HOCHSCHILD COHOMOLOGY AND CATEGORICAL TORELLI THEOREMS FOR GUSHEL–MUKAI THREEFOLDS

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Abstract. The Kuznetsov component $\mathcal{K}u(X)$ of a Gushel–Mukai (GM) threefold has two numerical (-1)-classes with respect to the Euler form. We describe the Bridgeland moduli spaces for stability conditions on Kuznetsov components with respect to each of the (-1)-classes and prove refined and birational categorical Torelli theorems in terms of $\mathcal{K}u(X)$. We also prove a categorical Torelli theorem for special GM threefolds. We study the smoothness and singularities on Bridgeland moduli spaces for all smooth GM threefolds and use this to prove a conjecture of Kuznetsov–Perry in dimension three under a mild assumption. Finally, we use our moduli spaces to restate a conjecture of Debarre–Iliev–Manivel regarding fibers of the period map for ordinary GM threefolds. We also prove the restatement of this conjecture infinitesimally using Hochschild (co)homology.

1. Introduction

In recent times, derived categories have played an important role in algebraic geometry; in many cases, much of the geometric information of a variety/scheme X is encoded by its bounded derived category of coherent sheaves $\mathrm{D}^b(X)$. In this setting, one of the most fundamental questions that can be asked is whether $\mathrm{D}^b(X)$ recovers X up to isomorphism, in other words, whether a *derived Torelli theorem* holds for X. For varieties with ample or anti-ample canonical bundle (which include Fano varieties and varieties of general type), this question was answered affirmatively by Bondal–Orlov in [BO01].

1.1. Kuznetsov components and categorical Torelli theorems.

Therefore, for the class of varieties above, it is natural to ask whether they are also determined up to isomorphism by *less* information than the whole derived category $\mathrm{D}^b(X)$. A natural candidate for this is a subcategory $\mathcal{K}u(X)$ of $\mathrm{D}^b(X)$ called the *Kuznetsov component*. This subcategory has been studied extensively by Kuznetsov and others (e.g. [Kuz03, Kuz09a, KP18b]) for many classes of varieties, including Fano varieties and Gushel–Mukai (GM) varieties.

In fact, the question of whether $\mathcal{K}u(X)$ determines X up to isomorphism has been studied for certain cases in the setting of Fano threefolds. In [BMMS12], the authors show that the Kuznetsov component completely determines cubic threefolds up to isomorphism, in other words, a *categorical Torelli theorem* holds for cubic threefolds Y. This is proved by showing that moduli space of Bridgeland stable objects of class $[I_L]$ (where $L \subset Y$ is a line) in $\mathcal{K}u(Y)$ is isomorphic to the Hilbert scheme of

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lines on Y. Then the classical Torelli for cubic threefolds gives an isomorphism of the two cubic threefolds. This same result was also verified in [PY20].

Cubic threefolds are related to index 1 Picard rank 1 degree 14 Fano threefolds (V_{14} threefolds) in the sense that each cubic threefold Y is associated to a collection of V_{14} Fano threefolds X parameterized by the moduli space of instanton bundles on the cubic threefold. This along with a theorem of Kuznetsov [Kuz03] which states that $\mathcal{K}u(Y) \simeq \mathcal{K}u(X)$ for any X in the collection, and the classical Torelli theorem for cubic threefolds [CG72, Tyu71] gives a *birational categorical Torelli theorem* for V_{14} Fano threefolds, i.e. $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ implies that X and X' are birationally equivalent.

In [APR19], the authors show that a categorical Torelli theorem holds for index 2 Picard rank 1 degree 2 Fano threefolds, again utilising the framework of Bridgeland stability conditions on Kuznetsov components, and moduli spaces of stable objects with respect to these stability conditions [BLMS17]. A weaker version of this statement was also shown in [BT16], which requires an assumption that the equivalence of Kuznetsov components is Fourier–Mukai.

For GM threefolds – the focus of our paper – by [KP19] it is known that there are birational GM threefolds with equivalent Kuznetsov components. So there are two natural questions to ask in this setting:

Question 1.1.

(1) Does $\mathcal{K}u(X)$ determine the birational equivalence class of X?
(2) What extra data along with $\mathcal{K}u(X)$ do we need to identify a particular GM threefold X from its birational equivalence class?

1.2. **Hochschild cohomology and (categorical) Torelli theorems.** On the other hand, we have the setting of Hodge theory where the Torelli question classically originated. The global Torelli problem asks whether the intermediate Jacobian $J(X)$ can determine a variety X up to isomorphism via the period map. The infinitesimal Torelli problem asks whether the differential of the period map is injective (and so whether the intermediate Jacobian determines X at least locally). This is usually a starting point for studying the global Torelli problem.

As mentioned in the previous section, for Torelli problems in the categorical setting, one can use the Kuznetsov component as the invariant. Recently, the intermediate Jacobian of a scheme X was reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ in [Per20]. In particular, we have an imaginary commutative diagram of sets under certain assumptions [Per20, Proposition 5.3].

$$\begin{tikzcd}
\{\mathcal{K}u(X)\}/\sim \arrow[r] & \text{Abelian varieties}/\sim \\
\{X\}/\sim \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Therefore, along with Question 1.1 we can ask the following questions both on a global and infinitesimal (local) level:

Question 1.2.

(1) Does the infinitesimal categorical Torelli hold, or equivalently, is the map $H^1(X, T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$ injective?
(2) Does the intermediate Jacobian determine the Kuznetsov component, i.e. $J(X) \cong J(X') \implies \mathcal{K}u(X) \simeq \mathcal{K}u(X')$?

1.3. **Summary of results.**

1.3.1. *(Birational) categorical Torelli for X_{10}.* In the present paper, we deal with the case of index 1 Picard rank 1 degree 10 Fano threefolds, also known as GM threefolds, which are split into two types: ordinary GM threefolds and special GM threefolds. Our first main theorem is concerned with ordinary GM threefolds and gives an answer to Question 1.1 (2):

Theorem 1.3 (Theorem 7.17)**.** *Let X be a general ordinary GM threefold, and set $\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E}\rangle \subset \mathrm{D}^b(X)$ where $\mathcal{E}$ is the restriction of the tautological bundle on $\mathrm{Gr}(2,5)$ to X. Let $\pi : \mathcal{D} \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $\mathcal{K}u(X) \subset \mathcal{D}$. Then the data of $\mathcal{K}u(X)$ along with the object $\pi(\mathcal{E})$ is enough to recover X up to isomorphism.*

We prove the above theorem by considering the moduli space of Bridgeland stable objects in $\mathcal{A}_X$ (the alternative Kuznetsov component defined by the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee\rangle$; there is an equivalence $\Xi : \mathcal{K}u(X) \simeq \mathcal{A}_X$) with respect to $-x$, one of the two (-1)-classes in the numerical Grothendieck group of $\mathcal{A}_X$. In particular, we show that this moduli space is isomorphic to the minimal model $\mathcal{C}_m(X)$ of the Fano surface of conics (Theorem 7.13). In fact we show that the unique exceptional curve contracted in $\mathcal{C}(X)$ is the rational curve of conics whose ideal sheaf I_C is not in $\mathcal{A}_X$ and that the image is the smooth point represented by $\pi(\mathcal{E})$ (Proposition 7.1). Thus the data $(\mathcal{K}u(X), \pi(\mathcal{E}))$ recovers $\mathcal{C}(X)$. A classical result of Logachev [Log12] states that X can be recovered up to isomorphism from $\mathcal{C}(X)$, and so the theorem is proved. On the other hand, for general special GM threefolds, we show that a categorical Torelli theorem holds.

Theorem 1.4 (Theorem 8.1)**.** *Let X and X' be general special GM threefolds, and assume that there is an equivalence of categories $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then X and X' are isomorphic.*

We prove this theorem by considering the equivariant Kuznetsov components $\mathcal{K}u(X)^{\mu_2}$, first discussed in [KP18a], and exploiting the fact that X is the double cover of a degree 5 index 2 Picard rank 1 Fano threefold Y, branched over a quadric hypersurface $\mathcal{B} \subset Y$. In this case, the equivariant Kuznetsov component is equivalent to $\mathrm{D}^b(\mathcal{B})$ where $\mathcal{B}$ is a K3 surface. Therefore, a number of results concerning the Fourier–Mukai partners of K3 surfaces can be used to deduce that $\mathcal{K}u(X)^{\mu_2} \simeq \mathcal{K}u(X')^{\mu_2}$ implies $\mathcal{B} \cong \mathcal{B}'$. Then the fact that the moduli space of Fano threefolds of the type Y is a point can be used to deduce that indeed, $X \cong X'$.

Next, returning to the setting of ordinary GM threefolds, we show that a birational categorical Torelli theorem holds for general ordinary GM threefolds, as in the case of V_{14}, which answers Question 1.1 (2).

Theorem 1.5 (Theorem 10.1)**.** *Let X and X' be general ordinary GM threefolds, and suppose that there is an equivalence of categories $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X is birationally equivalent to X'.*

To prove this result, first we study the Bridgeland moduli space of stable objects $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ in $\mathcal{A}_X$ with respect to $y - 2x$, the other (-1)-class in the numerical Grothendieck group of $\mathcal{A}_X$. We show that the projection functor $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$ sends each object in the moduli space of $M_G^X(2,1,5)$ of Gieseker semistable sheaves to a stable object in $\mathcal{A}_X$, thus inducing a morphism from $M_G^X(2,1,5)$ to $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$. The morphism is injective and étale at each point. All of this enables us to identify the moduli space $M_G^X(2,1,5)$ with the Bridgeland moduli space $M_\sigma(\mathcal{A}_X, y-2x)$ (Theorem 9.7). We then invoke a few more results from [DIM12]. More precisely, an equivalence of categories $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ identifies the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ with either $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ or $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$. The former case gives an isomorphism of minimal surfaces $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$. Blowing $\mathcal{C}_m(X)$ up at the smooth point associated to $\pi(\mathcal{E})$ gives $\mathcal{C}(X)$, and blowing up $\mathcal{C}_m(X')$ at the image of $\pi(\mathcal{E})$ under Φ gives $\mathcal{C}(X'_c)$, where X'_c is certain birational transformation of X', associated with a conic $c \subset X'$. Then by Logachev's Reconstruction Theorem on $\mathcal{C}(X)$, X is isomorphic to X'_c which is birational to X'. For the latter case, we start with the isomorphism $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. In fact, $M_G^{X'}(2,1,5)$ is birational to $\mathcal{C}(X'_L)$, where X'_L is another birational transformation of X', associated with a line $L \subset X'$. Since $\mathcal{C}(X'_L)$ is a surface of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$. Then by the same argument as in the previous case, X is isomorphic to some birational transformation of X'.

In [KP19], the authors studied GM varieties of arbitrary dimension and proved the Duality Conjecture [KP18b, Conjecture 3.7] for them, i.e. they showed that the period partner or period dual of a GM variety X shares the same Kuznetsov component $\mathcal{A}_X$ as X. Combining the earlier results [DK15, Theorem 4.20] on the birational equivalence of these varieties, this gives a strong evidence to the following conjecture:

Conjecture 1.6 ([KP19, Conjecture 1.7])**.** *If X and X' are GM varieties of the same dimension such that there is an equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then X and X' are birationally equivalent.*

By a careful study of Bridgeland moduli spaces of stable objects in the Kuznetsov components $\mathcal{A}_X$ for not only smooth ordinary GM threefolds, but also special GM threefolds X, we can prove Conjecture 1.6 in dimension three with assumptions on the genericity of the GM threefolds.

Theorem 1.7 (Theorem 11.18 and Theorem 11.19)**.**

(1) *If X and X' are smooth* general *GM threefolds such that there is an equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then X and X' are birationally equivalent.*
(2) *If X and X' are smooth GM threefolds with one of them a* general *ordinary GM threefold such that there is an equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then they are birationally equivalent.*

The proof is similar to that of Theorem 1.5. Firstly, we identify the Bridgeland moduli spaces of stable objects $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ on a special GM threefold X' with $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ respectively (Theorem 11.6 and Corollary 9.11), where $\mathcal{C}_m(X')$ is the contraction of the Fano surface $\mathcal{C}(X')$ of conics on X' along one of the components to a singular point. Then the equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify those moduli spaces on a general ordinary GM threefold X with those on a special GM threefold X'; we show that this is impossible since $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x) \cong \mathcal{C}_m(X')$ is always singular while $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G^X(2,1,5)$ are both smooth and irreducible. Then Theorem 1.7 (1) reduces to Theorem 1.5 and Theorem 1.4. Similarly, Theorem 1.7 (2) reduces to Theorem 1.5 via a comparison of the singularities on those moduli spaces on X and X' respectively (Table 11.4).

In [DIM12], the authors conjecture that the general fiber of the classical period map from the moduli space of ordinary GM threefolds to the moduli space of 10 dimensional principally polarised abelian varieties is birational to the union of $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$, both quotiented by involutions (which we call the *Debarre-Iliev-Manivel conjecture*). Within the moduli space of smooth GM threefolds, we define the fiber of the "categorical period map" through $[X]$ as all ordinary GM threefolds X' whose Kuznetsov components satisfy $\mathcal{A}_{X'} \simeq \mathcal{A}_X$. The fact that $\mathcal{A}_X$ has only two (-1)-classes, and our results on the Bridgeland moduli spaces with respect to these classes enable us to prove the following categorical analogue of their conjecture:

Theorem 1.8 (Theorem 12.2)**.** *Let X be an ordinary GM threefold. A general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ of the categorical period map is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$ where ι, ι' are geometrically meaningful involutions.*

By Theorem 1.3, an ordinary GM threefold X is uniquely determined by $\mathcal{K}u(X)$ and $\pi(\mathcal{E})$ (or equivalently, $\mathcal{A}_X$ and $\Xi(\pi(\mathcal{E}))$). Thus we can describe the general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ as the family $\mathcal{F}$ of $\Xi(\pi(\mathcal{E}')) \in \mathcal{A}_{X'}$ as X' varies. In fact, this description will extend to other classes of index one prime Fano threefolds. In the upcoming preprint [JLZ21] we show that for X_{14}, $\mathcal{F}$ is the moduli space $\mathcal{M}_Y^{\mathrm{inst}}$ of rank 2 instanton bundles of minimal charge on a cubic threefold Y. For X_{16}, $\mathcal{F}$ is the moduli space of rank 2 stable vector bundles with a certain special property over a genus 3 curve and for X_{12}, $\mathcal{F}$ consists of only one point. For X_{18}, $\mathcal{F}$ is the moduli space of rank 3 stable vector bundles with another certain special property over a genus 2 curve. In view of Conjecture 12.4 and Theorem 12.5, the notion of the fiber of the "categorical period" map not only recovers the classical results on fibers of period maps for X_{10} in [DIM12], X_{14} in [MT98] and [Kuz03], and X_{16} in [BF13], but also reproves the very recent result in [FV21] for X_{18}.

Remark 1.9. It is possible to prove Conjecture 1.6 in dimension three in full generality and describe the fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ for all smooth GM threefolds X. In Section 13, we outline a possible route by relating the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ to the double dual EPW surface $\tilde{Y}_{A^\perp}^{\geq 2}$ and double EPW surface $\tilde{Y}_A^{\geq 2}$. Then a number of results from a series of works by Debarre–Kuznetsov ([DK15], [DK19], [DK20a] and [DK20b]) would probably enable us to prove the conjecture. We give another proof of Theorem 1.5 in this fashion and the general case will be studied in [LZ21].

1.3.2. *Answers to Question 1.2 in the infinitesimal setting for other Fano threefolds of index* 1 *and* 2. In this paper, we propose infinitesimal versions of the Torelli type problems corresponding to the global Torelli problems in Question 1.2, using the machinery of Hochschild cohomology. The starting point of the story is the following theorem.

Theorem 1.10 (Theorem 15.11)**.** *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, \ldots, E_n \rangle$, where $\{E_1, \ldots, E_n\}$ is an exceptional collection.*

Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[rr, "\gamma"] & & \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[urr, "\tau"] & & \\
H^1(X, T_X) \arrow[u] \arrow[uurr, "d\mathcal{P}"'] & &
\end{tikzcd}$$

where τ is defined as a contraction of polyvector fields.

If an odd dimensional variety X ($\dim X = n$) satisfies $\mathrm{HH}_{2i+1}(X) = 0$ for $i \geq 1$, then by the Additivity Theorem of Hochschild Homology and Periodic Cyclic Homology, there is an isomorphism which preserves the two sided weight 1 Hodge structure of $\mathrm{ch}^{\mathrm{top}}(K_1^{\mathrm{top}}(\mathcal{K}u(X))) \cong H^n_f(X, \mathbb{Z})$ under further assumptions of the cohomology groups,

$$\mathrm{HP}_1(\mathcal{K}u(X)) \cong H^{\mathrm{odd}}_{\mathrm{dR}}(X, \mathbb{C}).$$

Also, the intermediate Jacobian $J(X)$ and $J(\mathcal{K}u(X))$ are isomorphic [Per20, Proposition 5.23], and their first order deformation spaces are isomorphic too. With the variety X satisfying the above, the diagram of the above theorem can be interpreted as the diagram of deformations

$$\begin{tikzcd}
\{\mathcal{K}u(X)\}/\sim \arrow[r] & \mathrm{Tori}/\sim \\
\mathcal{X}_{10} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

if we have good knowledge of the moduli spaces in question.

Though the following definition has a more geometric intuition if we assume vanishing of the higher Hochschild homology, we note here that it applies for smooth projective varieties which admit a Kuznetsov component, without the further assumptions.

Definition 1.11 (Definition 15.14)**.** Recall the diagram from Theorem 1.10. The variety X has the property of

(1) *infinitesimal Torelli* if
$$d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$
is injective,

(2) *infinitesimal categorical Torelli* if the composition
$$\eta : H^1(X, T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$$
is injective,

(3) *infinitesimal categorical Torelli of period type* if
$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$
is injective.

The main examples for our problems of infinitesimal categorical Torelli that we study are those of Fano threefolds of index 1 and 2. Note that Fano threefolds satisfy the assumption $\mathrm{HH}_{2i+1}(X) = 0$ for $i \geq 1$ and the cohomology assumption in [Per20, Proposition 5.23], see for example Remark 15.9. We summarise our results in the following theorem:

Theorem 1.12 (Theorems 16.12, 16.13, 16.7, 16.9, 16.4)**.** *Let X_{2g-2} be a Fano threefold of index 1 and degree $2g-2$, where g is its genus. Let Y_d be a Fano threefold of index 2 and degree d.*

- *For Y_d where $1 \leq d \leq 4$ we have infinitesimal categorical Torelli, and for $3 \leq d \leq 4$ we furthermore have infinitesimal categorical Torelli of period type.*
- *For X_{2g-2} where $2 \leq g \leq 10$ and $g \neq 4$, we have infinitesimal categorical Torelli of period type. Furthermore, for $g = 2, 3, 5, 7$ we have infinitesimal categorical Torelli.*

Due to the infinitesimal categorical Torelli of period type holding in numerous cases, one may be tempted to make the following conjecture based on Question 1.2 (2):

Conjecture 1.13. *Let X be an index one prime Fano threefold. The intermediate Jacobian $J(X)$ determines the Kuznetsov component $\mathcal{K}u(X)$.*

In fact, by already known results this conjecture is true for almost all cases of Fano threefolds of index one (summarised in Theorem 12.5), and since we show that the infinitesimal categorical Torelli of period type holds for GM threefolds, we have infinitesimal evidence for the conjecture. We also show the following corollary of Theorem 1.8:

Corollary 1.14 (Proposition 12.6)**.** *The Debarre–Iliev–Manivel Conjecture is equivalent to Conjecture 1.13 in the ordinary GM case.*

Remark 1.15. The status (to the best of our knowledge) of the global and infinitesimal versions of classical and categorical Torelli problems is summarised in the tables in Section 16.3.

1.4. Further questions.

1.4.1. *Categorical Torelli theorems for other classes of Fano threefolds.*

- A birational categorical Torelli theorem holds for V_{14} Fano threefolds, so one can ask whether an analogue of Theorem 1.3 holds, i.e. whether $\mathcal{K}u(X)$ and the analogous object $\pi(\mathcal{E})$ is enough to determine a V_{14} Fano threefold up to isomorphism. Since $\pi(\mathcal{E})$ is a rank -2 object (as opposed to a rank -1 object in the GM case), the situation is more complicated. One potential approach is to consider the locus of $\pi(\mathcal{E})$ objects in the moduli space $M_\sigma(\mathcal{K}u(X), [\pi(\mathcal{E})])$. This locus can then potentially be used to recover (via a Brill-Noether argument) X inside another moduli space related to projections of skyscraper sheaves into $\mathcal{K}u(X)$. This is work in preparation [JZ21] (where we also consider index 1 Fano threefolds of other degrees).

  Given the equivalence of categories $\Phi : \mathcal{K}u(X) \to \mathcal{K}u(Y)$ between the V_{14} threefold X and its associated cubic threefold Y, another possible approach is to directly study the image $\Phi(\pi(\mathcal{E}))$. If this can be shown to be isomorphic to an instanton bundle on Y, then we would be done since the instanton bundle is precisely the extra data along with Y, which is needed to determine X. This approach is implemented in an upcoming preprint [JLZ21].

- Whether a categorical Torelli theorem holds for index 2 Picard rank 1 degree 1 Fano threefolds is still an open question. The same methods as in [APR19] cannot be applied to this case, because the homological dimension of the heart of stability conditions on $\mathcal{K}u(X)$ is 3 as opposed to 2.
- Similar questions can be asked about whether categorical Torelli theorems/birational categorical Torelli theorems/extra data categorical Torelli theorems hold for index 1 Picard rank 1 degree 12, 16,18 and 22, etc. Fano threefolds. This is also work in preparation [JLZ21].
- In Theorem 1.5, we actually show that X' is either a conic transform of X or a conic transform of a line transform of X. It is an interesting question to ask what the precise birational equivalences between X_{2g-2} and X'_{2g-2} are whenever $\mathcal{K}u(X_{2g-2}) \simeq \mathcal{K}u(X'_{2g-2})$ for $g = 9, 10, 12$ (note that $X_{2g-2} \simeq X'_{2g-2}$ since both of them are rational).

1.4.2. *The Debarre–Iliev–Manivel conjecture.* Since our work allows us to state the Debarre–Iliev–Manivel conjecture in the form of Conjecture 1.13, this potentially opens new directions to attack this conjecture.

1.4.3. *Non-general GM threefolds.* Theorems 1.3, 1.5, 1.4 and 1.7 are all stated for *general* GM threefolds. Therefore, it is natural to ask whether these results also hold for GM threefolds which are not general in their respective moduli spaces.

1.5. **Notation and conventions.**

- $\mathcal{K}u(X)$ is the ordinary Kuznetsov component of X, and $\mathcal{A}_X$ is the alternative Kuznetsov component of X (see Definition 3.1).
- We call slope stability μ-stability.
- $(\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$, $(\mathrm{Coh}^\mu_{\alpha,\beta}(X), Z^\mu_{\alpha,\beta})$, and $(\mathcal{A}(\alpha,\beta), Z(\alpha,\beta))$ denote once-tilted stability conditions, twice-tilted stability conditions, and Bridgeland stability conditions on the Kuznetsov component $\mathcal{K}u(X)$, respectively. $(\mathcal{A}'(\alpha,\beta), Z'(\alpha,\beta))$ denotes a stability condition on the alternative Kuznetsov component $\mathcal{A}_X$. Also, $(\mathcal{T}^\beta, \mathcal{F}^\beta)$ and $(\mathcal{T}^\mu_{\alpha,\beta}, \mathcal{F}^\mu_{\alpha,\beta})$ denote torsion pairs on the once-tilted and twice-tilted hearts, respectively.
- $\mathcal{H}^*_{\mathcal{A}}$ means cohomology with respect to the heart $\mathcal{A}$. When the $\mathcal{A}$ subscript is dropped, we take the heart to be $\mathrm{Coh}(X)$.
- The symbol $\simeq$ denotes an equivalence of categories, and a birational equivalence of varieties. The symbol $\cong$ denotes an isomorphism of varieties.

1.6. **Organization of the paper.** In Section 2, we collect basic facts about semiorthogonal decompositions. In Section 3, we introduce Gushel–Mukai threefolds and their Kuznetsov components. In Section 4, we introduce the definition of weak stability conditions on $\mathrm{D}^b(X)$, and then apply [BLMS17, Theorem 5.1] to induce stability conditions on the alternative Kuznetsov components $\mathcal{A}_X$ of GM threefolds. We then introduce Serre-invariant stability conditions on Kuznetsov components and show that they are contained in one $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ orbit. In Section 5, we introduce a distinguished object $\pi(\mathcal{E}) \in \mathcal{K}u(X)$ and its alternative Kuznetsov component analogue $\Xi(\pi(\mathcal{E})) \in \mathcal{A}_X$ and prove its stability. In Section 6 we discuss the geometry of the Fano surface of conics of an ordinary GM threefold. In Section 7, we construct the Bridgeland moduli space of σ-stable objects with class $-x = -(1-2L)$ in $\mathcal{A}_X$, and prove a refined categorical Torelli theorem (Theorem 1.3). In Section 8, we prove a categorical Torelli theorem (Theorem 1.4) for special GM threefolds. In Sections 9 and 10 we construct the Bridgeland moduli space of σ-stable objects with respect to the other (-1)-class in $\mathcal{A}_X$, and prove a birational categorical Torelli theorem (Theorem 1.5) and Conjecture 1.6 in dimension three with mild assumptions. In Section 11, we study the classical moduli space of sheaves $M_G(2,1,5)$ and the associated Bridgeland moduli space of σ-stable objects in the Kuznetsov component of a special GM threefold and describe their geometric properties. Combining with our results for ordinary GM threefolds, we prove the inverse of Duality Conjecture under mild assumptions. In Section 12, we describe the general fiber of the "categorical period map" for ordinary GM threefolds 1.8 and restate the *Debarre-Iliev-Manivel conjecture* in terms of Conjecture 1.13. In Section 13, we introduce the concepts of double EPW surfaces and double dual EPW surfaces, and identify them with $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$. We then reprove Theorem 10.1 in the language of [DK15] and [DK20b]. In Section 14, we prove connectedness of the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ for GM threefolds. In Section 15, we introduce the definition of Hochschild (co)homology for admissible subcategories of bounded derived categories $\mathrm{D}^b(X)$ of smooth projective varieties X. We then prove Theorem 1.10. In Section 16, we apply the techniques developed in Section 15 to prime Fano threefolds of index one and two. In particular, we show the infinitesimal version of Conjecture 1.13 for ordinary GM threefolds. We also summarize the status of classical and categorical (birational) Torelli theorems for Fano threefolds. In Appendix A, we state a theorem reconstructing a smooth GM threefold as the *Brill–Noether* locus of a certain Bridgeland moduli space of stable objects in the Kuznetsov component, which is proved in our upcoming work [JZ21]. As an application, we prove a refined categorical Torelli theorem for *any* GM threefold and use it to describe the fiber of the categorical period map over $\mathcal{A}_X$ for any smooth GM threefold.

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2. Semiorthogonal decompositions

In this section, we collect some useful facts/results about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06], for example. From now on, let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on X, and for $E, F \in \mathrm{D}^b(X)$, define

$$\mathrm{RHom}^\bullet(E,F) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(E,F)[-i].$$

2.1. **Exceptional collections and semiorthogonal decompositions.**

Definition 2.1. Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$. We say that E is an *exceptional object* if $\mathrm{RHom}^\bullet(E,E) = \mathbb{C}$. Now let $(E_1, \ldots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y,X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X,Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Definition 2.3. Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ be a collection of full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{C}_j \subset \mathcal{C}_i^\perp$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ generate $\mathcal{D}$, i.e. the category resulting from taking all shifts and cones of objects in the categories $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$ is equivalent to $\mathcal{D}$.

Definition 2.4. The *Serre functor* $S_{\mathcal{D}}$ of a triangulated category $\mathcal{D}$ is the autoequivalence of $\mathcal{D}$ such that there is a functorial isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{D}}(A,B) \cong \mathrm{Hom}_{\mathcal{D}}(B, S_{\mathcal{D}}(A))^\vee$$

for any $A, B \in \mathcal{D}$.

Proposition 2.5. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} \simeq \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

Example 2.6. Let $\mathcal{D} = \mathrm{D}^b(X)$. Then $S_{\mathcal{D}}(-) = (- \otimes \mathcal{O}(K_X))[\dim X]$.

2.2. **Mutations.** Let $\mathcal{C} \subset \mathcal{D}$ be an admissible triangulated subcategory. Then one has both left and right adjoints to the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ denoted i^* and $i^!$, respectively. Then the *left mutation functor* $\mathbf{L}_{\mathcal{C}}$ through $\mathcal{C}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \to \mathrm{id} \to \mathbf{L}_{\mathcal{C}}$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{C}}$ through $\mathcal{C}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{C}} \to \mathrm{id} \to ii^*.$$

When $E \in \mathrm{D}^b(X)$ is an exceptional object, and $F \in \mathrm{D}^b(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{RHom}^\bullet(E,F) \to F \to \mathbf{L}_E F,$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \to F \to E \otimes \mathrm{RHom}^\bullet(F,E)^\vee.$$

Furthermore, when $(E_1, \ldots, E_m)$ is an exceptional collection, for $i = 1, \ldots, m-1$ the collections

$$(E_1, \ldots, E_{i-1}, \mathbf{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \ldots, E_m)$$

and

$$(E_1, \dots, E_{i+1}, \mathbf{R}_{E_{i+1}} E_i, E_{i+2}, E_{i+3}, \dots, E_m)$$

are also exceptional.

Proposition 2.7. *Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then*

$$S_{\mathcal{B}} = \mathbf{R}_{\mathcal{A}} \circ S_{\mathcal{D}} \quad \textit{and} \quad S_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{B}} \circ S_{\mathcal{D}}^{-1}.$$

3. Gushel–Mukai threefolds and their derived categories

Let X be a prime Fano threefold of index one and degree $H^3 = 10$. Then X is either a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5)$, in which case it is called an ordinary Gushel–Mukai (GM) threefold, or X is a double cover of a degree 5 and index 2 Fano threefold Y_5 ramified in a quadric hypersurface, in which case it is called a special GM threefold. In the latter case, it has a natural involution $\tau : X \to X$ induced by the double cover $\pi : X \to Y_5$. There exists a unique stable vector bundle $\mathcal{E}$ of rank 2 with $c_1(\mathcal{E}) = -H$ and $c_2(\mathcal{E}) = 4L$, where L is the class of a line on X. In addition, $\mathcal{E}$ is exceptional and $H^\bullet(X, \mathcal{E}) = 0$. In fact, $\mathcal{E}$ is the restriction of the tautological bundle on the Grassmannian $\mathrm{Gr}(2,5)$. Furthermore, there is a standard short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to \mathcal{Q} \to 0 \tag{1}$$

where $\mathcal{Q}$ is the restriction of the tautological quotient bundle on $\mathrm{Gr}(2,5)$ to X.

Definition 3.1. Let X be a GM threefold.

- The *Kuznetsov component* of X is defined as $\mathcal{K}u(X) := \langle \mathcal{E}, \mathcal{O}_X \rangle^\perp$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$;
- The *alternative Kuznetsov component* of X [KP18b, Proposition 2.3] is defined as $\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{E}^\vee \rangle^\perp$. In particular, it fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$.

Definition 3.2. The left adjoint to the inclusion $\mathcal{A}_X \hookrightarrow \mathrm{D}^b(X)$ is given by $\mathrm{pr} := \mathbf{L}_{\mathcal{O}_X} \mathbf{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$. We call this the *natural projection functor*.

The analogous natural projection functor can be defined for $\mathcal{K}u(X)$ also, but we do not state it since we will not use it in this paper.

3.1. Kuznetsov components.

Let $K_0(\mathcal{D})$ denote the Grothendieck group of a triangulated category $\mathcal{D}$. We have the bilinear Euler form/characteristic

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{ext}^i(E, F)$$

for $E, F \in K_0(\mathcal{D})$. An application of the Hirzebruch-Riemann-Roch formula gives a convenient way of calculating Euler characteristics for GM threefolds. We have [Kuz09a, p. 5] $\chi(u, v) = \chi_0(u^* \cap v)$ where $u \mapsto u^*$ is an involution of $\oplus_{i=0}^3 H^i(X, \mathbb{Q})$ given by $(-1)^i$-multiplication of $H^{2i}(X, \mathbb{Q})$, and χ_0 is given by

$$\chi_0(x + yH + zL + wP) = x + \frac{17}{6} y + \frac{1}{2} z + w.$$

The *numerical Grothendieck group* of $\mathcal{D}$ is $\mathcal{N}(\mathcal{D}) = K_0(\mathcal{D}) / \ker \chi$.

Lemma 3.3 ([Kuz09a, p. 5]). *The numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X))$ of the Kuznetsov component is a rank 2 integral lattice generated by the basis elements $v = 1 - 3L + \frac{1}{2}P$ and $w = H - 6L + \frac{1}{6}P$. Computing χ on the basis vectors gives the matrix*

$$\begin{pmatrix} -2 & -3 \\ -3 & -5 \end{pmatrix}.$$

3.2. **Alternative Kuznetsov components.** One can also show that the numerical Grothendieck group of the alternative Kuznetsov component is also a rank 2 lattice.

Lemma 3.4. *The numerical Grothendieck group of $\mathcal{A}_X$ is a rank 2 lattice with basis vectors $x = 1 - 2L$ and $y = H - 4L - \frac{5}{6}P$, and the Euler form with respect to the basis is*

$$\begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix}.$$

Proof. The statement follows from [Kuz09a, Proposition 3.9], but we give another explicit computation. Note that $\mathrm{ch}(\mathcal{E}^\vee) = 2 + H + L - \frac{1}{3}P$ and $\mathrm{ch}(\mathcal{O}_X) = 1$. Also $\mathcal{N}(\mathrm{D}^b(X)) = \langle 1, H - 5L + \frac{5}{3}P, L + \frac{1}{2}P, P\rangle$. Thus $\mathcal{N}(\mathcal{A}_X) = \langle 1, 2 + H + L - \frac{1}{3}P\rangle^\perp$. Let $x, y \in \mathcal{N}(\mathcal{A}_X)$ be the basis elements. Then we have

$$\chi(\mathcal{E}^\vee, x) = 0,\ \chi(\mathcal{O}_X, x) = 0,$$
$$\chi(\mathcal{E}^\vee, y) = 0, \text{ and } \chi(\mathcal{O}_X, y) = 0.$$

Then $x = 1-2L$ and $y = H - 4L - \frac{5}{6}P$, thus $\mathcal{N}(\mathcal{A}_X) = \langle 1 - 2L, H - 4L - \frac{5}{6}p\rangle \cong \mathbb{Z}^2$. The Euler form follows from computing $\chi(x,x), \chi(y,x), \chi(x,y)$, and $\chi(y,y)$. ◻

Remark 3.5. It is straightforward to check that the (-1)-classes of $\mathcal{N}(\mathcal{A}_X)$ are $x = 1 - 2L$ and $2x - y = 2 - H + \frac{5}{6}P$ up to sign.

It is true that Kuznetsov components of Subsection 3.1 and alternative Kuznetsov components of this section are equivalent:

Lemma 3.6. *The original and alternative Kuznetsov components are equivalent, i.e. there is an equivalence of categories $\Xi : \mathcal{K}u(X) \xrightarrow{\sim} \mathcal{A}_X$ given by $E \mapsto \mathbf{L}_{\mathcal{O}_X}(E \otimes \mathcal{O}_X(H))$ in the forward direction, and $F \mapsto (\mathbf{R}_{\mathcal{O}_X} F) \otimes \mathcal{O}_X(-H)$ for the reverse direction.*

Proof. We manipulate the semiorthogonal decomposition as follows:

$$\begin{aligned}
\mathrm{D}^b(X) &= \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X\rangle \\
&\simeq \langle \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee, \mathcal{O}_X(H)\rangle \\
&\simeq \langle \mathcal{O}_X, \mathcal{K}u(X) \otimes \mathcal{O}_X(H), \mathcal{E}^\vee\rangle \\
&\simeq \langle \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H)), \mathcal{O}_X, \mathcal{E}^\vee\rangle.
\end{aligned}$$

Setting $\mathcal{A}_X := \mathbf{L}_{\mathcal{O}_X}(\mathcal{K}u(X) \otimes \mathcal{O}_X(H))$, the desired result follows. The reverse direction is similar. ◻

4. Bridgeland stability conditions

In this section, we recall (weak) Bridgeland stability conditions on $\mathrm{D}^b(X)$, and the notions of tilt stability, double-tilt stability, and stability conditions induced on Kuznetsov components from weak stability conditions on $\mathrm{D}^b(X)$. We follow [BLMS17, § 2].

4.1. **Weak stability conditions.** Let $\mathcal{D}$ be a triangulated category, and $K_0(\mathcal{D})$ its Grothendieck group. Fix a surjective morphism $v : K_0(\mathcal{D}) \to \Lambda$ to a finite rank lattice.

Definition 4.1. The *heart of a bounded t-structure* on $\mathcal{D}$ is an abelian subcategory $\mathcal{A} \subset \mathcal{D}$ such that the following conditions are satisfied:

(1) for any $E, F \in \mathcal{A}$ and $n < 0$, we have $\operatorname{Hom}(E, F[n]) = 0$;
(2) for any object $E \in \mathcal{D}$ there exist objects $E_i \in \mathcal{A}$ and maps

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

such that $\operatorname{cone}(\phi_i) = A_i[k_i]$ where $A_i \in \mathcal{A}$ and the k_i are integers such that $k_1 > k_2 > \cdots > k_m$.

Definition 4.2. Let $\mathcal{A}$ be an abelian category and $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a group homomorphism such that for any $E \in \mathcal{D}$ we have $\operatorname{Im} Z(E) \geq 0$ and if $\operatorname{Im} Z(E) = 0$ then $\operatorname{Re} Z(E) \leq 0$. Then we call Z a *weak stability function* on $\mathcal{A}$. If furthermore we have for any $0 \neq E$ that $\operatorname{Im} Z(E) \geq 0$, and $\operatorname{Im} Z(E) = 0$ implies that $\operatorname{Re} Z(E) < 0$, then we call Z a *stability function* on $\mathcal{A}$.

Definition 4.3. A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : \Lambda \to \mathbb{C}$ is a group homomorphism such that

(1) the composition $Z \circ v : K_0(\mathcal{A}) \cong K_0(\mathcal{D}) \to \mathbb{C}$ is a weak stability function on $\mathcal{A}$. From now on, we write $Z(E)$ rather than $Z(v(E))$.

Much like the slope from classical μ-stability, we can define a *slope* μ_σ for σ using Z. For any $E \in \mathcal{A}$, set

$$\mu_\sigma(E) := \begin{cases} -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)}, & \operatorname{Im} Z(E) > 0 \\ +\infty, & \text{else.} \end{cases}$$

We say an object $0 \neq E \in \mathcal{A}$ is σ-(semi)stable if $\mu_\sigma(F) < \mu_\sigma(E)$ (respectively $\mu_\sigma(F) \leq \mu_\sigma(E)$) for all proper subobjects $F \subset E$.

(2) Any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of σ-semistability defined above;
(3) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$ for all σ-semistable objects $E \in \mathcal{A}$. This is known as the *support property*.

If the composition $Z \circ v$ is a stability function, then σ is a *stability condition* on $\mathcal{D}$.

For this paper, we let Λ be the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ which is $K_0(\mathcal{D})$ modulo the kernel of the Euler form $\chi(E, F) = \sum_i (-1)^i \operatorname{ext}^i(E, F)$.

4.2. **Tilt stability.** Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on a triangulated category $\mathcal{D}$. Now consider the following subcategories[1] of $\mathcal{A}$:

$$\mathcal{T}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle$$
$$\mathcal{F}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leq \mu \rangle.$$

Then it is a result of [HRS96] that

[1]The angle brackets here mean extension closure.

Proposition 4.4. *The abelian category $\mathcal{A}^\mu_\sigma := \langle \mathcal{T}^\mu_\sigma, \mathcal{F}^\mu_\sigma[1]\rangle$ is the heart of a bounded t-structure on $\mathcal{D}$.*

We call $\mathcal{A}^\mu_\sigma$ the *tilt* of $\mathcal{A}$ around the torsion pair $(\mathcal{T}^\mu_\sigma, \mathcal{F}^\mu_\sigma)$. Let X be an n-dimensional smooth projective complex variety. Tilting can be applied to the heart $\mathrm{Coh}(X) \subset \mathrm{D}^b(X)$ to form the once-tilted heart $\mathrm{Coh}^\beta(X)$. Define for $E \in \mathrm{Coh}^\beta(X)$

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^n \mathrm{ch}_0^\beta(E) - H^{n-2}\mathrm{ch}_2^\beta(E) + \mathfrak{i} H^{n-1}\mathrm{ch}_1^\beta(E).$$

Proposition 4.5 ([BMT11, BMS16])**.** *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$. The quadratic form Q is given by the discriminant*

$$\Delta_H(E) = (H^{n-1}\mathrm{ch}_1(E))^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E).$$

The stability conditions $\sigma_{\alpha,\beta}$ vary continuously as $(\alpha,\beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies. Furthermore, for any $v \in \Lambda^2_H$ there is a locally-finite wall-and-chamber structure on $\mathbb{R}_{>0} \times \mathbb{R}$ which controls stability of objects with class v.

We now state a useful lemma which relates μ-stability and tilt stability.

Lemma 4.6 ([BMS16, Lemma 2.7])**.** *Let $E \in \mathrm{Coh}(X)$ be a μ-stable sheaf torsion-free sheaf.*

(1) If $H^2\mathrm{ch}_1^\beta(E) > 0$ then $E \in \mathrm{Coh}^\beta(X)$ and E is $\sigma_{\alpha,\beta}$-stable for $\alpha \gg 0$;
(2) If $H^2\mathrm{ch}_1^\beta(E) \leq 0$, then $E[1] \in \mathrm{Coh}^\beta(X)$. Moreover, if E is a vector bundle, then $E[1]$ is $\sigma_{\alpha,\beta}$-stable for $\alpha \gg 0$.

On the other hand, suppose $E \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable for $\alpha \gg 0$. Then one of the following conditions is satisfied:

(1) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a μ-semistable torsion-free sheaf,
(2) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a torsion sheaf,
(3) $\mathcal{H}^{-1}(E)$ is a μ-semistable sheaf, and $\mathcal{H}^0(E)$ is either 0 or supported in dimension ≤ 1.

4.3. **Stronger BG inequalities.** In this subsection, we state stronger Bogomolov-Gieseker (BG) style inequalities, which hold for tilt-semistable objects. These will be useful later on for ruling out potential walls for tilt-stability of objects in $\mathrm{D}^b(X)$. The first is a conjectured inequality from [BMS16, Conjecture 4.1] which was proved by Li in [Li16, Theorem 0.1] for Picard rank one Fano threefolds.

Lemma 4.7 (Stronger BG I)**.** *Let X be a smooth Fano threefold with Picard rank one, and E a $\sigma_{\alpha,\beta}$-stable object where $\alpha > 0$. Then*

$$\alpha^2\left((H^2\mathrm{ch}_1^\beta(E))^2 - 2H^3\mathrm{ch}_0^\beta(E)\mathrm{ch}_2^\beta(E)\right) + 4\left(H\mathrm{ch}_2^\beta(E)\right)^2 - 6H^2\mathrm{ch}_1^\beta(E)\mathrm{ch}_3^\beta(E) \geq 0.$$

The second is due to Naoki Koseki and Chunyi Li. It is based on [Kos20, Lemma 4.2, Theorem 4.3], however for our purposes we quote a reformulation for Fano threefolds from the upcoming paper [JZ21]. Chunyi Li also sent us a similar inequality from his upcoming paper [Li21].

Lemma 4.8 (Stronger BG II)**.** *Let X_{2g-2} be an index 1 Fano threefold of degree $d = 2g-2$, and $E \in \mathrm{Coh}^0(X)$ be a $\mu_{\alpha,0}$-semistable object for some $\alpha > 0$ with $\mu_H(E) \in [0,1]$ and $\mathrm{rk}(E) \geq 2$. Then*

$$\frac{H \cdot \mathrm{ch}_2(E)}{H^3 \cdot \mathrm{ch}_0(E)} \leq \max\left\{\frac{1}{2}\mu_H(E)^2 - \frac{3}{4d}, \mu_H(E)^2 - \frac{1}{2}\mu_H(E)\right\}.$$

4.4. **Double-tilting.** Now pick a weak stability condition $\sigma_{\alpha,\beta}$ and tilt the once-tilted heart $\mathrm{Coh}^{\beta}(X)$ with respect to the tilt slope $\mu_{\alpha,\beta}$ and some second tilt parameter μ. One gets a torsion pair $(\mathcal{T}^{\mu}_{\alpha,\beta}, \mathcal{F}^{\mu}_{\alpha,\beta})$ and another heart $\mathrm{Coh}^{\mu}_{\alpha,\beta}(X)$ of $\mathrm{D}^b(X)$. Now "rotate" the stability function $Z_{\alpha,\beta}$ by setting

$$Z^{\mu}_{\alpha,\beta} := \frac{1}{u} Z_{\alpha,\beta}$$

where $u \in \mathbb{C}$ such that $|u| = 1$ and $\mu = -\operatorname{Re} u / \operatorname{Im} u$. Then

Proposition 4.9 ([BLMS17, Proposition 2.15]). *The pair* $(\mathrm{Coh}^{\mu}_{\alpha,\beta}(X), Z^{\mu}_{\alpha,\beta})$ *defines a weak stability condition on* $\mathrm{D}^b(X)$*.*

4.5. **Stability conditions on the Kuznetsov component of a GM threefold.** Proposition 5.1 in [BLMS17] gives a criterion for checking when weak stability conditions on a triangulated category can be used to induce stability conditions on a subcategory. Each of the criteria of this proposition can be checked for $\mathcal{K}u(X) \subset \mathrm{D}^b(X)$ to give stability conditions on $\mathcal{K}u(X)$.

More precisely, let $\mathcal{A}(\alpha, \beta) = \mathrm{Coh}^{\mu}_{\alpha,\beta}(X) \cap \mathcal{K}u(X)$ and $Z(\alpha, \beta) = Z^{\mu}_{\alpha,\beta}|_{\mathcal{K}u(X)}$. Furthermore, let $0 < \epsilon < \frac{1}{10}$, $\beta = -1 + \epsilon$ and $0 < \alpha < \epsilon$. Also impose the following condition on the second tilt parameter μ:

$$\mu_{\alpha,\beta}(\mathcal{E}(-H)[1]) < \mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1]) < \mu < \mu_{\alpha,\beta}(\mathcal{E}) < \mu_{\alpha,\beta}(\mathcal{O}_X). \tag{2}$$

Then we get the following theorem.

Theorem 4.10 ([BLMS17, Theorem 6.9]). *Let $\epsilon, \alpha, \beta, \mu$ be as above. Then the pair $\sigma(\alpha, \beta) = (\mathcal{A}(\alpha, \beta), Z(\alpha, \beta))$ defines a Bridgeland stability condition on $\mathcal{K}u(X)$.*

4.6. **Stability conditions on the alternative Kuznetsov component of a GM threefold.** In this subsection, we show that stability conditions exist on $\mathcal{A}_X$ too, analogously to the previous subsection. Consider the following semiorthogonal decomposition:

$$\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^{\vee} \rangle.$$

Consult Section 3 for background on $\mathcal{A}_X$.

Theorem 4.11. *Let X be a GM threefold. Let $0 < \alpha < \epsilon$, $\beta = -\epsilon$ and $\epsilon > 0$ sufficiently small. Then there exists a stability condition $\sigma(\alpha, \beta)$ on $\mathcal{A}_X$.*

Sketch proof. It is sufficient to apply [BLMS17, Proposition 5.1] to the exceptional pair $(\mathcal{O}_X, \mathcal{E}^{\vee})$. Let $\beta = -\epsilon$. Then

$$\mu_H(\mathcal{O}_X) = 0 > \beta, \quad \mu_H(\mathcal{E}^{\vee}) = \frac{1}{2} > \beta,$$
$$\mu_H(\mathcal{O}_X(-H)) = -1 < \beta, \quad \mu_H(\mathcal{E}) = -\frac{1}{2} < \beta.$$

Thus $\mathcal{O}_X, \mathcal{E}^{\vee}, \mathcal{O}_X(-H)[1], \mathcal{E}[1] \in \mathrm{Coh}^{\beta}(X)$. By similar arguments to those in the proof of [BLMS17, Lemma 6.11], we show that $\mathcal{E}^{\vee}$ and $\mathcal{E}[1]$ are $\sigma_{\alpha,-\epsilon}$-stable for all $\alpha > 0$. It is not difficult to check that

$$\mu_{\alpha,\beta}(\mathcal{E}^{\vee}) > \mu_{\alpha,\beta}(\mathcal{O}_X) > 0 > \mu_{\alpha,\beta}(\mathcal{E}[1]) > \mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1]). \tag{3}$$

This implies that $\mathcal{O}_X, \mathcal{E}^{\vee}, \mathcal{E}[2], \mathcal{O}_X(-H)[2]$ are all contained in the second tilted heart $\mathrm{Coh}^{0}_{\alpha,\beta}(X)$ for $0 < \alpha < \epsilon$ and $0 < \epsilon < \frac{1}{10}$. Thus the weak stability condition $\sigma^{\mu}_{\alpha,\beta}$ gives a stability condition $\sigma(\alpha, \beta) = (\mathrm{Coh}^{0}_{\alpha,\beta}(X) \cap \mathcal{A}_X, Z^{0}_{\alpha,\beta}|_{\mathcal{A}_X})$ on $\mathcal{A}_X$. $\square$

4.7. **Parallelogram lemma.** In this subsection, we state a useful result for checking the stability of objects in $\mathcal{K}u(X)$ and $\mathcal{A}_X$.

Lemma 4.12. *Let $\sigma_{\alpha,\beta}$ be a tilt-stability condition on $\mathrm{D}^b(X)$, and $\sigma(\alpha,\beta)$ the stability condition on $\mathcal{K}u(X)$ induced by it. Let $E \in \mathcal{K}u(X)$ be an object such that there is a triangle*

$$A \to E \to B$$

where A and B are $\sigma_{\alpha,\beta}$-semistable objects in $\mathrm{D}^b(X)$. Furthermore, assume that $A, B \notin \mathcal{K}u(X)$. Consider the parallelogram in the complex plane defined by the vertices $0, Z_{\alpha,\beta}(A), Z_{\alpha,\beta}(B)$, and $Z_{\alpha,\beta}(E)$. Then the only potentially destabilising objects of E with respect to $\sigma(\alpha,\beta)$ will be integral linear combinations of the central charges of basis vectors of $\mathcal{N}(\mathcal{K}u(X))$ which lie in the interior of the parallelogram.

Remark 4.13. An analogue of Proposition 4.12 also holds for $\mathcal{A}_X$.

4.8. **Serre-invariant stability conditions on GM threefolds.**

Definition 4.14. Let σ be a stability condition on the alternative Kuznetsov component $\mathcal{A}_X$. It is called *Serre-invariant* if $S_{\mathcal{A}(X)} \cdot \sigma = \sigma \cdot g$ for some $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$

Remark 4.15. Since the Serre functor $S_{\mathcal{A}(X)} \cong \tau[2]$, the definition of Serre-invariance is equivalent to $\tau \cdot \sigma = \sigma \cdot g$, which we call *$\tau$-invariance*.

We recall several properties of Serre-invariant stability conditions from [PY20, Zha20] below:

Proposition 4.16. *Let σ be a Serre-invariant stability condition on $\mathcal{A}_X$. Then*

(1) the homological dimension of the heart of σ is 2,
(2) $\mathrm{ext}^1(A,A) \geq 2$ for every non-trivial object A in the heart of σ.

Proof. Denote by $\mathcal{A}'$ the heart of σ, and let $A, B \in \mathcal{A}'$. Then $\mathrm{Hom}(A, B[i]) = 0$ for $i < 0$. Note that the phase of the semistable factors of $\tau(A)$ is in the interval $(0,1)$, and the phase of the semistable factors of $B[i]$ is in $(i, i+1)$. Then $\mathrm{Hom}(A, B[i]) \cong \mathrm{Hom}(B[i], \tau(A)[2]) = 0$ if $i \geq 3$. This proves (1). For (2), note that $\chi(A,A) \geq -1$ for all non-zero objects $A \in \mathcal{A}_X$, so the result follows. ◻

To show an object in $\mathcal{A}_X$ is stable with respect to a Serre-invariant stability condition, we use the *Weak Mukai Lemma*, written below:

Lemma 4.17 (Weak Mukai Lemma)**.** *Let $A \to E \to B$ be an exact triangle in $\mathcal{A}_X$ with $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(A, \tau(B)) = 0$. Then*

$$\mathrm{hom}^1(A,A) + \mathrm{hom}^1(B,B) \leq \mathrm{hom}^1(E,E).$$

We recall a very recent result proved in [PR21].

Theorem 4.18. *Let X be an ordinary GM threefold and σ (or σ') a stability condition on $\mathcal{K}u(X)$ (or $\mathcal{A}_X$) defined by [BLMS17]. Then σ (or σ') is Serre-invariant.*

Proposition 4.19. *Let X be an ordinary GM threefold and E an object in $\mathcal{A}_X$ such that $\mathrm{ext}^1(E,E) = 2$ or 3 and $\chi(E,E) = -1$. Then E is σ-stable for each τ-invariant stability condition σ on $\mathcal{A}_X$.*

Proof. The proof is the same as in [Zha20, Lemma 9.12]. We omit the details. ◻

4.9. **Uniqueness of Serre-invariant stability conditions.** Let Y_d be smooth index two degree $d \geq 2$ prime Fano threefold and X_{4d+2} an index one degree $4d+2$ prime Fano threefold. In this section, we show that all Serre-invariant stability conditions on $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ (or $\mathcal{A}_{X_{4d+2}}$) are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit for each $d \geq 2$.

Lemma 4.20. *Let σ' be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ and $d \geq 2$. Then the heart of σ' has homological dimension at most 2.*

Proof. When $d = 2$, this follows from the same argument as in Proposition 4.16. When $d = 3$, this follows from [PY20, Lemma 5.10]. When $d = 4$ and 5, since $\mathcal{K}u(Y_4) \cong \mathrm{D}^b(C_2)$ and $\mathcal{K}u(Y_5) \cong \mathrm{D}^b(\mathrm{Rep}(K(3)))$ where C_2 is a genus 2 smooth curve and $\mathrm{Rep}(K(3))$ is the category of representations of the 3-Kronecker quiver ([KPS18a]), then in these two cases the heart has homological dimension 1. ◻

Lemma 4.21. *Let σ' be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ where $d \geq 2$. If E and F are two σ'-semistable objects with phases $\phi'(E) < \phi'(F)$, then* $\mathrm{Hom}(E, F[2]) = 0$.

Proof. When $d = 4$ and 5, this follows from the fact that the heart of σ' has homological dimension 1. When $d = 2$ and 3, this is by [PY20, Sec. 5, Sec. 6]. ◻

Lemma 4.22. (Weak Mukai Lemma) *Let σ' be a Serre-invariant stability condition on $\mathcal{K}u(Y_d), d \geq 2$. Let*

$$F \to E \to G$$

be an exact triangle in $\mathcal{K}u(Y_d)$ such that $\mathrm{Hom}(F, G) = \mathrm{Hom}(G, F[2]) = 0$. *Then we have*

$$\mathrm{ext}^1(F, F) + \mathrm{ext}^1(G, G) \leq \mathrm{ext}^1(E, E)$$

Lemma 4.23. *Let σ' be a Serre-invariant stability condition on $\mathcal{K}u(Y_d)$ and $d \geq 2$. Assume that there is a triangle of $E \in \mathcal{K}u(Y_d)$*

$$F \to E \to G$$

such that the phases of all σ'-semistable factors of F are greater than that of the σ'-semistable factors of G. Then we have $\mathrm{ext}^1(F, F) < \mathrm{ext}^1(E, E)$ *and* $\mathrm{ext}^1(G, G) < \mathrm{ext}^1(E, E)$

Proof. Since $\phi'(F) > \phi'(G)$, by Lemma 4.21 we have $\mathrm{Hom}(F, G) = 0$ and $\mathrm{Hom}(G, F[2]) = \mathrm{Hom}(F[2], S_{\mathcal{K}u(Y_d)}(E)) = 0$. Thus the result follows from Lemma 4.22. ◻

Let $\sigma = \sigma(\alpha, -\frac{1}{2})$ and $Y := Y_d$ where $d \geq 2$. As shown in [PY20, Section 4], the moduli spaces $\mathcal{M}_\sigma(\mathcal{K}u(Y), -v)$ and $\mathcal{M}_\sigma(\mathcal{K}u(Y), w - v)$ are non-empty. Let $A, B \in \mathcal{A}(\alpha, -\frac{1}{2})$ such that $[A] = -v, [B] = w - v$ are σ-stable objects. We denote the phase with respect to $\sigma = \sigma(\alpha, -\frac{1}{2})$ by $\phi(-) := \phi(\alpha, \beta)(-)$.

Now let σ_1 be any Serre-invariant stability condition on $\mathcal{K}u(Y)$. By [PY20, Remark 5.14], there is a $T = (t_{ij})_{1 \leq i,j \leq 2} \in \mathrm{GL}_2^+(\mathrm{R})$ such that $Z_1 = T \cdot Z(\alpha, -\frac{1}{2})$. Since A is stable with respect to every Serre-invariant stability condition by [PY20, Lemma 5.16], we can assume $A[m] \in \mathcal{A}_1$. Let $\sigma_2 = \sigma \cdot \tilde{g}$ for $\widetilde{g} := (g, T) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $A[m] \in \mathcal{A}_2$ and $Z_2 = Z_1$. Then we have $\phi_1(A) = \phi_2(A)$ and $\mathcal{A}_2 = \mathcal{P}(\alpha, -\frac{1}{2})((g(0), g(0) + 1])$.

Lemma 4.24. *Let the notation be as above. Then A and B are σ_1-stable with phase $\phi_1(A) = \phi_2(A)$ and $\phi_1(B) = \phi_2(B)$.*

Proof. The stability of A and B is from [PY20, Lemma 5.13]. By definition of σ_2, we know $\phi_1(A)=\phi_2(A)$ and $\phi_2(B)<\phi_2(A)<\phi_2(B)+1$. Also from [PY20, Remark 4.8] we know $\phi_1(B)<\phi_1(A)=\phi_2(A)<\phi_1(B)+1$. Thus $\phi_1(B)=\phi_2(B)$. ◻

Theorem 4.25. *All Serre-invariant stability conditions on $\mathcal{K}u(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit. Here $X:=X_{4d+2}$ or Y_d for all $d\geq 2$.*

Proof. Let the notation be as above. We are going to show $\sigma_1=\sigma_2$. Since $\mathcal{K}u(X_{12}),\mathcal{K}u(X_{16})$ and $\mathcal{K}u(X_{18})\simeq\mathcal{K}u(Y_4)$ are equivalent to the bounded derived categories of some smooth curves of positive genus, the results for these three cases follow from [Mac07, Theorem 2.7]. The results for X_{14} and X_{22} are from the results for $\mathcal{K}u(Y_3)$ and $\mathcal{K}u(Y_5)$ and the equivalences $\mathcal{K}u(Y_d)\simeq\mathcal{K}u(X_{4d+2})$, where $d\geq 3$ in [KPS18a]. Thus we only need to prove this for Y_d when $d\geq 2$ and $X:=X_{10}$.

We first prove this for Y_d when $d\geq 2$. Let $E\in\mathcal{A}(\alpha,-\frac{1}{2})$ be a σ-semistable object with $[E]=av+bw$. First we are going to show that if E is σ_1-semistable, then $\phi_2(E)=\phi_1(E)$. Note that we have following relations:

(1) $\chi(E,A)=a+(d-1)b$, $\chi(A,E)=a+b$; $\mu^0_{\alpha,-\frac{1}{2}}(E)>\mu^0_{\alpha,-\frac{1}{2}}(A)\iff b<0$

(2) $\chi(E,B)=-b$, $\chi(B,E)=-[(d-2)a+(d-1)b]$; $\mu^0_{\alpha,-\frac{1}{2}}(E)>\mu^0_{\alpha,-\frac{1}{2}}(B)\iff a+b<0$

From the definition of $\sigma=\sigma(\alpha,-\frac{1}{2})$-stability we have $a\leq 0$. When $a=0$, by definition of stability we have $b<0$. Thus in the case $b>0$ we always have $a<0$ and $\chi(D,E)<0$. Note that by definition of σ_2 we have $\phi_2(B)<\phi_2(A)$ and both of them lie in the interval $(g(0),g(0)+1]$.

- Assume that $b>0$ and $a+b>0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(E)<\mu^0_{\alpha,-\frac{1}{2}}(B)<\mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(E)<\phi_2(B)<\phi_2(A)$. We also have $\chi(E,A)>0$. Thus by Lemma 4.21 we know $\mathrm{Hom}(E,A[2])=0$. Thus $\chi(E,A)=\hom(E,A)-\hom(E,A[1])>0$ implies $\hom(E,A)>0$, and therefore $\phi_1(E)<\phi_1(A)$. Also from $\chi(B,E)<0$ and Lemma 4.20 we have $\phi_1(B)-1<\phi_1(E)$. Then we have $\phi_1(B)-1<\phi_1(E)<\phi_1(A)$. But by Lemma 4.24 we know $\phi_1(B)=\phi_2(B)$, $\phi_1(A)=\phi_2(A)$. Also, from the definition of σ_2 we have $|\phi_2(B)-\phi_2(A)|<1$ and $|\phi_2(A)-\phi_2(E)|<1$. Thus $\phi_2(E)-\phi_1(E)=0$ or 1. But if $\phi_2(E)=\phi_1(E)+1$, then $\phi_2(B)-1=\phi_1(B)-1<\phi_2(E)<\phi_1(B)=\phi_2(B)$. This implies $1=\phi_1(B)-\phi_1(B)+1>\phi_2(E)-\phi_1(B)+1=\phi_1(E)-\phi_1(B)+2$, which is impossible since $\phi_1(B)-1<\phi_1(E)$. Thus we have $\phi_1(E)=\phi_2(E)$.
- Assume that $b>0$ and $a+b<0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B)<\mu^0_{\alpha,-\frac{1}{2}}(E)<\mu^0_{\alpha,-\frac{1}{2}}(A)$ and hence $\phi_2(B)<\phi_2(E)<\phi_2(A)$. Since $\chi(A,E)<0$ and $\chi(E,B)<0$, from Lemma 4.20 we know $\hom(A,E[1])>0$ and $\hom(E,B[1])>0$, hence $\phi_1(A)-1<\phi_1(E)<\phi_1(B)+1$. This means $|\phi_1(E)-\phi_2(E)|=0$ or 1. But $|\phi_1(E)-\phi_2(E)|=1$ is impossible since $\phi_1(B)=\phi_2(B)<\phi_2(E)<\phi_2(A)=\phi_1(A)$. Therefore we have $\phi_1(E)=\phi_2(E)$.
- Assume that $b<0$. Then $\mu^0_{\alpha,-\frac{1}{2}}(B)<\mu^0_{\alpha,-\frac{1}{2}}(A)<\mu^0_{\alpha,-\frac{1}{2}}(E)$ and hence $\phi_2(B)<\phi_2(A)<\phi_2(E)$. Since $\chi(E,A)<0$, from Lemma 4.20 we have $\hom(E,A[1])>0$ and $\phi_1(E)<\phi_1(A)+1$. By Lemma 4.21, $\mu^0_{\alpha,-\frac{1}{2}}(B)<\mu^0_{\alpha,-\frac{1}{2}}(E)$ and $\chi(B,E)>0$ we know that $\hom(B,E)>0$. Thus $\phi_1(B)<\phi_1(E)<\phi_1(A)+1$. Hence $\phi_1(E)-\phi_2(E)=0$ or 1. But since $\mu^0_{\alpha,-\frac{1}{2}}(A)<$

$\mu^0_{\alpha,-\frac{1}{2}}(E)$, we have $\phi_2(A) = \phi_1(A) < \phi_2(E)$. Thus $\phi_1(A) < \phi_2(E) < \phi_1(A)+1$. Then $\phi_1(E) - \phi_2(E) = 1$ is impossible since $\phi_1(E) < \phi_1(A)+1$. Therefore we have $\phi_1(E) = \phi_2(E)$.

- When $b = 0$, we have $[E] = -a\cdot[A]$. Hence $\chi(E,A) = \chi(A,E) < 0$ and we have $\phi_1(A) - 1 \leq \phi_1(E) \leq \phi_1(A)+1$. But $\mu_1(E) = \mu_1(A)$, so we know $\phi_1(E) - \phi_1(A)$ is an integer. Thus $\phi_1(E) = \phi_1(A) \pm 1$. But from the definition of a stability function, we have $\mathrm{Im}(Z_1(E[\pm 1])) = -\mathrm{Im}(Z_1(A))$. Thus $\phi_1(E) = \phi_1(A) = \phi_2(E)$.
- When $a+b = 0$, we have $[E] = -a\cdot[B]$. Hence $\chi(E,B) = \chi(B,E) < 0$ and we have $\phi_1(B) - 1 \leq \phi_1(E) \leq \phi_1(B)+1$. But $\mu_1(E) = \mu_1(B)$, so we know $\phi_1(E) - \phi_1(B)$ is an integer. Thus $\phi_1(E) = \phi_1(B) \pm 1$. But from the definition of a stability function, we have $\mathrm{Im}(Z_1(E[\pm 1])) = -\mathrm{Im}(Z_1(B))$. Thus $\phi_1(E) = \phi_1(B) = \phi_2(E)$.

Next we show that $E \in \mathcal{A}_2$ is σ-semistable if and only if $E \in \mathcal{A}_1$ is σ_1-semistable. We prove this by induction.

If $\mathrm{ext}^1(E,E) < 2$, this is from [PY20, Sec. 5]. Now assume this is true for all $E \in \mathcal{A}_2$ σ_2-semistable such that $\mathrm{ext}^1(E,E) < N$.

When $E \in \mathcal{A}_2$ is σ_2-semistable and has $\mathrm{ext}^1(E,E) = N$, assume otherwise that E is not σ_1-semistable. Let A_0 be the first HN-factor of E with respect to σ_1 and A_n be the last one. Then $\phi_1(A_0) > \phi_1(A_n)$. By Lemma 4.23, $\mathrm{ext}^1(A_0,A_0) < N$ and $\mathrm{ext}^1(A_n,A_n) < N$. Thus A_0 and A_n are σ_2-semistable by the induction hypothesis and $\phi_2(A_0) > \phi_2(A_n)$ by the results above. Since $\mathrm{Hom}(A_0,E)$ and $\mathrm{Hom}(E,A_n)$ are both non-zero, we know that $\phi_2(A_0) \leq \phi_2(E)$ and $\phi_2(E) \leq \phi_2(A_n)$, which implies $\phi_2(A_0) \leq \phi_2(A_n)$ and gives a contradiction. Thus E is σ_1-semistable. When $E \in \mathcal{A}_1$ is σ_1-semistable, the same argument shows that $E \in \mathcal{A}_2$ is also σ_2-semistable.

Since every object in the heart is the extension of semistable objects, we have $\mathcal{A}_1 = \mathcal{A}_2$. And from $Z_1 = Z_2$, we in fact know that $\sigma_1 = \sigma_2 = \sigma\cdot\widetilde{g}$. Hence σ_1 is in the orbit of $\sigma = \sigma(\alpha, -\frac{1}{2})$.

Now we prove our statement for Serre-invariant stability conditions on the Kuznetsov component $\mathcal{A}_X$ of a GM threefold X. The argument is similar to the previous cases. Let C be a conic on X such that $I_C \notin \mathcal{A}_X$. If $E \in M_G(2,-1,5)$ is a locally free sheaf, then $E \in \mathcal{A}_X$ and by Lemma 4.26 we know $\hom(E, \mathrm{pr}(I_C)) > 0$ and $\hom(\mathrm{pr}(I_C), E[1]) > 0$. This means that $\phi_{\sigma'}(E) - 1 < \phi_{\sigma'}(\mathrm{pr}(I_C)) < \phi_{\sigma'}(E)$ for every Serre-invariant stability condition σ' on $\mathcal{A}_{X_{10}}$. We set $A = \mathrm{pr}(I_C)[1]$ and $B = E[1]$. Then $[A] = -x$, $[B] = y - 2x$ and $\phi_{\sigma'}(B) < \phi_{\sigma'}(A) < \phi_{\sigma'}(B) + 1$.

Let $\sigma := \sigma(\frac{1}{10}, -\frac{1}{10})$. Assume σ_1 is any Serre-invariant stability condition on $\mathcal{A}_{X_{10}}$. Then as in [PY20, Remark 5.14], there is a $T = (t_{ij})_{1\leq i,j\leq 2} \in \mathrm{GL}^+(2,\mathbb{R})$ such that $Z_1 = T\cdot Z(\frac{1}{10}, -\frac{1}{10})$. Since A is stable with respect to every Serre-invariant stability condition by [PY20, Lemma 5.16], we can assume $A[m] \in \mathcal{A}_1$. Let $\sigma_2 = \sigma\cdot\widetilde{g}$ for $\widetilde{g} := (g,T) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $A[m] \in \mathcal{A}_2$ and $Z_2 = Z_1$. Then we have $\phi_1(A) = \phi_2(A)$ and $\mathcal{A}_2 = \mathcal{P}(\frac{1}{10}, -\frac{1}{10})((g(0), g(0)+1])$. Now the same argument in Lemma 4.24 shows that $\phi_1(B) = \phi_2(B)$. Let $E \in \mathcal{A}(\frac{1}{10}, -\frac{1}{10})$ be a σ-semistable object with $[E] = ax + by$. Then by definition of stability we have $a + b \leq 0$. We have following relations:

(1) $\chi(E,A) = a + 2b, \chi(A,E) = a + 2b$; $\mu^0_{\frac{1}{10},-\frac{1}{10}}(E) > \mu^0_{\frac{1}{10},-\frac{1}{10}}(A) \iff b < 0$
(2) $\chi(E,B) = -b, \chi(B,E) = -b$; $\mu^0_{\frac{1}{10},-\frac{1}{10}}(E) > \mu^0_{\frac{1}{10},-\frac{1}{10}}(B) \iff a + 2b < 0$

Let $a' := a + b$. Then $a' \leq 0$ and these relations are of the same form as the index 2 case if we use a' and b. Since the above lemmas are all true for $\mathcal{A}_X$, the same argument works in this case too. $\square$

Lemma 4.26. *Let $E \in M_G(2, -1, 5)$ be a vector bundle and C be a conic on a GM threefold X such that $I_C \notin \mathcal{A}_X$. Then we have $\hom(E, \mathrm{pr}(I_C)) > 0$ and $\hom(\mathrm{pr}(I_C), E[1]) > 0$.*

Proof. Since $E^\vee \in M_G(2, 1, 5)$, we have $E \in \mathcal{A}_X$. Then there exists a non-locally free sheaf $F \otimes \mathcal{O}_X(H) \in M_G(2, 1, 5)$ such that $E = \ker(\mathrm{ev})$, where $\mathrm{ev} : \mathcal{O}_X^{\oplus 4} \to F \otimes \mathcal{O}_X(H)$ is the evaluation map. Then $\mathrm{pr}(F \otimes \mathcal{O}_X(H)) = E[1]$.

Consider the short exact sequence

$$0 \to F \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}(-1)|_L \to 0$$

and take cohomology. Since $\mathcal{E}(-1)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)$ and $h^1(\mathcal{E} \otimes \mathcal{E}) = 0$, $h^2(\mathcal{E} \otimes \mathcal{E}) = \hom(\mathcal{E}^\vee, \mathcal{E}[2]) = \hom(\mathcal{E}, \mathcal{E}[1]) = 0$ and $h^3(\mathcal{E} \otimes \mathcal{E}) = \hom(\mathcal{E}^\vee, \mathcal{E}[3]) = \hom(\mathcal{E}, \mathcal{E}) = 1$, we obtain $h^0(F \otimes \mathcal{E}) = h^1(F \otimes \mathcal{E}) = 0$ and $h^2(F \otimes \mathcal{E}) = h^3(F \otimes \mathcal{E}) = 1$. Thus applying $\mathrm{Hom}(F, -)$ to the triangle $\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$, we obtain a long exact sequence

$$0 \to \mathrm{Hom}(F, \mathcal{E}[1]) \to \mathrm{Hom}(F, \mathrm{pr}(I_C)) \to \mathrm{Hom}(F, \mathcal{Q}^\vee) \to \cdots$$

By Serre duality we have $\hom(F, \mathcal{E}[i]) = 0$ for $i \neq 0, 1$ and $\hom(F, \mathcal{E}[i]) = 1$ for $i = 0, 1$. Thus $\hom(F, \mathrm{pr}(I_C)) > 0$.

Similarly, consider the short exact sequence

$$0 \to F \otimes \mathcal{Q} \to \mathcal{E} \otimes \mathcal{Q} \to \mathcal{Q}(-1)|_L \to 0$$

and take cohomology. As $\mathcal{Q}(-1)|_L \cong \mathcal{O}_L^{\oplus 2}(-1) \oplus \mathcal{O}_L$, $h^2(\mathcal{E} \otimes \mathcal{Q}) = 1$ and $h^i(\mathcal{E} \otimes \mathcal{Q}) = 0, i \neq 2$, we get $h^1(F \otimes \mathcal{Q}) = h^2(F \otimes \mathcal{Q}) = 1$ and $h^0(F \otimes \mathcal{Q}) = h^3(F \otimes \mathcal{Q}) = 0$. Then by Serre duality we get $\hom^1(F(1), \mathcal{Q}^\vee) = \hom^2(F(1), \mathcal{Q}^\vee) = 1$ and $\hom(F(1), \mathcal{Q}^\vee) = \hom^3(F(1), \mathcal{Q}^\vee) = 0$.

Consider the exact sequence

$$0 \to F(1) \otimes \mathcal{Q} \to \mathcal{E}^\vee \otimes \mathcal{Q} \to \mathcal{Q}|_L \to 0$$

and take cohomology. As $\mathcal{Q}|_L \cong \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(1)$, we get $h^2(F(1) \otimes \mathcal{Q}) = h^3(F(1) \otimes \mathcal{Q}) = 0$, $h^0(F(1) \otimes \mathcal{Q}) \geq 22$, and $h^1(F(1) \otimes \mathcal{Q}) \leq 2$. Then by Serre duality we get $\hom(F(2), \mathcal{Q}^\vee[3]) \geq 22$, $\hom(F(2), \mathcal{Q}^\vee[2]) \leq 2$, and $\hom(F(2), \mathcal{Q}^\vee[k]) = 0$ for $k \neq 2, 3$. Now consider the short exact sequence

$$0 \to F(1) \otimes \mathcal{E}^\vee \to \mathcal{E}^\vee \otimes \mathcal{E}^\vee \to \mathcal{E}^\vee|_L \to 0$$

and take cohomology. We obtain $h^0(F(1) \otimes \mathcal{E}^\vee) \geq 20$ and $h^1(F(1) \otimes \mathcal{E}^\vee) \leq 3$. Thus by Serre duality again, we get $\hom(F(2), \mathcal{E}[3]) \geq 20$, $\hom(F(2), \mathcal{E}[2]) \leq 3$, and $\hom(F(2), \mathcal{E}[k]) = 0$ for $k \neq 2, 3$. Therefore by applying $\mathrm{Hom}(F(2), -)$ to the triangle $\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$, we obtain $\hom(F(2), \mathrm{pr}(I_C)[3]) = \hom(F(2), \mathcal{Q}^\vee[3]) \geq 22$. By Serre duality we have $\hom(\mathrm{pr}(I_C), F(1)) \geq 22$.

Finally, consider the short exact sequence

$$0 \to F \otimes \mathcal{E}^\vee \to \mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{E}^\vee(-1)|_L \to 0$$

and take cohomology. As $\mathcal{E}^\vee(-1)|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$, $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$, and $h^i(\mathcal{E} \otimes \mathcal{E}^\vee) = 0$ for $i \neq 0$, we obtain $h^i(F \otimes \mathcal{E}^\vee) = 0$ for all i. Thus by Serre duality we have $\hom(F(1), \mathcal{E}[i]) = 0$ for all i. Then we apply $\mathrm{Hom}(F(1), -)$ to the triangle $\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$ and we obtain $\hom(F(1), \mathrm{pr}(I_C)[2]) = \hom(F(1), \mathrm{pr}(I_C)[1]) =$

1. By Serre duality we get $\hom(F(1), \mathrm{pr}(I_C)[2]) = \hom(\mathrm{pr}(I_C), F[1]) = 1$ and $\hom(F(1), \mathrm{pr}(I_C)[1]) = \hom(\mathrm{pr}(I_C), F[2]) = 1$.

Thus by applying $\mathrm{Hom}(\mathrm{pr}(I_C), -)$ to the exact sequence $0 \to E(-1) \to \mathcal{O}_X^{\oplus 4}(-1) \to F \to 0$ and using Serre duality and the results above, we obtain $\hom(E, \mathrm{pr}(I_C)) = \hom(\mathrm{pr}(I_C), E(-1)[3]) = \hom(\mathrm{pr}(I_C), F[2]) = 1 > 0$.

Note that $\hom(\mathrm{pr}(I_C), \mathcal{O}_X) = 5, \hom(\mathrm{pr}(I_C), \mathcal{O}_X[1]) = 5$, and $\hom(\mathrm{pr}(I_C), \mathcal{O}_X[2]) = \hom(\mathrm{pr}(I_C), \mathcal{O}_X[3]) = 0$. Thus applying $\mathrm{Hom}(\mathrm{pr}(I_C), -)$ to the triangle $0 \to E \to \mathcal{O}_X^{\oplus 4} \to F(1) \to 0$ gives $\hom(\mathrm{pr}(I_C), E[1]) > 0$, since $\hom(\mathrm{pr}(I_C), \mathcal{O}_X^{\oplus 4}) = 20$ and $\hom(\mathrm{pr}(I_C), F(1)) \geq 22$. ◻

Remark 4.27. The idea of the proof of Theorem 4.25 was first explained by Arend Bayer to us. In [Zha20, Proposition 4.21], the author made an attempt to prove this statement but the argument is incomplete. Here, we fill the gaps and give a uniform argument for all $\mathcal{K}u(Y_d), \mathcal{K}u(X_{4d+2})$ when $d \geq 2$. In an upcoming paper [FP21, Theorem 3.1], the authors prove the uniqueness of Serre-invariant stability conditions for a general triangulated category satisfying a list of very natural assumptions which include Kuznetsov components of a series prime Fano threefolds.

5. Projection of $\mathcal{E}$ into $\mathcal{K}u(X)$

In this section, we consider the object that results from projecting the vector bundle $\mathcal{E}$ into $\mathcal{K}u(X)$, and its stability in $\mathcal{K}u(X)$.

5.1. The projection of $\mathcal{E}$ into $\mathcal{K}u(X)$.

The set-up is as follows. Let

$$\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E} \rangle = \langle \mathcal{O}_X \rangle^{\perp} \subset \mathrm{D}^b(X),$$

and let $\pi := i^! : \mathcal{D} \to \mathcal{K}u(X)$ be the right adjoint to the inclusion $i : \mathcal{K}u(X) \hookrightarrow \mathcal{D}$. Here $\mathcal{K}u(X)$ is the original Kuznetsov component, and v, w are the basis vectors of its numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X))$. We record here some useful Chern characters:

- $\mathrm{ch}(v) = (1, 0, -3L, \frac{1}{2}P) = (1, 0, -\frac{3}{10}H^2, \frac{1}{20}H^3)$
- $\mathrm{ch}(w) = (0, H, -6L, \frac{1}{6}P) = (0, H, -\frac{3}{5}H^2, \frac{1}{60}H^3$
- $\mathrm{ch}(\mathcal{E}) = (2, -H, L, \frac{1}{3}P) = (2, -H, \frac{1}{10}H^2, \frac{1}{30}H^3)$
- $\mathrm{ch}(\mathcal{Q}) = (3, H, -L, -\frac{1}{3}P) = (3, H, -\frac{1}{10}H^2, -\frac{1}{30}H^3)$
- $\mathrm{ch}(\mathcal{E}(-H)) = (2, -3H, 21L, -9P) = (2, -3H, \frac{21}{10}H^2, -\frac{9}{10}H^3)$
- $\mathrm{ch}(\mathcal{Q}(-H)) = (3, -2H, 4L, \frac{2}{3}P) = (3, -2H, \frac{2}{5}H^2, \frac{1}{15}H^3)$

Lemma 5.1. *The projection object $\pi(\mathcal{E})$ is given by $\mathbf{L}_{\mathcal{E}}\mathcal{Q}(-H)[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(\pi(\mathcal{E})) = \begin{cases} \mathcal{Q}(-H), & i = -1 \\ \mathcal{E}, & i = 0 \\ 0, & i \neq -1, 0. \end{cases}$$

Proof. Indeed, by e.g. [Kuz10, p. 4] we have the exact triangle

$$i\pi(\mathcal{E}) \to \mathcal{E} \to \mathbf{L}_{\mathcal{K}u(X)}\mathcal{E} \to .$$

But note that $\langle \mathcal{K}u(X), \mathcal{E} \rangle \simeq \langle S_{\mathcal{D}}(\mathcal{E}), \mathcal{K}u(X) \rangle \simeq \langle \mathbf{L}_{\mathcal{K}u(X)}\mathcal{E}, \mathcal{K}u(X) \rangle$. Therefore the triangle above becomes $i\pi(\mathcal{E}) \to \mathcal{E} \to S_{\mathcal{D}}(\mathcal{E})$. To find $S_{\mathcal{D}}(\mathcal{E})$ explicitly, note that

$S_{\mathcal{D}} \cong \mathbf{R}_{\mathcal{O}_X(-H)} \circ S_{\mathrm{D}^b(X)}$. Since $\mathbf{R}_{\mathcal{O}_X(-H)}\mathcal{E}(-H) \cong \mathcal{Q}(-H)[-1]$, we have $S_{\mathcal{D}}(\mathcal{E}) \cong \mathcal{Q}(-H)[2]$. So the triangle above becomes

$$i\pi(\mathcal{E}) \to \mathcal{E} \to \mathcal{Q}(-H)[2].$$

Applying $i^* = \mathbf{L}_{\mathcal{E}}$ to the triangle and using the fact that $i^*i \cong \mathrm{id}$ and $i^*\mathcal{E} = 0$ gives $\pi(\mathcal{E}) \cong \mathbf{L}_{\mathcal{E}}\mathcal{Q}(-H)[1]$, as required. Taking the long exact sequence with respect to $\mathcal{H}_{\mathrm{Coh}(X)}$ gives the cohomology objects. □

Remark 5.2. The object $\pi(\mathcal{E})$ lies in the exact triangle

$$\mathcal{Q}(-H)[1] \to \pi(\mathcal{E}) \to \mathcal{E}. \tag{4}$$

5.2. **Stability of $\pi(\mathcal{E})$.**

Proposition 5.3. *The vector bundle $\mathcal{Q}$ is μ-stable.*

Proof. We apply Hoppe's criterion [Hop84] to prove the μ-stability of $\mathcal{Q}^\vee$, then the dual of $\mathcal{Q}^\vee$, which is $\mathcal{Q}$ is also μ-stable. Indeed, $\bigwedge^2 \mathcal{Q}^\vee \cong \mathcal{Q}(-H)$, then we only need to show that $H^0(\mathcal{Q}^\vee) = H^0(\mathcal{Q}(-H)) = 0$ since rank of $\mathcal{Q}^\vee$ is three. Note that $H^0(\mathcal{Q}(-H)) \cong H^1(\mathcal{E}(-H)) \cong H^2(\mathcal{E}^\vee) = 0$. On the other hand, the fact that $H^0(\mathcal{Q}^\vee) = 0$ follows from the long exact sequence associated to $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$. Then $\mathcal{Q}$ is μ-stable. □

Lemma 5.4. *The vector bundle $\mathcal{Q}(-H)$ is $\sigma_{\alpha,-1+\epsilon}$-stable for $0 < \epsilon \leq \frac{1}{10}$ and for all $\alpha > 0$.*

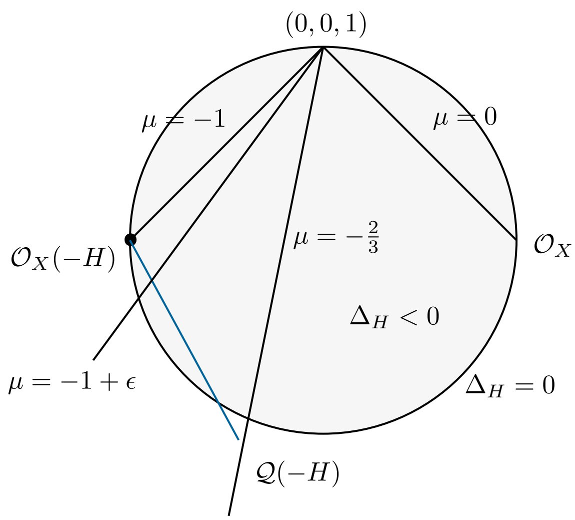

FIGURE 1.

Proof. The proof is the same as the one in [BLMS17, Lemma 6.11], but we write it for our case for the reader's convenience. Firstly, $\mathcal{Q}(-H)$ is μ-stable so $\mathcal{Q}(-H)$ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$ by [BLMS17, Proposition 2.13]. Now observe that

$$\operatorname{Im} Z_{\alpha,-1}(\mathcal{Q}(-H)) = H^2\mathrm{ch}_1^{-1}(\mathcal{Q}(-H)) = H^3.$$

Since $\operatorname{Im} Z_{\alpha,-1}(F) \in \mathbb{Z}_{\geq 0} \cdot H^3$ for all $F \in \operatorname{Coh}^{-1}(X)$, we cannot have $\sigma_{\alpha,-1}$-semistability of $\mathcal{Q}(-H)$ for any $\alpha > 0$. Then the wall and chamber structure of tilt-stability implies that $\mathcal{Q}(-H)$ is $\sigma_{\alpha,-1}$-stable for all $\alpha > 0$. The argument above shows that there are no walls in the interior of the triangle with vertices 0, $\mathcal{O}_X(-H)$, and $\mathcal{Q}(-H)$, so by local finiteness of walls, it is enough to show that the open line segment in the negative cone $\Delta_H < 0$ through $\mathcal{Q}(-H)$ and $\mathcal{O}_X(-H)$ is not a wall. Assume for a contradiction that there is a wall. Then there exists a destabilising short exact sequence

$$0 \to A \to \mathcal{Q}(-H) \to B \to 0.$$

When (α, β) lies on the wall the central charges $Z_{\alpha,\beta}(A)$ and $Z_{\alpha,\beta}(B)$ will be colinear in the open line segment through 0 and $Z_{\alpha,\beta}(\mathcal{Q}(-H))$ in the complex upper half plane. Continuity in the space of stability conditions implies that this must also hold at the point $(\alpha, \beta) = (0, -1)$. Then

$$H^3 = \operatorname{ch}_1^{-1}(\mathcal{Q}(-H)) = \operatorname{ch}_1^{-1}(A) + \operatorname{ch}_1^{-1}(B)$$

so by the fact that $\operatorname{Im} Z_{\alpha,-1}(F) \in \mathbb{Z}_{\geq 0} \cdot H^3$ for all $F \in \operatorname{Coh}^{-1}(X)$, we must have $\operatorname{ch}_1^{-1}(A) = 0$ or $\operatorname{ch}_1^{-1}(B) = 0$. Since they are on the same line segment, one of $Z_{0,-1}(A) = 0$ or $Z_{0,-1}(B) = 0$ holds. Say that $Z_{0,-1}(A) = 0$. Then $v_H^2(A)$ is proportional to $v_H^2(\mathcal{O}_X(-H))$ in the negative cone. Proposition 2.14 of [BLMS17] implies that $A \cong \mathcal{O}_X(-H)[1]$. But this is impossible, because $\operatorname{Hom}(\mathcal{O}_X(-H)[1], \mathcal{Q}(-H)) = \operatorname{Ext}^{-1}(\mathcal{O}_X, \mathcal{Q}) = 0$. Similarly, if $Z_{0,-1}(B) = 0$, then $B \cong \mathcal{O}_X(-H)[1]$. This is also impossible; applying $\operatorname{Hom}(\mathcal{O}_X, -)$ to the short exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 5} \to \mathcal{E}^\vee \to 0$ shows that $\operatorname{Hom}(\mathcal{Q}(-H), \mathcal{O}_X(-H)[1]) \cong \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{Q}^\vee) = 0$. Therefore, the wall cannot exist and $\mathcal{Q}(-H)$ is $\sigma_{\alpha,-1+\epsilon}$-stable for all $\alpha > 0$. ◻

Lemma 5.5. *We have $\pi(\mathcal{E}) \in \mathcal{A}(\alpha, -1+\epsilon)$ where $0 < \epsilon < \frac{1}{10}$ and for $0 < \alpha \ll 1$.*

Proof. Let ϵ be as above. By Lemma 5.4 $\mathcal{Q}(-H)$ is $\sigma_{\alpha,-1+\epsilon}$-stable for all $\alpha > 0$. Next, a computation shows that $\mu_{\alpha,-1+\epsilon}(\mathcal{Q}(-H)) < 0$ which implies that $\mathcal{Q}(-H)[1] \in \operatorname{Coh}^0_{\alpha,-1+\epsilon}(X)$ for $0 < \alpha \ll 1$. By [BLMS17] we also know that $\mathcal{E} \in \operatorname{Coh}^0_{\alpha,-1+\epsilon}(X)$ for $0 < \alpha \ll 1$. Therefore, because we have the triangle

$$\mathcal{Q}(-H)[1] \to \pi(\mathcal{E}) \to \mathcal{E},$$

we have $\pi(\mathcal{E}) \in \operatorname{Coh}^0_{\alpha,-1+\epsilon}(X)$ for $0 < \alpha \ll 1$. Finally, by construction $\pi(\mathcal{E}) \in \mathcal{K}u(X)$, so $\pi(\mathcal{E}) \in \mathcal{A}(\alpha, -1+\epsilon)$ for $0 < \alpha \ll 1$. ◻

Remark 5.6. The object $\pi(\mathcal{E})$ is *not* $\sigma^\mu_{\alpha,\beta}$-stable in the double-tilted heart $\operatorname{Coh}^\mu_{\alpha,\beta}(X)$, because it is destabilised by the object $\mathcal{Q}(-H)[1]$ in the triangle defining $\pi(\mathcal{E})$.

Since our projection object is in the heart of some stability condition on $\mathcal{K}u(X)$, we may now ask whether it is (semi)stable with respect to this stability condition. The answer is affirmative:

Lemma 5.7. *The object $\pi(\mathcal{E})$ is $\sigma(\alpha, -1+\epsilon)$-stable for $0 < \alpha \ll 1$.*

Proof. We use the "parallelogram method"; this is applied in a similar fashion in [Zha20, JZ21]. Consider the parallelogram in the upper half-plane whose non-origin vertices are given by the images under $Z_{\alpha,\beta}$ of each element in the triangle (4). Note that the second tilt parameter $\mu = 0$ corresponds to rotating the parallelogram in

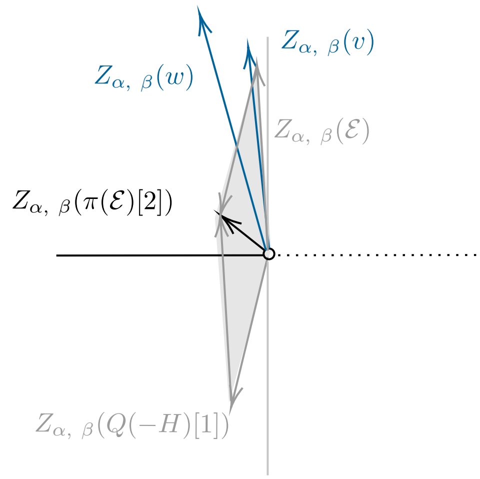

FIGURE 2. A cartoon depiction of the parallelogram associated to triangle (4) for $\alpha \sim 1/20$ and $\beta \sim -9/10$.

the figure by $\pi/2$ radians clockwise, which is what we require for the stability function $Z^0_{\alpha,\beta}|_{\mathcal{K}u(X)}$.

We now check that there are no integral linear combinations of $Z_{\alpha,\beta}(v)$ and $Z_{\alpha,\beta}(w)$ that lie in the interior of the parallelogram. Note the relation

$$Z_{\alpha,\beta}(\pi(\mathcal{E})) = Z_{\alpha,\beta}(\mathcal{E}) + Z_{\alpha,\beta}(\mathcal{Q}(-H)[1]). \tag{5}$$

For ease of notation, define $p_1 := Z_{\alpha,\beta}(\mathcal{E})$, $p_2 := Z_{\alpha,\beta}(\mathcal{Q}(-H)[1])$, and $p_3 := Z_{\alpha,\beta}(\pi(\mathcal{E}))$. The lines which define the parallelogram are

$$\begin{aligned}
L_1: &\quad y = \frac{\operatorname{Im} p_1}{\operatorname{Re} p_1} x \\
L_2: &\quad y = \frac{\operatorname{Im} p_2}{\operatorname{Re} p_2} x \\
L_3: &\quad y - \operatorname{Im} p_3 = \frac{\operatorname{Im} p_1 - \operatorname{Im} p_3}{\operatorname{Re} p_1 - \operatorname{Re} p_3}(x - \operatorname{Re} p_3) \\
L_4: &\quad y - \operatorname{Im} p_3 = \frac{\operatorname{Im} p_2 - \operatorname{Im} p_3}{\operatorname{Re} p_2 - \operatorname{Re} p_3}(x - \operatorname{Re} p_3)
\end{aligned}$$

where x is the real coordinate, and y is the imaginary coordinate. A linear integral combination of the basis vectors of $\mathcal{N}(\mathcal{K}u(X))$ may be given as $\mathcal{I}(m,n) := n \cdot Z_{\alpha,\beta}(v) + m \cdot Z_{\alpha,\beta}(w)$ where $m, n \in \mathbb{Z}$. Then asking for $\mathcal{I}(m,n)$ to lie in the interior of the parallelogram defined by the four equations above is equivalent to asking for the following two inequalities to be satisfied:

$$\begin{aligned}
&(\operatorname{Im}\mathcal{I}(m,n) - \operatorname{Im} p_3)\frac{\operatorname{Re} p_2 - \operatorname{Re} p_3}{\operatorname{Im} p_2 - \operatorname{Im} p_3} + \operatorname{Re} p_3 < \operatorname{Re}\mathcal{I}(m,n) < \frac{\operatorname{Re} p_1}{\operatorname{Im} p_1}\operatorname{Im}\mathcal{I}(m,n) \\
&\frac{\operatorname{Im} p_2}{\operatorname{Re} p_2}\operatorname{Re}\mathcal{I}(m,n) < \operatorname{Im}\mathcal{I}(m,n) < \frac{\operatorname{Im} p_1 - \operatorname{Im} p_3}{\operatorname{Re} p_1 - \operatorname{Re} p_3}(\operatorname{Re}\mathcal{I}(m,n) - \operatorname{Re} p_3) + \operatorname{Im} p_3.
\end{aligned}$$

It can be checked using e.g. *Wolfram Mathematica* that no integral solutions n, m exist. If the inequalities are made non-strict, then $n = -1$, $m = 1$ and $n = 0 = m$

are returned as solutions, as we would expect. Hence by Lemma 4.12, $\pi(\mathcal{E})$ is $\sigma(\alpha, -1+\epsilon)$-stable. ◻

5.3. **The analogous projection object for $\mathcal{A}_X$.** In this subsection, we state and prove the analogous results as in Subsection 5.1, except for $\mathcal{A}_X$ instead of $\mathcal{K}u(X)$. First note that

$$\begin{aligned} \mathrm{D}^b(X) &= \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle \\ &\simeq \langle \mathcal{A}_X, \mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee, \mathcal{O}_X \rangle \\ &\simeq \langle \mathcal{A}_X, \mathcal{Q}^\vee, \mathcal{O}_X \rangle. \end{aligned}$$

because $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee \cong \mathcal{Q}^\vee[1]$. Now let $\mathcal{D}' := \langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle$ and let $\pi' := (i')^! : \mathcal{D}' \to \mathcal{A}_X$ be the right adjoint to the inclusion $i' : \mathcal{A}_X \hookrightarrow \mathcal{D}'$.

Lemma 5.8. *The projection object $\pi'(\mathcal{Q}^\vee)$ is given by $\mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(\pi'(\mathcal{Q}^\vee)) = \begin{cases} \mathcal{E}, & i=-1 \\ \mathcal{Q}^\vee, & i=0 \\ 0, & i \neq -1, 0. \end{cases}$$

Proof. The proof is completely analogous to the proof of Lemma 5.1. As before, we have the exact triangle

$$i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathbf{L}_{\mathcal{A}_X}\mathcal{Q}^\vee \to .$$

But note that $\langle \mathcal{A}_X, \mathcal{Q}^\vee \rangle \simeq \langle S_{\mathcal{D}'}(\mathcal{Q}^\vee), \mathcal{A}_X \rangle \simeq \langle \mathbf{L}_{\mathcal{A}_X}\mathcal{Q}^\vee, \mathcal{A}_X \rangle$. Therefore the triangle above becomes $i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to S_{\mathcal{D}'}(\mathcal{Q}^\vee)$. To find $S_{\mathcal{D}'}(\mathcal{Q}^\vee)$ note that $S_{\mathcal{D}'}(\mathcal{Q}^\vee) = \mathbf{R}_{\mathcal{O}_X}(-H)(\mathcal{Q}^\vee(-H))[3]$. One can check that $\mathbf{R}_{\mathcal{O}_X}\mathcal{Q}^\vee = \mathcal{E}^\vee[-1]$, so $S_{\mathcal{D}'}(\mathcal{Q}^\vee) \cong \mathcal{E}[2]$. Hence our triangle becomes $i'\pi'(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathcal{E}[2]$. Now applying $(i')^* = \mathbf{L}_{\mathcal{Q}^\vee}$ to the triangle, we get $\pi'(\mathcal{Q}^\vee) = \mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}[1]$, as required. Since $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee, \mathcal{E}) = k[-2]$, we have the triangle $\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathbf{L}_{\mathcal{Q}^\vee}\mathcal{E}$. Taking the long exact sequence of this triangle with respect to $\mathcal{H}^*$ gives the required cohomology objects. ◻

Remark 5.9.

(1) It is not hard to check that $\Xi(\pi(\mathcal{E})) \cong \pi'(\mathcal{Q}^\vee)[1]$, where Ξ is the equivalence $\mathcal{K}u(X) \simeq \mathcal{A}_X$ from Lemma 3.6.
(2) Later in Section 7, we will see that we in fact have $\pi'(\mathcal{Q}^\vee) \cong \mathrm{pr}(I_C)[1]$ where $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ is the left projection, and $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$. In Section 7, we will also prove stability of $\mathrm{pr}(I_C)[1]$ which will complete the analogy between $\pi(\mathcal{E})$ and $\pi'(\mathcal{Q}^\vee)[1]$.

6. Hilbert schemes on ordinary GM threefolds

In this section, we collect some useful results regarding the birational geometry of ordinary GM threefolds and their Hilbert schemes of conics. The results in this section are all from [DIM12] and [Log12].

A conic means a subscheme $C \subset X$ with Hilbert polynomial $p_C(t) = 1 + 2t$, and a line means a subscheme $L \subset X$ with Hilbert polynomial $p_L(t) = 1 + t$. Denote their Hilbert schemes by $\mathcal{C}(X)$ and $\Gamma(X)$, respectively.

6.1. **Hilbert schemes of conics.** Let X be an ordinary GM threefold. Then recall that it is a quadric section of a linear section of codimension 2 of the Grassmannian $\mathrm{Gr}(2,5)=\mathrm{Gr}(2,V_5)$, where V_5 is a 5-dimensional complex vector space. Let V_i be an i-dimensional vector subspace of V_5. There are two types of 2-planes in $\mathrm{Gr}(2,5)$; *α-planes* are given set-theoretically as $\{[V_2] \mid V_1 \subset V_2 \subset V_4\}$, and *$\beta$-planes* are given by $\{[V_2] \mid V_2 \subset V_3\}$.

Definition 6.1 ([DIM12, p. 5])**.**

- A conic $C \subset X$ is called a *σ-conic* if the 2-plane $\langle C\rangle$ is an α-plane, and if there is a unique hyperplane $V_4 \subset V_5$ such that $C \subset \mathrm{Gr}(2,V_4)$ and the union of the corresponding lines in $\mathbb{P}(V_5)$ is a quadric cone in $\mathbb{P}(V_4)$.
- A conic $C \subset X$ is called a *ρ-conic* if the 2-plane $\langle C\rangle$ is a β-plane, and the union of corresponding lines in $\mathbb{P}(V_5)$ is this 2-plane.

It is a fact that there is a unique ρ-conic on X; denote it c_X. Logachev [Log12] shows that $S(X)$ is a smooth (Corollary 4.2 in *loc. cit.*) connected (page 41 in *loc. cit.*) surface. Furthermore, we have the following result which is a corollary of Logachev's Tangent Bundle Theorem (Section 4 in *loc. cit.*). Let $L_\sigma \subset S(X)$ be the curve of σ-conics as defined above.

Lemma 6.2 ([DIM12, p. 16])**.** *The only rational curve in $\mathcal{C}(X)$ is L_σ. Furthermore, there exists a minimal surface $\mathcal{C}_m(X)$ and a map $\mathcal{C}(X) \to \mathcal{C}_m(X)$ which contracts L_σ to a point.*

Another very useful result which we require for this text is Logachev's Reconstruction Theorem. This was originally proved in [Log12, Theorem 7.7], and then reproved later in [DIM12, Theorem 9.1].

Theorem 6.3 (Logachev's Reconstruction Theorem)**.** *Let X and X' be a general ordinary GM threefolds. If $\mathcal{C}(X) \cong \mathcal{C}(X')$, then $X \cong X'$.*

6.2. **Conic and line transformations.** For this section we follow [DIM12, § 6.1]. Let X be a general ordinary GM threefold, and let c be a smooth conic such that $c \neq c_X$. Let $\pi_c : \mathbb{P}^7 \to \mathbb{P}^4$ be the projection away from the 2-plane $\langle c\rangle$, and let $\epsilon : \widetilde{X} \to X$ be the blow-up of X in c with exceptional divisor E. Note that the composition $\pi_c \circ \epsilon$ is the morphism associated to the linear system $|-K_{\widetilde{X}}|$. This morphism has a Stein factorisation

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\phi_{|-K_{\widetilde{X}}|}"] \arrow[dr, "\phi"'] & & \mathbb{P}^4 \\ & \overline{X} \arrow[ur] & \end{tikzcd}$$

Since the conditions in [Isk99, Theorem 1.4.15] are all satisfied, there exists a $(-E)$-*flop*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "\phi"'] & & \widetilde{X}_c \arrow[dl, "\phi_c"] \\ & \overline{X} & \end{tikzcd}$$

A study of the properties of $-K_{\widetilde{X}_c}$ shows that there is a contraction $\epsilon_{c'} : \widetilde{X}_c \to X_c$, where X_c is an ordinary GM threefold, and $\epsilon_{c'} : \widetilde{X}_c \to X_c$ is the blow-up of X_c in a smooth conic c' with exceptional divisor $E' = -2K_{\widetilde{X}_c} - f(E)$. In summary, there exists a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "f"] \arrow[dd, "\epsilon"'] \arrow[dr, "\phi"] & & \widetilde{X}_c \arrow[dl, "\phi_c"'] \arrow[dd, "\epsilon_{c'}"] \\
& \overline{X} & \\
X \arrow[rr, "\psi_c"] \arrow[ur, "\pi_c"] & & X_c \arrow[ul, "\pi_{c'}"']
\end{tikzcd}$$

where $\psi_c : X \to X_c$ is the *elementary transformation of X along the conic c*. Note that the elementary transformation of X_c along the conic c' is $\psi_c^{-1} : X_c \to X$.

Remark 6.4. A similar flopping procedure can be done to construct the *elementary transformation of X along the line L*, which we denote as $\psi_L : X \to X_L$ (see [DIM12, § 6.2]).

Another important result which we require for this text is the following.

Theorem 6.5 ([DIM12, Theorem 6.4])**.** *Let X be a general ordinary GM threefold, and let $c \subset X$ be any conic. Then $\mathcal{C}(X_c)$ is isomorphic to $\mathcal{C}_m(X)$ blown up at the point $[c] \in \mathcal{C}_m(X)$, where $\mathcal{C}_m(X)$ is the minimal surface of $\mathcal{C}(X)$.*

7. A refined categorical Torelli theorem for general ordinary GM threefolds

In this section, we construct the moduli space of σ-stable objects of the (-1)-class $-x$ in the alternative Kuznetsov component $\mathcal{A}_X$ of an ordinary GM threefold X. Then we prove a refined categorical Torelli theorem for $\mathcal{A}_X$. For the definition of $\mathcal{A}_X$, see Definition 3.1.

Proposition 7.1. *Let $C \subset X$ be a conic on an ordinary GM threefold X. Then $I_C \notin \mathcal{A}_X$ if and only if there is a resolution of I_C of the form*

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

In particular, such a family of conics is parametrized by $\mathbb{P}^1$.

Proof.

(1) First we show that if there is a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0,$$

then $I_C \notin \mathcal{A}_X$. Indeed, applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to this exact sequence we get

$$\begin{aligned}
0 \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{Q}^\vee) &\to \mathrm{Hom}(\mathcal{E}^\vee, I_C) \to \\
\mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{Q}^\vee) &\to \mathrm{Ext}^1(\mathcal{E}^\vee, I_C) \to \\
\mathrm{Ext}^2(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E}^\vee, \mathcal{Q}^\vee) &\to \mathrm{Ext}^2(\mathcal{E}^\vee, I_C) \to \\
\mathrm{Ext}^3(\mathcal{E}^\vee, \mathcal{E}) \to \mathrm{Ext}^3(\mathcal{E}^\vee, \mathcal{Q}^\vee) &\to \mathrm{Ext}^3(\mathcal{E}^\vee, I_C) \to 0
\end{aligned}$$

Note that $\mathrm{Hom}(\mathcal{E}^\vee, \mathcal{Q}^\vee) = 0$ by μ-stability of $\mathcal{E}^\vee$ and $\mathcal{Q}^\vee$. Further note that $\mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{E}) \cong \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$, so $\mathrm{Hom}(\mathcal{E}^\vee, I_C) = 0$. It also follows that $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C) \cong \mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{Q}^\vee)$ since $\mathrm{Ext}^2(\mathcal{E}^\vee, \mathcal{E}) \cong \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$. Also note that $\mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$ since $\mathrm{Ext}^3(\mathcal{E}^\vee, \mathcal{Q}^\vee) \cong \mathrm{Hom}(\mathcal{Q}^\vee, \mathcal{E}) = 0$ again by stability. Applying $\mathrm{Hom}(-, \mathcal{E})$ to tautological short exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^{\oplus 5} \to \mathcal{Q} \to 0,$$

we get $\mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{Q}^\vee) \cong \mathrm{Ext}^1(\mathcal{Q}, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E}) = k$. But $\chi(\mathcal{E}^\vee, I_C) = 0$, thus $\mathrm{Ext}^2(\mathcal{E}^\vee, I_C) = k$. Then $I_C \notin \mathcal{A}_X$.

(2) Now we show that if $I_C \notin \mathcal{A}_X$, then there is the short exact sequence above. Note that $\mathrm{Hom}(\mathcal{E}^\vee, I_C) = \mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$. Indeed, applying $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the standard short exact sequence

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

we get $\mathrm{Hom}(\mathcal{E}^\vee, I_C) \hookrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}) = 0$ and $\mathrm{Ext}^3(\mathcal{E}^\vee, I_C) = 0$ since $\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{E}|_C) = 0$ and $\mathrm{Ext}^3(\mathcal{O}_X, \mathcal{E}) = 0$. But note that $\chi(\mathcal{E}^\vee, I_C) = 0$; this implies that $I_C \notin \mathcal{A}_X$ if and only if $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C) \neq 0$ since $\mathrm{RHom}^\bullet(\mathcal{O}_X, I_C) = 0$. Now we claim that $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. Otherwise if $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) = 0$ then $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C) = 0$. Indeed, applying $\mathrm{Hom}(-, I_C)$ to the dual of tautological short exact sequence

$$0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^5 \to \mathcal{E}^\vee \to 0,$$

we get

$$0 \to \mathrm{Hom}(\mathcal{Q}^\vee, I_C) \to \mathrm{Ext}^1(\mathcal{E}^\vee, I_C) \to \mathrm{Ext}^1(\mathcal{O}_X, I_C)^{\oplus 5} = 0,$$

so $\mathrm{Ext}^1(\mathcal{E}^\vee, I_C) = 0$. Thus, if $I_C \notin \mathcal{A}_X$ then $\mathrm{Hom}(\mathcal{Q}^\vee, I_C) \neq 0$. Let $\pi : \mathcal{Q}^\vee \to I_C$ be a non-zero map. Then we claim that π is surjective. Indeed, its image is the ideal sheaf $I_{C'}$ of the zero locus C' of a section s of $\mathcal{Q}$ containing the conic C. But it is known that the zero locus of s is either two points or a conic (by a similar argument to [San14, Lemma 2.18]). Hence C' must be C and the image of π is just I_C, hence π is surjective. Then we have a short exact sequence

$$0 \to \ker \pi \to \mathcal{Q}^\vee \to I_C \to 0.$$

Note that $\ker \pi$ is a semistable torsion-free sheaf with the same Chern character as $\mathcal{E}$. Indeed, any subsheaf destabilizing $\mathcal{E}$ would destabilize $\mathcal{Q}^\vee$. Then by [BF11, Theorem 3.2(4)], it must be the vector bundle $\mathcal{E}$, so we finally get the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0$$

and the proof is complete.

In particular, such conics are parametrized by $\mathbb{P}(\mathrm{Hom}(\mathcal{E}, \mathcal{Q}^\vee)) \cong \mathbb{P}^1$, by an application of the Borel–Bott–Weil theorem. ◻

Proposition 7.2. *Let X be a smooth ordinary GM threefold and $C \subset X$ a conic on X. If $I_C \notin \mathcal{A}_X$, then we have the exact triangle:*

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$$

where $\mathcal{Q}$ is the tautological quotient bundle.

Proof. By Proposition 7.1, I_C fits into the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Applying the projection functor to this short exact sequence, we get a triangle

$$\mathrm{pr}(\mathcal{E}) \to \mathrm{pr}(\mathcal{Q}^\vee) \to \mathrm{pr}(I_C),$$

where $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}$. Note that $\mathrm{pr}(\mathcal{Q}^\vee) = 0$. Indeed, $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{Q}^\vee) \cong k[-1]$. This is because $\mathrm{Hom}(\mathcal{E}^\vee, \mathcal{Q}^\vee) = \mathrm{Ext}^3(\mathcal{E}^\vee, \mathcal{Q}^\vee) = 0$ by stability of the vector bundles $\mathcal{E}^\vee$ and $\mathcal{Q}^\vee$ and comparison of their slopes. Also note that $\mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{Q}^\vee) \cong \mathrm{Ext}^1(\mathcal{Q}, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{Q}, \mathcal{Q}) = k$ and $\mathrm{Ext}^2(\mathcal{Q}, \mathcal{E}) \cong \mathrm{Ext}^1(\mathcal{Q}, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is an exceptional bundle. Then we have the triangle

$$\mathcal{E}^\vee[-1] \to \mathcal{Q}^\vee \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{Q}^\vee.$$

Applying $\mathbf{L}_{\mathcal{O}_X}$ to this triangle we get

$$\mathcal{Q}^\vee \to \mathcal{Q}^\vee \to \mathrm{pr}(\mathcal{Q}^\vee),$$

so $\mathrm{pr}(\mathcal{Q}^\vee) = 0$. Then $\mathrm{pr}(I_C) \cong \mathrm{pr}(\mathcal{E})[1]$. Now we compute the projection $\mathrm{pr}(\mathcal{E}) = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. We have the triangle

$$\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \otimes \mathcal{E}^\vee \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}.$$

Since $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}) \cong k[-3]$, we get $\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{E}$. Now apply $\mathbf{L}_{\mathcal{O}_X}$ to this triangle. We get $\mathbf{L}_{\mathcal{O}_X}\mathcal{E}^\vee[-3] \to \mathcal{E} \to \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\mathcal{E} = \mathrm{pr}(\mathcal{E})$, which is equivalently

$$\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathrm{pr}(\mathcal{E}).$$

Therefore we obtain the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(\mathcal{E})[1] \to \mathcal{Q}^\vee$$

and the desired result follows. ∎

By [KP18b, Proposition 2.6], there is a natural involutive autoequivalence functor of $\mathcal{A}_X$, called an *involution* and denoted by $\tau_{\mathcal{A}}$. We describe the action of $\tau_{\mathcal{A}}$ on the projection into $\mathcal{A}_X$ of an ideal sheaf of a conic $\mathrm{pr}(I_C)$.

Proposition 7.3.

(1) Let C be a conic on X such that $I_C \in \mathcal{A}_X$. Then $\tau_{\mathcal{A}}(I_C)$ is either
- *$I_{C'}$ such that $C \cup C' = Z(s)$ for $s \in H^0(\mathcal{E}^\vee)$, where $Z(s)$ is the zero locus of the section s;*
- *or D, where D is given by the triangle*

$$\mathcal{E}[1] \to D \to \mathcal{Q}^\vee.$$

(2) If $I_C \notin \mathcal{A}_X$, then $\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong I_{C''}$ for some conic $C'' \subset X$.

We first state two technical lemmas which we require for the proof of the proposition above.

Lemma 7.4. *The morphsim $\mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2}$ is injective and there is a short exact sequence*

$$0 \to \mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2} \to I_C \to 0$$

for some conic $C \subset X$.

Proof. The argument that follows is a variant of [San14, Corollary 2.19]. Note that $\mathcal{Q}(-H)$ is μ-stable and $\mathcal{E}^{\oplus 2}$ is μ-strictly semistable, and $\mu(\mathcal{Q}(-H)) = -\frac{2}{3}$ and $\mu(\mathcal{E}^{\oplus 2}) = -\frac{1}{2}$. Also note again that $\mathrm{Hom}(\mathcal{Q}(-H), \mathcal{E}) \neq 0$, so take any non-zero map $p \in \mathrm{Hom}(\mathcal{Q}(-H), \mathcal{E}^{\oplus 2})$. If p is not injective, then either its image destablizes $\mathcal{E}^{\oplus 2}$ or its kernel destablizes $\mathcal{Q}(-H)$. Indeed, the only case to take care of is when the rank of the kernel of p is one, which means the rank of the image of p is two. In this case $c_1(\operatorname{im} p) \geq -H$, so $\mu(\operatorname{im} p) \geq -\frac{1}{2}$. On the other hand, $\operatorname{im} p$ is subsheaf of $\mathcal{E}^{\oplus 2}$ and $\mathcal{E}^{\oplus 2}$ is strictly semistable, thus $\mu(\operatorname{im} p) \leq -\frac{1}{2}$ giving $\mu(\operatorname{im} p) = -\frac{1}{2}$. Hence $\operatorname{im} p \cong \mathcal{E}$ and we have a short exact sequence

$$0 \to \ker p \to \mathcal{Q}(-H) \to \mathcal{E} \to 0.$$

Then $\ker p$ is a torsion-free sheaf of rank one with the same Chern character as $I_C \otimes \mathcal{O}_X(-H)$. By a standard argument, this torsion-free sheaf is exactly $I_C \otimes \mathcal{O}_X(-H)$. But this is impossible; $\mathcal{Q}$ and $\mathcal{E}^\vee$ are both vector bundles and the map p is surjective, so its kernel is still a vector bundle but $I_C \otimes \mathcal{O}_X(-H)$ is not a vector bundle on X. This in turn means that p must be injective. We also note that $\operatorname{im}(\mathcal{E}^{\oplus 2} \to \mathcal{O}_X)$ is the ideal sheaf of the zero locus of two linearly independent sections of $\mathcal{E}^\vee$, which is a conic C since X does not contain any planes or quadrics. Then we have the short exact sequence

$$0 \to \mathcal{Q}(-H) \to \mathcal{E}^{\oplus 2} \to I_C \to 0$$

as required. ◻

Lemma 7.5. *Consider the semiorthogonal decomposition*

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle.$$

Let C be a conic on X. Then

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E}), & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1] \end{cases}$$

where C' is the involutive conic of C.

Proof. Consider the standard short exact sequence of $C \subset X$,

$$0 \to I_C \to O_X \to \mathcal{O}_C \to 0.$$

Apply $\operatorname{Hom}(\mathcal{E}, -)$ to it to get an exact sequence

$$0 \to \operatorname{Hom}(\mathcal{E}, I_C) \to H^0(\mathcal{E}^\vee) \to H^0(\mathcal{E}^\vee|_C) \to \operatorname{Ext}^1(\mathcal{E}, I_C) \to 0.$$

But $H^0(\mathcal{E}^\vee) = k^5$ and $H^0(\mathcal{E}^\vee|_C) = k^4$, so $\hom(\mathcal{E}, I_C) \geq 1$. We claim that $\hom(\mathcal{E}, I_C) \leq 2$. Assume $\hom(\mathcal{E}, I_C) \geq 3$. Then C is contained in the zero locus of at least three linearly independent sections of $\mathcal{E}^\vee$, which are points; this is impossible. Since $\chi(\mathcal{E}, I_C) = 1$, we have that $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either k or $k^2 \oplus k[-1]$. If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then we have the triangle

$$\mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Taking cohomology with respect to the standard heart we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \xrightarrow{s} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to 0.$$

The image of the map s is the ideal sheaf of an elliptic quartic D, thus we have following two short exact sequences: $0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to I_D \to 0$ and $0 \to I_D \to I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C) \to 0$. Then $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C))$ is a torsion-free sheaf of rank 1 with the same Chern character as $\mathcal{O}_X(-H)$. It is easy to show that it must be $\mathcal{O}_X(-H)$. On the other hand $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C))$ is supported on the residual curve of C in D. Then $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{O}_{C'}(-H)$. Thus we have the triangle

$$\mathcal{O}_X(-H)[1] \to \mathbf{L}_{\mathcal{E}}(I_C) \to \mathcal{O}_{C'}(-H)$$

and we observe that $\mathbf{L}_{\mathcal{E}}(I_C)$ is exactly the twisted derived dual of the ideal sheaf $I_{C'}$ of a conic $C' \subset X$, i.e, $\mathbf{L}_{\mathcal{E}}(I_C) \cong \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1]$.

If $\mathrm{RHom}(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then we have the triangle

$$\mathcal{E}^2 \oplus \mathcal{E}[-1] \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C)$$

Taking the long exact sequence in cohomology with respect to the standard heart, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E}^2 \xrightarrow{s'} I_C \to \mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \to \mathcal{E} \to 0.$$

We claim that s' is surjective. Indeed the image of s' is the ideal sheaf I_D of the zero locus D of two linearly independent sections of $\mathcal{E}^\vee$ containing C. Note that D is just the quadric section of a linear section of $\mathrm{Gr}(2, 3)$, which is exactly a conic since X does not contain any planes or quadrics. But $C \subset D$, hence $C = D$ and then s' is surjective. This in turn implies that $\mathcal{H}^0(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{E}$. Then by Lemma 7.4 the cohomology object $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{E}}(I_C)) \cong \mathcal{Q}(-H)$, which implies that $\mathbf{L}_{\mathcal{E}}(I_C) \cong \pi(\mathcal{E})$. ◻

Proof of Proposition 7.3. Since $\tau_{\mathcal{A}} \circ \tau_{\mathcal{A}} \cong \mathrm{id}$, we have $\tau_{\mathcal{A}} \cong \tau_{\mathcal{A}}^{-1}$. It is easy to see $\tau_A^{-1} \cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(- \otimes \mathcal{O}_X(H))[-1]$. Then

$$\begin{aligned} \tau_A(I_C) &\cong \mathbf{L}_{\mathcal{O}_X} \circ \mathbf{L}_{\mathcal{E}^\vee}(I_C \otimes \mathcal{O}_X(H))[-1] \\ &\cong \mathbf{L}_{\mathcal{O}_X}(\mathbf{L}_{\mathcal{E}}(I_C) \otimes \mathcal{O}_X(H))[-1]. \end{aligned}$$

The left mutation $\mathbf{L}_{\mathcal{E}}(I_C)$ is given by

$$\mathrm{RHom}^\bullet(\mathcal{E}, I_C) \otimes \mathcal{E} \to I_C \to \mathbf{L}_{\mathcal{E}}(I_C).$$

Note that $\mathrm{RHom}^\bullet(\mathcal{E}, I_C)$ is either k or $k^2 \oplus k[-1]$. Then by Lemma 7.5,

$$\mathbf{L}_{\mathcal{E}}(I_C) = \begin{cases} \mathbb{D}(I_{C'}) \otimes \mathcal{O}_X(-H)[1], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k \\ \pi(\mathcal{E})[2], & \mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1] \end{cases}$$

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))$. We have the triangle

$$\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \otimes \mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))$$

Note that $\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathbb{D}(I_{C'})) \cong \mathrm{RHom}^\bullet(I_{C'}, \mathcal{O}_X) = k \oplus k[-1]$. Then we have the triangle

$$\mathcal{O}_X \oplus \mathcal{O}_X[-1] \to \mathbb{D}(I_{C'}) \to \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})). \tag{6}$$

The derived dual $\mathbb{D}(I_{C'})$ is given by the triangle $\mathcal{O}_X \to \mathbb{D}(I_{C'}) \to \mathcal{O}_{C'}[-1]$. Then taking cohomology with respect to the standard heart of triangle (6) we have the long exact sequence

$$0 = \mathcal{H}^{-1}(\mathbb{D}(I_{C'})) \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_X$$

$$\to \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to \mathcal{O}_X \to \mathcal{O}_{C'} \to \mathcal{H}^1(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \to 0$$

Thus we have $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$, $\mathcal{H}^0(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) = 0$, and $\mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'}))) \cong I_{C'}[1]$. Hence $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\mathbb{D}(I_{C'})) \cong I_{C'}$.

If $\mathrm{RHom}^\bullet(\mathcal{E}, I_C) = k^2 \oplus k[-1]$, then $\tau_{\mathcal{A}}(I_C) \cong \mathbf{L}_{\mathcal{O}_X}(\pi(\mathcal{E})[2] \otimes \mathcal{O}_X(H)[-1]) \cong \Xi(\mathcal{E}[2])[-1]$. Then the desired result follows from Theorem 7.17. Since $\tau_{\mathcal{A}}^2 = \mathrm{id}$, (2) follows from (1).

◻

Lemma 7.6. *If $C \subset X$ is a conic such that $I_C \notin \mathcal{A}_X$, then*

$$\mathrm{RHom}^\bullet(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k \oplus k^2[-1].$$

Proof. Recall the exact triangle $\mathcal{Q}^\vee[-1] \to \mathcal{E}[1] \to \mathrm{pr}(I_C)$ from Proposition 7.2. By Proposition 7.3(2), $\tau_{\mathcal{A}}(\mathrm{pr}(I_C)) \cong I_{C''}$ for some conic $C'' \subset X$ and $I_{C''} \in \mathcal{A}_X$. Then $\mathrm{Hom}(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) \cong \mathrm{Hom}(I_{C''}, I_{C''}) = k$ since $\tau_{\mathcal{A}}$ is an autoequivalence of $\mathcal{A}_X$ and $I_{C''}$ is μ-stable. On the other hand, since the Fano surface of conics on a general ordinary GM threefold is a smooth irreducible surface, we have $\mathrm{Ext}^2(I_C, I_C) = 0$ and $\mathrm{Ext}^1(I_C, I_C) = k^2$ for each conic $C \subset X$. The desired result follows. ◻

Corollary 7.7. *The projection object*[2] $\mathrm{pr}(I_C)[1] \in \mathcal{A}_X$ *is* σ*-stable for all* τ*-invariant stability conditions* σ*.*

7.1. **Projections of ideal sheaves of conics and their stability.** In this subsection, we give an alternative proof using explicit methods that the projections of ideal sheaves of conics on an ordinary GM threefold X are $\sigma(\alpha, -\epsilon)$-stable in $\mathcal{A}_X$. Recall the projection functor $\mathrm{pr} = \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$.

Proposition 7.8. *Let* $C \subset X$ *be a conic on an ordinary GM threefold such that* $I_C \in \mathcal{A}_X$*. Then* $I_C[1] \in \mathcal{A}_X$ *is* $\sigma(\alpha, -\epsilon)$*-stable if* $0 < \alpha < \epsilon$ *and* ϵ *is sufficiently small.*

Proof. The sheaf I_C is μ-stable since it is torsion-free of rank one. Since $\mu_H(I_C) = 0 > \beta$, we have $I_C \in \mathrm{Coh}^\beta(X)$. Note that $\mathrm{ch}(I_C) = 1 - 2L$, thus $H^2\mathrm{ch}_1^\beta(I_C) = H^2\epsilon H = 10\epsilon > 0$. Thus I_C is $\sigma_{\alpha,\beta}$-stable for $\alpha \gg 0$ by Lemma 4.6. Further note that $\mathrm{Im}Z_{\alpha,0}(I_C) = H^2\mathrm{ch}_1^0(I_C) = 0$, hence I_C is $\mu_{\alpha,0}$-stable since it has maximal slope. Note $\mu_H(I_C) = \mu_H(\mathcal{O}_X) = 0$ and the discriminant $\Delta_H(I_C) = 40 > 0$. By similar argument in [BLMS17, Lemma 6.11], I_C is $\sigma_{\alpha,-\epsilon}$-semistable for sufficiently small $\epsilon > 0$ (any wall intersecting the line segment of slope $\mu = -\epsilon$ would intersect the line segment of slope $\mu = 0$). We also note that

$$\mu_{\alpha,\beta}(I_C) = \frac{5(\epsilon^2 - \alpha^2) - 2}{10\epsilon} < 0$$

for ϵ sufficiently small. Thus $I_C[1] \in \mathcal{A}(\alpha, \beta)$ and $I_C[1]$ is $\sigma(\alpha, \beta)$-stable since the class $-(1 - 2L)$ is a primitive (-1)-class in $\mathcal{N}(\mathcal{A}_X)$. ◻

We have already shown in Proposition 7.8 that $I_C[1]$ is $\sigma(\alpha, -\epsilon)$-stable in $\mathcal{A}_X$. Our aim in this section is to show that $\mathrm{pr}(I_C)[1]$ for conics C whose ideal sheaf $I_C \notin \mathcal{A}_X$ is also stable in $\mathcal{A}_X$. It is easy to see that $I_C \notin \mathcal{A}_X$ iff $\mathrm{h}^0(\mathcal{E}|_C) = 1$. Note that $\mathrm{pr}(I_C)[1]$ fits into the triangle

$$\mathcal{E}[2] \to \mathrm{pr}(I_C)[1] \to \mathcal{Q}^\vee[1]. \tag{7}$$

We use the same method as for showing stability of $\pi(\mathcal{E})$ (as described in Section 5).

Proposition 7.9. *The vector bundle* $\mathcal{Q}^\vee$ *is* μ*-stable.*

Proof. See the proof of Proposition 5.3. ◻

Proposition 7.10. *The object* $\mathcal{Q}^\vee[1]$ *is* $\sigma_{\alpha,-\epsilon}$*-stable for all* $\alpha > 0$ *and* $0 < \epsilon < \frac{1}{10}$*.*

Proof. This is similar to the proof of Lemma 5.4 (which closely follows [BLMS17]), but we write it for the convenience of the reader. We first show that $\mathcal{Q}^\vee[1]$ is $\sigma_{\alpha,0}$-stable, and we will then argue that it is $\sigma_{\alpha,-\epsilon}$-stable.

---

[2] We shift by [1] here to have the correct phase, i.e. so that $\mathrm{pr}(I_C)[1]$ is in the heart $\mathcal{A}'(\alpha, \beta)$ of $\mathcal{A}_X$.

Firstly, since $\mathcal{Q}^\vee$ is a μ-stable vector bundle and $\mathrm{ch}_1^\beta(\mathcal{Q}^\vee) = -1 - 3\beta$, [BMS16, Lemma 2.7] implies that $\mathcal{Q}^\vee[1]$ is $\sigma_{\alpha,0}$-stable for $\alpha \gg 0$. Next, $\mathrm{ch}_1^0(\mathcal{Q}^\vee[1]) = 1$, and since $\operatorname{Im} Z_{\alpha,0}(F) \in \mathbb{Z}_{\geq 0} \cdot H^3$ for all $F \in \mathrm{Coh}^0(X)$, $\mathcal{Q}^\vee[1]$ cannot be strictly $\sigma_{\alpha,0}$-semistable for any $\alpha > 0$. The wall and chamber structure of stability conditions thus implies that $\mathcal{Q}^\vee[1]$ is $\sigma_{\alpha,0}$-stable for all $\alpha > 0$. Now we turn to showing $\sigma_{\alpha,-\epsilon}$-stability of $\mathcal{Q}^\vee[1]$. The possible walls for $\sigma_{\alpha,\beta}$-stability are the open line segments in the negative cone $\Delta_H < 0$ passing through $v_H^2(\mathcal{Q}^\vee[1])$. By the argument above, there can be no walls in the *interior* of the triangle with vertices $v_H^2(\mathcal{O}_X)$, $v_H^2(\mathcal{Q}^\vee[1])$, and 0. So local finiteness of walls implies that all we must show is that there is no wall passing through $v_H^2(\mathcal{O}_X)$ and $v_H^2(\mathcal{Q}^\vee[1])$.

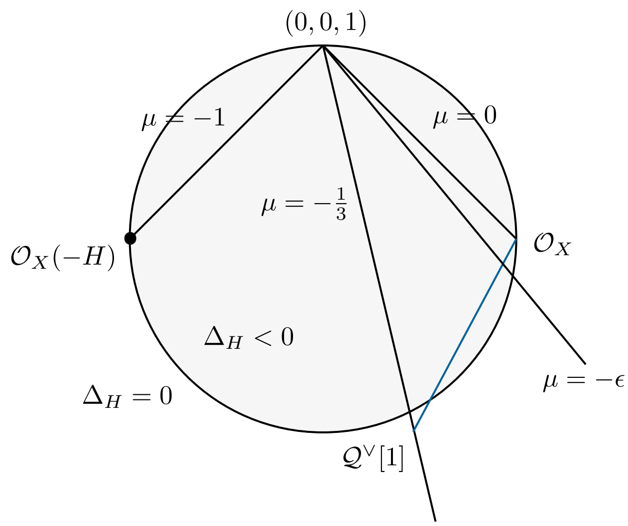

FIGURE 3.

Assume for a contradiction that there is. Then there is a destabilising short exact sequence

$$0 \to A \to \mathcal{Q}^\vee[1] \to B \to 0$$

such that $Z_{\alpha,\beta}(A)$ and $Z_{\alpha,\beta}(B)$ lie on the ray in the complex upper half plane passing through the origin and $Z_{\alpha,\beta}(\mathcal{Q}^\vee[1])$. Continuity of stability conditions implies that this will hold at the point $(\alpha,\beta) = (0,0)$. Then exactly the same integrality/central charge arguments as we saw before imply that either $A \cong \mathcal{O}_X$ or $B \cong \mathcal{O}_X$. But these are contradictions; $\mathrm{Hom}(\mathcal{O}_X, \mathcal{Q}^\vee[1]) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{Q}^\vee) = 0$ as we saw in the proof of Proposition 5.4, and $\mathrm{Hom}(\mathcal{Q}^\vee[1], \mathcal{O}_X) \cong \mathrm{Ext}^{-1}(\mathcal{Q}^\vee, \mathcal{O}_X) = 0$. So $\mathcal{Q}^\vee[1]$ is $\sigma_{\alpha,-\epsilon}$-stable for all $\alpha > 0$. ◻

Lemma 7.11. *When $I_C \notin \mathcal{A}_X$, we have $\mathrm{pr}'(I_C)[1] \in \mathcal{A}'(\alpha,-\epsilon)$ for $0 < \alpha < \epsilon$ and $0 < \epsilon < \frac{1}{10}$.*

Proof. This is similar to the proof of Lemma 5.5. First note by Proposition 7.10 that $\mathcal{Q}^\vee[1]$ is $\sigma_{\alpha,-\epsilon}$-stable for all $\alpha > 0$. We have $\mu_{\alpha,-\epsilon}(\mathcal{Q}^\vee[1]) > 0$, so $\mathcal{Q}^\vee[1] \in \mathcal{T}^0_{\alpha,-\epsilon} \subset \mathrm{Coh}^0_{\alpha,-\epsilon}(X)$. Furthermore, in the proof of Theorem 4.11 we have seen that $\mathcal{E}[2] \in \mathrm{Coh}^0_{\alpha,-\epsilon}(X)$. Since $\mathrm{pr}(I_C)[1]$ is an extension of $\mathcal{E}[2]$ and $\mathcal{Q}^\vee[1]$, we have

$\mathrm{pr}(I_C)[1] \in \mathrm{Coh}^0_{\alpha,-\epsilon}(X)$ for all $0 < \alpha < \epsilon$. Finally, $\mathrm{pr}(I_C)[1] \in \mathcal{A}_X$ by construction, so $\mathrm{pr}(I_C)[1] \in \mathcal{A}'_X(\alpha,-\epsilon)$ as required. □

Now that we know $\mathrm{pr}(I_C)[1]$ is in the heart. Then we can show it is stable in this heart:

Lemma 7.12. *If $I_C \notin \mathcal{A}_X$, the projection $\mathrm{pr}(I_C)[1]$ is $\sigma(\alpha,-\epsilon)$-stable in $\mathcal{A}_X$ for all $0 < \epsilon < \frac{1}{10}$ and $0 < \alpha < \frac{1}{10}$.*

Proof. We again use the parallelogram method (see Remark 4.13). Consider the diagram 7.1 which shows the interior of the parallelogram defined by having vertices central charges $Z_{\alpha,-\epsilon}$ of the elements of the triangle 7.

Note that the second tilt parameter of $\mu = 0$ which we have chosen to use for e.g. $\mathrm{Coh}^0_{\alpha,-\epsilon}(X)$ means rotating the diagram above $\pi/2$ radians clockwise, which lands the whole parallelogram in the complex upper half plane. This is a necessary condition for $Z^0_{\alpha,-\epsilon}|_{\mathcal{A}_X}$ to be a stability function.

Exactly the same argument as in the proof of Lemma 5.7 (except with $p_1 = Z_{\alpha,\beta}(\mathcal{Q}^\vee[1])$, $p_2 = Z_{\alpha,\beta}(\mathcal{E}[2])$, and $p_3 = Z_{\alpha,\beta}(\mathrm{pr}(I_C)[1])$ and v, w replaced by x, y, respectively) checked on *Mathematica* shows that there exist no integral linear combinations of $Z_{\alpha,-\epsilon}(x)$ and $Z_{\alpha,-\epsilon}(y)$ which lie in the interior of the parallelogram. Therefore, by Proposition 4.12 and Remark 4.13 $\mathrm{pr}(I_C)[1]$ is $\sigma(\alpha,-\epsilon)$-stable. □

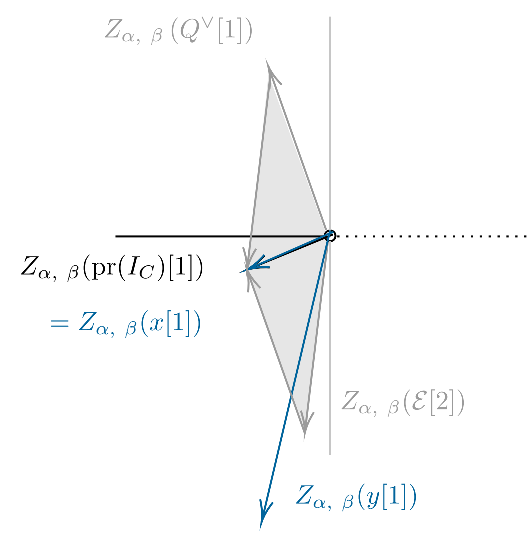

FIGURE 4. The parallelogram associated to triangle (7). The grey lines show its perimeter. A computation using elementary geometry shows that no linear combination of the basis vectors can lie inside the parallelogram.

Theorem 7.13. *Let X be a general smooth ordinary GM threefold. The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $p(\mathcal{C}(X)) = \mathcal{S} \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of σ-stable objects in $\mathcal{A}_X$, where $p : \mathcal{C}(X) \to \mathcal{S}$ is a blowing down morphism to a smooth point. In particular, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is isomorphic to the minimal model $\mathcal{C}_m(X)$ of Fano surface of conics on X.*

Proof. By Proposition 7.1, it is known that the family $\mathbb{L}$ of conics $C \subset X$ whose ideal sheaf $I_C \notin \mathcal{A}_X$ is parametrized by a $\mathbb{P}^1$. It is the fiber of the first quadric fibration over the scheme $\Sigma_1(X)$ in the language of [DK15]. In particular, $h^0(\mathcal{E}|_C) = k$ and $\mathrm{pr}(I_C) \in \mathcal{A}_X$ is given by the exact triangle in Proposition 7.2. The ideal sheaves I_C for all the conics $[C]$ in the complement of $\mathbb{L}$ in Fano surface $\mathcal{C}(X)$ of conics, are contained in $\mathcal{A}_X$. Then $\mathrm{pr}(I_C[1]) = I_C[1] \in \mathcal{A}_X$, they are σ-stable by Proposition 7.8, and $\mathrm{ext}^1(I_C, I_C) = 2$ and $\mathrm{ext}^2(I_C, I_C) = 0$ for all $[C] \in \mathcal{C}(X)$.

Let $\mathcal{C}$ be the universal family of conics on $X \times \mathcal{C}(X)$. The functor pr induces a morphism $p : \mathcal{C}(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ factoring through one of the irreducible components $\mathcal{S}$ of $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$. The complement of $\mathbb{L}$ in $\mathcal{C}(X)$ is a dense open subset $\mathcal{U}$ of $\mathcal{C}(X)$ since $\mathcal{C}(X)$ is irreducible. The morphism $p|_{\mathcal{U}}$ is injective and étale, so $p(\mathcal{U}) \subset \mathcal{S}$ is also a dense open subset of $\mathcal{S}$. But $\mathcal{C}(X)$ is a projective surface and p is a proper morphism, so $p(\mathcal{C}(X)) = \mathcal{S}$. By Lemma 7.6, $\mathcal{S}$ is smooth. Then p is a birational dominant proper morphism from $\mathcal{C}(X)$ to $\mathcal{S}$. In particular, $\mathbb{L} \cong \mathbb{P}^1$ is contracted by p to a smooth point by Lemma 7.6. Thus $\mathcal{S}$ is smooth surface obtained by blowing down a smooth rational curve on a smooth irreducible projective surface. This implies that $\mathcal{S}$ is also a smooth projective surface. Now from Proposition 14.5, we know $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$. On the other hand, it is known that there is a unique rational curve $L_\sigma \cong \mathbb{L} \subset \mathcal{C}(X)$ and it is the unique exceptional curve by Lemma 6.2. Thus $\mathcal{S}$ is the mininal model $\mathcal{C}_m(X)$ of Fano surface of conics on X. ∎

Remark 7.14. In Proposition 7.3, the categorical involution $\tau_{\mathcal{A}}$ induces the geometric involution τ on $\mathcal{C}_m(X)$. More precisely, if C is a τ-conic, then $\tau_{\mathcal{A}}(I_C) \cong I_{C'}$; if C is a ρ-conic, then $\tau_{\mathcal{A}}(I_C) \cong \pi$, and π represents the σ-conic.

Next we show that the pair $(\mathcal{K}u(X), \pi(\mathcal{E}))$ (see Section 5.1) can determine the isomorphism class of an ordinary GM threefold X.

Theorem 7.15. *A general ordinary GM threefold X can be reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ along with the additional data $\pi(\mathcal{E})$ (see Section 5) given by the projection of the rank 2 vector bundle $\mathcal{E}$ onto the Kuznetsov component.*

Proof. By Lemma 3.6, the equivalence $\Xi : \mathcal{K}u(X) \cong \mathcal{A}_X$ is given by $E \mapsto \mathbf{L}_{\mathcal{O}_X}(E \otimes \mathcal{O}_X(H))$. It is easy to see that $\Xi(\pi(\mathcal{E})) \in \mathcal{A}_X$ is given by the triangle

$$\mathcal{E}[2] \to \Xi(\pi(\mathcal{E})) \to Q^\vee[1]$$

which is exactly $\mathrm{pr}(I_C)[1]$, where $I_C \notin \mathcal{A}_X$. Thus, we get the data $(\mathcal{A}_X, \pi := \mathrm{pr}(I_C)[1])$. By Theorem 7.13, we get the minimal model $\mathcal{C}_m(X)$ of Fano surface of conics on X with the smooth point π, which is the image of the unique exceptional curve $L_\sigma \cong \mathbb{L}$ on $\mathcal{C}(X)$. Blowing up $\mathcal{C}_m(X)$ at this point, we recover $\mathcal{C}(X)$. Then by Theorem 6.3, we reconstruct X up to isomorphism. ∎

7.2. **The universal family for $\mathcal{C}_m(X)$.** In this section, we show that $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ admits a universal family, thus giving a fine moduli space. Let $\mathcal{I}$ be the universal ideal sheaf of conics on $X \times \mathcal{C}(X)$ and $\mathcal{I}_{\mathbb{L}}$ be the universal ideal sheaf of conics restricted to $X \times \mathbb{L}$. Let $q : X \times \mathcal{C}(X) \to X$ and $\pi : X \times \mathcal{C}(X) \to \mathcal{C}(X)$ be the projection maps on the first and second factors, respectively. Let $\mathcal{G}' := \mathrm{pr}(\mathcal{I}_{\mathbb{L}})$ be the projected family in $\mathcal{A}_{X \times \mathbb{L}}$. Let $t \in \mathbb{L} \cong \mathbb{P}^1$ be any point. Then $j_t^* \mathrm{pr}(\mathcal{I}_{\mathbb{L}}) \cong A$, where $j_t : X_t \to X_t \times \mathbb{L}$ and $A \in \mathcal{A}_X$ is a fixed object by Proposition 7.2. Then $\mathcal{G}' \cong q^*(A) \otimes \pi^* \mathcal{O}_{\mathbb{L}}(k)$ for some $k \in \mathbb{Z}$. Now let $\mathcal{G} := \mathrm{pr}(\mathcal{I}) \otimes \pi^* \mathcal{O}_{\mathcal{C}(X)}(kE)$, where $E \cong \mathbb{L} \cong \mathbb{P}^1$ is the unique exceptional curve on $\mathcal{C}(X)$.

Proposition 7.16. *$(p_X)_*\mathcal{G}$ is the universal family of $\mathcal{C}_m(X)$, where $p_X = \mathrm{id}_X \times p : X \times \mathcal{C}(X) \to X \times \mathcal{C}_m(X)$.*

Proof.

(1) If $s = [A] = \pi \in \mathcal{C}_m(X)$, s is contracted from the unique rational curve $\mathbb{L} \cong \mathbb{P}^1 \subset \mathcal{C}(X)$. Note that in this case $p_X|_{\mathbb{L}} = q$. Then

$$\begin{aligned} i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathcal{G}' \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong i_s^* q_*(q^*(A) \otimes \pi^*\mathcal{O}_{\mathbb{L}}(k) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong i_s^* q_*(q^*(A) \otimes (\pi^*\mathcal{O}_{\mathbb{L}}(k) \otimes \mathcal{O}_{\mathbb{L}}(kE))) \\ &\cong i_s^* q_*(q^*(A) \otimes \pi^*(\mathcal{O}_{\mathbb{L}}(k) \otimes \mathcal{O}_{\mathbb{L}}(-k))) \\ &\cong i_s^* q_*(q^*(A)) \\ &\cong i_s^*(A) \\ &\cong A \end{aligned}$$

(2) If $s = [I_C]$, then $\mathcal{C}_m(X)$ and $\mathcal{C}(X)$ are isomorphic outside $\mathbb{L}$. Note that p restricts to id on $\mathcal{C}(X) \smallsetminus \mathbb{L}$. Then

$$\begin{aligned} i_s^*(p_X)_*\mathcal{G} &\cong i_s^*(p_X)_*(\mathrm{pr}(\mathcal{I}) \otimes \pi^*\mathcal{O}_{\mathcal{C}(X)}(kE)) \\ &\cong j_s^*(\mathrm{pr}(\mathcal{I})) \otimes j_s^*\pi^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\ &\cong I_C \otimes (\pi \circ j_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\ &\cong I_C \otimes (i_s \circ \pi_s)^*\mathcal{O}_{\mathcal{C}(X)}(kE) \\ &\cong I_C. \end{aligned}$$

See below for commutative diagrams which summarise the maps in the proof:

$$\begin{tikzcd} X_s \arrow[r, "\cong"] \arrow[d, hook, "j_s"] & X_s \arrow[r, "\pi_s"] \arrow[d, hook, "i_s"] & \{s\} \arrow[d] \\ X \times \mathcal{C}(X) \arrow[r, "p_X"'] & X \times \mathcal{C}_m(X) \arrow[r] & \mathcal{C}_m(X) \end{tikzcd}$$

$$\begin{tikzcd} X_s \arrow[r, "j_s"] \arrow[d, "\pi_s"] & X \times \mathcal{C}(X) \arrow[d, "\pi"] \\ \{s\} \arrow[r, "i_s"'] & \mathcal{C}(X) \end{tikzcd}$$

$\square$

Now we prove a refined categorical Torelli theorem for ordinary GM threefolds.

Theorem 7.17. *Let X and X' be general ordinary GM threefolds such that $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is an equivalence and $\Phi(\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$. Then $X \cong X'$.*

Proof. Note that $\Xi(\pi(\mathcal{E})) = \mathrm{pr}(I_C)[1]$, where $I_C \notin \mathcal{A}_X$. Then the equivalence Φ induces an equivalence $\Psi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that $\Psi(\pi) = \pi'$, where $\pi := \mathrm{pr}(I_C)[1] \in \mathcal{A}_X$ and $\pi' := \mathrm{pr}(I_{C'})[1] \in \mathcal{A}_{X'}$.

$$\begin{tikzcd} \mathcal{K}u(X) \arrow[r, "\Phi"] \arrow[d, "\Xi"] & \mathcal{K}u(X') \arrow[d, "\Xi"] \\ \mathcal{A}_X \arrow[r, "\Psi"'] & \mathcal{A}_{X'} \end{tikzcd}$$

The existence of the universal family on $\mathcal{C}_m(X)$ guarantees a projective dominant morphism from $\mathcal{C}_m(X)$ to $\mathcal{C}_m(X')$, denoted by ψ, which is induced by Ψ (for more details on construction of the morphism ψ, see [APR19], [BT16] and [Zha20]). Since Ψ is an equivalence, ψ is bijective on closed points by Theorem 7.13 and Theorem 4.25. It also identifies tangent spaces of each point on $\mathcal{C}_m(X)$ and $\mathcal{C}_m(X')$, hence it is an isomorphism since both surfaces are smooth for general X and X'. On the other hand, we have $\psi(\pi) = \pi'$. Then ψ induces an isomorphism $\phi : \mathcal{C}(X) \cong \mathcal{C}(X')$ by blowing up $\pi \in \mathcal{C}_m(X)$ and $\pi' \in \mathcal{C}_m(X')$, respectively. Then we have $X \cong X'$ by Logachev's Reconstruction Theorem 6.3. ◻

8. A categorical Torelli theorem for generic special GM threefolds

In this section, we show that the Kuznetsov component of a special GM threefold X determines the isomorphism class of X.

Theorem 8.1. *Let X and X' be smooth general special GM threefolds with $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X \cong X'$.*

Recall from Section 3 that every special GM threefold X is a double cover of a degree 5 index 2 prime Fano threefold Y branched over a quadric hypersurface $\mathcal{B}$ in Y. Since X is smooth and general, $(\mathcal{B}, h)$ is a smooth degree $h^2 = 10$ K3 surface with Picard number one. There is a natural geometric involution τ on X induced by the double cover. The Serre functor on $\mathcal{K}u(X)$ is given as $S_{\mathcal{K}u(X)} = \tau \circ [2]$.

Proof of Theorem 8.1. By [KP18a, Theorem 1.1, Section 8.2], the equivariant triangulated category $\mathcal{K}u(X)^{\mu_2}$ is equivalent to $\mathrm{D}^b(\mathcal{B})$, where μ_2 is the group of square roots of 1 generated by the involution τ acting on $\mathcal{K}u(X)$. Assume there is an equivalence $\Phi : \mathcal{K}u(X) \cong \mathcal{K}u(X')$. Note that Φ commutes with involutions τ and τ' on $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$, respectively. Then we get induced equivalence

$$\Psi : \mathcal{K}u(X)^{\mu_2} \cong \mathcal{K}u(X')^{\mu_2'}$$

where $\mu_2 = \langle \tau \rangle$ and $\mu_2' = \langle \Phi \circ \tau \circ \Phi^{-1} = \tau' \rangle$, $\mu_2 \cong \mu_2'$. Thus we have $\Psi : D^b(\mathcal{B}) \cong D^b(\mathcal{B}')$. We know that $\mathcal{B}$ and $\mathcal{B}'$ are smooth projective surfaces with polarization h and h' respectively, so Ψ is a Fourier–Mukai functor by Orlov's representability theorem [Orl97, Theorem 2.2]. Moreover, $(\mathcal{B}, h)$ and $(\mathcal{B}', h')$ are both Picard number 1 smooth projective K3 surfaces of degree $h^2 = h'^2 = 10 = 2 \times 5$. Then by [Ogu02, Theorem 1.10] and [HLOY02, Corollary 1.7], there is an isomorphism $\phi : \mathcal{B} \cong \mathcal{B}'$. Note that $\phi^*(h')$ is an ample divisor on $\mathcal{B}$, thus $\phi^*(h') = ah$ for some $a \in \mathbb{Z}_+$. Similarly, $(\phi^{-1})^*(h) = bh'$ for some $b \in \mathbb{Z}_+$. Thus $a = b = 1$ and $\phi^*(h') = h$. On the other hand Y_5 is rigid [Kuz09a, § 4.1], which implies $X \cong X'$ ◻

Remark 8.2. Theorem 8.1 can also be proved via Bridgeland moduli spaces with respect to the Kuznetsov component $\mathcal{A}_X$. The details are contained in an upcoming preprint [JLZ21]; we only sketch the proof here. One can show that the Gieseker moduli space $M_G(2,1,5)$ for a general special GM threefold X is also a smooth projective surface. On the other hand, the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is a surface with a unique singular point represented by $\pi := \Xi(\pi(\mathcal{E}))$. In fact, $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is the contraction of a family of conics on $\mathcal{C}(X)$ parametrized by a $\mathbb{P}^2$ (instead of a $\mathbb{P}^1$ as in the ordinary GM case). Assume there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then it will induce an equivalence $\Psi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ such that the gluing data π is preserved by Ψ (because π is the unique singular point of the moduli space). Thus $\pi(\mathcal{E})$ is automatically preserved by Φ. Then X is reconstructed as the Brill-Noether locus of a Bridgeland moduli space of σ-stable objects in $\mathcal{K}u(X)$ with respect to $\pi(\mathcal{E})$.

9. The moduli space $M_G(2,1,5)$ on ordinary GM threefolds

In this section we investigate the moduli space of rank 2 Gieseker-semistable torsion-free sheaves on an ordinary GM threefold X with Chern classes $c_1 = H$ and $c_2 = 5L$, denoted $M_G^X(2,1,5)$. We drop X from the notation when it is clear from context on which threefold we work. Note that if $F \in M_G(2,1,5)$, then

$$\mathrm{ch}(F) = (2, H, 0, -\frac{5}{6}P).$$

Recall the following theorem [DIM12, Section 8]:

Theorem 9.1. *Let $F \in M_G(2,1,5)$. Then F is either a*

(1) globally generated locally free sheaf (vector bundle). These fit into the short exact sequence

$$0 \to \mathcal{O}_X \to F \to I_Z(H) \to 0$$

where Z is an elliptic quintic curve;

(2) non-locally free sheaf with a short exact sequence

$$0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$$

where L is a line on X.

(3) non-globally generated vector bundle which fits into the exact sequence

$$0 \to \mathcal{E} \to H^0(X,F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

Furthermore, in all cases we have $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$.

Proof. The proofs for statements in this theorem can be found in [DIM12, Section 8]. We recall the proof of $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = k^4$. When F is a vector bundle, we have $\chi(\mathcal{O}_X, F) = 4$ and $H^3(X,F) = 0$ by Serre duality. The vanishing of $H^2(X,F)$ is proved by restricting F to a hyperplane section (K3 surface) of X (see [DIM12, p. 28]). When F is non-locally free, apply $\mathrm{Hom}(\mathcal{O}_X, -)$ to the sequence in (2). By [DIM12, p. 29] we have the identification $H^0(X,F)^\vee \cong H^0(X, F_{\mathrm{ng}}) = k^4$, where F_{ng} is some non-globally generated vector bundle. Now apply $\mathrm{Hom}(\mathcal{O}_X, -)$ to the short exact sequence in part (2) of the theorem. We get the exact sequence

$$0 \to k^4 \to k^5 \to k^1 \to \mathrm{Ext}^1(\mathcal{O}_X, F) \to 0$$

so we conclude that $\mathrm{RHom}^\bullet(\mathcal{O}_X, F) = 0$ here too. ∎

Proposition 9.2. *Let $F \in M_G(2,1,5)$. Then we have*

$$\mathrm{RHom}^\bullet(F,F) = k \oplus k^2[-1].$$

Proof. Let $F \in M_G(2,1,5)$ and let $E \subset F$ be a subsheaf of F with $\mathrm{rk}(E) < \mathrm{rk}(F) = 2$. Then E is a torsion-free sheaf of rank one on X. But $\rho(X) = 1$, so $E \cong I_Z \otimes \mathcal{O}_X(aH)$ for some codimension two subscheme $Z \subset X$ and $a \in \mathbb{Z}$. Since F is μ_H-semistable, $\mu_H(E) = a \leq \mu_H(F) = \frac{1}{2}$. Thus $a \leq 0$, which implies that $\mu_H(E) < \mu_H(F)$. Thus F is a μ_H-stable. Hence $\hom(F,F) = 1$. Next, $\mathrm{ext}^3(F,F) = \hom(F, F \otimes \mathcal{O}_X(-H)) = 0$ by Serre duality and Gieseker stability of F. By [DIM12, Theorem 8.2], $\mathrm{ext}^2(F,F) = 0$. Finally note that $\chi(F,F) = -1$, so $\mathrm{ext}^1(F,F) = 2$. ∎

A natural question to ask is what Bridgeland moduli space we get after projecting an object from $M_G(2,1,5)$ into the Kuznetsov component. Since it is easier in this setting, we will work with the alternative Kuznetsov component $\mathcal{A}_X$ in this section. Our analysis of the projections of objects in $M_G(2,1,5)$ is based on the three cases listed in Theorem 9.1. We begin with a Hom-vanishing result.

Proposition 9.3. *Let $F \in M_G(2,1,5)$. Then we have the vanishing* $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$.

Proof. Firstly, let F be globally generated. Apply $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence in Theorem 9.1 (1). We have the following long exact sequence

$$\begin{aligned}(8)\qquad & 0 \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{E}^\vee, F) \to \mathrm{Hom}(\mathcal{E}, I_Z) \to \\ & \to \mathrm{Ext}^1(\mathcal{E}^\vee, \mathcal{O}_X) \to \cdots\end{aligned}$$

It is true that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$ since $(\mathcal{E}, \mathcal{O}_X)$ is an exceptional pair. Now we turn to $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_Z(1)) \cong \mathrm{RHom}^\bullet(\mathcal{E}, I_Z)$. We have $\mathrm{Hom}(\mathcal{E}, I_Z) = 0$ because it is the sections of $\mathcal{E}^\vee$ whose zero locus contains Z, and a degree 4 curve (because $c_2(\mathcal{E}) = 4L$) cannot contain a degree 5 curve Z. For the Ext's, apply $\mathrm{Hom}(\mathcal{E}, -)$ to the ideal sheaf sequence $0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. We get the long exact sequence

$$\begin{aligned}(9)\qquad & 0 \to \mathrm{Hom}(\mathcal{E}, I_Z) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_Z) \to \\ & \to \mathrm{Ext}^1(\mathcal{E}, I_Z) \to \cdots\end{aligned}$$

We have $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_X) = k^5$. We also claim that

$$\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_Z) = \mathrm{RHom}^\bullet(\mathcal{O}_Z, \mathcal{E}^\vee|_Z) = k^5.$$

Indeed, by Atiyah's classification of vector bundles on elliptic curves [Ati57] and the case described in e.g. [IM05, § 5.2], we have that $\mathcal{E}^\vee|_Z$ can only split as the direct sum of line bundles with degrees $(2,3)$ or $(0,5)$. The second case in [IM05, § 5.2] is not possible because $\mathcal{E}^\vee|_Z$ has odd degree. But as shown in *loc. cit.*, $\mathcal{E}^\vee|_Z$ cannot split as the sum of line bundles with degrees $(0,5)$, otherwise Z would not be projectively normal which is a contradiction. So $\mathcal{E}^\vee|_Z \cong \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)$ where $p \in Z$ is a point. Then a simple cohomology computation shows that $H^0(Z, \mathcal{O}_Z(2p) \oplus \mathcal{O}_Z(3p)) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}^\vee|_Z) = k^5$. Finally, an Euler characteristic computation shows that

$$\chi(\mathcal{E}, \mathcal{O}_Z) = 5 = \hom(\mathcal{E}, \mathcal{O}_Z) - \mathrm{ext}^1(\mathcal{E}, \mathcal{O}_Z),$$

as required for the claim. Then $\chi(\mathcal{E}, I_Z) = 0$ implies the vanishing

$$\mathrm{RHom}^\bullet(\mathcal{E}^\vee, I_Z(1)) = 0$$

from the long exact sequence (9). From the long exact sequence (8) we get at the beginning of the proof, it follows that $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

Now let F be non-locally free. Apply $\mathrm{Hom}(\mathcal{E}^\vee, -)$ to the sequence from Theorem 9.1 (2). Because $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E}^\vee) = k$ by exceptionality, and $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L) = k$ by a cohomology calculation ($\mathcal{E}|_L$ splits as $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$), we get the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}^\vee, F) \to k \to k \to \mathrm{Ext}^1(\mathcal{E}^\vee, F) \to 0.$$

The map $\mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}^\vee) \to \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{O}_L)$ is surjective. Indeed, any map $s : \mathcal{E}^\vee \to \mathcal{O}_L$ is the composition of identity map $\mathcal{E}^\vee \to \mathcal{E}^\vee$ and s itself. Hence $\mathrm{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$ as required.

Now let F be non-globally generated. Recall from Theorem 9.1 (3) the exact sequence

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-1) \to 0.$$

Letting $G := \operatorname{im}(H^0(X, F) \otimes \mathcal{O}_X \to F)$ this can be split up into the short exact sequences

$$0 \to \mathcal{E} \to H^0(X, F) \otimes \mathcal{O}_X \to G \to 0 \tag{10}$$

and

$$0 \to G \to F \to \mathcal{O}_L(-1) \to 0. \tag{11}$$

Apply $\operatorname{Hom}(\mathcal{E}^\vee, -)$ to sequence (10). We have the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{E}^\vee, \mathcal{E}) \to \operatorname{Hom}(\mathcal{E}^\vee, \mathcal{O}_X^{\oplus m}) \to \operatorname{Hom}(\mathcal{E}^\vee, G) \to \\ \to \operatorname{Ext}^1(\mathcal{E}^\vee, \mathcal{E}) \to \cdots \end{aligned}$$

where $m := h^0(X, F)$. First we know that $\operatorname{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_X) = 0$. Next we find $\operatorname{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{E})$. By Serre duality, $\operatorname{Ext}^i(\mathcal{E}^\vee, \mathcal{E}) \cong \operatorname{Ext}^{3-i}(\mathcal{E}, \mathcal{E})$ which is k for $i = 3$ and 0 else by exceptionality of $\mathcal{E}$. Hence $\operatorname{RHom}^\bullet(\mathcal{E}^\vee, G) = k[-2]$.

Next apply $\operatorname{Hom}(\mathcal{E}^\vee, -)$ to the sequence (11). We get the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{E}^\vee, G) \to \operatorname{Hom}(\mathcal{E}^\vee, F) \to \operatorname{Hom}(\mathcal{E}^\vee, \mathcal{O}_L(-1)) \to \\ \to \operatorname{Ext}^1(\mathcal{E}^\vee, G) \to \cdots \end{aligned}$$

Since $\mathcal{E}|_L(-1)$ splits as $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)$, cohomology computations show that $\operatorname{RHom}^\bullet(\mathcal{E}^\vee, \mathcal{O}_L(-1)) = k[-1]$, so the resulting long exact sequence and the paragraph above gives that $\operatorname{RHom}^\bullet(\mathcal{E}^\vee, F) = 0$. ◻

9.1. **Involutions on $M_G(2, 1, 5)$.** In this subsection, we briefly recall the involutions which exist on $M_G(2, 1, 5)$. Let F be a globally generated vector bundle, and consider the short exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0.$$

Note that $\ker(\mathrm{ev})$ is a rank 2 vector bundle with $c_1 = -H$ and $c_2 = 5L$ and no global sections, hence $\ker(\mathrm{ev})^\vee \in M_G(2, 1, 5)$. Define $\iota F := \ker(\mathrm{ev})^\vee$. This is globally generated, and we have $H^0(X, \iota F) \cong H^0(X, F)^\vee$ [DIM12, p. 29]. If F is a non-locally free sheaf, then the same construction gives $\iota F = \ker(\mathrm{ev})^\vee$.

9.2. **An explicit description of $\operatorname{pr}(F)$.** We are now ready to give an explicit description of $\operatorname{pr}(F)$, for all objects $F \in M_G(2, 1, 5)$.

Lemma 9.4. *Let $F \in M_G(2, 1, 5)$. Then we have*

$$\operatorname{pr}(F) = \begin{cases} (\iota F)^\vee[1] \cong \ker(\mathrm{ev})[1], & F \text{ globally generated or} \\ & \text{non-locally free} \\ \mathcal{E}[1] \to \operatorname{pr}(F) \to \mathcal{O}_L(-1), & F \text{ non-globally generated} \end{cases}$$

where ι is the involution on $M_G(2, 1, 5)$.

Proof. Let F be globally generated or non-locally free or non-globally generated. As a result of Proposition 9.3, $\mathbf{L}_{\mathcal{E}} F = F$, so $\operatorname{pr}(F) = \mathbf{L}_{\mathcal{O}_X} F$. By Theorem 9.1 we have $\operatorname{RHom}^\bullet(\mathcal{O}_X, F) = k^4$, and the triangle defining the left mutation is

$$\mathcal{O}_X^{\oplus 4} \xrightarrow{\mathrm{ev}} F \to \operatorname{pr}(F). \tag{12}$$

In the cases where F is globally generated or non-locally free, the evaluation map ev is surjective, so $\mathrm{pr}(F) = \ker(\mathrm{ev})[1]$. Subsection 9.1 relates $\ker(\mathrm{ev})$ to ιF as required.

If F is non-globally generated, ev is not surjective. So take the long exact sequence in cohomology with respect to $\mathrm{Coh}(X)$ of triangle (12). This gives the exact sequence

$$0 \to \mathcal{H}^{-1}(\mathrm{pr}(F)) \to \mathcal{O}_X^{\oplus 4} \to F \to \mathcal{H}^0(\mathrm{pr}(F)) \to 0. \tag{13}$$

Comparing the sequence (13) with the sequence in (3) of Theorem 9.1 gives that

$$\mathcal{H}^i(\mathrm{pr}(F)) = \begin{cases} \mathcal{E}, & i=-1 \\ \mathcal{O}_L(-1), & i=0 \\ 0, & \text{else.} \end{cases}$$

Thus $\mathrm{pr}(F)$ in this case fits in the triangle

$$\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1)$$

as required. □

Remark 9.5. Let F be a non-globally generated vector bundle and E the associated non-locally free sheaf. We have the short exact sequence [DIM12, p. 31]

$$0 \to F^\vee \to H^0(X,E)\otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} E \to 0 \tag{14}$$

From the sequence (14) we see that $F^\vee = \ker(\mathrm{ev}) = \mathrm{pr}(E)[-1]$, i.e. $F = \mathrm{pr}'(E)^\vee[1]$ (because F is reflexive).

9.3. **Compatibility of categorical and classical involutions.** Let $\tau_{\mathcal{A}}$ be the involution of $\mathcal{A}_X$, τ be the geometric involution of $\mathcal{C}_m(X)$, and ι be the geometric involution of $M_G(2,1,5)$. Then $\tau_{\mathcal{A}}$ induces involutions on closed points of the Bridgeland moduli spaces of σ-stable objects, $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)$. In Proposition 7.3, we already showed that the action of $\tau_{\mathcal{A}}$ on $\mathcal{M}_\sigma(\mathcal{A}_X,-x)$ induces a geometric involution τ on $\mathcal{C}_m(X)$. In this section, we show that $\tau_{\mathcal{A}}$ is also compatible with ι on $M_G(2,1,5)$.

Proposition 9.6. *Let $F \in M_G(2,1,5)$. Then $\tau_{\mathcal{A}}\mathrm{pr}(F) \cong \mathrm{pr}(\iota(F))$.*

Proof.

(1) If F is a non-globally generated vector bundle, then by Corollary 9.4, we have the triangle $\mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1)$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F))$ is given by a triangle

$$\mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee) \to \tau_{\mathcal{A}}(\mathrm{pr}(F)) \to \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1].$$

Note that $\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{E}^\vee) = 0$, hence $\tau_A(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}(\mathcal{O}_L)[-1]$. Also, it is easy to see $\mathrm{RHom}^\bullet(\mathcal{E}^\vee,\mathcal{O}_L) = k$, therefore we have $\mathcal{E}^\vee \to \mathcal{O}_L \to \mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L$. Also, since $\mathcal{E}^\vee \to \mathcal{O}_L$ is surjective, we have $\mathbf{L}_{\mathcal{E}^\vee}\mathcal{O}_L \cong \ker(\mathcal{E}^\vee \to \mathcal{O}_L)[1]$, where $F' := \ker(\mathcal{E}^\vee \to \mathcal{O}_L)$ is a non-locally free sheaf in $M_G(2,1,5)$ by Theorem 9.1. Thus $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \ker(\mathrm{ev})[1]$, where ev is the evaluation morphism $\mathrm{Hom}(\mathcal{O}_X,F')\otimes\mathcal{O}_X \xrightarrow{\mathrm{ev}} F'$. But note that $\ker(\mathcal{E}^\vee \to \mathcal{O}_L) \cong \iota(F)$ since F' and F are associated with the same line L. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}(\iota F)$. Note that ιF is already a non-locally free sheaf and $\mathrm{RHom}(\mathcal{E}^\vee,\iota F) = 0$ by Proposition 9.3. Thus we have $\iota F \cong \mathbf{L}_{\mathcal{E}^\vee}\iota F$. Then $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \mathbf{L}_{\mathcal{O}_X}\mathbf{L}_{\mathcal{E}^\vee}\iota F \cong \mathrm{pr}(\iota F)$ as required.

(2) If F is a non-locally free sheaf in $M_G(2,1,5)$, then $F \cong \iota E$ for some non-globally generated vector bundle E. Thus we only need to check $\tau_{\mathcal{A}}(\mathrm{pr}(\iota E)) \cong \mathrm{pr}(\iota \circ \iota(E)) \cong \mathrm{pr}(E)$, but this is true by part (1) of the proof.

(3) If F is a globally generated vector bundle, consider the standard short exact exact sequence

$$0 \to \ker(\mathrm{ev}) \to H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to 0.$$

Dualizing the sequence and applying pr, we get the triangle

$$\mathrm{pr}(F) \to \mathrm{pr}(\mathcal{O}_X^{\oplus 4}) \to \mathrm{pr}(\ker(\mathrm{ev})^\vee) \cong \mathrm{pr}(\iota F).$$

Note that $F^\vee \in \mathcal{A}_X$ and $\mathrm{pr}(\mathcal{O}_X) = 0$, thus we get $\mathrm{pr}(\iota F) \cong F^\vee[1]$. Since $F \in M_G(2,1,5)$ is a globally generated vector bundle, we have $F \cong \iota E$ for some globally generated vector bundle E. Then $\mathrm{pr}(F) = \mathrm{pr}(\iota E) \cong E^\vee[1] \cong E \otimes \mathcal{O}_X(-H)[1]$, hence $\tau_{\mathcal{A}}(\mathrm{pr}(F)) \cong \tau_{\mathcal{A}}(E \otimes \mathcal{O}_X(-H))[1] \cong \mathrm{pr}(E) \cong \mathrm{pr}(\iota F)$. ∎

9.4. **Bridgeland moduli space interpretation of $M_G(2,1,5)$.** We arrive at the first of the main results of Section 9:

Theorem 9.7. *The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$.*

We split the proof of this theorem into a series of lemmas and propositions.

Proposition 9.8. *The functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ is injective on all objects in $M_G(2,1,5)$, i.e. if $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$, then $F_1 \cong F_2$.*

Proof. For the case of globally generated vector bundle, by Corollary 9.4, $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies that

$$(\iota F_1)^\vee \cong (\iota F_2)^\vee. \tag{15}$$

Note that $(\iota F_i)^\vee \cong \iota F_i \otimes \mathcal{O}_X(-H)$ for $i = 1, 2$. Then we get $\iota F_1 \cong \iota F_2$. Finally, apply ι to both sides; since it is an involution, $\iota^2 = 1$ so $F_1 \cong F_2$ as required.

For the case of non-globally generated vector bundle F, recall that $\mathcal{H}^0(\mathrm{pr}(F)) = \mathcal{O}_L(-H)$. In particular the line L is uniquely determined by F, so $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$ implies $F_1 \cong F_2$ as required.

For the case of non-locally free sheaf F, if F_1, F_2 are two non-locally free sheaves such that $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$, then by Proposition 9.6, $\mathrm{pr}(\iota F_1) \cong \tau_{\mathcal{A}}\mathrm{pr}(F_1) \cong \tau_{\mathcal{A}}\mathrm{pr}(F_2) \cong \mathrm{pr}(\iota F_2)$, where ιF_1 and ιF_2 are both non-globally generated vector bundle. Then it follows from the previous case that $\iota F_1 \cong \iota F_2$, then $F_1 \cong F_2$ by applying the involution ι on both sides. ∎

Proposition 9.9. *The functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces isomorphisms of $\mathrm{Ext}^i(\mathrm{pr}(F), \mathrm{pr}(F))$ and $\mathrm{Ext}^i(F, F)$ for all i and for all $F \in M_G(2,1,5)$.*

Proof. If $F \in M_G(2,1,5)$ is a globally generated vector bundle, then by Corollary 9.4,

$$\begin{aligned}
\mathrm{Ext}^i(\mathrm{pr}(F), \mathrm{pr}(F)) &= \mathrm{Ext}^i((\iota F)^\vee, (\iota F)^\vee) \\
&\cong \mathrm{Ext}^i(\iota F \otimes \mathcal{O}_X(-H), \iota F \otimes \mathcal{O}_X(-H)) \\
&\cong \mathrm{Ext}^i(\iota F, \iota F) \\
&\cong \mathrm{Ext}^i(F, F)
\end{aligned}$$

since ι is an involution on all $F \in M_G(2,1,5)$. If F is a non-globally generated vector bundle, apply $\mathrm{Hom}(F,-)$ to the exact triangle $\mathcal{O}_X^{\oplus 4} \to F \to \mathrm{pr}(F)$. We get a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{Ext}^1(\mathcal{O}_X, F^\vee)^4 \to \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^1(\mathrm{pr}(F),\mathrm{pr}(F)) \to \\ \to \mathrm{Ext}^2(\mathcal{O}_X, F^\vee)^4 \to \cdots\end{aligned}$$

Note that $\mathrm{Ext}^k(\mathcal{O}_X, F^\vee) = 0$ for all k. Thus the desired result follows.

If F is a non-locally free sheaf, then by Corollary 9.4 again,

$$\begin{aligned}\mathrm{Ext}^i(\mathrm{pr}(F),\mathrm{pr}(F)) &= \mathrm{Ext}^i(\tau_{\mathcal{A}}\mathrm{pr}(F), \tau_{\mathcal{A}}\mathrm{pr}(F)) \\ &\cong \mathrm{Ext}^i(\mathrm{pr}(\iota F),\mathrm{pr}(\iota F)) \\ &\cong \mathrm{Ext}^i(\iota F, \iota F) \\ &\cong \mathrm{Ext}^i(F,F)\end{aligned}$$

□

In what follows, we describe a method to show stability of $\mathrm{pr}(F)$ in $\mathcal{A}_X$.

Lemma 9.10. *For all $F \in M_G(2,1,5)$, the object $\mathrm{pr}(F)$ is stable in the heart $\mathcal{A}'(\alpha,\beta)$ for $\beta = -\epsilon$ where $0 < \epsilon < 1/10$, and $\alpha > 0$ is sufficiently small.*

Proof. Recall from Lemma 9.4 that if F is globally generated or non-locally free, we have $\mathrm{pr}(F) \cong \iota(F)^\vee[1] \cong \ker(\mathrm{ev})[1]$. In particular, if F is a globally generated vector bundle then $\ker(\mathrm{ev})^\vee$ is a vector bundle and so too is $\ker(\mathrm{ev})$. If F is locally free then $\ker(\mathrm{ev})$ is also a vector bundle.

We first claim that in the cases where $\ker(\mathrm{ev})$ is a vector bundle, $\ker(\mathrm{ev})[1]$ is $\sigma_{\alpha,\beta}$-stable in the heart $\mathrm{Coh}^\beta(X)$ for all $\alpha > 0$ and $\beta = -\epsilon$. Indeed, for our choice of β we have $\mathrm{ch}_1^\beta(\ker(\mathrm{ev})) < 0$, so by [BMS16, Lemma 2.7] $\ker(\mathrm{ev})[1]$ is stable in $\mathrm{Coh}^\beta(X)$ for $\alpha \gg 0$. To conclude tilt-stability for small α, we apply the same inequalities as [PY20] to $\mathrm{ch}_{\leq 2}(\ker(\mathrm{ev})[1]) = (-2, H, 0)$. One can compute that there are no solutions and hence no walls in tilt stability for all $\alpha > 0$.

Now pick a second tilt parameter μ' satisfying the list of inequalities (3) and such that $\mu_{\alpha,\beta}(\ker(\mathrm{ev})[1]) > \mu'$. It follows that $\ker(\mathrm{ev})[1] \in \mathcal{T}^{\mu'}_{\alpha,\beta} \subset \mathrm{Coh}^{\mu'}_{\alpha,\beta}(X)$. By the same reasoning as [PY20, p. 14], $\ker(\mathrm{ev})[1]$ is also stable in the double tilted heart $\mathrm{Coh}^{\mu'}_{\alpha,\beta}(X)$. Finally, $\mathrm{pr}(F) = \ker(\mathrm{ev})[1]$ is in $\mathcal{A}'(\alpha,\beta)$ by construction, so it is stable in this heart too as required. Note that stability inside $\mathcal{A}'(\alpha,\beta)$ can also be checked using the Parallelogram Lemma, as in the proof of Lemma 5.7.

Now let F be non-globally generated. Recall that $\tau_{\mathcal{A}_X}\mathrm{pr}(F) \cong \mathrm{pr}(\tau F)$ by Proposition 9.6. But by [DIM12], τF is a non-locally free sheaf. We have already seen earlier in the proof that the projection is $\sigma(\alpha,\beta)$-stable for non-locally free sheaves. So $\tau_{\mathcal{A}_X}$ is $\sigma(\alpha,\beta)$-stable. Finally, $\tau_{\mathcal{A}_X}$ preserves stability conditions on $\mathcal{A}_X$, so $\mathrm{pr}(F)$ is $\sigma(\alpha,\beta)$-stable. □

Corollary 9.11. *The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$.*

Proof. First note that $M_G(2,1,5)$ is a fine moduli space. This is a consequence of [HL10, Theorem 4.6.5]. Indeed, if L is any line on X, then $\mathrm{ch}(I_L) = 1 - L - \frac{1}{2}P$. Then we have $\chi((2+H-\frac{5}{6}P)(1-L-\frac{1}{2}P)) = \chi_0(2+H-2L-\frac{17}{6}P) = 1$. By the same argument as in [Zha20, Theorem 8.9], let $\mathcal{S} := p(M_G(2,1,5))$ be an irreducible component of $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$. The morphism

$$p : M_G(2,1,5) \to \mathcal{S}$$

(induced by pr) is bijective on points and bijective on tangent spaces. Hence $\mathcal{S} \cong M_G(2,1,5)$. Finally, the irreducibility of the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ follows from Theorem 14.2 ◻

10. Birational categorical Torelli for ordinary GM threefolds

In this section, we show a birational categorical Torelli theorem for ordinary GM threefolds, i.e. assuming the Kuznetsov components are equivalent leads to a birational equivalence of the ordinary GM threefolds.

Theorem 10.1. *Let X and X' be general ordinary GM threefolds such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then $X \simeq X'$.*

Proof. Assume that $\Phi : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$, and fix a (-1)-class $-x$ in $\mathcal{N}(\mathcal{A}_X)$. The equivalence Φ sends $-x$ to either itself or $y-2x$ in $\mathcal{N}(\mathcal{A}_{X'})$. By the same argument as in [PY20, Zha20], we thus get induced morphisms between Bridgeland moduli spaces

$$\begin{tikzcd} M_\sigma(\mathcal{A}_X, -x) \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & M_\sigma(\mathcal{A}_{X'}, -x) \\ & M_\sigma(\mathcal{A}_{X'}, y-2x) \end{tikzcd}$$

As we have seen in Sections 7 and 9, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G(2,1,5)$. So we have two cases: either $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$ or $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$.

For the first case, blow up $\mathcal{C}_m(X)$ at the distinguished point $[\pi] := [\Xi(\pi(\mathcal{E}))]$, and blow up $\mathcal{C}_m(X')$ at the point $[c] := [\Phi(\pi)]$. We have $\mathcal{C}(X) \cong \mathrm{Bl}_{[c]}\mathcal{C}_m(X)$ and we have $\mathrm{Bl}_{[c]}\mathcal{C}_m(X') \cong \mathcal{C}(X'_c)$ by Theorem 6.5. So $\mathcal{C}(X) \cong \mathcal{C}(X'_c)$, therefore by Logachev's Reconstruction Theorem 6.3 we have $X \cong X'_c$. But X'_c is birational to X', so X and X' are birational.

For the second case, we get $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$ but we have a birational equivalence $M_G^{X'}(2,1,5) \simeq \mathcal{C}(X'_L)$ of surfaces by [DIM12, Proposition 8.1]. Thus $\mathcal{C}_m(X)$ is birationally equivalent to $\mathcal{C}(X'_L)$. Let $\mathcal{C}_m(X'_L)$ be the minimal surface of $\mathcal{C}(X'_L)$. Note that the surfaces here are all smooth surfaces of general type. By the uniqueness of minimal models of surfaces of general type, we get $\mathcal{C}_m(X) \cong \mathcal{C}_m(X'_L)$, which implies $X \cong (X'_L)_c \simeq X'$. In this case, X is parameterised by $\mathcal{C}_m(X'_L)/\tau$, but note that $\mathcal{C}_m(X'_L) \simeq \mathcal{C}(X'_L) \simeq M_G(2,1,5)$ (compatible with their involutions). This means that X is parameterised by a surface birationally equivalent to $M_G(2,1,5)/\iota$. The desired result follows. ◻

Remark 10.2. Theorem 10.1 proves a conjecture [KP19, Conjecture 1.7] of Kuznetsov-Perry for *general* ordinary GM varieties of dimension 3.

11. Inverse of the Duality Conjecture of Kuznetsov-Perry

11.1. **Moduli spaces of sheaves and Bridgeland moduli spaces for special GM threefolds.** Let X be a special GM threefold, which is the double cover of a degree 5 del Pezzo threefold Y with branch locus a quadric hypersurface $\mathcal{B} \subset Y$. In this section, we study the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ of stable objects of class $-x$ and $y-2x$ in the Kuznetsov component $\mathcal{A}_X$ on X for every τ-invariant stability condition σ. We then combine corresponding results for ordinary GM threefolds from previous sections to study Conjecture 1.6. We start with a lemma which will be useful for our computations.

Lemma 11.1. *Let $\pi : X \to Y$ be the double cover map over Y. Let $\mathcal{U}$ and $\mathcal{V}$ be the tautological sub and quotient bundles on Y, respectively, and let $\mathcal{E} = \pi^*\mathcal{U}$ and $\mathcal{Q} = \pi^*\mathcal{V}$ the correspondent tautological sub and quotient bundles on X. Then the following statements hold:*

(1) $h^\bullet(\mathcal{U}) = h^\bullet(\mathcal{U}(-1)) = h^\bullet(\mathcal{V}^\vee) = h^\bullet(\mathcal{V}^\vee(-1)) = h^\bullet(\mathcal{O}_Y(-1)) = 0$,
(2) $h^\bullet(\mathcal{V}(-1)) = \hom(\mathcal{V}^\vee, \mathcal{V}(-1)) = 0$,
(3) $h^i(\mathcal{V}) = h^i(\mathcal{U}^\vee) = 5$ *if* $i = 0$ *and* 0 *if* $i \neq 0$,
(4) $\hom^i(\mathcal{U}, \mathcal{V}^\vee) = 3$ *if* $i = 0$ *and* 0 *if* $i \neq 0$,
(5) $\mathcal{U}, \mathcal{U}^\vee, \mathcal{V}$, *and* $\mathcal{V}^\vee$ *are all* μ*-stable,*
(6) $\hom^i(\mathcal{U},\mathcal{U}) = \hom^i(\mathcal{V},\mathcal{V}) = 1$ *if* $i = 0$ *and* 0 *if* $i \neq 0$,
(7) $\hom^i(\mathcal{V},\mathcal{U}) = 1$ *when* $i = 1$ *and* 0 *if* $i \neq 0$,
(8) $\hom^i(\mathcal{U},\mathcal{V}) = 24$ *when* $i = 0$ *and* 0 *if* $i \neq 0$,
(9) $\hom^i(\mathcal{U},\mathcal{U}^\vee) = 22$ *if* $i = 0$ *and* 0 *if* $i \neq 0$. *Also,* $\hom^i(\mathcal{U}^\vee,\mathcal{U}) = 0$ *for all* i,
(10) $\hom^i(\mathcal{U}(-1), \mathcal{V}^\vee) = 42$ *if* $i = 0$ *and* 0 *if* $i \neq 0$. *Also,* $\hom^i(\mathcal{U}^\vee, \mathcal{V}) = 0$ *for all* i,
(11) $\hom(\mathcal{Q}^\vee, \mathcal{E}) = 0$, $\hom^1(\mathcal{Q}^\vee, \mathcal{E}) = 0$, $\hom^2(\mathcal{Q}^\vee, \mathcal{E}) = 1$, *and* $\hom^3(\mathcal{Q}^\vee, \mathcal{E}) = 0$,
(12) $\hom(\mathcal{E}, \mathcal{Q}^\vee) = 3$, $\hom^1(\mathcal{E}, \mathcal{Q}^\vee) = 1$, *and* $\hom^2(\mathcal{E}, \mathcal{Q}^\vee) = \hom^3(\mathcal{E}, \mathcal{Q}^\vee) = 0$.

Proof. (1) to (5) are from [San14, Lemma 2.14]; (6) is by exceptionality; (7), (8) and (9) are from previous results and the tautological short exact sequence.

For (10), as noted in [San14, Proposition 2.15], there is an exact sequence

$$0 \to \mathcal{V}(-1) \to \mathcal{U}^{\oplus 3} \to \mathcal{V}^\vee \to 0.$$

We apply $\mathrm{Hom}(\mathcal{U}(-1), -)$ to this sequence. Since $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-H)$, we have $\hom^i(\mathcal{Q}^\vee, \mathcal{E}) = h^i(\mathcal{Q} \otimes \mathcal{E}) = h^i(\pi^*(\mathcal{U} \otimes \mathcal{V})) = h^i(\pi_*\pi^*(\mathcal{U} \otimes \mathcal{V})) = h^i((\mathcal{U} \otimes \mathcal{V}) \oplus (\mathcal{U} \otimes \mathcal{V}(-1))) = \hom^i(\mathcal{U}^\vee, \mathcal{V}) + \hom^i(\mathcal{U}^\vee(1), \mathcal{V}) = \hom^i(\mathcal{U}^\vee, \mathcal{V}) + \hom^{3-i}(\mathcal{V},\mathcal{U})$. Also $\hom^i(\mathcal{E}, \mathcal{Q}^\vee) = h^i(\mathcal{E}^\vee \otimes \mathcal{Q}^\vee) = h^i(\pi^*(\mathcal{U}^\vee \otimes \mathcal{V}^\vee)) = h^i(\pi_*\pi^*(\mathcal{U}^\vee \otimes \mathcal{V}^\vee)) = h^i((\mathcal{U}^\vee \otimes \mathcal{V}^\vee)) \oplus (\mathcal{U}^\vee \otimes \mathcal{V}^\vee(-1)))) = \hom^i(\mathcal{U}, \mathcal{V}^\vee) + \hom^i(\mathcal{V},\mathcal{U})$. Thus the results follow from the previous computations. ◻

Similar to Proposition 7.1 for ordinary GM threefolds, we prove almost the same statement for special GM threefolds X.

Proposition 11.2. *Let $C \subset X$ be a conic on a special GM threefold X. Then $I_C \notin \mathcal{A}_X$ if and only if there is a resolution of I_C of the form*

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

In particular, such a family of conics is parametrized by $\mathbb{P}^2$.

Proof. The proof is almost the same as the proof of Proposition 7.1. The same argument shows that if there is such an exact sequence, then $I_C \in \mathcal{A}_X$. For the other direction, by the same argument, one shows that there is a non-trivial map $p : \mathcal{Q}^\vee \to I_C$. Then we claim that this map is also surjective on a special GM threefold X. The image of π is the ideal sheaf of the zero locus of a section s of $\mathcal{Q}$, which is the π-preimage of the zero locus of a section $s \in H^0(\mathcal{V})$. By [San14, Lemma 2.18], the zero locus of a section of $\mathcal{V}$ is either a line or a point. Thus the zero locus of a section of $\mathcal{Q}$ is either a conic or two points. But this zero locus contains a conic $C \subset X$, so the map p is surjective. Then the rest of the argument is the same as in the proof of Proposition 7.1. In particular, such conics are parametrized by $\mathbb{P}(\mathrm{Hom}(\mathcal{E}, \mathcal{Q}^\vee)) \cong \mathbb{P}^2$ by Lemma 11.1. We also mention that these conics are the π-preimages of lines on Y. ◻

Next, we show an analogue of Proposition 7.2

Proposition 11.3. *Let X be a smooth special GM threefold and $C \subset X$ a conic on X. If $I_C \notin \mathcal{A}_X$, then we have the exact triangle*

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee$$

where $\mathcal{Q}$ is the tautological quotient bundle.

Proof. By Proposition 11.2, I_C fits into the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{Q}^\vee \to I_C \to 0.$$

Now apply the projection functor to this short exact sequence; the same argument as in Proposition 7.2 shows that we have a triangle

$$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee.$$

◻

Corollary 11.4.

(1) *If X is general, then*

$$\mathrm{Ext}^2(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \begin{cases} 0, & I_C \in \mathcal{A}_X \\ k, & I_C \notin \mathcal{A}_X. \end{cases}$$

(2) *Let X be non-general and $\pi(C)$ a conic $c \subset \mathcal{B}$ where $\mathcal{B}$ is the branch locus. Then*

$$\mathrm{Ext}^2(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \begin{cases} k, & I_C \in \mathcal{A}_X \\ k, & I_C \notin \mathcal{A}_X. \end{cases}$$

Proof. If $I_C \in \mathcal{A}_X$, then C is contained in the preimage of a conic on Y that is bitangent to $\mathcal{B}$. If X is general, then the branch locus does not contain any conics, so $\tau(C) \neq C$ and I_C is not isomorphic to $I_{\tau(C)}$. This menas $\mathrm{Hom}(I_C, I_{\tau(C)}) = 0$. Then we have $\mathrm{Ext}^2(I_C, I_C) = \mathrm{Hom}(I_C, \tau(I_C)) = \mathrm{Hom}(I_C, I_{\tau(C)}) = 0$. If X is not general and $\mathcal{B}$ contains a conic c, then its π-preimage is a conic fixed by the involution τ. Then $\mathrm{Ext}^2(I_C, I_C) \cong \mathrm{Hom}(I_C, \tau I_C) \cong \mathrm{Hom}(I_C, I_C) = k$. In the first case, $\mathrm{Ext}^1(I_C, I_C) = k^2$ and in the latter case, $\mathrm{Ext}^1(I_C, I_C) = k^3$.

If $I_C \notin \mathcal{A}_X$, then by Proposition 11.2, the conic C is the π-preimage of a line on Y and $\tau(C) \cong C$. Then we have $\mathrm{Ext}^2(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \mathrm{Hom}(\mathrm{pr}(I_C), \tau(\mathrm{pr}(I_C))) = \mathrm{Hom}(\mathrm{pr}(I_C), \mathrm{pr}(\tau(I_C))) = \mathrm{Hom}(\mathrm{pr}(I_C), \mathrm{pr}(I_{\tau(C)})) = \mathrm{Hom}(\mathrm{pr}(I_C), \mathrm{pr}(I_C))$. Thus we only need to show $\mathrm{Hom}(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k$. To this end, we consider the triangle

$$\mathcal{Q}^\vee[-1] \to \mathcal{E}[1] \to \mathrm{pr}(I_C)$$

and applying spectral sequence in [Zha20, Lemma 3.9], from Lemma 11.1 we obtain $\mathrm{ext}^2(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = \mathrm{hom}(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = 1$ and $\mathrm{ext}^3(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = 0$. Thus $\mathrm{Ext}^1(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = k^3$. ◻

Corollary 11.5. *Let C be a conic on X. Then $\mathrm{pr}(I_C)[1]$ is σ-stable for every τ-invariant stability condition σ on $\mathcal{A}_X$.*

Proof. By Corollary 11.4 and [Zha20, Corollary 4.15], $\mathrm{pr}(I_C)$ is in the heart of σ up to shift. If $\mathrm{pr}(I_C)$ is unstable, then there's a destabilizing sequence

$$A \to \mathrm{pr}(I_C) \to B$$

such that A and B is σ-semistable and $\phi(A) > \phi(B)$. Thus $\mathrm{Hom}(A, B) = 0$. On the other hand, $\tau(B)$ is also σ-stable with the same phase as B since σ is τ-invariant. Then $\mathrm{Hom}(A, \tau(B)) = 0$. Therefore by weak Mukai Lemma [Zha20, Lemma 4.14], we have $\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leq \mathrm{ext}^1(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) \leq 3$. But $\mathrm{ext}^1(A, A)$ and $\mathrm{ext}^1(B, B)$ are both greater or equal to 2 by [Zha20, Proposition 4.13], which makes a contradiction. Now assume $\mathrm{pr}(I_C)$ is strictly σ-semistable, then since $\chi(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) = -1$, the same argument in [Zha20, Corollary 4.16] shows that this is impossible. Thus $\mathrm{pr}(I_C)[1]$ is σ-stable. ◻

Next, we construct the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects of class $-x = -(1 - 2L)$ in $\mathcal{A}_X$ with respect to any τ-invariant stability condition σ.

Theorem 11.6. *Let X be a special GM threefold. The projection functor* pr *identifies $p(\mathcal{C}(X)) := \mathcal{S}$ with the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$, where $p : \mathcal{C}(X) \to \mathcal{S}$ is the contraction of the component $\mathbb{P}^2$ to a singular point $q \in \mathcal{S}$. Moreover $\mathcal{S} \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ is always singular and has at most finitely many singular points. If $\mathcal{B}$ does not contain any conics, q is the unique singular point of $\mathcal{S}$.*

Proof. By [Ili94, Proposition 2.1.2], the Fano surface $\mathcal{C}(X)$ of conics on X has two irreducible components $\mathcal{C}(X) = \overline{\mathcal{F}} \cup \pi^*\Sigma(Y)$. The first component $\overline{\mathcal{F}}$ is the closure of the double cover of the family of conics on Y which are bitangent to the branch locus $\mathcal{B}$. The other component is the π-preimage of lines on Y. The intersection $\overline{\mathcal{F}} \cap \pi^*\Sigma(Y) = \pi^*(p) \cup \pi^*\{L \subset \mathcal{B}\}$, where p is a smooth curve of $(-1, 1)$-lines and L is a line in the branch locus. Let $\mathcal{C}$ be the universal family of conics on $X \times \mathcal{C}(X)$ and denote its restrictions to $\overline{\mathcal{F}}$ and $\pi^*\Sigma(X)$ by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. As the projection functor pr is of Fourier–Mukai type, by a similar argument to [Zha20, Lemma 6.1], the functor pr induces a morphism $p : \mathcal{C}(X) \to \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ since $\mathrm{pr}(I_C)[1]$ is σ-stable in $\mathcal{A}_X$. Note that $\overline{\mathcal{F}}$ is projective and irreducible by [Ili94]. Then $p|_{\overline{\mathcal{F}}}$ factors through one of the irreducible components, say $\mathcal{S}$ in $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$. The complement of ρ in $\overline{\mathcal{F}}$ is a dense open set $\mathcal{U}$ and it is known that $\overline{\mathcal{F}}$ is smooth away from ρ. Note that any conic $[C] \subset \mathcal{U}$ is the π-preimage of a conic which is double tangent to $\mathcal{B}$ on Y. Then $I_C \in \mathcal{A}_X$ so that $\mathrm{pr}(I_C) \cong I_C$ and p is injective over $\mathcal{U}$. Further note that p is also étale over $\mathcal{U}$ since the tangent map $dp : \mathrm{Ext}^1(\mathrm{pr}(I_C), \mathrm{pr}(I_C)) \to \mathrm{Ext}^1(I_C, I_C)$ is an isomorphism at each point in $\mathcal{U}$. So $p(\mathcal{U}) \subset \mathcal{S}$ is also a dense open subset. But p is a proper morphism, so $p(\overline{\mathcal{F}}) = \mathcal{S}$. Now by Proposition 11.2, p contracts the whole component $\pi^*\Sigma(X) \cong \mathbb{P}^2$ to a fixed point. Since the image of $\rho \subset \mathbb{P}^2$ lies in $p(\overline{\mathcal{F}}) = \mathcal{S}$, the map p contracts $\pi^*\Sigma(X)$ to a fixed point $q \in \mathcal{S}$, which implies $p(\mathcal{C}(X))$ is an irreducible component of $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ with a singular point q. If X is general where the branch locus does not contain line or conic, then q is the unique singular point. If $\mathcal{B}$ contains a conic C, then $\mathcal{S}$ has an extra singular point. In particular, since $\mathcal{B}$ can only contain finitely many lines and conics, $\mathcal{S}$ only admits finitely many singular points. Finally $\mathcal{S} = \mathcal{M}_\sigma(\mathcal{A}_X, -x)$ follows from Theorem 14.5. ◻

11.2. **The moduli space $M_G(2,1,5)$ on special GM threefolds.** Let X be a special GM threefold with branch locus $\mathcal{B}$. In this section, we show that the moduli space $M_G(2,1,5)$ of semistable torsion-free sheaves of rank 2, $c_1 = 1, c_2 = 5, c_3 = 0$ is isomorphic to the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$, and is a smooth irreducible surface when the branch locus $\mathcal{B}$ does not contain a line or conic.

First, as in the ordinary GM case [DIM12, Section 8], there are three types of sheaves in the moduli space $M_G(2,1,5)$.

Proposition 11.7. *Let X be a special GM threefold and $F \in M_G(2,1,5)$. Then either*

(1) F is locally free and globally generated, and F fits into

$$0 \to \mathcal{O}_X \to F \to I_Z(H) \to 0$$

where Z is an elliptic quintic curve,

(2) or F is not locally free, and F fits into

$$0 \to F \to \mathcal{E}^\vee \to \mathcal{O}_L \to 0$$

for a unique line L on X,

(3) or F is locally free but not globally generated, and F fits into

$$0 \to \mathcal{E} \to H^0(F) \otimes \mathcal{O}_X \to F \to \mathcal{O}_L(-H) \to 0$$

for a unique line L on X, and

$$0 \to F^\vee \to H^0(F) \otimes \mathcal{O}_X \to E \to 0$$

where $E \in M_G(2,1,5)$ is a non-locally free sheaf defined by L as in (2).

Next, we study the geometric properties of the moduli space $M^X_G(2,1,5)$ on a special GM threefold X and construct the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ of σ-stable objects of class $y-2x$ in the alternative Kuznetsov component $\mathcal{A}_X$ of X.

Proposition 11.8. *Let $F \in M_G(2,1,5)$.*

(1) If F is a globally generated vector bundle, then we have

$$\mathrm{RHom}^\bullet(F,F) = k \oplus k^2[-1].$$

(2) If F is a non-globally generated vector bundle or a non-locally free sheaf, then

- $\mathrm{RHom}^\bullet(F,F) = k \oplus k^2[-1]$ *if the normal bundle is given by* $\mathcal{N}_{L|X} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$,
- $\mathrm{RHom}^\bullet(F,F) = k \oplus k^3[-1] \oplus k[-2]$ *if* $\tau(L) = L$, *where τ is the involution induced by the double cover.*

Proof.

(1) If F is a non-globally generated vector bundle, then by Proposition 11.12

$$\begin{aligned}\mathrm{Ext}^2(F,F) &\cong \mathrm{Ext}^2(\mathrm{pr}(F), \mathrm{pr}(F)) \\ &\cong \mathrm{Hom}(\mathrm{pr}(F), \tau_{\mathcal{A}}(\mathrm{pr}(F))) \\ &\cong \mathrm{Hom}(\mathrm{pr}(F), \mathrm{pr}(\tau F)),\end{aligned}$$

where $\tau_{\mathcal{A}} = \tau$ is the geometric involution induced by the double cover. Note that τF is also a globally generated vector bundle in $M_G(2,1,5)$ since τ preserves coherent sheaves and it is an exact functor. Then $\mathrm{pr}(\tau F) \cong \ker(\mathrm{ev}')[1]$, where ev' is the evaluation map $\mathrm{Hom}(\mathcal{O}_X, \tau F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}'} \tau F \to 0$. Now we compute $\mathrm{Hom}(\mathrm{pr}(F), \mathrm{pr}(\tau F))$ via the spectral sequence in [Zha20, Lemma 3.9] with respect to the two triangles $\mathcal{O}_X^{\oplus 4} \to F \to \mathrm{pr}(F)$ and $\mathcal{O}_X^{\oplus 4} \to \tau F \to \mathrm{pr}(\tau F)$. The first page is given by

$$E_1^{p,q} = \begin{cases} \mathrm{Ext}^1(F, \mathcal{O}_X^{\oplus 4}) = 0, & p = -1 \\ \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \mathcal{O}_X^{\oplus 4}) \oplus \mathrm{Hom}(F, \tau F), & p = 0 \\ \mathrm{Hom}^{-1}(\mathcal{O}_X^{\oplus 4}, \tau F) = 0, & p = 1 \\ 0, & p \notin [-1, 1] \end{cases}$$

Further note that we have the following sequence

$$\mathrm{Hom}(F, \mathcal{O}_X^{\oplus 4}) = E_1^{-1,0} \to E_1^{0,0} = \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \mathcal{O}_X^{\oplus 4}) \oplus \mathrm{Hom}(F, \tau F)$$
$$\to \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \tau F) = E_1^{1,0}.$$

It is known that $\mathrm{Hom}(F, \mathcal{O}_X^{\oplus 4}) = 0$ and $\mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \tau F) \cong k^{16}$. Then the sequence above becomes

$$0 \to \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \mathcal{O}_X^{\oplus 4}) \oplus \mathrm{Hom}(F, \tau F) \xrightarrow{t} \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \tau F) = k^{16}.$$

It is easy to see that the map $v : \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \mathcal{O}_X^{\oplus 4}) \to \mathrm{Hom}(\mathcal{O}_X^{\oplus 4}, \tau F)$ is surjective hence an isomorphism since the domain and codomain have the same dimension 16. Thus the map t is a projection with $\ker(t)$ being $\mathrm{Hom}(F, \tau F)$. This means that $E_2^{0,0} \cong \mathrm{Hom}(F, \tau F)$. Also note that $E_2^{2,-1} = E_2^{-2,1} = 0$, so $\mathrm{Hom}(\mathrm{pr}(F), \mathrm{pr}(\tau F)) \cong \mathrm{Hom}(F, \tau F)$. Since F is a globally generated vector bundle, we have the short exact sequence

$$0 \to \mathcal{O}_X \to F \to I_{\Gamma_5} \otimes \mathcal{O}_X(H) \to 0,$$

where Γ_5 is a smooth irreducible elliptic quintic curve, appearing as the general locus of a section of F. For τF we also have following short exact sequence

$$0 \to \mathcal{O}_X \to \tau F \to I_{\tau\Gamma_5} \otimes \mathcal{O}_X(H) \to 0,$$

since τ is exact and it preserves all numerical classes. Thus $\mathrm{Hom}(F, \tau F) \neq 0$ if and only if $F \cong \tau F$ if and only if $\tau\Gamma_5 = \Gamma_5$. On the other hand, the image $\pi(\Gamma_5) \subset Y_5$ is also a smooth irreducible elliptic quintic γ_5 since Γ_5 itself is. Then $\tau\Gamma_5 = \Gamma_5$ if and only if γ_5 lies in branch locus $\mathcal{B}$. Note that $\mathcal{B}$ is a smooth degree 10 K3 surface and γ_5 is degree 5 and genus 1. But then by [Knu02, Theorem 1.1], $\mathcal{B}$ does not contain γ_5. This implies $\mathrm{Ext}^2(F, F) = 0$. It is clear that $\mathrm{Hom}(F, F) = 1$ and $\mathrm{Ext}^3(F, F) \cong \mathrm{Hom}(F, F \otimes \mathcal{O}_X(-H)) = 0$; the desired result follows.

(2) If F is a non-locally free sheaf or a non-globally generated vector bundle, the argument follows from a similar argument as in [DIM12, Theorem 8.2] via the spectral sequence in [IM05, Proposition 5.12].

□

Proposition 11.9. *Let X be a special GM threefold. Then the singularities of $M_G(2,1,5)$ consist of only isolated singular points and the number of them is finite* *and even. Moreover, these singular points are permuted by the involution ι. If $\mathcal{B}$ does not contain any lines, then $M_G(2,1,5)$ is a smooth irreducible surface.*

Proof. The first statement follows from Proposition 11.8 and the fact that $\mathcal{B}$ at most contains finitely many lines.

Now assume $\mathcal{B}$ does not contain any lines. Then $\Sigma(X)$ is a smooth irreducible curve. Note that all non-locally free sheaves in $M_G(2,1,5)$ are parametrized by a curve C_1 and all locally free but non-globally generated vector bundles in $M_G(2,1,5)$ are parametrized by a curve C_2, with $C_1 \cong C_2 \cong \Sigma(X)$ being irreducible curves. Thus if $M_G(2,1,5) = M_1 \cup M_2$ such that M_1, M_2 are non-empty disjoint closed subsets, then each C_i is contained in a M_j for $i, j = 1, 2$. Thus we only need to show $M^0 := M_G(2,1,5) - (C_1 \cup C_2)$ is irreducible.

As argued in [DIM12, Proposition 8.1], if we fixed a line L on X, then every $E \in M^0$ is a globally generated vector bundle and corresponds to a unique rational quartic Γ_4^0 bisecant to L. As in [Isk99, Section 4.3], we have the commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi"] \arrow[dd, "\epsilon"'] & & \widetilde{X^+} \arrow[dl, "\varphi^+"'] \arrow[dd, "\epsilon^+"] \\
& \overline{X} & \\
X \arrow[rr, dashed, "\psi_L"] \arrow[ur, dashed, "\pi_L"] & & X^+ \arrow[ul, dashed, "\pi_{L^+}"']
\end{tikzcd}$$

that has the following properties:

(1) X^+ is a smooth prime Fano threefold of degree 10,
(2) π_L and π_{L^+} are induced by the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^5$ from the lines L and L^+ respectively,
(3) ϵ is the blow-up of X centered at L and ψ is induced by the free linear series $|\epsilon^*H - E|$, where $E = \mathrm{Exc}(\epsilon)$,
(4) ϵ^+ is the blow-up of X^+ centered at L^+ with exceptional divisor $E^+ = -K_{\widetilde{X^+}} - \chi_*(E)$,
(5) $-K_{\widetilde{X}} = \epsilon^*(H) - E = \varphi^*\overline{H}$ and $-K_{\widetilde{X^+}} = \epsilon^{+*}(H^+) - E^+ = \varphi^{+*}\overline{H}$
(6) $\chi^*\epsilon^{+*}H^+ = 2\epsilon^*H - 3E$,
(7) the non-trivial fibers of π_L (respectively π_{L^+}) are lines on X (respectively X^+) intersecting with L (respectively L^+).

Now from computations of intersection numbers with hyperplanes, we know $\varphi_*\epsilon_*\Gamma_4^0 = \pi_{L*}\Gamma_4^0$ is a conic on $\overline{X}$, and $\psi_{L*}\Gamma_4^0$ is also a conic on X^+. This defines a morphism $T : M^0 \to \mathcal{C}(X^+)$. If X^+ is ordinary, then $F(X^+)$ is irreducible. Then the same argument as in [DIM12, Proposition 8.1] shows that T is birational, which implies M^0 is irreducible.

Now assume that X^+ is special. Let $\mathcal{H}$ be the Hilbert scheme of rational quartics on X. Let $\mathcal{H}^0 \subset \mathcal{H}$ be the locus parametrizing quartics that are the union of four lines. Define $\mathcal{H}^1 := \mathcal{H} - \mathcal{H}^0$ and $M^1 \subset M^0$ whose points correspond to quartics in $\mathcal{H}^1$. We claim that $\mathcal{H}^1$ is dense in $\mathcal{H}$. Indeed, we only need to prove that $\mathcal{H}^0$ cannot contain any component of $\mathcal{H}$. Consider the incidence variety $\mathcal{I} \subset \Sigma(X)^{\times 4}$. Since $\Sigma(X)$ is an irreducible curve, $\Sigma(X)^{\times 4}$ is irreducible and has dimension 4, and we know $\dim \mathcal{I} \leq 3$. Hence $\dim \mathcal{H}^0 \leq 3$. But $\mathrm{ext}^1(I_C, I_C) \geq 4$ for any rational quartic

C on X, hence $\mathcal{H}^0$ cannot contain any component of $\mathcal{H}$. Thus $\mathcal{H}^1$ is dense in $\mathcal{H}$ and therefore M^1 is dense in M^0. Thus we only need to show M^1 is irreducible.

To this end, if C^+ is a reducible conic (union of two lines intersecting at a point) on X^+ such that $C^+ \cap L^+ = \varnothing$, then $C' = \pi_{L^+ *}C^+$ is also a union of two lines that intersect at a point. Thus if $\pi_L^* C' = C$ for a rational quartic C on X bisecant to L, C' is obtained by contracting C via π_L. Since π_L only contracts lines intersecting with L, this means that C is the union of four lines on X. Therefore, $T^{-1}(\mathcal{F}_2) \cap M^1 = \varnothing$ and hence $T(M^1)$ is contained in $\mathcal{F}_1$. Thus T induces a morphism $T' : M^1 \to \overline{\mathcal{F}_1}$. Now as in [DIM12, Proposition 8.1], the preimage of a general conic in $\mathcal{F}_1$ is a rational quartic bisecant to L, which corresponds to a bundle in M^1. Therefore, T' has an inverse and is birational. Now since $\overline{\mathcal{F}_1}$ is irreducible [Ili94], M^1 is irreducible. Hence $\overline{M^1} = M^0$ is irreducible, which shows that $M_G(2,1,5)$ is irreducible. $\square$

Next we find the images of sheaves in the moduli space $M_G(2,1,5)$ under projection into the Kuznetsov component $\mathcal{A}_X$. The statements and proofs are entirely the same as Proposition 9.3, Lemma 9.4, Proposition 9.8, and Proposition 9.9.

Lemma 11.10. *Let $F \in M_G(2,1,5)$. Then we have*

$$\mathrm{pr}(F) = \begin{cases} (\iota F)^\vee[1] \cong \ker(\mathrm{ev})[1], & F \text{ globally generated or} \\ & \text{non-locally free} \\ \mathcal{E}[1] \to \mathrm{pr}(F) \to \mathcal{O}_L(-1), & F \text{ non-globally generated} \end{cases}$$

where ι is the involution on $M_G(2,1,5)$.

Proposition 11.11. *The functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ is injective on all sheaves in $M_G(2,1,5)$, i.e. if $\mathrm{pr}(F_1) \cong \mathrm{pr}(F_2)$, then $F_1 \cong F_2$.*

Proposition 11.12. *The functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces isomorphisms of $\mathrm{Ext}^i(\mathrm{pr}(F), \mathrm{pr}(F))$ and $\mathrm{Ext}^i(F, F)$ for all i and for all $F \in M_G(2,1,5)$.*

Now the same argument as in Corollary 9.11 shows:

Theorem 11.13. *Let X be a special GM threefold. The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{A}_X$ induces an isomorphism $M_G(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$. In particular if $\mathcal{B}$ does not contain any lines, then it is a smooth surface. If the branch locus contains a line, then it is singular with isolated singular points permuted by the involution ι.*

11.3. **Smoothness and singularities for moduli spaces.** To get a better understanding of the various Bridgeland moduli spaces of stable objects in Kuznetsov components $\mathcal{A}_X$, we give following definitions on genericity conditions for a smooth GM threefold.

Definition 11.14.

(1) We call a special GM threefold X
- *general* if the branch locus $\mathcal{B}$ is general,
- *M-general* if X is not general but the branch locus $\mathcal{B}$ does not contain any lines,
- *C-general* if X is not general but the branch locus $\mathcal{B}$ does not contain any conics.

(2) We call an ordinary GM threefold X
- *general* if X is general in its deformation family,

- *M-general* if X is not general but the Hilbert scheme $\Sigma(X)$ is smooth.

Proposition 11.15.

(1) If X is a smooth special GM threefold and it is general, then the moduli space $M_G^X(2,1,5)$ is smooth and $\mathcal{C}_m(X)$ has a unique singular point.
(2) If X is a M-general special GM threefold, then $M_G^X(2,1,5)$ is smooth and $\mathcal{C}_m(X)$ has at least one singular point.
(3) If X is a C-general special GM threefold, then $\mathcal{C}_m(X)$ has a unique singular point. If $\mathcal{B}$ contain lines, then it must contain one line and $M_G^X(2,1,5)$ has only two singular points.

Proof.

(1) If X is general, then the branch locus $\mathcal{B}$ does not contain any lines or conics. Then $\mathcal{C}_m(X)$ only has one singular point q coming from the contraction of $\mathbb{P}^2$. By Proposition 11.9, $M_G^X(2,1,5)$ is also smooth.
(2) If X is M-general, then by the same argument as above $M_G^X(2,1,5)$ is smooth and possible additional singular points on $\mathcal{C}_m(X)$ come from conics $c \subset \mathcal{B}$.
(3) If X is C-general, then any point $[c] \in \mathcal{C}_m(X)$ other than q is not fixed by τ, so $\mathcal{C}_m(X)$ has unique singular point q. If $\mathcal{B}$ contains a line, then it must contain only one line, otherwise it would contain a conic. In this case, the only two singular points in $M_G^X(2,1,5)$ are represented by a non-locally free sheaf F and a non-globally generated vector bundle ιF.

$\square$

Proposition 11.16.

(1) If X is a general ordinary GM threefold, then both $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$ are smooth.
(2) If X is not general and $\mathcal{C}_m(X)$ is singular, the only singularities are given by τ-conics which are fixed by ι defined in [DIM12, Remark 5.2]. If in addition, the Hilbert scheme $\Sigma(X)$ of lines is smooth, then the only singular points on $\mathcal{C}_m(X)$ are the smooth τ-conics fixed by ι.
*(3) If X is M-*general*, then the moduli space $M_G^X(2,1,5)$ is smooth.*

Proof.

(1) If X is general, then the result follows from [Log12].
(2) By the proof of Proposition 7.3 and Remark 7.14, the only singular points come from the τ-conic c fixed by the involution. In this case, the double conic $2c$ is the zero locus of a section of $\mathcal{E}^\vee$. If $\Sigma(X)$ is smooth, then there is no double line on X. Then the only singular points are the smooth τ-conics fixed by ι.
(3) As $\Sigma(X)$ is smooth, the normal bundle of any line $L \subset X$ is $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Then by the same argument as in [IM05, Proposition 5.12], $\mathrm{Ext}^2(F,F) = 0$ for every non-locally free sheaf or non-globally generated vector bundle F. On the other hand, $\mathrm{Ext}^2(F,F)$ is always 0 for globally generated vector bundles F. The result follows.

$\square$

We summarize the various Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ for GM threefolds in Table 11.4 below.

11.4. **Inverse of the Duality Conjecture for GM threefolds.** In [KP19], the authors proved the Duality Conjecture for GM varieties stated in [KP18b, Conjecture 3.7]. In [KP19], they proposed the following conjecture, which is the inverse of the Duality Conjecture.

Conjecture 11.17. ([KP19, Conjecture 1.7]) *If X and X' are GM varieties of the same dimension such that there is an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, then X and X' are birationally equivalent.*

By a careful study of the smoothness and singularities of Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, u)$ of σ-stable objects of a (-1)-class u in $\mathcal{N}(\mathcal{A}_X)$ for all GM threefolds, we can prove Conjecture 11.17 with certain genericity assumptions.

Theorem 11.18 ([KP19, Conjecture 1.7])**.** *Let X and X' be general GM threefolds such that their Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ are equivalent. Then X is birationally equivalent to X'.*

If X and X' are general GM threefolds such that $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$, then both X and X' are ordinary or special simultaneously. Indeed, we may assume X' is ordinary and X is special. Then the equivalence would identify the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ of stable objects of class $-x$ in $\mathcal{A}_X$ with either the moduli space $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, -x)$ or $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, y-2x)$. Then the minimal Fano surface $\mathcal{C}_m(X)$ for a special GM threefold X would be identified with the minimal Fano surface $\mathcal{C}_m(X')$ or the moduli space $M_G^{X'}(2,1,5)$ for a general ordinary GM threefold X'. But $\mathcal{C}_m(X)$ has a unique singular point and both $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ are smooth and irreducible for X' general. This means that neither identification is possible. so the claim follows. But below, we argue in an alternative way.

Proof of Theorem 11.18. If X and X' are general GM threefolds such that Φ : $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, then both X and X' are ordinary or special simultaneously. Indeed, we may assume X' is ordinary and X is special. Then the equivalence Ψ would identify the moduli space $M_G^{X'}(2,1,5)$ with either $\mathcal{C}_m(X)$ or $M_G^X(2,1,5)$, but $\mathcal{C}_m(X)$ is singular and $M_G^X(2,1,5)$ is smooth, so $M_G^{X'}(2,1,5) \cong M_G^X(2,1,5)$. Meanwhile, the equivalence Φ identifies another pair of moduli spaces: $\mathcal{C}_m(X')$ with $\mathcal{C}_m(X)$ or $\mathcal{C}_m(X')$ with $M_G^X(2,1,5)$. But $\mathcal{C}_m(X)$ is singular by Theorem 11.6, so $\mathcal{C}_m(X') \cong M_G^X(2,1,5)$. Then we get isomorphisms $M_G^{X'}(2,1,5) \cong M_G^X(2,1,5) \cong \mathcal{C}_m(X')$, in particular $M_G^{X'}(2,1,5) \cong \mathcal{C}_m(X')$. We claim that these two surfaces are not isomorphic to each other. If they were, then $\mathcal{C}_m(X') \cong M_G^{X'}(2,1,5) \simeq \mathcal{C}_m(X'_L)$ and hence $\mathcal{C}_m(X') \cong \mathcal{C}_m(X'_L)$, where X'_L is the line transform of X'. Then we would have $X' \cong (X'_L)_c$ by Logachev's Reconstruction Theorem 6.3. This means that the point $[X']$ and $[(X'_L)_c]$ are the same point in the moduli stack $\mathcal{X}_{10}$ of smooth GM threefolds. But the line transform switches the connected components of the fiber of period map while the conic transform preserves it. Thus $[X']$ and $[(X'_L)_c]$ are in $\mathcal{C}_m(X')$ and $M_G^{X'}(2,1,5)$ respectively, but these two surfaces are disjoint by [DIM11, Section 7.3]. This means that X' is not isomorphic to $(X'_L)_c$, and this is a contradiction. Therefore X and X' are both general ordinary GM threefolds or special GM threefolds.

(1) If X and X' are general ordinary GM threefolds with $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, then X and X' are birationally equivalent by Theorem 10.1.
(2) If X and X' are general special GM threefolds with $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, then X and X' are isomorphic by Theorem 8.

In both cases, X is birationally equivalent to X'. ◻

Theorem 11.19. *Let X and X' be smooth GM threefolds and X a general ordinary GM threefold such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X' is also a general ordinary GM threefold and hence X and X' are birational.*

Proof. First, we show that X' is also a general GM threefold. Assuming otherwise, X' can be one of the following:

- X' is a special GM threefold but not general;
- X' is an ordinary GM threefold but not general.

Since X is a general ordinary GM threefold, the associated (-1)-class Bridgeland moduli space in the Kuznetsov component $\mathcal{A}_X$ is either $\mathcal{C}_m(X)$ or $M_G^X(2,1,5)$ and they are both smooth and irreducible. Then as in the proof of Theorem 11.18, the equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify two pairs of moduli spaces:

- $\mathcal{C}_m(X)$ with either $\mathcal{C}_m(X')$ or $M_G^{X'}(2,1,5)$;
- $M_G^X(2,1,5)$ with either $\mathcal{C}_m(X')$ or $M_G^{X'}(2,1,5)$.

(1) If X' is a special GM threefold, then $\mathcal{C}_m(X')$ is always singular and $M_G^{X'}(2,1,5)$ could be smooth or singular (see Table 11.4).
   - If $M_G^{X'}(2,1,5)$ is singular, then neither identification is possible.
   - If $M_G^{X'}(2,1,5)$ is smooth, then $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5) \cong M_G^X(2,1,5)$. Then by the same argument in Theorem 11.18, this is also impossible.

   This means that X' cannot be a special GM threefold.

(2) If X' is an ordinary GM threefold but not general, then $\mathcal{C}_m(X')$ would be singular at the point represented by a smooth τ-conic c whose normal bundle is $\mathcal{N}_{c|X} \cong \mathcal{O}_c(2) \oplus \mathcal{O}_c(-2)$ ([DIM12, Remark 5.2]). Then the equivalence $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify $\mathcal{C}_m(X')$ with either $M_G^X(2,1,5)$ or $\mathcal{C}_m(X)$, which is impossible since both moduli spaces on X are smooth.

Therefore, X' must be general, but the proof of Theorem 11.18 tells us that X' must be also an ordinary GM threefold, therefore X' must be a general ordinary GM threefold. Then by Theorem 11.18, X and X' are birationally equivalent. ◻

Remark 11.20. One can also prove the theorem by directly comparing the singularities of the moduli spaces for X and X'. Indeed, if X' is a special GM threefold but not general, then the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x) \cong \mathcal{C}_m(X')$ is singular. Then the equivalence Φ would identify $\mathcal{C}_m(X')$ with either $\mathcal{C}_m(X)$ or $M_G^X(2,1,5)$ on a general ordinary GM threefold X. But both of these moduli spaces are smooth surfaces, so neither of the identifications is possible. Thus X' must be general and ordinary.

12. The Debarre-Iliev-Manivel conjecture

In [DIM12, pp. 3-4], the authors make the following conjecture regarding the general fiber of the period map:

Conjecture 12.1 ([DIM12, pp. 3-4])**.** *A general fiber $\mathcal{P}^{-1}([J(X)])$ of the period map $\mathcal{P} : \mathcal{X}_{10} \to \mathcal{A}_{10}$ through a ordinary GM threefold X is the union of $\mathcal{C}_m(X)/\iota$ and a surface birationally equivalent to $M_G(2,1,5)/\iota'$, where ι, ι' are geometrically meaningful involutions.*

Table 1. Bridgeland moduli spaces of (-1)-class on GM threefolds

Types	$\mathcal{M}_\sigma(\mathcal{A}_X,-x)\cong \mathcal{C}_m(X)$	$\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)\cong M_G(2,1,5)$
General ordinary	Smooth	Smooth
M-general ordinary	Singular	Smooth
Non-M-general ordinary	Singular	At most finitely many and even number singular points
General special	Unique singular point	Smooth
M-general and non-C-general special	Finitely many, more than 1 singular point	Smooth
Non-M-general and C-general special	Unique singular point	Two singular points
Non-M-general and non-C-general	Finitely many, more than 1 singular points	Finitely many singular points, appear in pairs

We will prove a categorical analogue of this conjecture. Consider the "categorical period map"

$$\mathcal{P}_{\mathrm{cat}}:\mathcal{X}_{10}\to\{\mathcal{A}_X\}/\sim,\ X\mapsto\mathcal{A}_X$$

Note that a global description of a "moduli of Kuznetsov components" $\{\mathcal{K}u\}/\sim$ is not known, however local deformations are controlled by the second Hochschild cohomology $\mathrm{HH}^2(\mathcal{A}_X)$. The fiber of the "categorical period map" $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for an ordinary GM threefold is defined as all ordinary GM threefolds X' such that $\mathcal{A}_{X'}\simeq\mathcal{A}_X$.

Theorem 12.2. *The general fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ of the categorical period map over the alternative Kuznetsov component of an ordinary GM threefold X is the union of $\mathcal{C}_m(X)/\iota$ and $M^X_G(2,1,5)/\iota'$ where ι,ι' are geometrically meaningful involutions.*

Proof. The general fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ of the categorical period map consists of GM threefolds X' such that there is an equivalence of Kuznetsov components $\mathcal{A}_{X'}\simeq\mathcal{A}_X$. Then by Proposition 11.19, X' is also general ordinary GM threefold. But by the Refined Categorical Torelli Theorem 7.17, X' is determined by $\mathcal{A}_{X'}$ along with its gluing data $\Xi(\pi'(\mathcal{E}))\in\mathcal{A}_{X'}$. Denote by Φ the equivalences between the Kuznetsov components $\mathcal{A}_{X'}\xrightarrow{\Phi}\mathcal{A}_X$. Then the fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ is given by the family of objects $\Phi(\Xi(\pi'(\mathcal{E})))\in\mathcal{A}_X$ as X' varies. Further note that by Theorem 4.25, $\Phi(\Xi(\pi'(\mathcal{E})))$ are σ-stable objects of (-1)-class in $\mathcal{A}_X$. So they are either points in $\mathcal{M}_\sigma(\mathcal{A}_X,-x)\cong\mathcal{C}_m(X)$ or points in $\mathcal{M}_\sigma(\mathcal{A}_X,y-2x)\cong M_G(2,1,5)$. Thus the general fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ is a subset of the union of the two surfaces $\mathcal{C}_m(X)/\iota\cup M^X_G(2,1,5)/\iota'$ (since $X_c\cong X_{\iota(c)}$, quotienting by involutions is necessary). On the other hand, by [KP19, Corollary 6.5], these two surfaces parametrizing period partners (conic transformations) and period duals (conic transformations of line transformations) of X are a subset of the fiber of $\mathcal{P}_{\mathrm{cat}}$. Thus the result follows. ∎

Remark 12.3. The Kuznetsov components of prime Fano threefolds of index 1 and 2 are often regarded as categorical analogues of the intermediate Jacobians of these threefolds, and it is known that if there is a Fourier–Mukai type equivalence

$\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ (or $\mathcal{A}_X \simeq \mathcal{A}_{X'}$), then $J(X) \cong J(X')$ by [Per20]. For the converse, we have the following conjecture.

Conjecture 12.4. *The intermediate Jacobian $J(X)$ uniquely determines the Kuznetsov component $\mathcal{K}u(X)$, i.e, $J(X) \cong J(X') \Longrightarrow \mathcal{K}u(X) \simeq \mathcal{K}u(X')$.*

Theorem 12.5. *Conjecture 12.4 is true for smooth prime Fano threefolds X if X is one of the following:*

- Y_d, $\quad 2 \leq d \leq 5$
- X_{2g-2}, $\quad g = 5, 7, 8, 9, 10, 12$.

Proof. If X is an index 2 prime Fano threefold Y_d where $2 \leq d \leq 5$, then the statement follows from the Torelli theorems for Y_d. If $X = X_8$, the statement follows from its Torelli theorem. If $X = X_{12}, X_{18}, X_{16}$, their intermediate Jacobians are Jacobians of curves: $J(X_{12}) \cong J(C_7)$, $J(X_{16}) \cong J(C_3)$, and $J(X_{18}) \cong J(C_2)$. But $\mathcal{K}u(X_{12}) \simeq \mathrm{D}^b(C_7)$, $\mathcal{K}u(X_{16}) \simeq \mathrm{D}^b(C_3)$, and $\mathcal{K}u(X_{18}) \simeq \mathrm{D}^b(C_2)$. Thus the statement follows from the classical Torelli theorem for curves. If $X = X_{14}$, the statement follows from the Kuznetsov conjecture for the pair (Y_3, X_{14}) [Kuz03] and the Torelli theorem for cubic threefolds. If $X = X_{22}$, the statement is trivial since $\mathcal{K}u(X_{22}) \cong \mathcal{K}u(Y_5)$ ([KPS18b]) and Y_5 is rigid, so $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is always true. $\square$

In the case of general ordinary GM threefolds X_{10} we show that

Proposition 12.6. *The Debarre-Iliev-Manivel Conjecture 12.1 is equivalent to Conjecture 12.4.*

Proof. Assume Conjecture 12.4. Then by Theorem 11.19, the Debarre-Iliev-Manivel Conjecture 12.1 holds. On the other hand, assume the Debarre-Iliev-Manivel Conjecture 12.1. Then for any X and X' such that $J(X) \cong J(X')$, X is either a conic transform of X', or X is a conic transform of a line transform of X'. In both cases, $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ by the Duality Conjecture [KP19, Theorem 1.6]. Thus Conjecture 12.4 holds. $\square$

Remark 12.7. In Theorem 16.7, we will prove the infinitesimal version of Conjecture 12.4.

12.1. **Period map for a larger moduli space.** In [DIM12], the authors defined the period map on the moduli space $\mathcal{X}_{10}$ of smooth ordinary GM threefolds. But as is shown in [KP18b, Proposition A.2], the moduli stack $\mathcal{M}_3$ of all smooth GM threefolds is a smooth and irreducible Deligne-Mumford stack of dimension 22. In [DK20a, Remark 6.2], the authors defined period map $\mathcal{P} : \mathcal{M}_3 \to \mathcal{A}_{10}$ with $X \mapsto J(X)$ on this larger moduli space. Thus one can state Conjecture 12.1 for this larger moduli space $\mathcal{M}_3$. It is easy to see that Theorem 1.7 and Theorem 11.19 already prove Theorem 12.2 for $\mathcal{M}_3$. In fact, we prove the following theorem.

Theorem 12.8. *The general fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ of the "categorical period map" $\mathcal{P}_{\mathrm{cat}} : \mathcal{M}_3 \to \mathcal{A}_{10}$ with $X \mapsto \mathcal{A}_X$ over the alternative Kuznetsov component of a* general *special GM threefold X is the surface $\mathcal{C}_m(X)/\iota$.*

Proof. The fiber $\mathcal{P}^{-1}_{\mathrm{cat}}([\mathcal{A}_X])$ of the "categorical period map" over the alternative Kuznetsov component of a general special GM threefold X is all GM threefolds X' such that $\Phi : \mathcal{A}_{X'} \simeq \mathcal{A}_X$. The only possibilities for X' are a special GM threefold with the Picard number of the branch locus being one, or a non-general ordinary GM threefold. Indeed, X' cannot be a special GM threefold with $\rho(\mathcal{B}') \geq 2$. Otherwise the equivalence Φ would induce an equivalence of equivariant triangulated categories $\Psi : \mathrm{D}^b(\mathcal{B}') \simeq \mathrm{D}^b(\mathcal{B})$, where $\mathcal{B}$ and $\mathcal{B}'$ are smooth K3 surfaces of degree 10. Then by Lemma A.1, their Picard numbers satisfy $\rho(\mathcal{B}') = \rho(\mathcal{B}) = 1$. But $\rho(\mathcal{B}') \geq 2$. In the first case, we get $X' \cong X$, giving a point in the fiber, which is represented by the unique singular point q on $\mathcal{C}_m(X)$. If X' is an ordinary GM threefold, then the equivalence Φ identifies the moduli space $\mathcal{C}_m(X)$ with $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x) \cong \mathcal{C}_m(X')$. By Corollary A.3, X' is uniquely determined by the gluing data $\Xi(\pi(\mathcal{E})) \in \mathcal{A}_{X'}$ since the Kuznetsov component $\mathcal{A}_{X'}$ is fixed by Φ. So all such X' in the fiber are parametrized by the points $[\Xi(\pi(\mathcal{E}))] \in \mathcal{M}_\sigma(\mathcal{A}_{X'}, -x) \xrightarrow{\phi} \mathcal{C}_m(X)$. Note that these points are smooth by Lemma 7.6. This means that $\phi([\Xi(\pi(\mathcal{E}))])$ can be any point in $\mathcal{C}_m(X)$ except the singular point q. Therefore the general fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ of the "categorical period map" through a general special GM threefold X is a subset of $\{q\} \cup \mathcal{C}_m(X) \smallsetminus \{q\} = \mathcal{C}_m(X)$ up to involution. On the other hand, by [DK20b], $\mathcal{C}_m(X) \cong \tilde{Y}^{\geq 2}_{A^\perp}$ (see Section 13). Thus by [DK15, Theorem 3.25], every point on the surface $\mathcal{C}_m(X)/\iota \cong \tilde{Y}^{\geq 2}_{A^\perp}/\iota \cong Y^{\geq 2}_{A^\perp}$ is in one-to-one correspondence with a period partner of X, where the smooth loci of $Y^{\geq 2}_{A^\perp}$ parametrize ordinary GM threefolds while the singular loci parametrize special GM threefolds. Their Kuznetsov components are equivalent to $\mathcal{A}_X$ by [KP19, Corollary 6.5]. This implies $\mathcal{C}_m(X)/\iota \subset \mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$. ◻

Remark 12.9. There are two key points to proving that the fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ is equal to the surface $\mathcal{C}_m(X)/\iota$. The first one is the "Bridgeland Brill-Noether" reconstruction of X (see Theorem A.2 and the associated refined categorical Torelli Theorem A.3). The second point is that for any point in the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ or $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$, one can define a birational model of X, in particular the period partner or period partner of the period dual of X. Then by [KP19, Corollary 6.5], the result follows.

In view of Theorem 12.8, we make the following conjecture:

Conjecture 12.10. *Let $\mathcal{M}_3$ be the moduli space of all smooth GM threefolds and $\mathcal{P} : \mathcal{M}_3 \to \mathcal{A}$ be the period map. Then the general fiber of $\mathcal{P}$ over $J(X)$ through a special GM threefold $[X]$ is the surface $\mathcal{C}_m(X)/\iota$.*

13. Double (dual) EPW surfaces as Bridgeland moduli spaces

In this section, we provide another possible approach to reproving Theorem 10.1 and Theorem 12.2, which could be the potential way to prove Kuznetsov-Perry's Conjecture 1.6 in full generality.

Let X be a smooth GM threefold and $A(X) \subset V^6 := H^0(\mathbb{P}^7, I_X(2))$ its associated Lagrangian vector space defined in [DK15]. Let $Y^{\geq k}_A = \{[v] \in \mathbb{P}(V_6) \mid \dim(A \cap (v \wedge \bigwedge^2 V_6)) \geq k\}$ and $Y^{\geq l}_{A^\perp} = \{V_5 \in \mathrm{Gr}(5, V_6) \mid \dim(A \cap \bigwedge^3 V_5) \geq l\}$ be its associated Eisenbud-Popescu-Walter (EPW for short) and dual EPW varieties. Then we have two chains of closed subvarieties:

$$Y^{\geq 3}_A \subset Y^{\geq 2}_A \subset Y^{\geq 1}_A \subset Y^{\geq 0}_A = \mathbb{P}(V_6),$$

and

$$Y^{\geq 3}_{A^\perp} \subset Y^{\geq 2}_{A^\perp} \subset Y^{\geq 1}_{A^\perp} \subset Y^{\geq 0}_{A^\perp} = \mathbb{P}(V_6^\vee).$$

In [DK19, Theorem 5.20(2)], the authors defined a canonical double covering

$$\tilde{Y}^{\geq 2}_{A} \to Y^{\geq 2}_{A},$$

étale away from the finite set $Y^{\geq 3}_{A}$, where $\tilde{Y}^{\geq 2}_{A}$ is an integral normal surface, which is called a double EPW surface. If X is general, then $Y^{\geq 3}_{A} = \varnothing$ and $\tilde{Y}^{\geq 2}_{A}$ is smooth. Similarly, the *double dual EPW* surface $\tilde{Y}^{\geq 2}_{A^\perp}$ is smooth if X is general.

Proposition 13.1. *Let X be a general ordinary GM threefold. Then the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \tilde{Y}^{\geq 2}_{A^\perp}$.*

Proof. By Theorem 7.13, $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$. But by [DK20b], $\tilde{Y}^{\geq 2}_{A^\perp}$ is a blow down of the Fano surface $\mathcal{C}(X)$ of conics. Note that X is general, and there is a unique rational curve on $\mathcal{C}(X)$ by [DIM12, p.16]. Then $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X) \cong \tilde{Y}^{\geq 2}_{A^\perp}$ by Lemma 14.3 and Lemma 14.4. ∎

Corollary 13.2. *Let X be a general ordinary GM threefold. Then the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x)$ is birationally equivalent to $\tilde{Y}^{\geq 2}_{A}$.*

Proof. By [DIM12, Section 8], the moduli space $M^X_G(2,1,5)$ of semistable sheaves is birationally equivalent to the Fano surface $\mathcal{C}_m(X_l)$ of conics of a line transform of X. Note that the Lagrangian vector space $A(X_l) \cong A(X)^\perp$. Then by [DK20b] again, we have $\mathcal{C}_m(X_l) \cong \tilde{Y}^{\geq 2}_{A}$, thus $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M^X_G(2,1,5) \simeq \mathcal{C}_m(X_l) \cong \tilde{Y}^{\geq 2}_{A}$ by Theorem 14.2. ∎

Remark 13.3. It is interesting to compare these two identifications with [PPZ19, Proposition 5.17].

Theorem 13.4. *Let X and X' be very general ordinary GM threefolds such that their Kuznetsov components are equivalent via $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X and X' are birationally equivalent.*

Proof. As in the argument of Theorem 10.1, the equivalence Φ induces morphisms γ or γ' between Bridgeland moduli spaces

$$\begin{tikzcd} M_\sigma(\mathcal{A}_X, -x) \arrow[r, "\gamma"] \arrow[dr, "\gamma'"] & M_\sigma(\mathcal{A}_{X'}, -x) \\ & M_\sigma(\mathcal{A}_{X'}, y-2x) \end{tikzcd}$$

For the first case, by Proposition 13.1, we get an isomorphism $\gamma : \tilde{Y}^{\geq 2}_{A^\perp} \cong \tilde{Y}^{\geq 2}_{A'^\perp}$. For the second case, we get an isomorphism $\gamma' : \tilde{Y}^{\geq 2}_{A^\perp} \cong \tilde{Y}^{\geq 2}_{A'}$ since X is general and both surfaces are minimal surfaces of general type. Note that the equivalence Φ commutes with involutions $\tau_{\mathcal{A}_X}$ and $\tau_{\mathcal{A}_{X'}}$ on the Kuznetsov components $\mathcal{A}_X$ and $\mathcal{A}_{X'}$ respectively. From Proposition 7.3 and Theorem 9.6 we know that the involutions $\tau_{\mathcal{A}_X}$ and $\tau_{\mathcal{A}_{X'}}$ induce geometrically meaningful involutions ι and τ on $\mathcal{C}_m(X) \cong \tilde{Y}^{\geq 2}_{A^\perp}$ and $M^X_G(2,1,5)$, respectively. Thus there is an isomorphism $\psi_1 : \tilde{Y}^{\geq 2}_{A^\perp}/\iota \cong \tilde{Y}^{\geq 2}_{A'^\perp}/\iota'$ or an isomorphism $\psi_2 : \tilde{Y}^{\geq 2}_{A^\perp}/\iota \cong \tilde{Y}^{\geq 2}_{A'}/\tau'$. On the other hand, the surfaces $\tilde{Y}^{\geq 2}_{A^\perp}$ and $\tilde{Y}^{\geq 2}_{A}$ admit non-trivial involutions coming from the double covers $\tilde{Y}^{\geq 2}_{A^\perp} \xrightarrow{\pi} Y^{\geq 2}_{A^\perp}$ and $\tilde{Y}^{\geq 2}_{A} \xrightarrow{\pi} Y^{\geq 2}_{A}$, but by [DIM12, Corollary 9.3], these involutions must be ι and τ respectively. Then we have an isomorphism $\psi_1 : Y^{\geq 2}_{A^\perp} \cong Y^{\geq 2}_{A'^\perp}$ or an isomorphism $\psi_2 : Y^{\geq 2}_{A^\perp} \cong Y^{\geq 2}_{A'}$ since the double covers

π are unramified double covers (the branch loci $Y^3_A = \varnothing$ and $Y^3_{A^\perp} = \varnothing$ for very general X). By [DK15, Theorem 3.25] and [DK15, Theorem 3.27], there is a bijection between the set of isomorphism classes of period partners of X and the set $(Y^2_{A^\perp} \coprod Y^3_{A^\perp})/\mathrm{PGL}(V_6)_A$ and there is a bijection between the set of isomorphism classes of period duals of X and the set $(Y^2_A \coprod Y^3_A)/\mathrm{PGL}(V_6)_A$. Since X is very general, the group $\mathrm{PGL}(V_6)_A$ is trivial by [DK15, Proposition B.9]. Then the sets $(Y^2_{A^\perp} \coprod Y^3_{A^\perp})/\mathrm{PGL}(V_6)_A = Y^{\geq 2}_{A^\perp}$ and $(Y^2_{A^\perp} \coprod Y^3_{A^\perp})/\mathrm{PGL}(V_6)_A = Y^{\geq 2}_A$. Let $c \in Y^{\geq 2}_{A^\perp}$ be a point, then it uniquely determines a period partner of X, denoted by X_c, up to isomorphism. By the refined categorical Torelli Theorem 7.17 or A.3, it is determined by the pair $\langle \mathcal{K}u(X_c), \pi(\mathcal{E}_c)\rangle$. Then $\psi_1(c) \in Y^{\geq 2}_{A'^\perp}$, denoted by c'. Then c' uniquely determines a period partner $X'_{c'}$ of X' up to isomorphism. By the refined categorical Torelli Theorem again, it is determined by the pair $\langle \mathcal{K}u(X'_{c'}), \pi(\mathcal{E}_{c'})\rangle$. Then we have

$$\psi_1[\langle \mathcal{K}u(X_c), \pi(\mathcal{E}_c)\rangle] = [\langle \Phi(\mathcal{K}u(X_c)), \Phi(\pi(\mathcal{E}_c))\rangle] = [\langle \mathcal{K}u(X'_{c'}), \pi(\mathcal{E}_{c'})\rangle].$$

As the sets $Y^{\geq 2}_{A^\perp}$ and $Y^{\geq 2}_{A'^\perp}$ admit moduli space structures and there are chains of equivalences of Kuznetsov components $\mathcal{A}_{X_c} \simeq \mathcal{A}_X \xrightarrow{\Phi} \mathcal{A}_{X'} \simeq \mathcal{A}_{X'_{c'}}$, we have $\Phi(\pi(\mathcal{E}_c)) \cong \pi(\mathcal{E}_{c'})$. This implies the equivalence $\Phi : \mathcal{K}u(X_c) \simeq \mathcal{K}u(X'_{c'})$ preserves the gluing data: $\Phi(\pi(\mathcal{E}_c)) \cong \pi(\mathcal{E}_{c'})$. Then by the refined categorical Torelli Theorem 7.17 or A.3, we get an isomorphism $X_c \cong X'_{c'}$, but by [DK15, Theorem 4.15] $X \simeq X_c$ and $X' \simeq X'_{c'}$, so we conclude that $X \simeq X'$. Similarly, the isomorphism $\psi_2(c) \in Y^{\geq 2}_{A'}$, denoted by c''. Then c'' uniquely determines a period dual X'_l of X'. Running the same argument as above, we get

$$\psi_2[\langle \mathcal{K}u(X_c), \pi(\mathcal{E}_c)\rangle] = [\langle \Phi(\mathcal{K}u(X_c), \Phi(\pi(\mathcal{E}_c))\rangle] = [\langle \mathcal{K}u(X'_l), \pi(\mathcal{E}_l)\rangle].$$

Then we have $\Phi : \mathcal{K}u(X_c) \simeq \mathcal{K}u(X'_l)$ and $\Phi(\pi(\mathcal{E}_c)) \cong \pi(\mathcal{E}_l)$, which implies that $X_c \cong X'_l$ by the refined categorical Torelli theorem. Thus $X \simeq X_c \cong X'_l \simeq X'$ and we conclude that X is birationally equivalent to X'. ◻

Remark 13.5.

(1) The argument above actually cheats a little bit as we know that when X is a general ordinary GM threefold, the set of isomorphism classes of period partners of X is not only bijective to $Y^{\geq 2}_{A^\perp}$ but the surface itself by [DIM12, Theorem 6.4]. Thus to make the arguments above rigorously work for all GM threefolds, one has to show that the equivalence induced by period partners $\mathcal{K}u(X_c) \simeq \mathcal{K}u(X)$ maps $\pi(\mathcal{E}_c)$ to $c \in Y^{\geq 2}_{A^\perp}$, or we need a non-naive version of [DK15, Theorem 3.25, Theorem 3.27].

(2) We expect the arguments would generalize to all smooth GM threefolds and the Kuznetsov-Perry Conjecture 1.6 could be proved in full generality. If X is not general, then the double dual EPW surface $\tilde{Y}^{\geq 2}_{A^\perp}$ and double EPW surface $\tilde{Y}^{\geq 2}_A$ may have singularities. We expect the equivalence of Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ would identify the smooth loci of $Y^{\geq 2}_{A^\perp}$ and $Y^{\geq 2}_{A'^\perp}$. Then applying [DK15, Theorem 3.25, Theorem 3.27], the result follows. We are now studying this in an ongoing work [LZ21].

14. Irreducibility of Bridgeland moduli spaces

14.1. **The moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$.** In this subsection, we show that $M_G(2, 1, 5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$. We begin with a lemma.

Lemma 14.1. *The only rank 2 μ-semistable reflexive sheaf E with Chern character $\mathrm{ch}(E) = 2 - H + aL + (\frac{5}{6} + \frac{b}{2})P$ for $a = 1, 2$ and $b \in \mathbb{Z}$ is $E \cong \mathcal{E}$.*

Proof. First we assume that $a = 2$. Then $c_1(E) = -1$ and $c_2(E) = 3$. By Maruyama's Restriction Theorem, we can choose a general section S of $\mathcal{O}_X(H)$ such that $E|_S$ is μ-semistable. Since $(\mathrm{rk}(E|_S), c_1(E|_S)) = 1$, we know $E|_S$ is μ-stable. In particular, $E|_S$ is simple. But by [HL10, Chapter 6], the dimension of the moduli space of rank 2 simple sheaves with Chern classes $c_1 = -1$ and $c_2 = 3$ on S is negative, hence this is impossible.

Now we assume that $a = 1$. Then $c_1(E) = -1, c_2(E) = 4$ and $\chi(E) = \frac{b+1}{2}$. Again by the restriction theorem, we can choose a general section S of $\mathcal{O}_X(H)$ such that $E|_S$ is μ-semistable. Since E is reflexive, we can furthermore assume that $E|_S$ is locally free. Since $(\mathrm{rk}(E|_S), c_1(E|_S)) = 1$, we know $E|_S$ is μ-stable. By [Muk02], we know $E|_S \cong \mathcal{E}|_S = T|_S$, where $T^\vee$ is the tautological bundle on $\mathrm{Gr}(2,5)$. Since S is a linear section of $\mathrm{Gr}(2,5)$, we have a Koszul complex

$$0 \to T(-4) \to \dots \to T(-1)^{\oplus 4} \to T \to E|_S \to 0$$

Thus $h^1(E(kH)|_S) = 0$ for all $k \in \mathbb{Z}$. Now consider the exact sequence

$$0 \to E(-H) \to E \to E|_S \to 0.$$

twisted by $\mathcal{O}_X(kH)$, which together with the vanishing of $H^1(E(kH)|_S)$ for all k, implies that $H^2(E((k-1)H))$ embeds inside $H^2(E(kH))$. Hence they are both always zero since $H^2(E(kH)) = 0$ for $k \gg 0$. In particular we have $h^2(E) = 0$. Thus $\chi(E) = -h^1(E) \leq 0$. This means $b \leq -1$. But from E being reflexive we know that $c_3(E) \geq 0$. Therefore $b = -1$ and we have $c_1(E) = -1, c_2(E) = 4$, and $c_3(E) = 0$. Thus E is furthermore locally free and by [Muk02] we know $E \cong \mathcal{E}$. ◻

We outline the steps to prove irreducibility of the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$ as follows.

(1) First, by a similar argument as in [PY20, Proposition 4.6], we show that F is in the first tilted heart $\mathrm{Coh}^0(X)$ and it is $\sigma_{\alpha,0}$-semistable for all $\alpha > 0$. By [BBF$^+$20, Proposition 4.9], the cohomology sheaf $\mathcal{H}^{-1}(F)$ is a μ-semistable reflexive sheaf and $\mathcal{H}^0(F)$ is a torsion sheaf supported in dimension ≤ 1.
(2) If $\mathcal{H}^0(F) = 0$, we show that $F[-1]$ is a rank 2 stable vector bundle with Chern classes $c_1 = -1$ and $c_2 = 5$. Then by the classification of sheaves in the moduli space $M_G(2,1,5)$, we show that it must be either the projection of a globally generated vector bundle or non-locally free sheaf.
(3) If $\mathcal{H}^0(F) \neq 0$, we use the Bogomolov-Gieseker inequality and Lemma 14.1 to show $\mathcal{H}^{-1}(F) \cong \mathcal{E}$ and $\mathcal{H}^0(F) \cong \mathcal{O}_L(-1)$, which means F is the projection of some sheaf in $M_G(2,1,5)$ by Lemmas 9.4 and 11.10.

Theorem 14.2. *Let $F \in \mathcal{A}(\alpha,\beta)$ be a $\sigma(\alpha,\beta)$-stable object with numerical class $y - 2x$. Then $F = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$. Here we set (α,β) as in Theorem 4.11.*

Proof. First we argue as in [PY20, Proposition 4.6]. When $(\alpha_0,\beta_0) = (0,0)$, we have $\mu^0_{\alpha_0,\beta_0}(F) = +\infty$. Since there is no wall intersecting with $\beta = 0$, we know that F is $\sigma^0_{\alpha,0}$-semistable for all $\alpha > 0$. By the definition of the double tilted heart, we have a triangle

$$A[1] \to F \to B$$

such that A (resp. B) is in $\mathrm{Coh}^0(X)$ with $\sigma_{\alpha,0}$-semistable factors having slope $\mu_{\alpha,0} \leq 0$ (resp. $\mu_{\alpha,0} > 0$). Since F is $\sigma^0_{\alpha,0}$-semistable and $\mu^0_{\alpha,0}(F) < 0$, we have that $\mathrm{ch}(A[1]) = (0,0,0,-mP)$ for $m \geq 0$. Thus $\mathrm{ch}(B) = (-2, H, 0, (-\frac{5}{6}+m)P)$. Moreover, B is $\sigma^0_{\alpha,0}$-semistable, and since $B \in \mathrm{Coh}^0(X)$, we have that B is $\sigma_{\alpha,0}$-semistable. From the non-existence of walls for $\beta = 0$, we know B is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. Thus by [Li16], [BMS16, Conjecture 4.1] holds for $(\alpha, \beta) = (0,0)$. Hence we have $m \leq \frac{5}{6}$, which implies $m = 0$ and $A[1] = 0$. Therefore $F = B$ is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. Thus by [BBF+20, Proposition 4.9], $\mathcal{H}^{-1}(F)$ is a μ-semistable reflexive sheaf and $\mathcal{H}^0(F)$ is 0 or supported in dimension ≤ 1. If $\mathcal{H}^0(F)$ is supported in dimension 0, then $\mathrm{ch}(\mathcal{H}^0(F)) = bP$ for $b \geq 1$. But this is impossible since $\mathcal{H}^{-1}(F)$ is reflexive and we have $c_3(\mathcal{H}^0(F)) \geq 0$. If $\mathcal{H}^0(F)$ supported in dimension 1, we can assume $\mathrm{ch}(\mathcal{H}^0(F)) = aL + \frac{b}{2}P$ where $a \geq 1$ and b are integers. Thus $\mathrm{ch}(\mathcal{H}^{-1}(F)) = 2 - H + aL + (\frac{5}{6} + \frac{b}{2})P$. Using the Bogomolov-Gieseker inequality on $\mathcal{H}^{-1}(E)$ we know $a = 1$ or 2. But from Lemma 14.1, we know $\mathcal{H}^{-1}(E) \cong \mathcal{E}$ and $\mathrm{ch}(\mathcal{H}^0(E)) = L - \frac{P}{2}$. Thus $\mathcal{H}^0(F) \cong \mathcal{O}_L(-1)$ for some line L on X. Therefore we have a triangle

$$\mathcal{E}[1] \to F \to \mathcal{O}_L(-1).$$

In this case we have $\mathrm{Hom}(\mathcal{O}_L(-1), \mathcal{E}[2]) = \mathrm{Hom}(\mathcal{E}^\vee(1), \mathcal{O}_L[1]) = H^1(L, \mathcal{E}(-1)|_L) = H^1(L, \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) = k$. Hence by Lemmas 9.4 and 11.10, $F \cong \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is locally free but not globally generated.

If $\mathcal{H}^0(F) = 0$, we have $F[-1] \cong \mathcal{H}^{-1}(F)$. Then $F[-1]$ is a μ-semistable sheaf. Since $F[-1]$ is reflexive and $c_3(F[-1]) = 0$, $F[-1] \in M_G(2,-1,5)$ is a stable vector bundle. Thus by Lemmas 9.4 and 11.10, we know $F[-1] = \mathrm{pr}(E)$ for some $E \in M_G(2,1,5)$ such that E is a globally generated vector bundle or non-locally free sheaf. ◻

14.2. **The moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$.** In this subsection we show that $\mathcal{C}_m(X) \cong \mathcal{M}_\sigma(\mathcal{A}_X, -x)$.

The key observation in the lemma below is that $\beta = 0$ is the vertical numerical wall for the class $-x = -(1-2L)$. But we start with an object in $\mathcal{A}_X$, so this will never be an actual wall.

Lemma 14.3. *If $F \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha,\beta)$-stable such that $[F] = -x$ and F is $\sigma^0_{\alpha_0,0}$-semistable for some $0 < \alpha_0 < \frac{1}{10}$, then $F \cong I_C[1]$ for some conic C on X.*

Proof. If $\beta = 0$ is not the actual wall for F, then F is $\sigma^0_{\alpha,0}$-stable for all $\alpha > 0$. Thus a similar argument as in [PR20, Lemma 5.7] shows that $F[-1] \in \mathrm{Coh}^0(X)$ is $\sigma_{\alpha,0}$-semistable for every $\alpha > 0$. By [BMS16, Lemma 2.7], $F[-1]$ is a torsion-free μ-semistable sheaf. Since $\mathrm{ch}(I_C) = \mathrm{ch}(F[-1])$ and $\mathrm{Pic}(X) \cong \mathbb{Z}$, we know $F[-1] \cong I_C$ for some conic C on X.

If $\beta = 0$ is an actual wall, then it is given by a triangle in $\mathrm{Coh}^0_{\alpha_0,0}(X)$

$$A \to F \to B$$

such that $\mu^0_{\alpha_0,0}(A) = \mu^0_{\alpha_0,0}(F) = \mu^0_{\alpha_0,0}(B)$ and $A, B \in \mathrm{Coh}^0_{\alpha_0,0}(X)$ are both $\sigma^0_{\alpha_0,0}$-semistable. Thus we have:

(1) $\mathrm{Im}(Z^0_{\alpha_0,0})(A) \geq 0, \mathrm{Im}(Z^0_{\alpha_0,0})(B) \geq 0$
(2) $\mu^0_{\alpha_0,0}(A) = \mu^0_{\alpha_0,0}(F) = \mu^0_{\alpha_0,0}(B)$
(3) $0 \leq \Delta(A) + \Delta(B) \leq \Delta(F)$ and equality holds if and only if one of A and B is supported in dimension zero.

Now we assume that

$$\mathrm{ch}(A) = (a, bH, \frac{c}{10}H^2, \frac{d}{20}H^3) \text{ and } \mathrm{ch}(B) = (-a-1, -bH, \frac{2-c}{10}H^2, \frac{-d}{20}H^3)$$

for $a, b, c, d \in \mathbb{Z}$. We have following relations:

(1) $5\alpha_0^2(a+1) + 2 \geq c \geq 5\alpha_0^2 a$
(2) $b = 0$
(3) $-2 \leq ac \leq 0, -2 \leq (a+1)(c-2) \leq 0$

It is not hard to see that the only possible solutions to the inequalities above are $\mathrm{ch}_{\leq 2}(A) = (0, 0, 2L)$, $\mathrm{ch}_{\leq 2}(B) = (-1, 0, 0)$ or $\mathrm{ch}_{\leq 2}(B) = (0, 0, 2L)$, $\mathrm{ch}_{\leq 2}(A) = (-1, 0, 0)$. In both cases there is no wall when $\beta = 0$ for A and B. Thus A and B are $\sigma^0_{\alpha,0}$-stable for all $\alpha > 0$. Since $\mathrm{ch}_3(A) + \mathrm{ch}_3(B) = 0$, using Lemma 4.7 and a similar argument as in [PR20, Lemma 5.7], we know that A and B are $\sigma_{\alpha,0}$-stable for every $\alpha > 0$ and $d = 0$. Thus by [BMS16, Lemma 2.7], we have $A \cong \mathcal{O}_C$, $B \cong \mathcal{O}_X[1]$ or $B \cong \mathcal{O}_C$, $A \cong \mathcal{O}_X[1]$ for some conics C on X. From $F \in \mathcal{A}_X$ we know that the only possible case is $A \cong \mathcal{O}_C$ and $B \cong \mathcal{O}_X[1]$. Thus we have a triangle $F \to \mathcal{O}_X[1] \to \mathcal{O}_C[1]$. Since $\hom(\mathcal{O}_X, \mathcal{O}_C) = 1$, this implies $F \cong I_C[1]$. $\square$

When F is not $\sigma_{\alpha,0}$-semistable, the argument is slightly more complicated. Our main tools are inequalities in [PR20], [PY20, Proposition 4.1], and [Li16, Proposition 3.2], which allow us to bound the rank and first two Chern characters $\mathrm{ch}_1, \mathrm{ch}_2$ of the destabilizing objects and their cohomology objects. Since $F \in \mathcal{A}_X$, by using the Euler characteristics $\chi(\mathcal{O}_X[1], -)$ and $\chi(-, \mathcal{O}_X(-1)[2])$ we can obtain a bound on ch_3. Finally, via a similar argument as in Lemma 14.1 we deduce that the Harder–Narasimhan factors of F are the ones we expect.

Lemma 14.4. *If $F \in \mathcal{A}(\alpha, \beta)$ is $\sigma(\alpha, \beta)$-stable such that $[F] = -x$ and F is not $\sigma^0_{\alpha,0}$-semistable for every $0 < \alpha < \frac{1}{10}$, then F fits into a triangle*

$$\mathcal{E}[2] \to F \to \mathcal{Q}^\vee[1].$$

Proof. Let $0 \to A \to F \to B \to 0$ be the destabilizing sequence of F in $\mathrm{Coh}^0_{\alpha,0}(X)$ such that $A, B \in \mathrm{Coh}^0_{\alpha,0}(X)$ are $\sigma^0_{\alpha,0}$-semistable with $\mu^0_{\alpha,0}(A) > \mu^0_{\alpha,0}(F) = 0 > \mu^0_{\alpha,0}(B)$ and α sufficiently small. By [BLMS17, Remark 5.12], we have $\mu^0_{\alpha,0}(B) \geq \min\{\mu^0_{\alpha,0}(F), \mu^0_{\alpha,0}(\mathcal{O}_X), \mu^0_{\alpha,0}(\mathcal{E}^\vee)\}$. By continuity we have

(1) $\mu^0_{0,0}(A) \geq \mu^0_{0,0}(F) = 0 \geq \mu^0_{0,0}(B)$,
(2) $\mathrm{Im}(Z^0_{0,0}(A)) \geq 0$, $\mathrm{Im}(Z^0_{0,0}(B)) \geq 0$,
(3) $\mu^0_{0,0}(B) \geq \min\{\mu^0_{0,0}(F), \mu^0_{0,0}(\mathcal{O}_X), \mu^0_{0,0}(\mathcal{E}^\vee)\}$,
(4) $\Delta(A) \geq 0$, $\Delta(B) \geq 0$.

Assume $[A] = a[\mathcal{O}_X] + b[\mathcal{O}_H] + c[\mathcal{O}_L] + d[\mathcal{O}_P]$. Then $[B] = (-1-a)[\mathcal{O}_X] - b[\mathcal{O}_H] + (2-c)[\mathcal{O}_L] - (1+d)[\mathcal{O}_P]$. Then $\mathrm{ch}(A) = (a, bH, \frac{c-5b}{10}H^2, \frac{\frac{5}{3}b + \frac{c}{2} + d}{10}H^3)$ and $Z^0_{0,0}(A) = bH^3 + (\frac{c-5b}{10}H^3) \cdot i$ and $\mu^0_{0,0}(A) = \frac{10b}{5b-c}, \mu^0_{0,0}(B) = \frac{-10b}{c-5b-2}$. Note that $[F] = -[\mathcal{O}_X] + 2[\mathcal{O}_L] - [\mathcal{O}_P]$. From (2) we know $c - 5b = 1$, 0 or 2.

First we assume that $c - 5b = 1$. Then we have

(1) $-1 \leq b \leq 0$,
(2) $b^2 - \frac{a}{5} \geq 0$,
(3) $b^2 + \frac{a+1}{5} \geq 0$.

If $b = 0$, then $c = 1$. Therefore $-1 \leq a \leq 0$. If $a = 0$, since A is $\sigma^0_{\alpha,0}$-semistable, we know $\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A)$ is either 0 or supported on points. Thus $\mathrm{ch}_{\leq 2}(\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)) =$

$(0,0,-L)$. But $\mathrm{Re}(Z_{\alpha,0}(\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)))>0$ which is impossible since $\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)\in\mathrm{Coh}^0(X)$. The symmetric argument also shows $a=-1$ is impossible.

Therefore we have $b=-1$ and $c=-4$. Hence $-6\leq a\leq 5$. As in the previous case we know $\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A)$ ($\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(B)$) is either 0 or supported on points. Thus we know $\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)$ and $\mathcal{H}^0_{\mathrm{Coh}^0(X)}(B)$ are $\sigma_{\alpha,0}$-semistable. Using [BMS16, Corollary 3.10] and arguments in [PY20, Proposition 4.1], it is not hard to see that they are both $\sigma_{\alpha,0}$-stable. Thus by [Li16, Proposition 3.2] we know $-1\leq a\leq 2$.

If $a=0$, then we have $\mathrm{ch}(A)=(0,-H,L,(d-\frac{11}{3})P)$. Thus $\chi(\mathcal{O}_X[1],A)=6-d$. Since $\mathcal{O}_X[1]\in\mathrm{Coh}^0_{\alpha,0}(X)$ is $\sigma^0_{\alpha,0}$-stable with $\mu^0_{\alpha,0}(\mathcal{O}_X[1])=0$ and $\mathcal{O}_X(-1)[2]\in\mathrm{Coh}^0_{\alpha,0}(X)$ is $\sigma^0_{\alpha,0}$-stable with $\mu^0_{\alpha,0}(\mathcal{O}_X(-1)[2])=\frac{2}{1-\alpha^2}$, using Serre duality, from a elementary computations and comparisons of $\mu^0_{\alpha,0}$, we have

- $\chi(\mathcal{O}_X[1],A)>0\Rightarrow \hom(\mathcal{O}_X[1],A)>0$,
- $\chi(B,\mathcal{O}_X(-1)[2])>0\Rightarrow\hom(B,\mathcal{O}_X(-1)[2])>0$.

By Serre duality we have $\chi(B,\mathcal{O}_X(-1)[2])=\chi(\mathcal{O}_X[1],B)$. Thus from $\chi(\mathcal{O}_X,F)=0$ we know that if $d\neq 6$, then either $0<\hom(\mathcal{O}_X[1],A)\leq\hom(\mathcal{O}_X[1],F)$ or $0<\hom(B,\mathcal{O}_X(-1)[2])\leq\hom(F,\mathcal{O}_X(-1)[2])$. Both cases are contradictions to $F\in\mathcal{A}_X$. Thus the only possible case is $d=6$, where $\mathrm{ch}(A)=(0,-H,L,\frac{7}{3}P)$ and $\mathrm{ch}(B)=(-1,H,L,-\frac{7}{3}P)$. Now since $\chi(\mathcal{O}_X[1],A)=\chi(B,\mathcal{O}_X(-1)[2])=0$ and $\chi(\mathcal{O}_X[1],\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A))\leq 0$ and $\chi(\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(B)[1],\mathcal{O}_X(-1)[2])\leq 0$, we have $\chi(\mathcal{O}_X[1],\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)[1])\geq 0$ and $\chi(\mathcal{H}^0_{\mathrm{Coh}^0(X)}(B),\mathcal{O}_X(-1)[2])\geq 0$. Thus for the same reasons as above, the only possible case is $\chi(\mathcal{O}_X[1],\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A))=0$ and $\chi(\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(B)[1],\mathcal{O}_X(-1)[2])=0$, which means $\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A)=\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(B)=0$. Therefore $A[-1]\in\mathrm{Coh}^0(X)$ and $B\in\mathrm{Coh}^0(X)$ is $\sigma_{\alpha,0}$-stable. In this case $\mathrm{ch}(A)=(0,-H,L,\frac{7}{3}P)$ and $\mathrm{ch}(B)=(-1,H,L,-\frac{7}{3}P)$. Now since $\mathcal{E}^\vee,\mathcal{E}[2]\in\mathrm{Coh}^0_{\alpha,0}(X)$ and $\hom(\mathcal{E}^\vee,A[2])=\hom(A,\mathcal{E}[1])=0$, we have the relation

$$\chi(\mathcal{E}^\vee,A)>0\Rightarrow\hom(\mathcal{E}^\vee,A)>0.$$

In our case we have $\chi(\mathcal{E}^\vee,A)=9$, hence $0<\hom(\mathcal{E}^\vee,A)\leq\hom(\mathcal{E}^\vee,F)$, which contradicts $F\in\mathcal{A}_X$.

Thus the only possible case is $a=2,b=-1$ and $c=-4$. This means $\mathrm{ch}_{\leq 2}(A)=\mathrm{ch}_{\leq 2}(\mathcal{E}[2])$ and $\mathrm{ch}_{\leq 2}(B)=\mathrm{ch}_{\leq 2}(\mathcal{Q}^\vee[1])$. Using inequalities in Lemma 14.3, it is not hard to see that A and B are $\sigma^0_{\alpha,0}$-stable for every $\alpha>0$. Since $F\in\mathcal{A}_X$, from reasons above we have $A[-1]\in\mathrm{Coh}^0(X)$ and $B\in\mathrm{Coh}^0(X)$ are $\sigma_{\alpha,0}$-semistable for every $\alpha>0$. Thus by [BBF+20, Proposition 4.9] $\mathcal{H}^{-1}(A[-1])$ and $\mathcal{H}^{-1}(B)$ are μ-semistable reflexive sheaves and, $\mathcal{H}^0(A[-1])$ and $\mathcal{H}^0(B)$ are supported in dimension ≤ 1. By Lemma 14.1 we know that actually $\mathcal{H}^{-1}(A[-1])\cong A[-2]\cong\mathcal{E}$. Now assume $\mathrm{ch}(\mathcal{H}^0(B))=pL+\frac{q}{2}P$ for $p,q\in\mathbb{Z}$. If $p\neq 0$, then $p>0$. Applying the Bogomolov-Gieseker inequality to $\mathcal{H}^{-1}(B)$, we obtain $p=1$. But this is impossible by Koseki's Stronger BG inequality 4.8.

Thus we have $\mathrm{ch}(\mathcal{H}^0(B))=qP$ for $q\in\mathbb{Z}_{\geq 0}$. In this case $\mathrm{ch}(\mathcal{H}^{-1}(B))=(3,-H,-L,(\frac{1}{3}+q)P)$. Applying $\mathrm{Hom}(\mathcal{O}_X,-)$ to the triangle $\mathcal{H}^{-1}(B)[1]\to B\to\mathcal{H}^0(B)$, we obtain $h^0(\mathcal{H}^{-1}(B))=h^1(\mathcal{H}^{-1}(B))=0$. From $\mathcal{O}_X(H)\in\mathrm{Coh}^0_{\alpha,0}(X)$ and $\mu^0_{\alpha,0}(\mathcal{O}_X(1))>\mu^0_{\alpha,0}(B)$, we know $\hom(\mathcal{O}(H),B)=0$. Then applying $\mathrm{Hom}(\mathcal{O}_X(H),-)$ to the triangle again, we obtain $\hom(\mathcal{O}_X(H),\mathcal{H}^{-1}(B)[1])=0$. Since $\mathcal{H}^{-1}(B)$ is reflexive, we can find a general smooth section $S\in|H|$ such that $\mathcal{H}^{-1}(B)|_S$ is locally free. We claim that $\mathcal{H}^{-1}(B)|_S$ is μ-stable. Indeed, we have an exact sequence $0 \to \mathcal{H}^{-1}(B)(-1) \to \mathcal{H}^{-1}(B) \to \mathcal{H}^{-1}(B)|_S \to 0$. Then taking cohomology we have $h^0(\mathcal{H}^{-1}(B)|_S) = h^1(\mathcal{H}^{-1}(B)(-1)) = 0$. By the inclusion $\wedge^{p-1}\mathcal{H}^{-1}(B)|_S \hookrightarrow \wedge^p\mathcal{H}^{-1}(B)|_S$, we obtain $h^0(\wedge^p\mathcal{H}^{-1}(B)|_S) = 0$ for all $p \geq 0$. Then from Hoppe's criterion we know $\mathcal{H}^{-1}(B)|_S$ is a stable vector bundle. Therefore $\mathcal{H}^{-1}(B)|_S \cong \mathcal{Q}^\vee|_S$. Hence using the Koszul resolution, a similar argument as in Lemma 14.1 shows that $\chi(\mathcal{H}^{-1}(B)) \leq 0$, which implies $q = 0$. Then $c_3(\mathcal{H}^{-1}(B)) = 0$, which implies $\mathcal{H}^{-1}(B) \cong B[-1]$ is a stable vector bundle. Now applying $\mathrm{Hom}(B[-1], -)$ to the sequence $0 \to \mathcal{Q}^\vee(-1) \to \mathcal{O}_X^{\oplus}(-1) \to \mathcal{E} \to 0$, since $\mathrm{RHom}^\bullet(B, \mathcal{O}_X(-1)) = 0$, we obtain $\mathrm{ext}^3(B[-1], \mathcal{Q}^\vee(-1)) = \mathrm{ext}^2(B[-1], \mathcal{E}) = \hom(B, \mathcal{E}[3])$. But from $A \cong \mathcal{E}[2]$ and the triangle $A \to F \to B$, we have $\hom(B, \mathcal{E}[3]) = \mathrm{ext}^3(B[-1], \mathcal{Q}^\vee(-1)) > 0$. Thus from Serre duality we obtain $\hom(\mathcal{Q}^\vee, B[-1]) > 0$. Since $B[-1]$ and $\mathcal{Q}^\vee$ are both stable vector bundles of rank 3, they are isomorphic $B[-1] \cong \mathcal{Q}^\vee$.

Next we assume that $c - 5b = 0$. We have

(1) $-2 \leq b \leq 0$,
(2) $b^2 + \frac{2a+2}{5} \geq 0$.

If $b = 0$, then $c = 0$ and $a \geq -1$. For the same reason as above we know $\mathcal{H}^0_{\mathrm{Coh}^0(X)}(A)$ is either 0 or supported on points. Thus $\mathrm{ch}_{\leq 2}(\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A)) = (-a, 0, 0)$ and it is $\sigma_{\alpha,0}$-semistable. A standard argument shows that $\mathcal{H}^{-1}_{\mathrm{Coh}^0(X)}(A) \cong \mathcal{O}_X$. But since $F \in \mathcal{A}_X$, this is impossible.

If $b = -1$, we have $c = -5$ and $a \geq -3$. Since $A \in \mathrm{Coh}^0_{\alpha,0}(X)$, we have $-3 \leq a \leq 0$. Then as in the previous case we know $d = 7 - a$, $A[-1] \in \mathrm{Coh}^0(X)$ and $B \in \mathrm{Coh}^0(X)$ are $\sigma_{\alpha,0}$-stable. By [Li16, Proposition 3.2] we know $-2 \leq a \leq 2$. From the inequalities in [PY20, Proposition 4.1] it is not hard to see that $A[-1]$ and B are $\sigma_{\alpha,0}$-stable for every $\alpha > 0$. In this case $\mathrm{ch}(A) = (a, -H, 0, (\frac{17}{6} - a)P)$ and $\mathrm{ch}(B) = (-1 - a, H, 2L, (a - \frac{17}{6})P)$. In this case, we have $\chi(\mathcal{E}^\vee, A) = 4 - 2a$. As previous cases, the only possible case is $\chi(\mathcal{E}^\vee, A) \leq 0$, which is impossible.

If $b = -2$, we have $c = -10$ and $a \geq -11$. Since $A \in \mathrm{Coh}^0_{\alpha,0}(X)$, we have $-11 \leq a \leq 0$. Then a similar argument as in the previous cases shows that $A[-1] \in \mathrm{Coh}^0(X)$ and $B \in \mathrm{Coh}^0(X)$ are $\sigma_{\alpha,0}$-semistable. The Chern character is $\mathrm{ch}(A) = (a, -2H, 0, (\frac{17}{3} - a)P)$ and $\mathrm{ch}(B) = (-1 - a, 2H, 2L, (a - \frac{17}{3})P)$. In this case $\chi(\mathcal{E}^\vee, A) = 8 - 2a$. As in the previous cases, we have $\chi(\mathcal{E}^\vee, A) > 0$, which gives a contradiction. □

Theorem 14.5. *Let X be a GM threefold. Then the the irreducible component $\mathcal{S}$ in Theorem 7.13 is the whole moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$.*

Proof. Note that $\hom(\mathcal{Q}^\vee[1], \mathcal{E}[2]) = 1$. Then the result follows from Lemma 14.3 and Lemma 14.4. □

15. Hochschild (co)homology and infinitesimal Torelli theorems

15.1. **Definitions.** In this subsection, we recall basic Hochschild (co)homology of admissible subcategories of $\mathrm{D}^b(X)$, where X is a smooth projective variety. We refer the reader to [Kuz09b] for more details. For Hochschild (co)homology of dg-categories, we refer to Bernhard Keller's papers and surveys [Kel98, Kel06].

Definition 15.1 ([Kuz09b])**.** Let X be a smooth projective variety, and $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$. Consider any semiorthogonal decomposition of

$\mathrm{D}^b(X)$ that contains $\mathcal{A}$ as a component. Let P be the kernel of the projection to $\mathcal{A}$. The *Hochschild cohomology* of $\mathcal{A}$ is defined as

$$\mathrm{HH}^*(\mathcal{A}) := \mathrm{Hom}^*(P, P).$$

The *Hochschild homology* of $\mathcal{A}$ is defined as

$$\mathrm{HH}_*(\mathcal{A}) := \mathrm{Hom}^*(P, P \circ S_X).$$

Remark 15.2. The definition of Hochschild cohomology is independent of the semiorthogonal decomposition. There is a natural identification of $\mathrm{HH}_*(\mathcal{A})$ in $\mathrm{HH}_*(X)$, and the subspace is independent of the semiorthogonal decomposition.

Lemma 15.3 ([Kuz09b, Theorem 4.5, Proposition 4.6])**.** *Choose a strong compact generator E of $\mathcal{A}$, and define $A = \mathrm{RHom}^\bullet(E, E)$. Then there are isomorphisms*

$$\mathrm{HH}^*(A) \cong \mathrm{HH}^*(\mathcal{A}) \quad \textit{and} \quad \mathrm{HH}_*(A) \cong \mathrm{HH}_*(\mathcal{A}).$$

Remark 15.4.

(1) Let $\mathrm{Per}_{\mathrm{dg}}(X)$ be a dg-enhancement of $\mathrm{Perf}(X)$ whose objects are K-injective perfect complexes, and $\mathcal{A}_{\mathrm{dg}}$ be a dg-subcategory of $\mathrm{Per}_{\mathrm{dg}}(X)$ whose objects are in $\mathcal{A}$. Then

$$\mathrm{HH}_*(\mathcal{A}) \cong \mathrm{HH}_*(\mathcal{A}_{\mathrm{dg}}) \quad \text{and} \quad \mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HH}^*(\mathcal{A}_{\mathrm{dg}})$$

because the morphism of dg-categories $* \to \mathcal{A}_{\mathrm{dg}}$ is a derived Morita equivalence. Here $*$ is a dg-category with one object and the endomorphism of this unique object is the dg-algebra A; the morphism sends the unique object to an K-injective resolution of E.

(2) Since $\mathrm{Perf}(X)$ is a unique enhanced triangulated category [LO10], the Hochschild (co)homology of admissible subcategories of $\mathrm{Perf}(X)$ defined by Kuznetsov coincides with that of the dg-enhancement that naturally comes from dg-enhancements of $\mathrm{Perf}(X)$.

Let A be a k-algebra. Note that the Hochschild homology $\mathrm{HH}_*(A)$ is a graded $\mathrm{HH}^*(A)$-module. The module structure is easily described by the definition of Hochschild (co)homology via the Ext and Tor functors. It encodes deformation information of the algebra and its invariant Hochschild homology. For example, when considering a variety X, the degree 2 Hochschild cohomology has a factor $H^1(X, T_X)$ which is the first order deformations of X. The action of $H^1(X, T_X)$ on the Hochschild homology via the module structure can be imagined as deformations of a certain invariant with respect to deformation of X. Here, we have the invariant $\mathrm{HH}_\bullet(X)$ that is closely related to the intermediate Jacobian of X. When X is a Fano threefold, the action of $H^1(X, T_X)$ on $\mathrm{HH}_{-1}(X) = H^{2,1}(X)$ is the derivative of period map.

In the case of admissible subcategories of derived categories, we can describe the module structure by kernels.

Definition 15.5. Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$, and P the kernel of left projection to $\mathcal{A}$. Take $\alpha \in \mathrm{HH}^{t_1}(\mathcal{A})$, and $\beta \in \mathrm{HH}_{t_2}(\mathcal{A})$. The action of α on β is the composition

$$P \xrightarrow{\beta} P \circ S_X[t_2] \xrightarrow{\alpha \otimes \mathrm{id}} P \circ S_X[t_1 + t_2].$$

Proposition 15.6. *Let $\mathcal{A}$ be an admissible subcategory of* $\mathrm{Perf}(X)$*. Let E be a strong compact generator of $\mathcal{A}$, and $A := \mathrm{RHom}^\bullet(E, E)$. The isomorphisms $\mathrm{HH}^*(A) \cong \mathrm{HH}^*(\mathcal{A})$ and $\mathrm{HH}_*(A) \cong \mathrm{HH}_*(\mathcal{A})$ in Lemma 15.3 preserve both sides of the obvious module structure and algebra structure of Hochschild cohomology.*

Proof. These facts can be proved via the description in [Kuz09b, Proposition 4.6]. ◻

Theorem 15.7. *Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$, and $\mathcal{B}$ be an admissible subcategory of $\mathrm{D}^b(Y)$. Suppose the Fourier–Mukai functor $\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ induces an equivalence of subcategories $\mathcal{A}$ and $\mathcal{B}$. Then we have isomorphisms of Hochschild cohomology*

$$\mathrm{HH}^\bullet(\mathcal{A}) \cong \mathrm{HH}^\bullet(\mathcal{B})$$

and Hochschild homology

$$\mathrm{HH}_\bullet(\mathcal{A}) \cong \mathrm{HH}_\bullet(B)$$

which preserve both sides of the module structure and algebra structure.

Proof. We have $\mathcal{A}_{\mathrm{dg}} \cong \mathcal{B}_{\mathrm{dg}}$ in the homotopy category whose weak equivalences are Morita equivalences [BT14, Section 9]. Hence, there is an isomorphism $\mathrm{Per}_{\mathrm{dg}}(\mathcal{A}_{\mathrm{dg}}) \cong \mathrm{Per}_{\mathrm{dg}}(\mathcal{B}_{\mathrm{dg}})$ in Hqe [Tab05]. That is, $\mathcal{A}_{\mathrm{dg}}$ and $\mathcal{B}_{\mathrm{dg}}$ are connected by a chain of Morita equivalences

$$\begin{tikzcd}
 & \mathrm{Per}_{\mathrm{dg}}(\mathcal{A}_{\mathrm{dg}}) \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathrm{Per}_{\mathrm{dg}}(\mathcal{B}_{\mathrm{dg}}) & \\
\mathcal{A}_{\mathrm{dg}} \arrow[ur] & & C_0 & \cdots & C_n \arrow[ur] & & \mathcal{B}_{\mathrm{dg}} \arrow[ul]
\end{tikzcd}$$

According to [AK19, Theorem 3.1], if two dg-categories are derived equivalent induced by a bi-module (Morita equivalence), then the equivalence induces an isomorphism of Hochschild (co)homology and preserves the module structure. ◻

Let X be a smooth algebraic variety. Classically we have the HKR isomorphisms [Kuz09b, Theorem 8.3] given by

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_{p+q=*} H^p(X, \wedge^q T_X)$$

and

$$\mathrm{Hom}^{-*}(X \times X, \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) \cong \bigoplus_{p-q=*} H^p(X, \Omega_X^q).$$

However, the original HKR isomorphisms may not preserve the obvious two-sided algebra structure and the module structure. After some twisting of the HKR isomorphisms which we will denote as IK, it was originally conjectured by in [Căl05, Conjecture 5.2] and proved in [CRVdB12, Theorem 1.4] that the new HKR isomorphisms preserve the obvious module structure.

15.2. **Deformations and infinitesimal (categorical) Torelli theorems.**

Definition 15.8 ([Per20, Definition 5.24])**.** Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^b(X)$ and consider the diagram

$$\begin{tikzcd} K_1^{\mathrm{top}}(\mathcal{A}) \arrow[r, "\mathrm{ch}_1^{\mathrm{top}}"] \arrow[drr, "P'"'] & \mathrm{HP}_1(\mathcal{A}) \arrow[r, "\cong"] & \bigoplus_n \mathrm{HH}_{2n-1}(\mathcal{A}) \arrow[d, "P"] \\ & & \mathrm{HH}_1(\mathcal{A}) \oplus \mathrm{HH}_3(\mathcal{A}) \oplus \cdots \end{tikzcd}$$

Define the *intermediate Jacobian of* $\mathcal{A}$ as

$$J(\mathcal{A}) = (\mathrm{HH}_1(\mathcal{A}) \oplus \mathrm{HH}_3(\mathcal{A}) \oplus \cdots)/\Gamma,$$

where Γ is the image of P'. Note that Γ is a lattice.

Remark 15.9. In general, $J(\mathcal{A})$ is a complex torus. When X is a Fano threefold of index 1 or 2, we have a non-trivial admissible subcategory $\mathcal{K}u(X)$, called the Kuznetsov component, see the survey [Kuz16] or Section 3. Clearly $H^1(X, \mathbb{Z}) = 0$. According to [Per20, Lemma 5.2, Proposition 5.3], there is a Hodge isometry $K_{-3}^{\mathrm{top}}(\mathcal{K}u(X))_{\mathrm{tf}} \cong H^3(X, \mathbb{Z})_{\mathrm{tf}}$ which preserves both pairings. The left hand side pairing is the Euler paring, and the right hand side is the cohomology paring. Then, $J(\mathcal{K}u(X))$ is an abelian variety with a polarization induced from the Euler paring. Moreover, we have an isomorphism of abelian varieties $J(\mathcal{K}u(X)) \cong J(X)$.

Proposition 15.10. *Assume there is a semiorthogonal decomposition* $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, \ldots, E_n \rangle$ *where* $\{E_1, \ldots, E_n\}$ *is an exceptional collection. Also assume that* $\mathrm{HH}_{2n+1}(X) = 0$ *for* $n \geq 1$. *The first order deformation space of* $J(\mathcal{K}u(X))$ *is*

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

Proof. Write $V := \mathrm{HH}_1(\mathcal{K}u(X))$. Since $V = \mathrm{HH}_1(\mathcal{K}u(X)) \cong \mathrm{HH}_1(X)$, there is a natural conjugation $\overline{V} := \mathrm{HH}_{-1}(\mathcal{K}u(X)) \cong \mathrm{HH}_{-1}(X)$. Since the tangent bundle of a torus is trivial, we have

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong H^1(V/\Gamma, V \otimes \mathcal{O}_{V/\Gamma}) \cong V \otimes H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}).$$

Since $H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}) \cong \mathrm{Hom}_{\mathrm{anti-linear}}(V, k) \cong \mathrm{Hom}_k(\overline{V}, k)$, we finally get the equality

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X)))$$

as required. ◻

When $\mathrm{HH}_{2n+1}(X) = 0$, $n \geq 1$, we define a linear map from the deformations of $\mathcal{K}u(X)$ to the deformations of its intermediate Jacobian $J(\mathcal{K}u(X))$ by the action of cohomology:

$$\mathrm{HH}^2(\mathcal{K}u(X)) \longrightarrow \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

This map can be imagined as the derivative of the following map of "moduli spaces"

$$\{\mathcal{K}u(X)\}/\sim \longrightarrow \{J(\mathcal{K}u(X))\}/\sim .$$

This map makes sense since it recovers the information of the derivative of the period map when $\mathcal{K}u(X) = \mathrm{D}^b(X)$ by using the new HKR isomorphisms.

In the theorem below we write

$$\mathrm{HH}_*(X) := \bigoplus_{p-q=*} H^p(X, \Omega_X^q).$$

$$\mathrm{HH}^*(X) := \bigoplus_{p+q=*} H^p(X, \wedge^q T_X).$$

Theorem 15.11. *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, E_2, ..., E_n\rangle$ where $\{E_1, E_2, ..., E_n\}$ is an exceptional collection. Then we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[ur, "\tau"] & \\
H^1(X, T_X) \arrow[u] \arrow[uur] &
\end{tikzcd}$$

where τ is defined as a contraction of polyvector fields.

Proof. We write $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{A}\rangle$ as the semiorthogonal decomposition, where $\mathcal{A} = \langle E_1, E_2, ..., E_n\rangle$. Let P_1 be the kernel of the projection to $\mathcal{K}u(X)$, and P_2 the kernel corresponding to the projection to $\mathcal{A}$. There are triangles

$$P_2 \longrightarrow \mathcal{O}_\Delta \longrightarrow P_1 \longrightarrow P_2[1] \tag{16}$$

$$P_2 \circ S_X \longrightarrow \mathcal{O}_\Delta \circ S_X \longrightarrow P_1 \circ S_X \longrightarrow P_2 \circ S_X[1]\ . \tag{17}$$

Applying $\mathrm{Hom}(-, P_1)$ to the triangle (16), we get an isomorphism

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^*(P_1, P_1)$$

because $\mathrm{Hom}^*(P_2, P_1) = 0$. Applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to the triangle (16), we get a long exact sequence

$$\mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \longrightarrow \mathrm{Hom}^t(\mathcal{O}_\Delta, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2)\ . \tag{18}$$

Since $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^*(P_1, P_1)$, we get a new long exact sequence

$$\mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{\alpha} \mathrm{Hom}^t(P_1, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2)\ . \tag{19}$$

Again, applying the functor $\mathrm{Hom}(-, P_1 \circ S_X)$ to the triangle (16), we obtain an isomorphism $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^*(P_1, P_1 \circ S_X)$ because $\mathrm{Hom}^*(P_2, P_1 \circ S_X) = 0$ [Kuz09b, Cor 3.10]. Applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to triangle (17), we again obtain a long exact sequence

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X) \tag{20}$$

By the isomorphism $\mathrm{Hom}^*(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^*(P_1, P_1 \circ S_X)$, we get a new long exact sequence

$$\mathrm{Hom}^*(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^*(P_1, P_1 \circ S_X) \tag{21}$$

Thus, these natural procedures induce a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{t_1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \times \mathrm{Hom}^{t_2}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[r] \arrow[d, "(\alpha,\beta)"] & \mathrm{Hom}^{t_1+t_2}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[d, "\beta"] \\
\mathrm{Hom}^{t_1}(P_1, P_1) \times \mathrm{Hom}^{t_2}(P_1, P_1 \circ S_X) \arrow[r] & \mathrm{Hom}^{t_1+t_2}(P_1, P_1 \circ S_X)
\end{tikzcd}$$

The morphisms in the rows are the composition maps described in Definition 15.5. To explain the commutative diagram, we take $t_1 = t_2 = 0$; the general cases are similar. Let $f \in \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$, and $g \in \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X)$. We write L as the natural morphism $\mathcal{O}_\Delta \to P_1$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_\Delta \arrow[r, "g"] \arrow[d, "L"'] & \mathcal{O}_\Delta \circ S_X \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "L\otimes\mathrm{id}"] & \mathcal{O}_\Delta \circ S_X \arrow[d, "L\otimes\mathrm{id}"] \\
P_1 \arrow[r, "g'"] & P_1 \circ S_X \arrow[r, "f'\otimes\mathrm{id}"] & P_1 \circ S_X
\end{tikzcd}$$

The composition $(L \otimes \mathrm{id}) \circ g$ gives an element g' in $\mathrm{Hom}(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}(P_1, P_1 \circ S_X)$, that is $\beta(g) = g'$. Similarly, $\alpha(f) = f'$. By the uniqueness of the isomorphism $\mathrm{Hom}(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}(P_1, P_1 \circ S_X)$, we have

$$\beta((f \otimes \mathrm{id}) \circ g) = (f' \otimes \mathrm{id}) \circ g'.$$

Taking $t_1 = 2$ and $t_2 = -1$, in our examples, β becomes an isomorphism. The proof of this fact is as follows. Let $h \in \mathrm{Hom}^*(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X \circ S_X)$. Then according to [Kuz09b, Lemma 5.3], there is a commutative diagram.

$$\begin{tikzcd}
\Delta_*\mathcal{O}_X \arrow[r, "L"] \arrow[d, "h"'] & P_1 \arrow[d, "\gamma_{P_1}(h)"] \\
\Delta_*\mathcal{O}_X \circ S_X[*] \arrow[r, "L\otimes\mathrm{id}"] & P_1 \circ S_X[*]
\end{tikzcd}$$

Hence, $\beta = \gamma_{P_1}$ (see [Kuz09b, Section 5] for the definition of γ_{P_1}). Therefore, by the Theorem of Additivity [Kuz09b, Theorem 7.3], β is an isomorphism when $t_2 = -1$.

After applying the new HKR isomorphisms, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{HH}^2(X) \times \mathrm{HH}_{-1}(X) \arrow[r, "\gamma'"] \arrow[d, "(\alpha'{,}\mathrm{id})"] & \mathrm{HH}_1(X) \arrow[d, "\mathrm{id}"] \\
\mathrm{HH}^2(\mathcal{K}u(X)) \times \mathrm{HH}_{-1}(X) \arrow[r, "\gamma"] & \mathrm{HH}_1(X)
\end{tikzcd}$$

The map α' is the composition of the maps

$$\mathrm{HH}^2(X) \xrightarrow{\mathrm{IK}^{-1}} \mathrm{Hom}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{\alpha} \mathrm{Hom}^2(P_1, P_1) = \mathrm{HH}^2(\mathcal{K}u(X)).$$

The map in the row γ' is the natural action of polyvectors to the forms, when restricting to $H^1(X, T_X)$, it is exactly the derivative of the period map. The map γ is defined by cohomology action as follows: let $w \in \mathrm{Hom}^2(P_1, P_1)$, then $\gamma(w)\colon \mathrm{HH}_{-1}(X) \to \mathrm{HH}_1(X)$ is defined by the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{-1}(P_1, P_1 \circ S_X) \arrow[r, "w"] & \mathrm{Hom}^1(P_1, P_1 \circ S_X) \arrow[d, "\beta^{-1}"] \\
\mathrm{Hom}^{-1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[u, "\beta"] & \mathrm{Hom}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \arrow[d, "\mathrm{IK}"] \\
\mathrm{HH}_{-1}(X) \arrow[u, "\mathrm{IK}^{-1}"] \arrow[r, "\gamma(w)"] & \mathrm{HH}_1(X)
\end{tikzcd}$$

Thus, we obtain a commutative diagram, η is defined to be the composition $\alpha' \circ$ (inclusion).

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X)) \\
\mathrm{HH}^2(X) \arrow[u, "\alpha'"] \arrow[ur] & \\
H^1(X, T_X) \arrow[u, "\text{inclusion}"] \arrow[uur, "d\mathcal{P}"'] &
\end{tikzcd}$$

$\square$

Corollary 15.12. *Let X be Fano threefolds of index 1 or 2. Note here that we have $\mathrm{HH}_{-1}(X) = H^{2,1}(X)$ and $\mathrm{HH}_1(X) = H^{1,2}(X)$. Then, there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \\
H^1(X, T_X) \arrow[u, "\eta"] \arrow[ur, "d\mathcal{P}_{\mathrm{period}}"] &
\end{tikzcd}$$

Remark 15.13. The commutative diagram can be regarded as the infinitesimal version of the following imaginary maps

$$\begin{tikzcd}
\{\mathcal{K}u(X)\}/\sim \arrow[r] & \{J(X)\}/\sim \\
\{X\}/\sim \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Definition 15.14. Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, E_2, \dots, E_n \rangle$ where $\{E_1, E_2, \dots, E_n\}$ is an exceptional collection. The variety X has the property of

(1) *infinitesimal Torelli* if
$$d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$
is injective,

(2) *infinitesimal categorical Torelli* if the composition
$$\eta : H^1(X, T_X) \to \mathrm{HH}^2(\mathcal{K}u(X))$$
is injective,

(3) *infinitesimal categorical Torelli of period type* if
$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \to \mathrm{Hom}(\mathrm{HH}_{-1}(X), \mathrm{HH}_1(X))$$
is injective.

Remark 15.15. The definitions (2) and (3) depend on the choice of the Kuznetsov component $\mathcal{K}u(X)$.

16. Infinitesimal categorical Torelli for Fano threefolds of index 1 and 2

In this section, we use the definition of the Kuznetsov component of a Fano threefold of index 1 or 2 from the survey paper [Kuz16]. We study the commutative diagram constructed in Corollary 15.12, and investigate the infinitesmal categorical Torelli properties defined in Definition 15.14. We refer to [PB21] and [Bel21] for the dimensions of $H^1(X, T_X)$ and $H^1(X, \Omega^2_X)$, where X is Fano threefold of index 1 or 2.

16.1. Kuznetsov components of Fano threefolds of index 2.

Theorem 16.1 ([Kuz09b, Theorem 8.8]). *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E, \mathcal{O}_X\rangle$ where $E^\vee$ is a global generated rank r vector bundle with vanishing higher cohomology. There is a map $\phi : X \to \mathrm{Gr}(r, V)$. Define $\mathcal{N}^\vee_{X/\,\mathrm{Gr}}$ as the shifted cone lying in the triangle*

$$\mathcal{N}^\vee_{X/\,\mathrm{Gr}} \to \phi^*\Omega_{Gr(r,V)} \to \Omega_X.$$

We have the following long exact sequences:

(1)

$$\cdots \to \bigoplus_{p=0}^{n-1} H^{t-p}(X, \Lambda^p T_X) \to \mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \to$$

$$\to H^{t-n+2}(X, E^\perp \otimes E \otimes \omega_X^{-1}) \xrightarrow{\alpha} \bigoplus_{p=0}^{n-1} H^{t+1-p}(X, \Lambda^p T_X) \to \cdots$$

(2)

$$\cdots \to \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^p T_X) \to \mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \to$$

$$\to H^{t-n+2}(X, \mathcal{N}^\vee_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1}) \xrightarrow{\nu} \bigoplus_{p=0}^{n-2} H^{t+1-p}(X, \Lambda^p T_X) \to \cdots$$

(3) If E is a line bundle, then $\nu = 0$ and

$$\mathrm{HH}^t(\langle E, \mathcal{O}_X\rangle^\perp) \cong \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^q T_X) \oplus H^{t-n+2}(X, \mathcal{N}^\vee_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1})$$

An application of part (3) of Theorem 16.1 to the case of index 2 Fano threefolds of degree d, i.e. when $\mathcal{K}u(Y_d) = \langle \mathcal{O}_{Y_d}(-H), \mathcal{O}_{Y_d}\rangle^\perp$ gives the following theorem.

Theorem 16.2 ([Kuz09b, Theorem 8.9]). *The second Hochschild cohomology of Kuznetsov component of an index 2 Fano threefold of degree d is given by*

$$\mathrm{HH}^2(\mathcal{K}u(Y_d)) = \begin{cases} 0, & d = 5 \\ k^3, & d = 4 \\ k^{10}, & d = 3 \\ k^{20}, & d = 2 \\ k^{35}, & d = 1. \end{cases}$$

Theorem 16.3. *Let $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E, \mathcal{O}_X\rangle$ be a semiorthogonal decomposition where E is an exceptional vector bundle. Then, the map η constructed in Theorem 15.11 is isomorphic to the corresponding map in the long exact sequence in Theorem 16.1.*

Proof. We use the same notation as in the proof of Theorem 16.1 [Kuz09b, Theorem 8.8]. Let P_1 be the kernel of the projection to $\mathcal{O}_X^{\perp}$, and P_2 be the kernel of the projection to $\langle E, \mathcal{O}_X\rangle^{\perp}$. There is a triangle

$$E^{\perp} \boxtimes E[1] \to P_1 \to P_2.$$

Applying the functor $\Delta^!$ to the triangle, we have map $\Delta^! P_1 \to \Delta^! P_2$. According to [Kuz09b, Theorem 8.5], $\Delta^! P_1 \cong \bigoplus_{p=0}^{n-1} \wedge^p T_X[-p]$. Then, applying the functor $\mathrm{Hom}^*(\mathcal{O}_X, -)$, we get the map $\eta' \colon \bigoplus_{p=0}^{n-1} H^{t-p}(X, \wedge^p T_X) \to \mathrm{HH}^t(\mathcal{K}u(X))$ in the long exact sequence. Since we have the isomorphism

$$\mathrm{Hom}^*(\mathcal{O}_X, \Delta^! P_1) \cong \mathrm{Hom}^*(\Delta_* \mathcal{O}_X, P_1),$$

the map η' is isomorphic to the map

$$\mathrm{Hom}(\Delta_* \mathcal{O}_X, P_1) \to \mathrm{Hom}(\Delta_* \mathcal{O}_X, P_2)$$

given by the composition $P_1 \to P_2$. Consider the filtration

$$0 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \Delta_* \mathcal{O}_X$$

with respect to the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E, \mathcal{O}_X\rangle$. Taking cones, we get a commutative diagram whose morphisms correspond to the following triangle of projections:

$$\begin{tikzcd}
\Delta_* \mathcal{O}_X \arrow[r] \arrow[dr] & P_1 \arrow[d] \\
 & P_2
\end{tikzcd}$$

Hence, applying the functor $\mathrm{Hom}^*(\Delta_* \mathcal{O}_X, -)$ to the above triangle, we have a new commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^*(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \arrow[r, "F"] \arrow[dr, "\alpha"'] & \mathrm{Hom}^*(\Delta_* \mathcal{O}_X, P_1) \arrow[d] \\
 & \mathrm{Hom}^*(\Delta_* \mathcal{O}_X, P_2)
\end{tikzcd}$$

When $* = 2$, the map F is an isomorhism. This can be checked by the HKR isomorphism. Thus, η is naturally isomorphic to η'. ∎

Theorem 16.4. *Let Y_d be Fano vareities of index 2 of degree $3 \le d \le 4$. The commutative diagrams of Corollary 15.12 for Y_d are as follows:*

(1) Y_4, η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd}
k^3 \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\
k^3 \arrow[r, "\gamma"] & k^4
\end{tikzcd}$$

(2) Y_3, η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{10} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{10} \arrow[r, "\gamma"] & k^{25} \end{tikzcd}$$

(3) Y_2, η is injective, $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{19} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

(4) Y_1, η is injective, $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{34} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{35} \arrow[r, "\gamma"] & k^{441} \end{tikzcd}$$

Proof. The statement that η is injective is true for all Y_d, and the proof is as follows. We write X for Y_d. According to Theorem 16.1 (3), v is zero and then there is an exact sequence

$$0 \to \bigoplus_{p=0}^{n-2} H^{t-p}(X, \wedge^p T_X) \to \mathrm{HH}^t(\langle \mathcal{O}(-1), \mathcal{O}\rangle^{\perp}) \to H^{t-n+2}(X, \mathcal{N}^{\vee}_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1}) \to 0.$$

Taking $t = 2$ and $n = 3$, the short exact sequence becomes

$$0 \to H^2(X, \mathcal{O}_X) \oplus H^1(X, T_X) \to \mathrm{HH}^2(\langle \mathcal{O}(-1), \mathcal{O}\rangle^{\perp}) \to H^1(X, \mathcal{N}^{\vee}_{X/\,\mathrm{Gr}} \otimes \omega_X^{-1}) \to 0.$$

According to Theorem 16.3, η is just the map form $H^1(X, T_X)$ to $\mathrm{HH}^2(\mathcal{K}u(X))$ in the above short exact sequence, hence η is always injective. For dimension reasons, η is an isomorphism for the cases Y_3 and Y_4.

The map $d\mathcal{P}$ is injective for $d = 1, 2, 3, 4$. Indeed, for $d = 4$ the moduli space of Fano threefolds Y_4 is isomorphic to the moduli space of genus 2 curves C, thus $H^1(Y, T_Y) \cong H^1(C, T_C)$. Also note that the intermediate Jacobian $J(Y_4)$ is isomorphic to $J(C)$. Then the map $d\mathcal{P}$ is just the map $d\mathcal{P}_C : H^1(C, T_C) \to \mathrm{Hom}(H^{1,0}(C), H^{0,1}(C))$, which is injective since infinitesimal Torelli holds for genus 2 curves. For $d = 3$, $d\mathcal{P}$ is injective since infinitesimal Torelli holds for Y_3 by [Fle86]. In both cases, since η is an isomorphism, γ is injective because $d\mathcal{P}$ is injective. For $d = 2$, $d\mathcal{P}$ is injective for Y_2 by [FRZ19, Theorem 3.4.1] and $d\mathcal{P}$ is also injective for Y_1 by [FRZ19, Section 3.4.1]. ◻

Remark 16.5. We do not know whether γ is injective for the cases Y_1 and Y_2.

16.2. Kuznetsov components of Fano threefolds of index 1.

16.2.1. *The ordinary GM case.* We write $X = X_{10}^{o}$ for ordinary GM varieties. Let $\phi : X \hookrightarrow \mathrm{Gr}(2, 5)$ be the Gushel map. Write $\mathcal{E} = \phi^* T$, where T is the tautological rank 2 bundle on $\mathrm{Gr}(2, 5)$. There is a semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^{\vee}\rangle$ [KP18b, Proposition 2.3]. There is also a semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}_2, \mathcal{O}\rangle,$$

where $\mathcal{K}u(X) \cong \mathcal{A}_X$. In this subsection, we do not distinguish $\mathcal{K}u(X)$ and $\mathcal{A}_X$ since the Hochschild (co)homology of them is the same.

Proposition 16.6 ([KP18b, Proposition 2.12])**.** *Let X be an odd dimensional GM variety, and let $\mathcal{A}_X$ be the nontrivial component of $\mathrm{D}^b(X)$ as in [KP18b, Proposition 2.3]. Then*

$$\mathrm{HH}^\bullet(\mathcal{A}_X) = k \oplus k^{20}[-2] \oplus k[-4].$$

Theorem 16.7. *Let X be an ordinary GM threefold. The commutative diagram of deformations in Corollary 15.12 is as follows:*

$$\begin{tikzcd} k^{22} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

Moreover, γ is injective and η is surjective. In particular, X_{10}^o has property of infinitesimal categorical Torelli of period type.

Proof. The moduli $\mathcal{X}$ of Fano threefolds of index 1 and degree 10 is a 22 dimensional smooth irreducible algebraic stack by [DIM12, p. 13]. By §7 of [DIM12] we have that the differential

$$d\mathcal{P} : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

of the period map $\mathcal{P} : \mathcal{X} \to \mathcal{A}_{10}$ has 2 dimensional kernel. Consider the kernel of η. Clearly, $\ker \eta \subset \ker d\mathcal{P}$ hence $\dim \ker \eta \leq 2$. Since the dimension of the image of η is less than or equal to 20, the dimension of $\ker \eta$ must be 2 and η is surjective. Finally, since image of $d\mathcal{P}$ is 20 dimensional, so is γ, and then γ is injective. $\square$

Remark 16.8. There is an another proof which uses the long exact sequence in Proposition 16.6 part (2):

$$\begin{aligned} 0 \to H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) \to H^1(X, T_X) \xrightarrow{\eta} \mathrm{HH}^2(\mathcal{K}u(X)) \xrightarrow{\nu} \\ \xrightarrow{\nu} H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) \to H^2(X, T_X) \to 0. \end{aligned}$$

It suffices to compute $H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(1))$ and $H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(1))$. Since $\mathcal{N}_{X/\mathrm{Gr}}$ is the restriction of $\mathcal{O}_G(1) \oplus \mathcal{O}_G(1) \oplus \mathcal{O}_G(2)$, we have that $\mathcal{N}^\vee_{X/\mathrm{Gr}}(1)$ is the restriction of $\mathcal{O}_G \oplus \mathcal{O}_G \oplus \mathcal{O}_G(-1)$, which is $\mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X(-H)$. Then $H^0(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) = k^2$ and $H^1(X, \mathcal{N}^\vee_{X/\mathrm{Gr}}(H)) = 0$ by the Kodaira vanishing theorem. Thus, η is surjective with 2-dimensional kernel, and then γ is injective.

16.2.2. *The special GM case.* In this subsection, we write $X = X_{10}^s$ for special GM threefolds.

Theorem 16.9. *The diagram in Corollary 15.12 for a special GM threefold X is as follows:*

$$\begin{tikzcd} k^{22} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{20} \arrow[r, "\gamma"] & k^{100} \end{tikzcd}$$

The map γ is injective and in particular, X has the property of infinitesimal categorical Torelli of period type.

Proof. Since the kernel of $d\mathcal{P}$ is 2 dimensional, γ is injective. It is similar to the proof of Theorem 16.7. ◻

16.2.3. *The cases of X_{18}, X_{16}, X_{14}, and X_{12}.*

Theorem 16.10. *Let X and Y be smooth projective varieties with the Kuznetsov components $\mathcal{K}u(X)$ and $\mathcal{K}u(Y)$. Suppose there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ which is induced by a Fourier–Mukai functor. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X))), \mathrm{HH}_1(\mathcal{K}u(X))) \arrow[d, "\cong"] \\
\mathrm{HH}^2(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(Y)), \mathrm{HH}_1(\mathcal{K}u(Y)))
\end{tikzcd}$$

Proof. According to Theorem 15.7, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \times \mathrm{HH}_{-1}(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{HH}_1(\mathcal{K}u(X)) \arrow[d, "\cong"] \\
\mathrm{HH}^2(\mathcal{K}u(Y)) \times \mathrm{HH}_{-1}(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{HH}_1(\mathcal{K}u(Y))
\end{tikzcd}$$

The maps in the rows are defined as the cohomology action on homology. Hence the commutative diagram in the theorem follows. ◻

Remark 16.11. When X and Y are Fano threefolds of index 1 and 2, respectively, we have $\mathrm{HH}_{-1}(X) \cong H^{2,1}(X)$, $\mathrm{HH}_{-1}(Y) \cong H^{2,1}(Y)$. Hence we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma_X"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \arrow[d, "\cong"] \\
\mathrm{HH}^2(\mathcal{K}u(Y)) \arrow[r, "\gamma_Y"] & \mathrm{Hom}(H^{2,1}(Y), H^{1,2}(Y))
\end{tikzcd}$$

where γ_X and γ_Y are the maps constructed in Theorem 15.12.

Theorem 16.12. *The diagrams of X_{18}, X_{16}, X_{14}, and X_{12} in Corollary 15.12 are as follows:*

(1) *X_{18}, γ is injective.*

$$\begin{tikzcd}
k^{10} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\
k^3 \arrow[r, "\gamma"] & k^4
\end{tikzcd}$$

(2) *X_{16}, γ is injective.*

$$\begin{tikzcd}
k^{12} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\
k^6 \arrow[r, "\gamma"] & k^9
\end{tikzcd}$$

(3) *X_{14}, γ is injective.*

$$\begin{tikzcd}
k^{15} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\
k^{10} \arrow[r, "\gamma"] & k^{25}
\end{tikzcd}$$

(4) X_{12}, γ is injective, $d\mathcal{P}$ is injective, and η is an isomorphism.

$$\begin{tikzcd} k^{18} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{18} \arrow[r, "\gamma"] & k^{49} \end{tikzcd}$$

Proof. In the cases of X_{18}, X_{16}, and X_{12} we always have $\mathcal{K}u(X) \cong \mathrm{D}^b(C)$ for some curve C. The semiorthogonal decompositions are

$$\begin{aligned} \mathrm{D}^b(X_{18}) =& \langle \mathrm{D}^b(C_2), \mathcal{E}_2, \mathcal{O}_{X_{18}} \rangle \\ \mathrm{D}^b(X_{16}) =& \langle \mathrm{D}^b(C_3), \mathcal{E}_3, \mathcal{O}_{X_{16}} \rangle \\ \mathrm{D}^b(X_{12}) =& \langle \mathrm{D}^b(C_7), \mathcal{E}_5, \mathcal{O}_{X_{12}} \rangle \end{aligned}$$

where C_i is a curve of genus i and $\mathcal{E}_i$ is a vector bundle of rank i. We write X for X_{18}, X_{16}, and X_{12}. By the HKR isomorphism, $\mathrm{HH}^2(C) \cong H^2(C, \mathcal{O}_C) \oplus H^1(C, T_C) \oplus H^0(C, \wedge^2 T_C) = H^1(C, T_C)$. The second equality follows from the fact that C is of dimension 1. Note that we always refer to the HKR twisted by IK as the twisted HKR. This IK isomorphism preserves the module structure, where the geometric side is the action of polyvector fields on differential forms. Thus by Theorem 16.10 there is a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X)) \arrow[r, "\gamma"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \arrow[d, "\cong"] \\ H^1(C, T_C) \arrow[r, "d\mathcal{P}_C"] & \mathrm{Hom}(H^{1,0}(C), H^{0,1}(C)) \end{tikzcd}$$

Therefore γ is injective for each X since $d\mathcal{P}_C$ is injective for each $C := C_i$. Indeed, C_3 is a plane quartic curve, which is a canonical curve in $\mathbb{P}^2$. Similarly, C_7 is also a canonical curve in $\mathbb{P}^6$ by [IM07]. Thus they are both non-hyperelliptic.

For the case X_{14}, it is known that $\mathcal{K}u(X_{14}) \simeq \mathcal{K}u(Y_3)$ by [Kuz09a]. Then there is a commutative diagram (by Theorem 16.10 again)

$$\begin{tikzcd} \mathrm{HH}^2(\mathcal{K}u(X_{14})) \arrow[r, "\gamma_{X_{14}}"] \arrow[d, "\cong"] & \mathrm{Hom}(H^{2,1}(X_{14}), H^{1,2}(X_{14})) \arrow[d, "\cong"] \\ \mathrm{HH}^2(\mathcal{K}u(Y_3)) \arrow[r, "\gamma_{Y_3}"] & \mathrm{Hom}(H^{2,1}(Y_3), H^{1,2}(Y_3)) \end{tikzcd}$$

Then $\gamma_{X_{14}}$ is injective since γ_{Y_3} is injective, by Theorem 16.4. ∎

16.2.4. *The cases of X_8, X_4, and X_2.* In these cases, the Kuznetsov components are defined as $\langle \mathcal{O}_X \rangle^{\perp}$.

Theorem 16.13. *The diagrams of X_8, X_4, and X_2 in Corollary 15.12 are as follows:*

(1) X_8, γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{27} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{27} \arrow[r, "\gamma"] & k^{196} \end{tikzcd}$$

(2) X_4, γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{45} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{45} \arrow[r, "\gamma"] & k^{900} \end{tikzcd}$$

(3) X_2, γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{tikzcd} k^{68} \arrow[d, "\eta"] \arrow[dr, "d\mathcal{P}"] & \\ k^{68} \arrow[r, "\gamma"] & k^{2704} \end{tikzcd}$$

Proof. First, we prove that η is an isomorphism in each case. We write X for X_8, X_4 and X_2. Note that $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X \rangle$. Denote by P_1 the kernel of the left projection to $\mathcal{K}u(X)$, and P_2 the kernel of the right projection to $\langle \mathcal{O}_X \rangle$. There is a triangle

$$P_2 \to \Delta_* \mathcal{O}_X \to P_1 \to P_2[1].$$

Applying the functor $\Delta^!$, we obtain

$$\begin{tikzcd} \Delta^! P_2 \arrow[r] \arrow[d, "\cong"] & \Delta^! \Delta_* \mathcal{O}_X \arrow[r, "L"] \arrow[d, "\cong"] & \Delta^! P_1 \arrow[d, "\mathrm{id}"] \\ \omega_X^{-1}[-3] \arrow[r, "w"] & \bigoplus_{p=0}^{3} \Lambda^p T_X[-p] \arrow[r] & \Delta^! P_1 \end{tikzcd}$$

According to [Kuz09b, Theorem 8.5], the map w is an isomorphism onto the third summand. Applying $\mathrm{Hom}^2(\mathcal{O}_X, -)$, we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! \Delta_* \mathcal{O}_X) \arrow[r, "L"] \arrow[d, "\cong"] & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \arrow[d, "\mathrm{id}"] \\ \mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^{3} \Lambda^p T_X[-p]) \arrow[r] & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \\ \mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^{2} \Lambda^p T_X[-p]) \arrow[u, "\cong"] \arrow[ur, "\cong"] & \end{tikzcd}$$

Thus the morphism L is an isomorphism. However, L is naturally isomorphic to the morphism

$$\mathrm{Hom}^2(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \to \mathrm{Hom}^2(\Delta_* \mathcal{O}_X, P_1) \cong \mathrm{Hom}^2(P_1, P_2)$$

That is to say the map $\alpha' : \mathrm{HH}^2(X) \to \mathrm{HH}^2(\mathcal{K}u(X))$ constructed in Theorem 15.11 is an isomorphism. According to [PB21, Appendix A], $H^2(X, \mathcal{O}_X) = 0$ and $H^0(X, \Lambda^2 T_X) = 0$, hence η is an isomorphism.

The map $d\mathcal{P}$ is injective for X_8, X_4 by [Fle86] and X_2 by [Cle83]. Then γ is injective for these cases because η is an isomorphim and $d\mathcal{P}$ is injective. ∎

Corollary 16.14. *Infinitesimal classical Torelli implies infinitesimal categorical Torelli.*

Proof. Suppose $d\mathcal{P}$ is injective. Then the fact that we have a composition $d\mathcal{P} = \gamma \circ \eta$ implies that η is injective too. ∎

16.3. **A summary of the status of classical and categorical Torelli theorems for Fano threefolds.** In the tables below, we summarise what is known in terms of the intermediate Jacobian $J(X)$ (classical Torelli statements) and in terms of the Kuznetsov component $\mathcal{K}u(X)$ (categorical Torelli statements: categorical Torelli means $\mathcal{K}u(X)$ determines X up to isomorphism; birational categorical Torelli means $\mathcal{K}u(X)$ determines X up to birational equivalence; refined categorical Torelli means $\mathcal{K}u(X)$ along with some extra data determines X up to isomorphism).

The first and second tables are for index 2 and 1 Fano threefolds, respectively. For each degree, the grey row represents global Torelli statements and the white row represents infinitesimal Torelli statements. The letters d and g are the degree and genus, respectively. Also, OGM and SGM mean ordinary GM and special GM threefolds (of genus 6), respectively.

d	$J(Y_d)$	$\mathcal{K}u(Y_d)$
5	Classical Torelli; trivial proof	Categorical Torelli, trivial proof because the moduli space of index two degree 5 prime Fano threefolds is a point.
	Unknown	Unknown
4	Classical Torelli [Mer85]	Since $\mathcal{K}u(Y_4) \simeq \mathrm{D}^b(C_2)$, categorical Torelli follows from [BO01] (see the discussion in [BLMS17, p. 24]), also [BT16] when $\mathcal{K}u(Y_4) \simeq \mathcal{K}u(Y_4')$ is a FM equivalence
	Unknown	Theorem 16.4
3	Classical Torelli [CG72], [Tyu71]	Categorical Torelli; [BMMS12], [PY20], also [BT16] when $\mathcal{K}u(Y_3) \simeq \mathcal{K}u(Y_3')$ is a FM equivalence
	Unknown	Theorem 16.4
2	Classical Torelli; generic Torelli for double covers of $\mathbb{P}^n$ follows from same reasoning used in [Don83] (cf. p. 327)	Categorical Torelli; [APR19]; also [BT16] when $\mathcal{K}u(Y_2) \simeq \mathcal{K}u(Y_2')$ is a FM equivalence
	Unknown	Theorem 16.4
1	$\Phi(F(Y_1))$ recovers Y_1, where $\Phi : F(Y_1) \to J(Y_1)$ is the Abel-Jacobi map, and $F(Y_1)$ is the Fano surface of lines [Tih82]	Unknown
	Unknown	Theorem 16.4

g	$J(X_{2g-2})$	$\mathcal{K}u(X_{2g-2})$
12	Birational Torelli	Refined categorical Torelli, [JLZ21] and [JZ21]
	Unknown	Unknown
10	Birational Torelli	Refined categorical Torelli, [JLZ21], [JZ21] and [FV21]
	Unknown	Period type (Theorem 16.12)
9	Birational Torelli	Refined categorical Torelli [Muk01], [BF13] (reproved in [JLZ21] and [JZ21])
	Unknown	Period type (Theorem 16.12)
8	Birational Torelli, [CG72] and [IM05]	Birational categorical Torelli ([Kuz03], [BMMS12]) and refined categorical Torelli ([JLZ21] and [JZ21])
	Unknown	Period type (Theorem 16.12)
7	Classical Torelli [Muk01]	Categorical Torelli ([BF14])
	Theorem 16.12	Period and geometric type (Theorem 16.12)
OGM	Birational Torelli conjectured, [DIM12] (cf. Proposition 12.6)	Birational categorical Torelli (Theorem 10.1) and refined categorical Torelli (Theorem 7.17, and [JZ21])
	Unknown	Period type (Theorem 16.7)
SGM	Unknown	Categorical Torelli (Theorem 8.1) and another proof via Brill-Noether locus [JZ21]
	Unknown	Period type (Theorem 16.9)
5	Classical Torelli [FS82]	Categorical Torelli [BT16], [Per20] for F-M type equivalence
	[Fle86]	Period and geometric type (Theorem 16.13)
4	Unknown	Pre-categorical Torelli
	[Fle86]	Unknown
3	Generic Classical Torelli[Don83]	Pre-categorical Torelli
	[Fle86]	Period and geometric type (Theorem 16.13)
2		Pre-categorical Torelli
	Unknown	Period and geometric type (Theorem 16.13)

In the table above, pre-categorical Torelli means that X can be reconstructed as a Bridgeland moduli space of σ-stable objects in $\mathcal{O}_X^{\perp}$ with respect to the class of the shift of an ideal sheaf I_p of a point. The actual categorical Torelli statements for X_2, X_4 and X_8 are still work in progress in [JLZ21].

Appendix A. Fiber of the categorical period map: general case

In this section, we make an attempt to describe the fiber of the "categorical period map" $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for *any* smooth GM threefold X in the moduli space $\mathcal{M}_3$. We start with several lemmas, propositions and theorems which will be frequently used.

Lemma A.1 ([Huy16, Chapter 16])**.** *Let S and S' be two smooth K3 surfaces such that $\mathrm{D}^b(S) \simeq \mathrm{D}^b(S')$. Then their Picard numbers are equal, i.e. $\rho(S) = \rho(S')$.*

Theorem A.2 ([JZ21])**.** *Let X be a smooth GM threefold. Then X can be reconstructed as a Brill-Noether locus inside a Bridgeland moduli space of stable objects in the Kuznetsov component $\mathcal{K}u(X)$, i.e.*

$$X \cong \mathcal{BN}_X := \{F \in \mathcal{M}_\sigma(\mathcal{K}u(X), [\mathrm{pr}(k(x))[-1]] \mid \mathrm{Ext}^1(F, i^!\mathcal{E}) = k^3\},$$

where σ is a τ-invariant stability condition.

Corollary A.3 (Refined categorical Torelli for *any* GM threefold)**.** *Let X and X' be smooth GM threefolds and suppose there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ such that $\Phi(\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$. Then $X \cong X'$.*

Proof. First we show that X and X' must be ordinary GM threefolds or special GM threefolds simultaneously. Indeed, we may assume X is an ordinary GM threefold and X' is a special GM threefold. Then $\mathrm{Ext}^2(\pi(\mathcal{E}), \pi(\mathcal{E})) = 0$ (Lemma 7.6) and $\mathrm{Ext}^2(\pi(\mathcal{E}'), \pi(\mathcal{E}')) = k$ (Corollary 11.4), which is impossible since Φ is an equivalence. Now we assume X and X' are both ordinary GM threefolds; the case of special GM threefolds can be argued in a similar way. By Theorem A.2, $X \cong \mathcal{BN}_X$. Let $F \in \mathcal{BN}_X$. Then $\Phi(F) \in \mathcal{BN}_{X'}$.

Indeed, $\Phi(\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$ and the vector $[\pi(\mathcal{E})]$ is one of two (-1)-classes in $\mathcal{N}(\mathcal{K}u(X))$ (up to sign). This implies the linear isometry $[\Phi] : \mathcal{N}(\mathcal{K}u(X)) \cong \mathcal{N}(\mathcal{K}u(X'))$ preserves the standard basis. Thus $\mathrm{ch}(\Phi(F)) = \mathrm{ch}(\mathrm{pr}(k(x)[-1]))$. As $\mathrm{Ext}^1(\Phi(F), i^!\mathcal{E}') \cong \mathrm{Ext}^1(\Phi(F), \Phi(\pi(\mathcal{E}))) \cong \mathrm{Ext}^1(F, i^!\mathcal{E}) \cong k^3$, it remains to show that $\Phi(F)$ is a σ'-stable object in $\mathcal{K}u(X')$ for every τ'-invariant stability condition σ'. Then it is sufficient to show $\Phi^{-1} \circ \Phi(F) \cong F$ is a $\Phi^{-1}(\sigma')$-stable object in $\mathcal{K}u(X)$. As the equivalence Φ^{-1} commutes with Serre functors of $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$ respectively, then $\tau(\Phi^{-1}(\sigma')) = \Phi^{-1}(\tau'(\sigma')) = \Phi^{-1}(\sigma')$. This implies $\Phi^{-1}(\sigma')$ is a τ-invariant stability condition on $\mathcal{K}u(X)$. On the other hand, F is σ-stable and σ is also a τ-invariant stability condition. Thus F is also $\Phi^{-1}(\sigma')$-stable by Theorem 4.25. Thus Φ induces a bijection of closed points of $\mathcal{BN}_X$ and that of $\mathcal{BN}_{X'}$. On the other hand, it is clear that the BN-locus $\mathcal{BN}$ admits the universal family coming from X (argued similarly as in [APR19, Section 5.2]). Then the induced morphism $\phi : \mathcal{BN}_X \to \mathcal{BN}_{X'}$ is an étale map and a bijection, thus $\mathcal{BN}_X \cong \mathcal{BN}_{X'}$, which implies $X \cong X'$. $\square$

By comparing the singularities of the Bridgeland moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x)$ for different GM threefolds X, we can compute the fiber of the "categorical Period map" $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for all $X \in \mathcal{M}_3$.

Proposition A.4. *The fiber of $\mathcal{P}_{\mathrm{cat}}$ over $\mathcal{A}_X$ for the following smooth GM threefolds of six types is the union of two surfaces except for general special GM threefolds, where the fiber is one surface. They are shown in Figure 5.*

We give a proof for one case, the other cases can be argued similarly. Let X be a non-general ordinary GM threefold, so that the moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_X, y-2x) \cong M_G^X(2,1,5)$ are both singular.

Proof. (Fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{A}_X])$ for non-general GM threefolds X.) Similar arguments as in Theorem 12.8 apply. Let X' be a GM threefold such that $\Phi : \mathcal{A}_{X'} \simeq \mathcal{A}_X$ is an equivalence. Then Φ identifies $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x) \cong \mathcal{C}_m(X')$ with either $\mathcal{C}_m(X)$ or $M_G^X(2,1,5)$.

(1) If X' is an ordinary GM threefold, then its gluing data $\Xi(\pi(\mathcal{E}'))$ is a smooth point in the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$. The image $\Phi(\Xi(\pi(\mathcal{E}')))$ may land anywhere in $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$ except their singular points.

(2) If X' is a special GM threefold, then its gluing data $\Xi(\pi(\mathcal{E}'))$ is a singular point in the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$. Then $\Phi(\Xi(\pi(\mathcal{E}')))$ can only go to singular points in $\mathcal{C}_m(X)$ and $M_G^X(2,1,5)$.

On the other hand, from [DK15, Theorem 3.25, Theorem 3.27], [DK20a] and [DK20b], we know that (up to an action of $\mathrm{PGL}(V_6)_A$) the smooth locus of $\mathcal{C}_m(X)/\iota \cong Y_{A^\perp}^{\geq 2}$ parametrizes ordinary GM threefolds whose period partners are X and the smooth locus of $M_G(2,1,5)/\iota'$ parametrizes ordinary GM threefolds whose period duals are X, while the singular loci of them parametrize special GM threefolds whose period partners or period duals are X. Then by [KP19, Corollary 6.5], the result follows. ∎

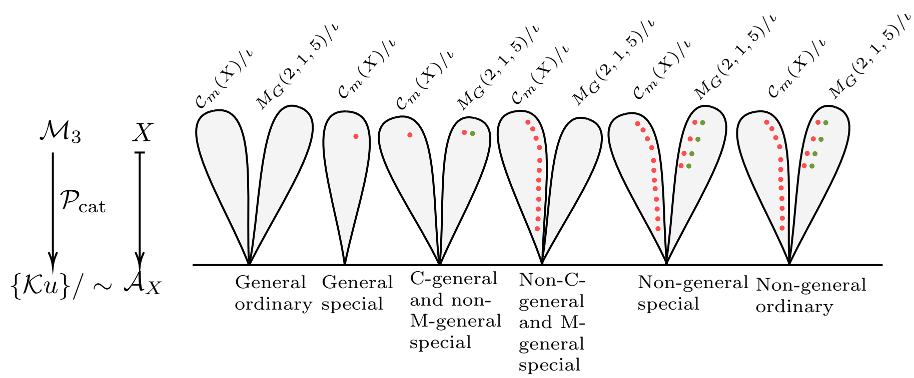

FIGURE 5. The fibers of the categorical period map for each type of GM threefold appearing in the moduli space $\mathcal{M}_3$. The dots indicate the singularities. Where there is a line of dots, we mean that finitely many (more than one) singularities appear.

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