INFINITESIMAL CATEGORICAL TORELLI THEOREMS FOR FANO THREEFOLDS

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Abstract. Let X be a smooth Fano variety and $\mathcal{K}u(X)$ its Kuznetsov component. A Torelli theorem for $\mathcal{K}u(X)$ states that $\mathcal{K}u(X)$ is uniquely determined by a certain polarized abelian variety associated to it. An infinitesimal Torelli theorem for X states that the differential of the period map is injective. A categorical variant of the infinitesimal Torelli theorem for X states that the morphism $\gamma : H^1(X, T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$ is injective. In the present article, we use the machinery of Hochschild (co)homology to relate the aforementioned three Torelli-type theorems for smooth Fano varieties via a commutative diagram. As an application, we prove an infinitesimal categorical Torelli theorem for a class of prime Fano threefolds. We then prove, infinitesimally, a restatement of the Debarre–Iliev–Manivel conjecture regarding the general fiber of the period map for Gushel–Mukai threefolds.

1. Introduction

Torelli problems are some of the oldest and the most classical problems in various aspects of algebraic geometry, including Hodge theory, birational geometry, moduli spaces of algebraic varieties, etc. The classical Torelli question asks whether an algebraic variety X is uniquely determined by an abelian variety associated to it. Denote by $\mathcal{P}$ the period map $\mathcal{P} : \mathcal{M} \to \mathcal{D}$, where $\mathcal{M}$ is the moduli space of some class of algebraic varieties up to isomorphism, and $\mathcal{D}$ is the period domain. A *Torelli theorem* holds for X if and only if $\mathcal{P}$ is injective. An *infinitesimal Torelli theorem* holds for X if and only if the differential $d\mathcal{P}$ of the period map is injective. Now let X be a smooth Fano threefold of Picard rank one. The period map $\mathcal{P}$ is given by $X \mapsto J(X)$, where $J(X)$ is the intermediate Jacobian of X.

On the other hand, the seminal work [BO01] shows that the bounded derived category $\mathrm{D}^b(X)$ of a smooth projective Fano variety determines the isomorphism class of X. In other words, a *derived Torelli theorem* holds for X. It is natural to ask for a class of Fano varieties whether they are also determined by *less* information than the whole derived category. A natural candidate is a subcategory $\mathcal{K}u(X) \subset \mathrm{D}^b(X)$ called the *Kuznetsov component*, which is defined as orthogonal complement of an exceptional collection of vector bundles on X. It is widely believed that the Kuznetsov component $\mathcal{K}u(X)$ encodes the essential geometric information of X, and it has been studied extensively by Kuznetsov and others (e.g. in [Kuz03, Kuz09a, KP18]) for many Fano varieties. In particular, for cubic threefolds, the Kuznetsov component determines their isomorphism classes (see [BMMS12] and [PY20]). This is known in the literature as a *categorical Torelli theorem*.

Recently, the intermediate Jacobian of a smooth Fano variety X was reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ in [Per20].

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Example 1.1 (Remark 3.9)**.** When X is a Fano threefold of index one or two, it is clear that $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) = 0$ since X is simply connected. According to [Per20, Lemma 5.2, Proposition 5.23], there is a Hodge isometry $K_{-3}^{\mathrm{top}}(\mathcal{K}u(X))_{\mathrm{tf}} \cong H^3(X,\mathbb{Z})_{\mathrm{tf}}$ which preserves both pairings. The left hand side pairing is the Euler paring, and the right hand side is the cohomology paring. Then, $J(\mathcal{K}u(X))$ is an abelian variety with a polarization induced from the Euler paring. Moreover, we have an isomorphism of abelian varieties $J(\mathcal{K}u(X)) \cong J(X)$. If there is a Fourier–Mukai equivalence $\mathcal{K}u(X_1) \cong \mathcal{K}u(X_2)$, where X_1 and X_2 are Fano threefolds of index one or two, then $J(X_1) \cong J(X_2)$.

As in the case of the classical Torelli problem, one could imagine that there is a "categorical period map" $d\mathcal{P}_{\mathrm{cat}}$ from the moduli space $\mathcal{M}$ of smooth Fano threefolds up to isomorphism, to a certain period domain $\mathcal{D}'$ containing the corresponding Kuznetsov components $\mathcal{K}u(X)$. But since there is no good notion of moduli space of semiorthogonal components of $\mathrm{D}^b(X)$, we cannot make sense of $\mathcal{P}_{\mathrm{cat}}$ mathematically. Nevertheless, its differential $\eta : H^1(X,T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$ is well-defined. We say an *infinitesimal categorical Torelli theorem* holds for X if the map η is injective.

1.1. **Main Results.**

1.1.1. *Infinitesimal Torelli vs. infinitesimal categorical Torelli.* In the present article, we relate *infinitesimal Torelli theorems* and *infinitesimal categorical Torelli theorems* for a class of Fano threefolds of Picard rank one, using the machinery of Hochschild (co)homology. We prove the following theorem:

Theorem 1.2 (Theorem 3.11)**.** *Let X be a smooth projective variety. Assume there is a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, \dots, E_n\rangle$, where $\{E_1,\dots,E_n\}$ is an exceptional collection. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[rr, "\gamma"] & & \mathrm{Hom}(\mathrm{HH}_{-1}(X),\mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[urr, "\tau"] & & \\
H^1(X,T_X) \arrow[u, "\text{inclusion}"] \arrow[uurr, "d\mathcal{P}"'] & &
\end{tikzcd}$$

where τ is defined as a contraction of polyvector fields.

Remark 1.3. $\eta : H^1(X,T_X) \longrightarrow \mathrm{HH}^2(\mathcal{K}u(X))$ is defined as the composition of the vertical maps in the commutative diagram.

If we assume the vanishing of the higher odd degree Hochschild homology of X, the deformations of $J(\mathcal{K}u(X))$ are given by $\mathrm{Hom}(\mathrm{HH}_{-1}(X),\mathrm{HH}_1(X))$ by Proposition 3.10. If we assume further that $J(\mathcal{K}u(X)) \cong J(X)$ as abelian varieties, the diagram from the theorem above can be interpreted as the diagram of deformations

$$\begin{tikzcd}
\{\mathcal{K}u(X)\}/\sim \arrow[r] & \text{Abelian Varieties}/\sim \\
\{X\}/\sim \arrow[u] \arrow[ur] &
\end{tikzcd}$$

if we have good knowledge of the moduli spaces in question.

Corollary 1.4. *Infinitesimal classical Torelli for X implies infinitesimal categorical Torelli for the Kuznetsov component $\mathcal{K}u(X)$.*

Proof. Suppose $d\mathcal{P}$ is injective. Then the fact that we have a composition $d\mathcal{P} = \gamma \circ \eta$ implies that η is injective too. ◻

The main examples for the problems of infinitesimal categorical Torelli that we study are those of Fano threefolds of Picard rank one of index one and two. Recall that Fano threefolds satisfy the assumption $\mathrm{HH}_{2i+1}(X) = 0$ for $i \geq 1$ and $J(\mathcal{K}u(X)) \cong J(X)$ as abelian varieties (see Example 1.1 or Remark 3.9). We summarise our results in the following theorem.

Theorem 1.5 (Theorems 4.10, 4.11, 4.5, 4.7, 4.3)**.** *Let X_{2g-2} be a Fano threefold of index one and degree $2g-2$, where g is its genus. Let Y_d be a Fano threefold of index two and degree d.*

- *For Y_d where $1 \leq d \leq 4$, the infinitesimal categorical Torelli theorem holds.*
- *For X_{2g-2} where $g = 2, 3, 5, 7$, the infinitesimal categorical Torelli theorem holds.*

1.1.2. *Debarre–Iliev–Manivel Conjecture.* In [DIM12], the authors conjecture that the general fiber of the classical period map from the moduli space of ordinary GM threefolds to the moduli space of 10 dimensional principally polarised abelian varieties is the disjoint union of $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$, both quotiented by involutions. We call this the *Debarre–Iliev–Manivel Conjecture.*

Within the moduli space of smooth GM threefolds, we define the fiber of the "categorical period map" through $[X]$ as the isomorphism classes of all ordinary GM threefolds X' whose Kuznetsov components satisfy $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$. In our recent work [JLLZ21], we prove the categorical analogue of the Debarre–Iliev–Manivel conjecture:

Theorem 1.6 ([JLLZ21, Theorem 1.7])**.** *A general fiber of the "categorical period map" through an ordinary GM threefold X is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$, where ι, ι' are geometrically meaningful involutions.*

As an application, the Debarre–Iliev–Manivel conjecture can be restated in an equivalent form as follows:

Conjecture 1.7. *Let X be a general ordinary GM threefold. Then the intermediate Jacobian $J(X)$ determines the Kuznetsov component $\mathcal{K}u(X)$.*

Although we are not able to prove the conjecture, we are able to show an infinitesimal version:.

Theorem 1.8. *Let X be an ordinary GM threefold. Then the map*

$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$

is injective.

1.2. **Organization of the paper.** In Section 2, we collect basic facts about semiorthogonal decompositions. In Section 3, we introduce the definition of Hochschild (co)homology for admissible subcategories of bounded derived categories $\mathrm{D}^b(X)$ of smooth projective varieties X. We then prove Theorem 1.2. In Section 4, we apply the techniques developed in Section 3 to prime Fano threefolds of index one and two. In particular, we show the infinitesimal version (Theorem 1.8) of Conjecture 1.7 for ordinary GM threefolds.

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2. Semiorthogonal decompositions

In this section, we collect some useful facts/results about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06], for example. From now on, let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on a smooth projective variety X.

2.1. Exceptional collections and semiorthogonal decompositions.

Definition 2.1. Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$. We say that E is an *exceptional object* if $\mathrm{RHom}^\bullet(E, E) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(E, E)[-i] = k$. Now let $(E_1, \dots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Definition 2.3. Let $\mathcal{D}$ be a triangulated category. We say a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is *admissible*, if the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ has left adjoint i^* and right adjoint $i^!$.

Definition 2.4. Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ be a collection of full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{C}_j \subset \mathcal{C}_i^\perp$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ generate $\mathcal{D}$, i.e. the category resulting from taking all shifts and cones of objects in the categories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ is equivalent to $\mathcal{D}$.

Definition 2.5. The *Serre functor* $S_\mathcal{D}$ of a triangulated category $\mathcal{D}$ is the autoequivalence of $\mathcal{D}$ such that there is a functorial isomorphism of vector spaces

$$\mathrm{Hom}_\mathcal{D}(A, B) \cong \mathrm{Hom}_\mathcal{D}(B, S_\mathcal{D}(A))^\vee$$

for any $A, B \in \mathcal{D}$.

Proposition 2.6. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} \simeq \langle S_\mathcal{D}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_\mathcal{D}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

Example 2.7. Let X be a smooth projective variety and $\mathcal{D} = \mathrm{D}^b(X)$. Then $S_X := S_\mathcal{D}(-) = (- \otimes \mathcal{O}(K_X))[\dim X]$.

3. Hochschild (co)homology and infinitesimal Torelli theorems

3.1. **Definitions.** In this subsection, we recall some basics on Hochschild (co)homology of admissible subcategories of $\mathrm{D}^b(X)$, where X is a smooth projective variety. We refer the reader to [Kuz09b] for more details. For Hochschild (co)homology of dg-categories, we refer the reader to to Bernhard Keller's papers and surveys [Kel98, Kel06].

Definition 3.1 ([Kuz09b])**.** Let X be a smooth projective variety, and $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$. Consider any semiorthogonal decomposition of $\mathrm{D}^b(X)$ that contains $\mathcal{A}$ as a component. Let P be the kernel of the projection to $\mathcal{A}$. The *Hochschild cohomology* of $\mathcal{A}$ is defined as

$$\mathrm{HH}^*(\mathcal{A}) := \mathrm{Hom}^*(P, P).$$

The *Hochschild homology* of $\mathcal{A}$ is defined as

$$\mathrm{HH}_*(\mathcal{A}) := \mathrm{Hom}^*(P, P \circ S_X).$$

Remark 3.2. The definition of Hochschild cohomology is independent of the semiorthogonal decomposition. There is a natural identification of $\mathrm{HH}_*(\mathcal{A})$ in $\mathrm{HH}_*(X)$, and the subspace is independent of the semiorthogonal decomposition.

Lemma 3.3 ([Kuz09b, Theorem 4.5, Proposition 4.6])**.** *Choose a strong compact generator E of $\mathcal{A}$, and define $A = \mathrm{RHom}^\bullet(E, E)$. Then there are isomorphisms*

$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(\mathcal{A}) \quad \textit{and} \quad \mathrm{HH}_*(A) \cong \mathrm{HH}_*(\mathcal{A}).$$

Remark 3.4.

(1) Let $\mathrm{Per}_{\mathrm{dg}}(X)$ be a dg-enhancement of $\mathrm{Perf}(X)$ whose objects are K-injective perfect complexes, and let $\mathcal{A}_{\mathrm{dg}}$ be a dg-subcategory of $\mathrm{Per}_{\mathrm{dg}}(X)$ whose objects are in $\mathcal{A}$. Then we have the isomorphisms

$$\mathrm{HH}_*(\mathcal{A}) \cong \mathrm{HH}_*(\mathcal{A}_{\mathrm{dg}}) \quad \text{and} \quad \mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HH}^*(\mathcal{A}_{\mathrm{dg}})$$

because the morphism of dg-categories $* \to \mathcal{A}_{\mathrm{dg}}$ is a derived Morita equivalence. Here $*$ is a dg-category with one object and the endomorphism of this unique object is the dg-algebra A; the morphism sends the unique object to an K-injective resolution of E.

(2) Since $\mathrm{Perf}(X)$ is a unique enhanced triangulated category [LO10], the Hochschild (co)homology of admissible subcategories of $\mathrm{Perf}(X)$ defined by Kuznetsov coincides with that of the dg-enhancement that naturally comes from the dg-enhancement of $\mathrm{Perf}(X)$.

Let A be a k-algebra. Note that the Hochschild homology $\mathrm{HH}_*(A)$ is a graded $\mathrm{HH}^*(A)$-module. The module structure is easily described by the definition of Hochschild (co)homology via the Ext and Tor functors. It encodes deformation information of the algebra and its invariant Hochschild homology. For example, when considering a variety X, the degree 2 Hochschild cohomology has a factor $H^1(X, T_X)$ which is the first order deformations of X. The action of $H^1(X, T_X)$ on the Hochschild homology via the module structure can be imagined as deformations of a certain invariant with respect to the deformations of X. Here, we have the invariant $\mathrm{HH}_\bullet(X)$ that is closely related to the intermediate Jacobian of X. When X is a Fano threefold, the action of $H^1(X, T_X)$ on $\mathrm{HH}_{-1}(X) = H^{2,1}(X)$ is the derivative of period map.

In the case of admissible subcategories of derived categories, we can describe the module structure by kernels.

Definition 3.5. Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$, and let P be the kernel of the left projection to $\mathcal{A}$. Take $\alpha \in \mathrm{HH}^{t_1}(\mathcal{A})$, and $\beta \in \mathrm{HH}_{t_2}(\mathcal{A})$. The action of α on β is the composition

$$P \xrightarrow{\beta} P \circ S_X[t_2] \xrightarrow{\alpha \otimes \mathrm{id}} P \circ S_X[t_1 + t_2].$$

Proposition 3.6. *Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{Perf}(X)$. Let E be a strong compact generator of $\mathcal{A}$, and $A := \mathrm{RHom}^\bullet(E, E)$. The isomorphisms $\mathrm{HH}^*(A) \cong \mathrm{HH}^*(\mathcal{A})$ and $\mathrm{HH}_*(A) \cong \mathrm{HH}_*(\mathcal{A})$ from Lemma 3.3 preserve both sides of the obvious module structure and algebra structure of Hochschild cohomology.*

Proof. These facts can be proved via the description in [Kuz09b, Proposition 4.6]. ◻

Theorem 3.7. *Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$, and $\mathcal{B}$ be an admissible subcategory of $\mathrm{D}^b(Y)$. Suppose the Fourier–Mukai functor $\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ induces an equivalence of subcategories $\mathcal{A}$ and $\mathcal{B}$. Then we have isomorphisms of Hochschild cohomology*

$$\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HH}^*(\mathcal{B})$$

and Hochschild homology

$$\mathrm{HH}_*(\mathcal{A}) \cong \mathrm{HH}_*(\mathcal{B})$$

which preserve both sides of the module structure and algebra structure.

Proof. We have $\mathcal{A}_{\mathrm{dg}} \cong \mathcal{B}_{\mathrm{dg}}$ in the homotopy category whose weak equivalences are Morita equivalences [BT14, Section 9]. Hence, there is an isomorphism $\mathrm{Per}_{\mathrm{dg}}(\mathcal{A}_{\mathrm{dg}}) \cong \mathrm{Per}_{\mathrm{dg}}(\mathcal{B}_{\mathrm{dg}})$ in Hqe [Tab05]. That is, $\mathcal{A}_{\mathrm{dg}}$ and $\mathcal{B}_{\mathrm{dg}}$ are connected by a chain of Morita equivalences

$$\begin{tikzcd}
 & \mathrm{Per}_{\mathrm{dg}}(\mathcal{A}_{\mathrm{dg}}) \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathrm{Per}_{\mathrm{dg}}(\mathcal{B}_{\mathrm{dg}}) & \\
\mathcal{A}_{\mathrm{dg}} \arrow[ur] & & C_0 & \cdots & C_n \arrow[ur] & & \mathcal{B}_{\mathrm{dg}} \arrow[ul]
\end{tikzcd}$$

According to [AK19, Theorem 3.1], if two dg-categories are derived equivalent induced by a bi-module (Morita equivalence), then the equivalence induces an isomorphism of Hochschild (co)homology and preserves the module structure. ◻

Let X be a smooth algebraic variety. Classically we have the HKR isomorphisms [Kuz09b, Theorem 8.3] given by

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_{p+q=*} H^p(X, \wedge^q T_X)$$

and

$$\mathrm{Hom}^{-*}(X \times X, \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) \cong \bigoplus_{p-q=*} H^p(X, \Omega_X^q).$$

However, the original HKR isomorphisms may not preserve the obvious two-sided algebra structure and the module structure. After some twisting of the HKR isomorphisms which we will denote as IK, it was originally conjectured by in [Căl05, Conjecture 5.2] and proved in [CRVdB12, Theorem 1.4] that the new HKR isomorphisms preserve the obvious module structure.

3.2. Deformations and infinitesimal (categorical) Torelli theorems.

Definition 3.8 ([Per20, Definition 5.24])**.** Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$ and consider the diagram

$$\begin{tikzcd} K_1^{\mathrm{top}}(\mathcal{A}) \arrow[r, "\mathrm{ch}_1^{\mathrm{top}}"] \arrow[drr, "P'"'] & \mathrm{HP}_1(\mathcal{A}) \arrow[r, "\cong"] & \bigoplus_n \mathrm{HH}_{2n-1}(\mathcal{A}) \arrow[d, "P"] \\ & & \mathrm{HH}_1(\mathcal{A}) \oplus \mathrm{HH}_3(\mathcal{A}) \oplus \cdots \end{tikzcd}$$

Define the *intermediate Jacobian of* $\mathcal{A}$ as

$$J(\mathcal{A}) = (\mathrm{HH}_1(\mathcal{A}) \oplus \mathrm{HH}_3(\mathcal{A}) \oplus \cdots)/\Gamma,$$

where Γ is the image of P'. Note that Γ is a lattice.

Remark 3.9. In general, $J(\mathcal{A})$ is a complex torus. When X is a Fano threefold of index one or two, we have a non-trivial admissible subcategory $\mathcal{K}u(X)$ called the Kuznetsov component (see the survey [Kuz16]). Clearly $H^1(X,\mathbb{Z}) = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) = 0$ since X is simply connected. According to [Per20, Lemma 5.2, Proposition 5.3], there is a Hodge isometry $K_{-3}^{\mathrm{top}}(\mathcal{K}u(X))_{\mathrm{tf}} \cong H^3(X,\mathbb{Z})_{\mathrm{tf}}$ which preserves both pairings. The left hand side pairing is the Euler paring, and the right hand side is the cohomology paring. Then, $J(\mathcal{K}u(X))$ is an abelian variety with a polarization induced from the Euler paring. Moreover, we have an isomorphism of abelian varieties $J(\mathcal{K}u(X)) \cong J(X)$. If there is an Fourier–Mukai equivalence $\mathcal{K}u(X_1) \cong \mathcal{K}u(X_2)$ for Fano threefolds X_1 and X_2, then $J(X_1) \cong J(X_2)$.

Proposition 3.10. *Assume there is a semiorthogonal decomposition* $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, \ldots, E_n\rangle$ *where* $\{E_1, \ldots, E_n\}$ *is an exceptional collection. Also assume that* $\mathrm{HH}_{2n+1}(X) = 0$ *for* $n \geq 1$*. The first order deformation space of* $J(\mathcal{K}u(X))$ *is*

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

Proof. Write $V := \mathrm{HH}_1(\mathcal{K}u(X))$. Since $V \cong \mathrm{HH}_1(X)$, there is a natural conjugation $\overline{V} := \mathrm{HH}_{-1}(\mathcal{K}u(X)) \cong \mathrm{HH}_{-1}(X)$. Since the tangent bundle of a torus is trivial, we have

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong H^1(V/\Gamma, V \otimes \mathcal{O}_{V/\Gamma}) \cong V \otimes H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}).$$

Since $H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}) \cong \mathrm{Hom}_{\mathrm{anti-linear}}(V, k) \cong \mathrm{Hom}_k(\overline{V}, k)$, we finally get the equality

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X)))$$

as required. ◻

When $\mathrm{HH}_{2n+1}(X) = 0$, $n \geq 1$, we define a linear map from the deformations of $\mathcal{K}u(X)$ to the deformations of its intermediate Jacobian $J(\mathcal{K}u(X))$ by the action of cohomology:

$$\mathrm{HH}^2(\mathcal{K}u(X)) \longrightarrow \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

This map can be imagined as the derivative of the following map of "moduli spaces"

$$\{\mathcal{K}u(X)\}/\sim \longrightarrow \{J(\mathcal{K}u(X))\}/\sim.$$

This map makes sense since it recovers the information of the derivative of the period map when $\mathcal{K}u(X) = \mathrm{D}^b(X)$ by using the new HKR isomorphisms.

In the theorem below we write

$$\mathrm{HH}_*(X) := \bigoplus_{p-q=*} H^p(X, \Omega_X^q).$$

$$\mathrm{HH}^*(X) := \bigoplus_{p+q=*} H^p(X, \wedge^q T_X).$$

Theorem 3.11. *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, E_2, ..., E_n\rangle$ where $\{E_1, E_2, ..., E_n\}$ is an exceptional collection. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[ur, "\tau"] & \\
H^1(X, T_X) \arrow[u, "\text{inclusion}"] \arrow[uur] &
\end{tikzcd}$$

where τ is defined as a contraction of polyvector fields.

Proof. We write $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{A}\rangle$ as the semiorthogonal decomposition, where $\mathcal{A} = \langle E_1, E_2, ..., E_n\rangle$. Let P_1 be the kernel of the projection to $\mathcal{K}u(X)$, and P_2 the kernel corresponding to the projection to $\mathcal{A}$. There are triangles

$$P_2 \longrightarrow \mathcal{O}_\Delta \longrightarrow P_1 \longrightarrow P_2[1] \tag{1}$$

$$P_2 \circ S_X \longrightarrow \mathcal{O}_\Delta \circ S_X \longrightarrow P_1 \circ S_X \longrightarrow P_2 \circ S_X[1]\ . \tag{2}$$

Applying $\mathrm{Hom}(-, P_1)$ to the triangle (1), we get an isomorphism

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^*(P_1, P_1)$$

because $\mathrm{Hom}^*(P_2, P_1) = 0$. Applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to the triangle (1), we get a long exact sequence

$$\mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \longrightarrow \mathrm{Hom}^t(\mathcal{O}_\Delta, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2)\ . \tag{3}$$

Since $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^*(P_1, P_1)$, we get a new long exact sequence

$$\mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{\alpha} \mathrm{Hom}^t(P_1, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2)\ . \tag{4}$$

Again, applying the functor $\mathrm{Hom}(-, P_1 \circ S_X)$ to the triangle (1), we obtain an isomorphism $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^*(P_1, P_1 \circ S_X)$ because $\mathrm{Hom}^*(P_2, P_1 \circ S_X) = 0$ [Kuz09b, Cor 3.10]. Applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to triangle (2), we again obtain a long exact sequence

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X)\ . \tag{5}$$

By the isomorphism $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^*(P_1, P_1 \circ S_X)$, we get a new long exact sequence

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^*(P_1, P_1 \circ S_X)\ . \tag{6}$$

Thus, these natural procedures induce a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{t_1}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)\times\mathrm{Hom}^{t_2}(\mathcal{O}_\Delta,\mathcal{O}_\Delta\circ S_X) \arrow[r] \arrow[d, "(\alpha,\beta)"] & \mathrm{Hom}^{t_1+t_2}(\mathcal{O}_\Delta,\mathcal{O}_\Delta\circ S_X) \arrow[d, "\beta"] \\
\mathrm{Hom}^{t_1}(P_1,P_1)\times\mathrm{Hom}^{t_2}(P_1,P_1\circ S_X) \arrow[r] & \mathrm{Hom}^{t_1+t_2}(P_1,P_1\circ S_X)
\end{tikzcd}$$

The morphisms in the rows are the composition maps described in Definition 3.5. To explain the commutative diagram, we take $t_1=t_2=0$; the general cases are similar. Let $f\in\mathrm{Hom}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$, and $g\in\mathrm{Hom}(\mathcal{O}_\Delta,\mathcal{O}_\Delta\circ S_X)$. We write L as the natural morphism $\mathcal{O}_\Delta\to P_1$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_\Delta \arrow[r, "g"] \arrow[d, "L"'] & \mathcal{O}_\Delta\circ S_X \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "L\otimes\mathrm{id}"] & \mathcal{O}_\Delta\circ S_X \arrow[d, "L\otimes\mathrm{id}"] \\
P_1 \arrow[r, "g'"] & P_1\circ S_X \arrow[r, "f'\otimes\mathrm{id}"] & P_1\circ S_X
\end{tikzcd}$$

The composition $(L\otimes\mathrm{id})\circ g$ gives an element g' in $\mathrm{Hom}(\mathcal{O}_\Delta,P_1\circ S_X)\cong\mathrm{Hom}(P_1,P_1\circ S_X)$, that is $\beta(g)=g'$. Similarly, $\alpha(f)=f'$. By the uniqueness of the isomorphism $\mathrm{Hom}(\mathcal{O}_\Delta,P_1\circ S_X)\cong\mathrm{Hom}(P_1,P_1\circ S_X)$, we have

$$\beta((f\otimes\mathrm{id})\circ g)=(f'\otimes\mathrm{id})\circ g'.$$

Taking $t_1=2$ and $t_2=-1$, in our examples, β becomes an isomorphism. The proof of this fact is as follows. Let $h\in\mathrm{Hom}^*(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X\circ S_X)$. Then according to [Kuz09b, Lemma 5.3], there is a commutative diagram.

$$\begin{tikzcd}
\Delta_*\mathcal{O}_X \arrow[r, "L"] \arrow[d, "h"'] & P_1 \arrow[d, "\gamma_{P_1}(h)"] \\
\Delta_*\mathcal{O}_X\circ S_X[*] \arrow[r, "L\otimes\mathrm{id}"] & P_1\circ S_X[*]
\end{tikzcd}$$

Hence, $\beta=\gamma_{P_1}$ (see [Kuz09b, Section 5] for the definition of γ_{P_1}). Therefore, by the Theorem of Additivity [Kuz09b, Theorem 7.3], β is an isomorphism when $t_2=-1$.

After applying the new HKR isomorphisms, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{HH}^2(X)\times\mathrm{HH}_{-1}(X) \arrow[r, "\gamma'"] \arrow[d, "(\alpha',\mathrm{id})"] & \mathrm{HH}_1(X) \arrow[d, "\mathrm{id}"] \\
\mathrm{HH}^2(\mathcal{K}u(X))\times\mathrm{HH}_{-1}(X) \arrow[r, "\gamma"] & \mathrm{HH}_1(X)
\end{tikzcd}$$

The map α' is the composition of the maps

$$\mathrm{HH}^2(X)\xrightarrow{\mathrm{IK}^{-1}}\mathrm{Hom}^2(\mathcal{O}_\Delta,\mathcal{O}_\Delta)\xrightarrow{\alpha}\mathrm{Hom}^2(P_1,P_1)=\mathrm{HH}^2(\mathcal{K}u(X)).$$

The map in the row γ' is the natural action of polyvectors on the forms: when restricting to $H^1(X,T_X)$, it is exactly the derivative of the period map. The map γ is defined by the cohomology action as follows. Let $w\in\mathrm{Hom}^2(P_1,P_1)$. Then $\gamma(w)\colon\mathrm{HH}_{-1}(X)\to\mathrm{HH}_1(X)$ is defined by the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{-1}(P_1, P_1 \circ S_X) \arrow[r, "w"] & \mathrm{Hom}^1(P_1, P_1 \circ S_X) \arrow[d, "\beta^{-1}"] \\
\mathrm{Hom}^{-1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[u, "\beta"] & \mathrm{Hom}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[d, "\mathrm{IK}"] \\
\mathrm{HH}_{-1}(X) \arrow[u, "\mathrm{IK}^{-1}"] \arrow[r, "\gamma(w)"] & \mathrm{HH}_1(X)
\end{tikzcd}$$

Thus, we obtain a commutative diagram where η is defined to be the composition $\alpha' \circ$(inclusion):

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[ur] & \\
H^1(X, T_X) \arrow[u, "\text{inclusion}"] \arrow[uur, "d\mathcal{P}"'] &
\end{tikzcd}$$

$\square$

Corollary 3.12. *Let X be Fano threefolds of index 1 or 2. Note here that we have $\mathrm{HH}_{-1}(X) = H^{2,1}(X)$ and $\mathrm{HH}_1(X) = H^{1,2}(X)$. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \\
H^1(X, T_X) \arrow[u, "\eta"] \arrow[ur, "d\mathcal{P}"] &
\end{tikzcd}$$

Remark 3.13. The commutative diagram above can be regarded as the infinitesimal version of the following "imaginary" maps

$$\begin{tikzcd}
\{\mathcal{K}u(X)\}/\sim \arrow[r] & \{J(X)\}/\sim \\
\{X\}/\sim \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Definition 3.14. Let X be a smooth projective variety. Assume there is a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, E_2, \dots, E_n \rangle$ where $\{E_1, E_2, \dots, E_n\}$ is an exceptional collection.

(1) The variety X satisfies *infinitesimal Torelli* if

$$d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$

is injective,

(2) The variety X satisfies *infinitesimal categorical Torelli* if the composition

$$\eta : H^1(X, T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$$

is injective,

(3) The Kuznetsov component $\mathcal{K}u(X)$ satisfies *infinitesimal categorical Torelli* if

$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$

is injective.

Remark 3.15. The definitions (2) and (3) from Definition 3.14 depend on the choice of the Kuznetsov component $\mathcal{K}u(X)$.

4. Infinitesimal categorical Torelli for Fano threefolds of index 1 and 2

In this section, we use the definition of the Kuznetsov component of a Fano threefold of index 1 or 2 from the survey paper [Kuz16]. We study the commutative diagram constructed in Corollary 3.12, and investigate the infinitesimal categorical Torelli properties defined in Definition 3.14. We refer to [PB21] and [Bel21] for the dimensions of $H^1(X, T_X)$ and $H^1(X, \Omega^2_X)$, where X is Fano threefold of index 1 or 2.

Proposition 4.1 ([Kuz09b, Theorem 8.8])**.** *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E, \mathcal{O}_X\rangle$ where $E^\vee$ is a global generated rank r vector bundle with vanishing higher cohomology. There is a map $\phi : X \to \mathrm{Gr}(r, V)$. Define $\mathcal{N}^\vee_{X/\,\mathrm{Gr}}$ as the shifted cone lying in the triangle*

$$\mathcal{N}^\vee_{X/\,\mathrm{Gr}} \to \phi^*\Omega_{Gr(r,V)} \to \Omega_X.$$

We have the following long exact sequences:

(1)

$$\cdots \to \bigoplus_{p=0}^{n-1} H^{t-p}(X, \Lambda^p T_X) \to \mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \to$$
$$\to H^{t-n+2}(X, E^\perp \otimes E \otimes \omega_X^{-1}) \xrightarrow{\alpha} \bigoplus_{p=0}^{n-1} H^{t+1-p}(X, \Lambda^p T_X) \to \cdots$$

(2)

$$\cdots \to \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^p T_X) \to \mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \to$$
$$\to H^{t-n+2}(X, \mathcal{N}^\vee_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1}) \xrightarrow{\nu} \bigoplus_{p=0}^{n-2} H^{t+1-p}(X, \Lambda^p T_X) \to \cdots$$

(3) If E is a line bundle, then $\nu = 0$ and

$$\mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \cong \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^q T_X) \oplus H^{t-n+2}(X, \mathcal{N}^\vee_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1}).$$

4.1. **Kuznetsov components of Fano threefolds of index 2.** An application of part (3) of Theorem 4.1 to the case of index 2 Fano threefolds of degree d, i.e. when $\mathcal{K}u(Y_d) = \langle \mathcal{O}_{Y_d}(-H), \mathcal{O}_{Y_d}\rangle^\perp$, gives the following result:

Proposition 4.2 ([Kuz09b, Theorem 8.9]). *The second Hochschild cohomology of Kuznetsov component of an index 2 Fano threefold of degree d is given by*

$$\mathrm{HH}^2(\mathcal{K}u(Y_d)) = \begin{cases} 0, & d=5 \\ k^3, & d=4 \\ k^{10}, & d=3 \\ k^{20}, & d=2 \\ k^{35}, & d=1. \end{cases}$$

Theorem 4.3. *Let Y_d be Fano varieties of index 2 of degree $3 \leq d \leq 4$. The commutative diagrams of Corollary 3.12 for Y_d are as follows:*

(1) Y_4: η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^3 \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^3 \arrow[r, "\gamma"] & k^4 \end{tikzcd}$$

(2) Y_3: η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{10} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{10} \arrow[r, "\gamma"] & k^{25} \end{tikzcd}$$

(3) Y_2: η is injective, $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{19} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

(4) Y_1: η is injective, $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{34} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{35} \arrow[r, "\gamma"] & k^{441} \end{tikzcd}$$

Proof. The map $d\mathcal{P}$ is injective for $d = 1, 2, 3, 4$. Indeed, for $d = 4$ the moduli space of Fano threefolds Y_4 is isomorphic to the moduli space of genus 2 curves C, thus $H^1(Y, T_Y) \cong H^1(C, T_C)$. Also note that the intermediate Jacobian $J(Y_4)$ is isomorphic to $J(C)$. Then the map $d\mathcal{P}$ is just the map $d\mathcal{P}_C : H^1(C, T_C) \to \mathrm{Hom}(H^{1,0}(C), H^{0,1}(C))$, which is injective since infinitesimal Torelli holds for genus 2 curves. For $d = 3$, $d\mathcal{P}$ is injective since infinitesimal Torelli holds for Y_3 by [Fle86]. For $d = 2$, $d\mathcal{P}$ is injective for Y_2 by [FRZ19, Theorem 3.4.1] and $d\mathcal{P}$ is also injective for Y_1 by [FRZ19, Section 3.4.1].

For $d = 1, 2, 3, 4$, since $d\mathcal{P}$ is injective, according to Corollary 1.4, η is injective. Thus for the cases $d = 3, 4$, η is an isomorphism, hence γ is injective. ◻

4.2. **Kuznetsov components of Fano threefolds of index 1.**

4.2.1. *The ordinary GM case.* For Section 4.2.1, let X be an ordinary GM threefold, i.e. X is a a quadric section of a linear section of codimension two of the Grassmannian $\mathrm{Gr}(2,5)$. Let $\phi : X \hookrightarrow \mathrm{Gr}(2,5)$ be the embedding, which is called the *Gushel map*. Write $\mathcal{E} = \phi^*T$, where T is the tautological rank 2 bundle on $\mathrm{Gr}(2,5)$. There is a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee\rangle$ [KP18, Proposition 2.3]. There is also a semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X\rangle,$$

and $\mathcal{K}u(X) \simeq \mathcal{A}_X$. In this subsection, we do not distinguish between $\mathcal{K}u(X)$ and $\mathcal{A}_X$ since the Hochschild (co)homology of them is the same.

Proposition 4.4 ([KP18, Proposition 2.12])**.** *Let X be an odd dimensional GM variety, and let $\mathcal{A}_X$ be the nontrivial component of $\mathrm{D}^b(X)$ as in [KP18, Proposition 2.3]. Then*

$$\mathrm{HH}^*(\mathcal{A}_X) = k \oplus k^{20}[-2] \oplus k[-4].$$

Theorem 4.5. *Let X be an ordinary GM threefold. The commutative diagram of deformations in Corollary 3.12 is as follows:*

$$\begin{tikzcd} k^{22} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

Moreover, γ is injective and η is surjective. In particular, the Kuznetsov component $\mathcal{K}u(X)$ of an ordinary GM threefold satisfies categorical Torelli.

Proof. The moduli $\mathcal{X}$ of Fano threefolds of index 1 and degree 10 is a 22 dimensional smooth irreducible algebraic stack by [DIM12, p. 13]. By Section 7 of [DIM12] we have that the differential

$$d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

of the period map $\mathcal{P} : \mathcal{X} \to \mathcal{A}_{10}$ has 2-dimensional kernel. Consider the kernel of η. Clearly, $\ker \eta \subset \ker d\mathcal{P}$ hence $\dim \ker \eta \leq 2$. Since the dimension of the image of η is less than or equal to 20, the dimension of $\ker \eta$ must be 2 and η is surjective. Finally, since image of $d\mathcal{P}$ is 20-dimensional, so is γ, hence γ is injective. □

Remark 4.6. There is an another proof which uses the long exact sequence in part (2) of Proposition 4.4:

$$0 \to H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) \to H^1(X, T_X) \xrightarrow{\eta} \mathrm{HH}^2(\mathcal{K}u(X)) \xrightarrow{\nu}$$
$$\xrightarrow{\nu} H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) \to H^2(X, T_X) \to 0.$$

It suffices to compute $H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(1))$ and $H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(1))$. Since $\mathcal{N}_{X/\mathrm{Gr}}$ is the restriction of $\mathcal{O}_G(1)\oplus\mathcal{O}_G(1)\oplus\mathcal{O}_G(2)$, we have that $\mathcal{N}^\vee_{X/\mathrm{Gr}}(1)$ is the restriction of $\mathcal{O}_G\oplus\mathcal{O}_G\oplus\mathcal{O}_G(-1)$, which is $\mathcal{O}_X\oplus\mathcal{O}_X\oplus\mathcal{O}_X(-H)$. Then $H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) = k^2$ and $H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) = 0$ by the Kodaira Vanishing Theorem. Thus, η is surjective with 2-dimensional kernel, hence γ is injective.

4.2.2. *The special GM case.* For Section 4.2.2, let X be a special GM threefold, i.e. X is a double cover of a degree 5 and index 2 Fano threefold Y_5 ramified in a quadric hypersurface.

Theorem 4.7. *The diagram in Corollary 3.12 for a special GM threefold X is as follows:*

$$\begin{tikzcd} k^{22} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

The map γ is injective and in particular, $\mathcal{K}u(X)$ satisfies categorical Torelli.

Proof. Since the kernel of $d\mathcal{P}$ is 2 dimensional, γ is injective. The rest is similar to the proof of Theorem 4.5. ∎

4.2.3. *The cases of X_{18}, X_{16}, X_{14}, and X_{12}.*

Theorem 4.8. *Let X and Y be smooth projective varieties. Suppose there is an equivalence of their Kuznetsov components $\mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ which is induced by a Fourier–Mukai functor. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X))), \mathrm{HH}_1(\mathcal{K}u(X))) \arrow[d, "\cong"] \\ \mathrm{HH}^2(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(Y)), \mathrm{HH}_1(\mathcal{K}u(Y))) \end{tikzcd}$$

Proof. According to Theorem 3.7, there is a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X)) \times \mathrm{HH}_{-1}(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{HH}_1(\mathcal{K}u(X)) \arrow[d, "\cong"] \\ \mathrm{HH}^2(\mathcal{K}u(Y)) \times \mathrm{HH}_{-1}(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{HH}_1(\mathcal{K}u(Y)) \end{tikzcd}$$

The maps in the rows are defined as the cohomology action on homology. Hence the commutative diagram in the theorem follows. ∎

Remark 4.9. When X and Y are Fano threefolds of index 1 and 2, respectively, we have $\mathrm{HH}_{-1}(X) \cong H^{2,1}(X)$ and $\mathrm{HH}_{-1}(Y) \cong H^{2,1}(Y)$, respectively. Hence we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \arrow[d, "\cong"] \\ \mathrm{HH}^2(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{Hom}(H^{2,1}(Y), H^{1,2}(Y)) \end{tikzcd}$$

where γ_X and γ_Y are the maps constructed in Theorem 3.12.

Theorem 4.10. *The diagrams in Corollary 3.12 for X_{18}, X_{16}, X_{14}, and X_{12} are as follows:*

(1) X_{18}: γ is injective.

$$\begin{tikzcd} k^{10} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{3} \arrow[r, "\gamma"] & k^{4} \end{tikzcd}$$

(2) X_{16}: γ is injective.

$$\begin{tikzcd} k^{12} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{6} \arrow[r, "\gamma"] & k^{9} \end{tikzcd}$$

(3) X_{14}: γ is injective.

$$\begin{tikzcd} k^{15} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{10} \arrow[r, "\gamma"] & k^{25} \end{tikzcd}$$

(4) X_{12}: γ is injective, $d\mathcal{P}$ is injective, and η is an isomorphism.

$$\begin{tikzcd} k^{18} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{18} \arrow[r, "\gamma"] & k^{49} \end{tikzcd}$$

Proof. In the cases of X_{18}, X_{16}, and X_{12} we always have $\mathcal{K}u(X) \simeq \mathrm{D}^b(C)$ for some curve C. The semiorthogonal decompositions are

$$\begin{aligned} \mathrm{D}^b(X_{18}) =& \langle \mathrm{D}^b(C_2), \mathcal{E}_2, \mathcal{O}_{X_{18}} \rangle \\ \mathrm{D}^b(X_{16}) =& \langle \mathrm{D}^b(C_3), \mathcal{E}_3, \mathcal{O}_{X_{16}} \rangle \\ \mathrm{D}^b(X_{12}) =& \langle \mathrm{D}^b(C_7), \mathcal{E}_5, \mathcal{O}_{X_{12}} \rangle \end{aligned}$$

where C_i is a curve of genus i and $\mathcal{E}_j$ is a vector bundle of rank j. We write X for X_{18}, X_{16}, and X_{12}. By the HKR isomorphism, $\mathrm{HH}^2(C) \cong H^2(C, \mathcal{O}_C) \oplus H^1(C, T_C) \oplus H^0(C, \wedge^2 T_C) = H^1(C, T_C)$. The second equality follows from the fact that C is of dimension 1. Note that we always refer to the version of HKR twisted by IK, as the "twisted HKR". This IK isomorphism preserves the module structure, where the geometric side is the action of polyvector fields on differential forms. Thus by Theorem 4.8 there is a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \arrow[d, "\cong"] \\ H^1(C, T_C) \arrow[r, "d\mathcal{P}_C"] & \mathrm{Hom}(H^{1,0}(C), H^{0,1}(C)) \end{tikzcd}$$

Therefore γ is injective for each X since $d\mathcal{P}_C$ is injective for each $C := C_i$. Indeed, C_3 is a plane quartic curve, which is a canonical curve in $\mathbb{P}^2$. Similarly, C_7 is also a canonical curve in $\mathbb{P}^6$ by [IM07]. Thus they are both non-hyperelliptic.

For the case X_{14}, it is known that $\mathcal{K}u(X_{14}) \simeq \mathcal{K}u(Y_3)$ by [Kuz09a]. Then by Theorem 4.8 there is a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X_{14})) \arrow[r, "\gamma_{X_{14}}"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X_{14}), H^{1,2}(X_{14})) \arrow[d, "\cong"] \\ \mathrm{HH}^2(\mathcal{K}u(Y_3)) \arrow[r, "\gamma_{Y_3}"] & \mathrm{Hom}(H^{2,1}(Y_3), H^{1,2}(Y_3)) \end{tikzcd}$$

Then $\gamma_{X_{14}}$ is injective since γ_{Y_3} is injective, by Theorem 4.3. ∎

4.2.4. *The cases of X_8, X_4, and X_2.* In these cases, the Kuznetsov components are defined as $\langle \mathcal{O}_X \rangle^{\perp}$.

Theorem 4.11. *The diagrams in Corollary 3.12 for X_8, X_4, and X_2 are as follows:*

(1) X_8: γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{27} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{27} \arrow[r, "\gamma"] & k^{196} \end{tikzcd}$$

(2) X_4: γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{45} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{45} \arrow[r, "\gamma"] & k^{900} \end{tikzcd}$$

(3) X_2: γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{68} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{68} \arrow[r, "\gamma"] & k^{2704} \end{tikzcd}$$

Proof. First, we prove that η is an isomorphism in each case. We write X for X_8, X_4 and X_2. Note that $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X \rangle$. Denote by P_1 the kernel of the left projection to $\mathcal{K}u(X)$, and P_2 the kernel of the right projection to $\langle \mathcal{O}_X \rangle$. There is a triangle

$$P_2 \to \Delta_* \mathcal{O}_X \to P_1 \to P_2[1].$$

Applying the functor $\Delta^!$ to the triangle, we obtain the diagram

$$\begin{tikzcd} \Delta^! P_2 \arrow[r] \arrow[d, "\cong"] & \Delta^! \Delta_* \mathcal{O}_X \arrow[r, "L"] \arrow[d, "\cong"] & \Delta^! P_1 \arrow[d, "\mathrm{id}"] \\ \omega_X^{-1}[-3] \arrow[r, "w"] & \bigoplus_{p=0}^3 \Lambda^p T_X[-p] \arrow[r] & \Delta^! P_1 \end{tikzcd}$$

According to [Kuz09b, Theorem 8.5], the map w is an isomorphism onto the third summand. Applying $\mathrm{Hom}^2(\mathcal{O}_X, -)$, we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! \Delta_* \mathcal{O}_X) \arrow[r, "L"] \arrow[d, "\cong"] & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \arrow[d, "\mathrm{id}"] \\ \mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^3 \Lambda^p T_X[-p]) \arrow[r] & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \\ \mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^2 \Lambda^p T_X[-p]) \arrow[u, "\cong"] \arrow[ur, "\cong"] & \end{tikzcd}$$

Thus the morphism L is an isomorphism. However, L is naturally isomorphic to the morphism

$$\mathrm{Hom}^2(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \to \mathrm{Hom}^2(\Delta_* \mathcal{O}_X, P_1) \cong \mathrm{Hom}^2(P_1, P_2).$$

That is to say the map $\alpha' : \mathrm{HH}^2(X) \to \mathrm{HH}^2(\mathcal{K}u(X))$ constructed in Theorem 3.11 is an isomorphism. According to [PB21, Appendix A], $H^2(X, \mathcal{O}_X) = 0$ and $H^0(X, \Lambda^2 T_X) = 0$, hence η is an isomorphism.

The map $d\mathcal{P}$ is injective for X_8, X_4 by [Fle86] and X_2 by [Cle83]. Then γ is injective for these cases because η is an isomorphism and $d\mathcal{P}$ is injective. $\square$

References

[AK19] Marco Antonio Armenta and Bernhard Keller. Derived invariance of the Tamarkin–Tsygan calculus of an algebra. *Comptes Rendus Mathematique*, 357(3):236–240, 2019.

[Bel21] Pieter Belmans. Fanography. `https://fanography.pythonanywhere.com/`, 2021. Accessed 09/07/2021.

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. *Advances in Mathematics*, 229(2):770–803, 2012.

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica*, 125(3):327–344, 2001.

[BT14] Marcello Bernardara and Goncalo Tabuada. From semi-orthogonal decompositions to polarized intermediate Jacobians via Jacobians of noncommutative motives. *arXiv preprint arXiv:1305.4687*, 2014.

[Căl05] Andrei Căldăraru. The Mukai pairing—II: the Hochschild–Kostant–Rosenberg isomorphism. *Advances in Mathematics*, 194(1):34–66, 2005.

[Cle83] C Herbert Clemens. Double solids. *Advances in mathematics*, 47(2):107–230, 1983.

[CRVdB12] Damien Calaque, Carlo A Rossi, and Michel Van den Bergh. Căldăraru's conjecture and Tsygan's formality. *Annals of Mathematics*, pages 865–923, 2012.

[DIM12] Olivier Debarre, Atanas Iliev, and Laurent Manivel. On the period map for prime Fano threefolds of degree 10. *J. Algebraic Geom*, 21(1):21–59, 2012.

[Fle86] H Flenner. The infinitesimal Torelli problem for zero sets of sections of vector bundles. *Math Z.*, 193:307–322, 1986.

[FRZ19] Enrico Fatighenti, Luca Rizzi, and Francesco Zucconi. Weighted Fano varieties and infinitesimal Torelli problem. *Journal of Geometry and Physics*, 139:1–16, 2019.

[Huy06] Daniel Huybrechts. *Fourier–Mukai transforms in algebraic geometry.* Oxford University Press on Demand, 2006.

[IM07] Atanas Iliev and Dimitri Markushevich. Parametrization of Sing(Theta) for a Fano 3-fold of genus 7 by moduli of vector bundles. *Asian Journal of Mathematics*, 11(3):427–458, 2007.

[JLLZ21] Augustinas Jacovskis, Xun Lin, Zhiyu Liu, and Shizhuo Zhang. Categorical Torelli theorems for Gushel-Mukai threefolds. *arxiv preprint arxiv: 2108.02946*, 2021.

[Kel98] Bernhard Keller. Invariance and localization for cyclic homology of DG algebras. *Journal of Pure and Applied Algebra*, 123(1):223–273, 1998.

[Kel06] B. Keller. On differential graded categories. *arXiv preprint arXiv:math/0601185*, 2006.

[KP18] Alexander Kuznetsov and Alexander Perry. Derived categories of Gushel–Mukai varieties. *Compositio Mathematica*, 154(7):1362–1406, 2018.

[Kuz03] Alexander Kuznetsov. Derived categories of cubic and V14 threefolds. *arXiv preprint math/0303037*, 2003.

[Kuz09a] Alexander Kuznetsov. Derived categories of Fano threefolds. *Proceedings of the Steklov Institute of Mathematics*, 264(1):110–122, 2009.

[Kuz09b] Alexander Kuznetsov. Hochschild homology and semiorthogonal decompositions. *arXiv preprint arXiv:0904.4330*, 2009.

[Kuz16] Alexander Kuznetsov. Derived Categories View on Rationality Problems. *Rationality Problems in Algebraic Geometry*, page 67–104, 2016.

[LO10] Valery A. Lunts and Dmitri O. Orlov. Uniqueness of enhancement for triangulated categories. *Journal of the American Mathematical Society*, 23(3):853–853, Sep 2010.

[PB21] F Tanturri P Belmans, E Fatighenti. Polyvector fields for Fano 3-folds. *arXiv preprint arXiv:2104.07626*, 2021.

[Per20] Alexander Perry. The integral Hodge conjecture for two-dimensional Calabi-Yau categories. *arXiv preprint arXiv:2004.03163*, 2020.

[PY20] Laura Pertusi and Song Yang. Some remarks on Fano threefolds of index two and stability conditions. *arXiv preprint arXiv:2004.02798*, 2020.

[Tab05] Gonçalo Tabuada. Invariants additifs de dg-catégories. *International Mathematics Research Notices*, 2005(53):3309–3339, 01 2005.

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