ by Corollary 10.6. For the converse, we have the following conjecture.

**Conjecture 11.5.** *The intermediate Jacobian $J(X)$ uniquely determines the Kuznetsov component $\mathcal{K}u(X)$, i.e. $J(X) \cong J(X') \Longrightarrow \mathcal{K}u(X) \simeq \mathcal{K}u(X')$.*

**Theorem 11.6.** *Conjecture 11.5 is true for smooth prime Fano threefolds $X$ if $X$ is one of the following:*

- $Y_d$, $\quad 2 \leq d \leq 5$
- $X_{2g-2}$, $\quad g = 5, 7, 8, 9, 10, 12.$

*Proof.* If $X$ is an index 2 prime Fano threefold $Y_d$ where $2 \leq d \leq 5$, then the statement follows from the Torelli theorems for $Y_d$. If $X = X_8$, the statement follows from its Torelli theorem. If $X = X_{12}, X_{18}, X_{16}$, their intermediate Jacobians are Jacobians of curves: $J(X_{12}) \cong J(C_7)$, $J(X_{16}) \cong J(C_3)$, and $J(X_{18}) \cong J(C_2)$. But $\mathcal{K}u(X_{12}) \simeq \mathrm{D}^b(C_7)$, $\mathcal{K}u(X_{16}) \simeq \mathrm{D}^b(C_3)$ and $\mathcal{K}u(X_{18}) \simeq \mathrm{D}^b(C_2)$. Thus the statement follows from the classical Torelli theorem for curves. If $X = X_{14}$, the statement follows from the Kuznetsov conjecture for the pair $(Y_3, X_{14})$ [Kuz03] and the Torelli theorem for cubic threefolds. If $X = X_{22}$, the statement is trivial since $\mathcal{K}u(X_{22}) \cong \mathcal{K}u(Y_5)$ ([KPS18b]) and $Y_5$ is rigid, so $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is always true. ◻

In the case of general ordinary GM threefolds $X_{10}$ we show that

**Proposition 11.7.** *The Debarre–Iliev–Manivel Conjecture 11.1 is equivalent to Conjecture 11.5.*

*Proof.* First we assume that Conjecture 11.5 holds. Then by Corollary 10.6 and Theorem 11.3, the Debarre–Iliev–Manivel Conjecture 11.1 holds.

On the other hand, we assume the Debarre–Iliev–Manivel Conjecture 11.1 holds. Then for any $X$ and $X'$ such that $J(X) \cong J(X')$, $X$ is either a conic transform of $X'$, or $X$ is a conic transform of a line transform of $X'$. In both cases, we have $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ by the Duality Conjecture Theorem 6.10. Thus Conjecture 11.5 holds. ◻

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