BRILL–NOETHER THEORY FOR KUZNETSOV COMPONENTS AND REFINED CATEGORICAL TORELLI THEOREMS FOR INDEX ONE FANO THREEFOLDS

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Abstract. We show by a uniform argument that every index one prime Fano threefold X of genus $g \geq 6$ can be reconstructed as a Brill–Noether locus inside a Bridgeland moduli space of stable objects in the Kuznetsov component $\mathcal{K}u(X)$. As an application, we prove refined categorical Torelli theorems for X and compute the fiber of the period map for each Fano threefold of genus $g \geq 7$ in terms of a certain gluing object associated with the subcategory $\langle \mathcal{O}_X \rangle^\perp$. This unifies results of Mukai, Brambilla-Faenzi, Debarre-Iliev-Manivel, Faenzi-Verra, Iliev-Markushevich-Tikhomirov and Kuznetsov.

1. Introduction

In the landmark paper [BO01], the authors show that the bounded derived category of coherent sheaves $\mathrm{D}^b(X)$ on a smooth projective variety X determines the isomorphism class of X if its anti-canonical divisor $-K_X$ (or its canonical divisor K_X) is ample. One of the most important ingredients of the proof is that the *point objects* can be intrinsically defined in the derived category $\mathrm{D}^b(X)$. In the case of smooth Fano threefolds X, tremendous work on the semiorthogonal decompositions of $\mathrm{D}^b(X)$ has been done, which enables us to recover geometric information about X from pieces of its derived category. Thus it is natural to ask whether a smooth Fano threefold X can be determined up to isomorphism by a particular piece of $\mathrm{D}^b(X)$. A natural candidate for this is a subcategory $\mathcal{K}u(X) \subset \mathrm{D}^b(X)$ called the *Kuznetsov component* of $\mathrm{D}^b(X)$. This subcategory has been studied extensively, e.g in [Kuz04], [Kuz16], [Kuz09], [KP18], [BF11], [BF13], and [BF14], for many classes of Fano varieties. It has been widely accepted that $\mathcal{K}u(X)$ encodes the most essential information of the birational geometry of X. In our previous work [JLLZ21], we show that the Kuznetsov component $\mathcal{K}u(X)$ of a general special GM threefold determines its isomorphism class while for a general ordinary GM threefold, $\mathcal{K}u(X)$ determines its birational isomorphism class. Furthermore, we show that for a general ordinary GM threefold, $\mathcal{K}u(X)$ together with certain extra data $i^!\mathcal{E}$ arising from the tautological sub-bundle $\mathcal{E}$ on X is enough to determine X up to isomorphism. We call this a *refined categorical Torelli theorem*. Therefore, a natural and interesting question is whether such refined categorical Torelli theorems hold for all index one prime Fano threefolds.

On the other hand, in the late 90s, in [Muk02] and [Muk01], Mukai considered Brill–Noether loci of the moduli space $\mathcal{M}_{\Gamma_g}(2,d)$ of rank two vector bundles on a curve Γ_g of genus g. He showed that every index one prime Fano threefold of degree 12 and 16 can be reconstructed as a Brill–Noether locus of the moduli space $\mathcal{M}_{\Gamma_g}(2,d)$ of rank two stable vector bundles and degree d, where Γ_g is either a curve of genus 7 (for X_{12}) or a curve of genus 3 (for X_{16}). It is interesting to note that $\mathcal{K}u(X_{12}) \simeq \mathrm{D}^b(\Gamma_7)$ and $\mathcal{K}u(X_{16}) \simeq \mathrm{D}^b(\Gamma_3)$ (see [BF13],[BF14]). Thus the moduli space $\mathcal{M}_{\Gamma_g}(2,d)$ is essentially a Bridgeland moduli space of stable objects in the Kuznetsov components of these Fano threefolds. Therefore it is reasonable to expect that a Brill–Noether reconstruction should work for all index one prime Fano threefolds. In this article, we prove the following theorem which simultaneously generalizes the refined categorical Torelli theorem for GM threefolds ([JLLZ21, Theorem 10.2]) and the Brill–Noether reconstruction of X_{12} and X_{16} ([Muk01]).

Let X be a prime Fano threefold of index one and genus $g \geq 6$. Consider its semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$$

where $\mathcal{E}$ is the pullback of the tautological sub-bundle on the Grassmannian. Let $i^!$ be the right adjoint to the inclusion functor $i : \mathcal{K}u(X) \hookrightarrow \mathrm{D}^b(X)$ and let i^* be the left adjoint of i. We define integers $n_g := \frac{g}{2}$ when g is even, $n_7 = 5$ and $n_9 = 3$. Then our main theorem is:

Theorem 1.1. *Let X be a prime Fano threefold of index one and genus $g \geq 6$ and σ be a Serre-invariant stability condition on $\mathcal{K}u(X)$. Then*

$$X \cong \mathcal{BN}_g := \{F \mid \mathrm{ext}^1(F, i^!\mathcal{E}) = n_g\} \subset \mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]]),$$

where $\mathcal{O}_x$ is the skyscraper sheaf supported on a point $x \in X$.

Here the stability condition σ is unique up to a $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action (c.f. Theorem 4.13). For the construction of σ, see Section 6.

Remark 1.2. For the other prime Fano threefolds, we show that they are isomorphic to Bridgeland moduli spaces $\mathcal{M}_{\sigma'}(\mathcal{O}_X^\perp, [I_x])$ of stable objects in the Kuznetsov component $\mathcal{O}_X^\perp$ with respect to stability conditions σ' in a family W. See Section 7.

The idea of the proof is the following. In one direction, we show that the projection of the shift of a skyscraper sheaf $i^* \mathcal{O}_x[-1] := \mathbf{L}_\mathcal{E} \mathbf{L}_{\mathcal{O}_X}(\mathcal{O}_x)[-1]$ is σ-stable. By construction, it fits into the triangle $\mathcal{E}^{n_g} \to I_x \to i^* \mathcal{O}_x[-1]$, from which $\mathrm{ext}^1(i^* \mathcal{O}_x[-1], i^!\mathcal{E}) = n_g$ follows. We then show that the morphism

$$p : X \to \mathcal{BN}_g$$

induced by the projection functor i^* is a closed embedding (see Proposition 6.4). The other direction is proved by contradiction. We begin with a σ-stable object $F \in \mathcal{K}u(X)$ satisfying the Brill–Noether condition, and we assume that F is not of the form $i^* \mathcal{O}_x[-1]$. Then we show that $G := F[-1]$ is a vector bundle (see Proposition 6.5), and form the cone C of the natural map $G = F[-1] \xrightarrow{s} \mathcal{E}^{\oplus n_g}$. Note that the character of C is the same as the ideal sheaf of a point on X. Then we show that C is semistable with respect to a suitable weak stability condition, which is the most technical part of our article (see Proposition 6.7). Then by a standard wall-crossing argument, we show that C is a torsion free sheaf and is therefore isomorphic to I_x for a point $x \in X$, which leads to a contradiction (c.f. Section 6.1).

The Brill–Noether reconstruction of X depends on an appropriately chosen induced stability condition σ on $\mathcal{K}u(X)$ a priori. In [PR21], the authors show that every induced stability condition on the Kuznetsov component of an index one prime Fano threefold of genus $g \geq 6$ is Serre-invariant (see Definition 4.10). Moreover, in [JLLZ21, Theorem 4.25] and [FP21, Theorem 3.1], the authors show the uniqueness of Serre-invariant stability conditions on $\mathcal{K}u(X)$. Thus if $E \in \mathcal{K}u(X)$ is a stable object for every Serre-invariant stability condition and $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is an equivalence of Kuznetsov components, then $\Phi(E)$ is also stable with respect to any Serre-invariant stability condition. As a result, we prove *refined categorical Torelli theorems* for all prime Fano threefolds of index one and genus $g \geq 6$:

Theorem 1.3. *Let X and X' be prime Fano threefolds of index one and genus $g \geq 6$ such that there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ with $\Phi(i^!\mathcal{E}) \cong i^!\mathcal{E}'$. Then $X \cong X'$.*

There is a very interesting application of Theorem 1.3. It is shown in Lemma 8.1 that the Kuznetsov component $\mathcal{K}u(X)$ and intermediate Jacobian $J(X)$ are mutually determined by each other for index one prime Fano threefolds of genus $g \geq 7$, and this is conjectured for $g = 6$. Hence one can determine the fiber of the period map over the intermediate Jacobian $J(X)$ for each of these $g \geq 7$ Fano threefolds by computing the fiber $\mathcal{P}_{\mathrm{cat}}^{-1}([\mathcal{K}u(X)])$ of the "categorical period map" $\mathcal{P}_{\mathrm{cat}}$ over their Kuznetsov components $\mathcal{K}u(X)$. It is the family of *gluing objects* $i^!\mathcal{E}' \in \mathcal{K}u(X')$ when X' varies but the equivalence class of Kuznetsov components is fixed. Since the object $i^!\mathcal{E}' \in \mathcal{K}u(X')$ is stable (c.f. Proposition 4.14), this makes the fiber of period map as an open subset of the union of the moduli spaces $\bigcup_{\chi(v,v)=\chi([i^!\mathcal{E}],[i^!\mathcal{E}])}\mathcal{M}_\sigma(\mathcal{K}u(X),v)$, where $\chi(-,-)\colon \mathcal{N}(X)\times\mathcal{N}(X)\to\mathbb{Z}$ is the Euler pairing on the numerical Grothendieck group $\mathcal{N}(X)$ of the Kuznetsov component $\mathcal{K}u(X)$.

As a result, we not only recover classical results on the fiber of the period map, but also reprove new results for X_{18} which were proved in [FV22] very recently.

Theorem 1.4. *Let X be a prime Fano threefold of index one and genus $g\geq 7$. Consider all index one prime Fano threefolds X' with the same genus g such that there is an equivalence $\Phi:\mathcal{K}u(X')\simeq\mathcal{K}u(X)$. Then the family of objects $\Phi(i^!\mathcal{E}')$ obtained as X' and Φ vary is given by*

(i) *a unique point if $g=7$;*
(ii) *the moduli space $\mathcal{M}_{\mathrm{vb}}\subset M_Y^{\mathrm{inst}}(2,0,2)$ of instanton bundles with minimal charge on a cubic threefold Y if $g=8$;*
(iii) *the moduli space of rank two vector bundles over a genus three curve C with a certain special property if $g=9$;*
(iv) *the Coble–Dolgachev sextic if $g=10$;*
(v) *the moduli space of plane quartics if $g=12$.*

Furthermore, in each case this describes the fiber of the period map.

Remark 1.5. One can apply Theorem 1.3 to obtain a description of the fiber of the "categorical period map" for index one prime Fano threefolds of genus $g=6$, which was previously described in [JLLZ21, Theorem 11.3].

1.1. **Related work.** The question of whether $\mathcal{K}u(X)$ determines X up to isomorphism, known as the *Categorical Torelli* question, has been studied for certain cases in the setting of Fano threefolds. There is a nice survey [PS22] on recent results. In the case of index two prime Fano threefolds, in [BMMS12], the authors show that the Kuznetsov component completely determines cubic threefolds up to isomorphism. The same result was also verified in [PY20]. In [BBF$^+$20], an alternative proof for this is provided. In [APR19], it is shown that the Kuznetsov component $\mathcal{K}u(X)$ determines X up to isomorphism if X is a smooth quartic double solid. A weaker version of this result was also shown in [BT16], where the equivalence between the Kuznetsov components is assumed to be Fourier–Mukai. In the case of index one prime Fano threefolds of genus ≥ 6, $\mathcal{K}u(X)$ usually does not determine the isomorphism class of X, while the whole derived category $\mathrm{D}^b(X)$ determines it by [BO01]. In our paper, we show that the subcategory $\langle\mathcal{K}u(X),\mathcal{E}\rangle\subset\mathrm{D}^b(X)$ does determine the isomorphism class of X. Our results not only recover the classical results in [Muk01] and [BF13], but also reprove the very recent results in [FV22] on reconstruction of X_{18} from the data $(\Gamma_2,\mathcal{V})$, where Γ_2 is a genus 2 curve and $\mathcal{V}$ is a rank 3 vector bundle with special properties over it. In the very recent work [BP22], the authors compute the fiber of the "categorical period map" for Gushel–Mukai threefolds, via a completely different method. In the upcoming work [FLZ22], we establish a Brill–Noether reconstruction theorem for index two prime Fano threefolds and give uniform proofs of categorical Torelli theorems for them.

1.2. **Organisation of the article.** In Section 2, we state some useful results on semiorthogonal decompositions. In section 3 we introduce semiorthogonal decompositions for all index one prime Fano threefolds, and define *gluing objects* for their Kuznetsov components. In Section 4, we recall weak stability conditions, and state the algorithm we frequently use to find and then discard solutions for potential walls in tilt stability. We also summarise results on the existence of Bridgeland stability conditions on Kuznetsov components of index one Fano threefolds. Furthermore, we prove stability of the *gluing object* introduced in Section 3. In Section 5, we embed our Fano threefolds into moduli spaces of stable objects in $\mathcal{K}u(X)$ with respect to the numerical classes of projections of skyscraper sheaves. In Section 6, we exhibit these embedded Fano threefolds as Brill–Noether loci. We also state the refined categorical Torelli theorem which follows as a corollary. In Section 7, we show that every index one prime Fano threefold is a Bridgeland moduli space of stable objects in $\mathcal{O}_X^\perp$. In Section 8, we compute the fiber of period map for all prime Fano threefolds of index one and genus $g \geq 6$ via Theorem 1.3. In Appendix A, we prove several lemmas used in the previous sections.

Notation and conventions.

- Throughout this article, g and d will mean the genus and degree of a Fano threefold, respectively. If it is relevant, we will state the genus/degree of the threefold we work with. If no assumptions on X are stated, then we will take it to be a smooth index one Picard rank one Fano threefold of genus $g \geq 6$. When X is written with a subscript, the subscript will indicate the degree of X.
- Let σ be a weak stability condition. Then the central charge and heart are denoted by Z_σ and $\mathcal{A}_\sigma$, respectively.
- The *projection functor* is defined by the left adjoint of the inclusion functor $i : \mathcal{K}u(X) \hookrightarrow \mathrm{D}^b(X)$.
- We denote the phase and slope with respect to σ by ϕ_σ and μ_σ, respectively. The maximal/minimal slope of the HN-factors of a given object will be denoted by μ^+_σ and μ^-_σ, respectively.
- We denote by $\mathcal{H}^i_\sigma$ the i-th cohomolgy object with respect to the heart $\mathcal{A}_\sigma$. If $\mathcal{A} = \mathrm{Coh}(X)$, we denote the cohomology objects by $\mathcal{H}^i$ for simplicity.
- We denote the numerical class in the numerical Grothendieck group by $[E]$ for any object E. In our setting, giving a numerical class is equivalent to giving a Chern character.

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2. Derived categories

In this section, we collect some useful facts/results about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06], for example. From now on, let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on X, and for $E, F \in \mathrm{D}^b(X)$, define

$$\mathrm{RHom}^\bullet(E, F) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i(E, F)[-i].$$

2.1. Exceptional collections and semiorthogonal decompositions.

Definition 2.1. Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}$. We say that E is an *exceptional object* if $\mathrm{RHom}^\bullet(E, E) = k$. Now let $(E_1, \ldots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{C}$ in $\mathcal{D}$ as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$${}^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Recall that a subcategory of $\mathcal{D}$ is called *admissible* if the inclusion functor has both left and right adjoint.

Definition 2.3. Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ be a collection of full admissible subcategories of $\mathcal{D}$. We say that $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{C}_j \subset \mathcal{C}_i^{\perp}$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ generate $\mathcal{D}$, i.e. the category resulting from taking all shifts and cones of objects in the categories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ is equivalent to $\mathcal{D}$.

Definition 2.4. The *Serre functor* $S_{\mathcal{D}}$ of a triangulated category $\mathcal{D}$ is the autoequivalence of $\mathcal{D}$ such that there is a functorial isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(B, S_{\mathcal{D}}(A))^{\vee}$$

for any $A, B \in \mathcal{D}$.

Proposition 2.5. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D} \simeq \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

2.2. **Mutations.** Let $\mathcal{C} \subset \mathcal{D}$ be an admissible triangulated subcategory. Then one has both left and right adjoints to the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$, denoted by i^* and $i^!$, respectively. Then the *left mutation functor* $\mathbf{L}_{\mathcal{C}}$ through $\mathcal{C}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \to \mathrm{id} \to \mathbf{L}_{\mathcal{C}}$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{C}}$ through $\mathcal{C}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{C}} \to \mathrm{id} \to ii^*.$$

When $E \in \mathrm{D}^b(X)$ is an exceptional object, and $F \in \mathrm{D}^b(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{RHom}^{\bullet}(E, F) \to F \to \mathbf{L}_E F,$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \to F \to E \otimes \mathrm{RHom}^{\bullet}(F, E)^{\vee}.$$

Furthermore, when $(E_1, \dots, E_m)$ is an exceptional collection, for $i = 1, \dots, m-1$ the collections

$$(E_1, \dots, E_{i-1}, \mathbf{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_m)$$

and

$$(E_1, \dots, E_{i+1}, \mathbf{R}_{E_{i+1}} E_i, E_{i+2}, E_{i+3}, \dots, E_m)$$

are also exceptional.

Proposition 2.6. *Let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition. Then*

$$S_{\mathcal{B}} = \mathbf{R}_{\mathcal{A}} \circ S_{\mathcal{D}} \quad \textit{and} \quad S_{\mathcal{A}}^{-1} = \mathbf{L}_{\mathcal{B}} \circ S_{\mathcal{D}}^{-1}.$$

3. Index one Prime Fano threefolds and their Kuznetsov components

A smooth projective variety with ample anticanonical bundle is called *Fano*. A Fano variety is called *prime* if it has Picard number one. For a prime Fano variety X, we can choose a unique ample divisor such that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H$. The *index* of a prime Fano variety is the least integer i_X such that $-K_X = i_X \cdot H$. The *degree* of a prime Fano threefold is $d_X := H^3$.

In this paper, we mainly consider prime Fano threefolds of index one. The prime Fano threefolds are classified in [VI99]. We list some of their properties below, see also [KPS18].

- X_2, $g = 2$: a double cover of $\mathbb{P}^3$ branched in a surface of degree six;
- X_4, $g = 3$: either a quartic threefold, or the double cover of a smooth quadric threefold branched in an intersection with a quartic;
- X_6, $g = 4$: a complete intersection of type $(2, 3)$;
- X_8, $g = 5$: a complete intersection of type $(2, 2, 2)$;
- X_{10}^O, $g = 6$: a section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear space and a quadric;
- X_{10}^S, $g = 6$: a double cover of a linear section of $\mathrm{Gr}(2, 5)$ of codimension 3, branched in an anticanonical divisor;

- X_{12}, $g=7$: a linear section of a connected component of the orthogonal Lagrangian Grassmannian $\mathrm{OGr}_+(5,10)\subset\mathbb{P}^{15}$;
- X_{14}, $g=8$: a linear section of $\mathrm{Gr}(2,6)\subset\mathbb{P}^{14}$;
- X_{16}, $g=9$: a linear section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6)\subset\mathbb{P}^{13}$;
- X_{18}, $g=10$: a linear section of the homogeneous space $G_2/P\subset\mathbb{P}^{13}$;
- X_{22}, $g=12$: a zero locus of three sections of the bundle $\wedge^2\mathcal{U}^\vee$, where $\mathcal{U}$ is the universal subbundle on $\mathrm{Gr}(3,7)$.

A prime Fano threefold of type X_{10}^S is called a *special Gushel–Mukai (GM) threefold*, and a threefold of type X_{10}^O is called an *ordinary Gushel–Mukai (GM) threefold*.

3.1. **Semiorthogonal decompositions.** A series of works of Bondal–Orlov and Kuznetsov show that the prime Fano threefolds of index one admit the following semiorthogonal decompositions.

Proposition 3.1. *Let $X:=X_{2g-2}$ be a prime Fano threefold of index one and genus g. Let $\Gamma_{g'}$ be a smooth curve of genus g'. Then we have*

- $\mathrm{D}^b(X_{2g-2})=\langle\mathcal{K}u(X_{2g-2}),\mathcal{O}_{X_{2g-2}}\rangle$ *for $g<6$;*
- $\mathrm{D}^b(X_{10})=\langle\mathcal{K}u(X_{10}),\mathcal{E}_6,\mathcal{O}_X\rangle$ *and $S_{\mathcal{K}u(X_{10})}=\tau[2]$, where τ is an involution on $\mathcal{K}u(X_{10})$;*
- $\mathrm{D}^b(X_{12})=\langle\mathcal{K}u(X_{12}),\mathcal{E}_7,\mathcal{O}_X\rangle$*, where $\mathcal{K}u(X_{12})\simeq\mathrm{D}^b(\Gamma_7)$;*
- $\mathrm{D}^b(X_{14})=\langle\mathcal{K}u(X_{14}),\mathcal{E}_8,\mathcal{O}_X\rangle$ *and $S^3_{\mathcal{K}u(X_{14})}=[5]$.*
- $\mathrm{D}^b(X_{16})=\langle\mathcal{K}u(X_{16}),\mathcal{E}_9,\mathcal{O}_X\rangle$*, where $\mathcal{K}u(X_{16})\simeq\mathrm{D}^b(\Gamma_3)$;*
- $\mathrm{D}^b(X_{18})=\langle\mathcal{K}u(X_{18}),\mathcal{E}_{10},\mathcal{O}_X\rangle$*, where $\mathcal{K}u(X_{18})\simeq\mathrm{D}^b(\Gamma_2)$;*
- $\mathrm{D}^b(X_{22})=\langle\mathcal{K}u(X_{22}),\mathcal{E}_{12},\mathcal{O}_X\rangle$*, where $\mathcal{K}u(X_{22})\simeq\mathrm{D}^b(\mathrm{Rep}(K(3)))$, where $K(3)$ is the 3-Kronecker quiver.*

Here $\mathcal{E}_g$ is a stable bundle of rank two when $g\neq 7$ and 9, rank five when $g=7$, and rank three when $g=9$.

When X_{10} is special, X_{10} is the double cover of a prime Fano threefold of index two and degree 5. Then τ is induced by the geometric involution on X_{10}, which we also denote by τ.

Note that when $g=4$, there is another semiorthogonal decomposition. The existence of stability conditions on this component is unknown, and we will not use this decomposition in our paper.

We call the subcategories $\mathcal{K}u(X_{2g-2})$ *Kuznetsov components* of X_{2g-2}. The left adjoint of the inclusion $i:\mathcal{K}u(X)\hookrightarrow\mathrm{D}^b(X)$ is called the *projection functor* and is denoted by i^*.

When the genus $g\geq 6$ is even, the rank two bundle $\mathcal{E}_g$ is the pullback of the tautological sub bundle on the Grassmannian $\mathrm{Gr}(2,\frac{g}{2}+2)$. When $g=7$ and 9, they are also pull-backs of the tautological sub-bundle on the corresponding Grassmannian. The Chern characters of each $\mathcal{E}_g$ are listed below:

$$\mathrm{ch}(\mathcal{E}_g)=\begin{cases}(2,-H,L,\frac{1}{3}P), & g=6\\ (5,-2H,0,P), & g=7\\ (2,-H,2L,\frac{1}{6}P), & g=8\\ (3,-H,0,\frac{1}{3}P), & g=9\\ (2,-H,3L,0), & g=10\\ (2,-H,4L,-\frac{1}{6}P), & g=12,\end{cases}$$

where L and P are the classes of a line and a point on X respectively. If the genus of X is clear, we will use $\mathcal{E}:=\mathcal{E}_g$ for simplicity. When $g\geq 6$ and g is even, by [Kuz04, Proposition 3.9] we know that the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(X_{2g-2}))$ is a rank two integral lattice and generated by

$$\mathcal{N}(\mathcal{K}u(X_{2g-2}))=\langle v:=1-\frac{g}{2}L+\frac{g-4}{4}P, w:=H-\frac{3g-6}{2}L+\frac{7g-40}{12}P\rangle \tag{1}$$

with Euler form given by

$$\begin{bmatrix} 1-\frac{g}{2} & -\frac{g}{2} \\ 3-g & 1-g \end{bmatrix}. \tag{2}$$

When $g=7$, the Todd class of X is given by $\mathrm{td}(X)=1+\frac{1}{2}H+3L+P$. Using Hirzebruch-Riemann-Roch theorem, one can verify with a direct computation that the numerical Grothendieck group is a rank two integral lattice generated by

$$\mathcal{N}(\mathcal{K}u(X_{12}))=\langle v:=2-5L+\frac{1}{2}P, w:=H-6L\rangle$$

with Euler form given by

$$\begin{bmatrix} -6 & -5 \\ -7 & -6 \end{bmatrix}. \tag{3}$$

When $g=9$, the Todd class of X is given by $\mathrm{td}(X)=1+\frac{1}{2}H+\frac{10}{3}L+P$. The numerical Grothendieck group is a rank two integral lattice generated by

$$\mathcal{N}(\mathcal{K}u(X_{16}))=\langle v:=1-3L+\frac{1}{2}P, w:=H-8L+\frac{2}{3}P\rangle$$

with Euler form given by

$$\begin{bmatrix} -2 & -3 \\ -5 & -8 \end{bmatrix}. \tag{4}$$

When the genus $g\geq 6$ is even, we will also use another semiorthogonal decomposition

$$\mathrm{D}^b(X_{2g-2})=\langle \mathcal{A}_{X_{2g-2}}, \mathcal{O}_{X_{2g-2}}, \mathcal{E}^\vee\rangle.$$

We call $\mathcal{A}_{X_{2g-2}}$ the *alternative Kuznetsov component*. The numerical Grothendieck group $\mathcal{N}(\mathcal{A}_{X_{2g-2}})$ of $\mathcal{A}_{X_{2g-2}}$ is a rank two integral lattice generated by

$$\mathcal{N}(\mathcal{A}_{X_{2g-2}})=\langle s:=1-2L, t:=H-(\frac{g}{2}+1)L-\frac{16-g}{12}P\rangle$$

with Euler form given by

$$\begin{bmatrix} -1 & -2 \\ -\frac{g}{2}+1 & -g+1 \end{bmatrix}. \tag{5}$$

We call $u\in\mathcal{N}(\mathcal{K}u(X_{2g-2}))$ (or $\mathcal{N}(\mathcal{A}_{2g-2})$) a $(-r)$*-class* if $\chi(u,u)=-r$.

Remark 3.2. When $g\geq 6$ is even, there is an equivalence between $\mathcal{K}u(X)$ and $\mathcal{A}_X$ which is given by $\Xi: E\mapsto \mathbf{L}_{\mathcal{O}_X}(E\otimes\mathcal{O}_X(H))$.

3.2. **Gluing objects in Kuznetsov components of even genus prime Fano threefolds.** In this subsection, we define *gluing objects* arising from Kuznetsov components $\mathcal{K}u(X)$ and $\mathcal{A}_X$ with respect to different semiorthogonal decompositions for derived categories of prime Fano threefolds of even genus $g\geq 6$. We furthermore investigate the relationship between the different gluing objects.

Let $\mathcal{D}:=\langle\mathcal{K}u(X),\mathcal{E}\rangle$ and $\mathcal{D}':=\langle\mathcal{A}_X,\mathcal{Q}^\vee\rangle$ where $\mathcal{K}u(X)$ and $\mathcal{A}_X$ are the original and alternative Kuznetsov components, respectively and $\mathcal{Q}$ is the tautological quotient bundle. We have the following inclusions $i_\mathcal{D}:\mathcal{K}u(X)\hookrightarrow\mathcal{D}$ and $i_{\mathcal{D}'}:\mathcal{A}_X\hookrightarrow\mathcal{D}'$. These inclusions have left adjoints $i^*_\mathcal{D}=\mathbf{L}_\mathcal{E}, i^*_{\mathcal{D}'}=\mathbf{L}_{\mathcal{Q}^\vee}$, respectively, and right adjoints $i^!_\mathcal{D}, i^!_{\mathcal{D}'}$, respectively.

Denote the functor $\langle\mathcal{E}\rangle\hookrightarrow\mathcal{D}$ by j_*. The functor $i^!_\mathcal{D}j_*[1]$ is called *gluing functor* in the sense of [KL15, Definition 2.4]. We define *gluing objects* for $\mathcal{D}$ and $\mathcal{D}'$ as $i^!_\mathcal{D}\mathcal{E}[1]$ and $i^!_{\mathcal{D}'}\mathcal{Q}^\vee[1]$, respectively. By [KL15, Lemma 2.5], $\mathcal{D}\simeq\{(F,V,\phi)\mid F\in\mathcal{K}u(X), V\in D^b(k);\phi: F\to i^!_\mathcal{D}\mathcal{E}\otimes V\}$, which is glued from $\mathcal{K}u(X)$ and $\mathrm{D}^b(k)$ generated by the exceptional object $\mathcal{E}$.

Recall that $i^!$ and $i'^!$ are the right adjoint of the inclusions $i:\mathcal{K}u(X)\hookrightarrow\mathrm{D}^b(X)$ and $i':\mathcal{A}_X\hookrightarrow\mathrm{D}^b(X)$, respectively. Since $\mathcal{E}\in\mathcal{D}$ and $\mathcal{Q}^\vee\in\mathcal{D}'$, we have $i^!\mathcal{E}=i^!_\mathcal{D}(\mathcal{E})$ and $i'^!\mathcal{E}=i^!_{\mathcal{D}'}(\mathcal{Q}^\vee)$.

Lemma 3.3. *The object $i^!\mathcal{E} = i^!_{\mathcal{D}}(\mathcal{E})$ is given by $\mathbf{L}_{\mathcal{E}}\,\mathcal{Q}(-H)[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(i^!(\mathcal{E})) = \begin{cases} \mathcal{Q}(-H), & i=-1 \\ \mathcal{E}, & i=0 \\ 0, & i\neq -1,0. \end{cases}$$

Proof. Indeed, by e.g. [Kuz10, Section 2] we have the exact triangle

$$i_{\mathcal{D}}i^!_{\mathcal{D}}(\mathcal{E}) \to \mathcal{E} \to \mathbf{L}_{\mathcal{K}u(X)}\,\mathcal{E} \to .$$

But note that $\langle \mathcal{K}u(X), \mathcal{E}\rangle \simeq \langle S_{\mathcal{D}}(\mathcal{E}), \mathcal{K}u(X)\rangle \simeq \langle \mathbf{L}_{\mathcal{K}u(X)}\,\mathcal{E}, \mathcal{K}u(X)\rangle$([Kuz10, Section 2]). Therefore the triangle above becomes $i_{\mathcal{D}}i^!_{\mathcal{D}}(\mathcal{E}) \to \mathcal{E} \to S_{\mathcal{D}}(\mathcal{E})$. To find $S_{\mathcal{D}}(\mathcal{E})$ explicitly, note that $S_{\mathcal{D}} \cong \mathbf{R}_{\mathcal{O}_X(-H)} \circ S_{\mathrm{D}^b(X)}$. Since $\mathbf{R}_{\mathcal{O}_X(-H)}\,\mathcal{E}(-H) \cong \mathcal{Q}(-H)[-1]$, we have $S_{\mathcal{D}}(\mathcal{E}) \cong \mathcal{Q}(-H)[2]$. So the triangle above becomes

$$i_{\mathcal{D}}i^!_{\mathcal{D}}(\mathcal{E}) \to \mathcal{E} \to \mathcal{Q}(-H)[2].$$

Applying $i^*_{\mathcal{D}} = \mathbf{L}_{\mathcal{E}}$ to the triangle and using the fact that $i^*_{\mathcal{D}}i_{\mathcal{D}} \cong \mathrm{id}$ and $i^*\mathcal{E} = 0$ gives $i^!_{\mathcal{D}}(\mathcal{E}) \cong \mathbf{L}_{\mathcal{E}}\,\mathcal{Q}(-H)[1]$, as required. Taking the long exact sequence with respect to $\mathsf{Coh}(X)$ gives the cohomology objects. □

Lemma 3.4 ([JLLZ21, Lemma 5.5])**.** *The object $i'^!(\mathcal{Q}^\vee) = i^!_{\mathcal{D}'}(\mathcal{Q}^\vee)$ is given by $\mathbf{L}_{\mathcal{Q}^\vee}\,\mathcal{E}[1]$. It is a two-term complex with cohomologies*

$$\mathcal{H}^i(i'^!(\mathcal{Q}^\vee)) = \begin{cases} \mathcal{E}, & i=-1 \\ \mathcal{Q}^\vee, & i=0 \\ 0, & i\neq -1,0. \end{cases}$$

Proof. The proof is completely analogous to the proof of Lemma 3.3. As before, we have the exact triangle

$$i_{\mathcal{D}'}i^!_{\mathcal{D}'}(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathbf{L}_{\mathcal{A}_X}\,\mathcal{Q}^\vee \to .$$

But note that $\langle \mathcal{A}_X, \mathcal{Q}^\vee\rangle \simeq \langle S_{\mathcal{D}'}(\mathcal{Q}^\vee), \mathcal{A}_X\rangle \simeq \langle \mathbf{L}_{\mathcal{A}_X}\,\mathcal{Q}^\vee, \mathcal{A}_X\rangle$. Therefore the triangle above becomes $i_{\mathcal{D}'}i^!_{\mathcal{D}'}(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to S_{\mathcal{D}'}(\mathcal{Q}^\vee)$. To find $S_{\mathcal{D}'}(\mathcal{Q}^\vee)$ note that $S_{\mathcal{D}'}(\mathcal{Q}^\vee) = \mathbf{R}_{\mathcal{O}_X}(-H)(\mathcal{Q}^\vee(-H))[3]$. One can check that $\mathbf{R}_{\mathcal{O}_X}\,\mathcal{Q}^\vee = \mathcal{E}^\vee[-1]$, so $S_{\mathcal{D}'}(\mathcal{Q}^\vee) \cong \mathcal{E}[2]$. Hence our triangle becomes $i_{\mathcal{D}'}i^!_{\mathcal{D}'}(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee \to \mathcal{E}[2]$. Now applying $i^*_{\mathcal{D}'} = \mathbf{L}_{\mathcal{Q}^\vee}$ to the triangle, we get $i^!_{\mathcal{D}'}(\mathcal{Q}^\vee) = \mathbf{L}_{\mathcal{Q}^\vee}\,\mathcal{E}[1]$, as required. Since $\mathrm{RHom}^\bullet(\mathcal{Q}^\vee, \mathcal{E}) = k[-2]$, we have the triangle $\mathcal{Q}^\vee[-2] \to \mathcal{E} \to \mathbf{L}_{\mathcal{Q}^\vee}\,\mathcal{E}$. Taking the long exact sequence of this triangle with respect to $\mathcal{H}^*$ gives the required cohomology objects and we have the following triangle:

$$\mathcal{E}[1] \to i^!_{\mathcal{D}'}(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee.$$

□

Remark 3.5. It is not hard to check that $\Xi(i^!(\mathcal{E})) \cong i'^!(\mathcal{Q}^\vee)[1]$, where Ξ is the equivalence $\mathcal{K}u(X) \simeq \mathcal{A}_X$ from Remark 3.2. Indeed, simply apply Ξ to the triangle defining $i^!(\mathcal{E})$. This yields the triangle defining $i'^!(\mathcal{Q}^\vee)$.

Remark 3.6. The gluing object in the Kuznetsov component of an odd genus prime Fano threefold is defined in a similar way. They were already defined in [BF13, Lemma 3.5] for genus 9 prime Fano threefolds, and in [BF14, Lemma 2.9]. Note that they define the Kuznetsov component $\mathcal{B}_X$ as ${}^\perp\langle \mathcal{O}_X, \mathcal{E}^\vee\rangle$ and $\mathcal{B}_X \simeq \mathcal{A}_X \otimes \mathcal{O}_X(H)$. It is easy to show that the gluing objects in $\mathcal{B}_X$ and $\mathcal{A}_X$ differ by the equivalence $-\otimes \mathcal{O}_X(H)$.

4. Bridgeland stability conditions

In this section, we recall the construction of (weak) Bridgeland stability conditions on $\mathrm{D}^b(X)$, and the notions of tilt stability, double-tilt stability, and stability conditions induced on Kuznetsov components from weak stability conditions on $\mathrm{D}^b(X)$. We follow [BLMS17, § 2].

4.1. **Weak stability conditions.** Let $\mathcal{D}$ be a triangulated category, and $K_0(\mathcal{D})$ its Grothendieck group. Fix a surjective morphism $v : K_0(\mathcal{D}) \to \Lambda$ to a finite rank lattice.

Definition 4.1. The *heart of a bounded t-structure* on $\mathcal{D}$ is an abelian subcategory $\mathcal{A} \subset \mathcal{D}$ such that the following conditions are satisfied:

(i) for any $E, F \in \mathcal{A}$ and $n < 0$, we have $\mathrm{Hom}(E, F[n]) = 0$;
(ii) for any object $E \in \mathcal{D}$ there exist objects $E_i \in \mathcal{A}$ and maps

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m = E$$

such that $\mathrm{cone}(\phi_i) = A_i[k_i]$ where $A_i \in \mathcal{A}$ and the k_i are integers such that $k_1 > k_2 > \cdots > k_m$.

Definition 4.2. Let $\mathcal{A}$ be an abelian category and $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a group homomorphism such that for any $E \in \mathcal{D}$ we have $\mathrm{Im}\, Z(E) \geq 0$ and if $\mathrm{Im}\, Z(E) = 0$ then $\mathrm{Re}\, Z(E) \leq 0$. Then we call Z a *weak stability function* on $\mathcal{A}$. If furthermore we have for $0 \neq E \in \mathcal{A}$ that $\mathrm{Im}\, Z(E) = 0$ implies that $\mathrm{Re}\, Z(E) < 0$, then we call Z a *stability function* on $\mathcal{A}$.

Definition 4.3. A *weak stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z \colon \Lambda \to \mathbb{C}$ is a group homomorphism such that

(i) the composition $Z \circ v \colon K_0(\mathcal{A}) \cong K_0(\mathcal{D}) \to \mathbb{C}$ is a weak stability function on $\mathcal{A}$. From now on, we write $Z(E)$ rather than $Z(v(E))$.

Much like the slope in classical μ-stability, we can define a *slope* μ_σ for σ using Z. For any $E \in \mathcal{A}$, set

$$\mu_\sigma(E) := \begin{cases} -\frac{\mathrm{Re}\, Z(E)}{\mathrm{Im}\, Z(E)}, & \text{if } \mathrm{Im}\, Z(E) > 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

We say an object $0 \neq E \in \mathcal{A}$ is σ-(semi)stable if $\mu_\sigma(F) < \mu_\sigma(E/F)$ (respectively $\mu_\sigma(F) \leq \mu_\sigma(E/F)$) for all proper subobjects $F \subset E$.

(ii) Any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of σ-semistability defined above.
(iii) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geq 0$ for all σ-semistable objects $E \in \mathcal{A}$. This is known as the *support property.*

If the composition $Z \circ v$ is a stability function, then σ is a *stability condition* on $\mathcal{D}$.

For this paper, we let Λ be the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ which is $K_0(\mathcal{D})$ modulo the kernel of the Euler form $\chi(E, F) = \sum_i (-1)^i \mathrm{ext}^i(E, F)$.

4.2. **Tilt stability.** Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on a triangulated category $\mathcal{D}$. Now consider the following subcategories of $\mathcal{A}$, where $\langle - \rangle$ denotes the extension closure:

$$\mathcal{T}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu \rangle$$
$$\mathcal{F}_\sigma^\mu = \langle E \in \mathcal{A} \mid E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leq \mu \rangle.$$

Then it is a result of [HRS96] that

Proposition 4.4. *The abelian category $\mathcal{A}_\sigma^\mu := \langle \mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu[1] \rangle$ is the heart of a bounded t-structure on $\mathcal{D}$.*

We call $\mathcal{A}_\sigma^\mu$ the *tilt* of $\mathcal{A}$ around the torsion pair $(\mathcal{T}_\sigma^\mu, \mathcal{F}_\sigma^\mu)$. Let X be an n-dimensional smooth projective complex variety. Tilting can be applied to the weak stability condition $(\mathsf{Coh}(X), Z_H)$ to form the once-tilted heart $\mathsf{Coh}^\beta(X)$, where $Z_H(E) := -c_1(E)H^{n-1} + \mathfrak{i}\,\mathrm{rk}(E)H^n$ for any $E \in \mathsf{Coh}(X)$. Define for $E \in \mathsf{Coh}^\beta(X)$

$$Z_{\alpha,\beta}(E) = \frac{1}{2}\alpha^2 H^n \mathrm{ch}_0^\beta(E) - H^{n-2}\mathrm{ch}_2^\beta(E) + \mathfrak{i} H^{n-1}\mathrm{ch}_1^\beta(E).$$

Proposition 4.5 ([BMT11, BMS16]). *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the pair $\sigma_{\alpha,\beta} = (\mathsf{Coh}^\beta(X), Z_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$. The quadratic form Q is given by the discriminant*

$$\Delta_H(E) = (H^{n-1}\mathrm{ch}_1(E))^2 - 2H^n\mathrm{ch}_0(E)H^{n-2}\mathrm{ch}_2(E).$$

The stability conditions $\sigma_{\alpha,\beta}$ vary continuously as $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies. Furthermore, for any $v \in \Lambda^2_H$ there is a locally finite wall-and-chamber structure on $\mathbb{R}_{>0} \times \mathbb{R}$ controlling stability of objects with class v.

We now state a useful lemma which relates 2-Giesesker-stability and tilt stability.

Lemma 4.6 ([BMS16, Lemma 2.7], [BBF+20, Proposition 4.8]). *Let $E \in \mathrm{D}^b(X)$ be an object. If $H^2\mathrm{ch}^\beta_1(E) > 0$, then $E \in \mathsf{Coh}^\beta(X)$ and E is $\sigma_{\alpha,\beta}$-(semi)stable for $\alpha \gg 0$ if and only if E is a 2-Gieseker-(semi)stable sheaf.*

4.3. **Finding solutions for walls in tilt stability.** In this subsection, we describe a way of finding (potential) walls in tilt stability with respect to objects in the derived category with a given truncated Chern character. This is similar to the method used in e.g. [PY20] to find walls for certain objects. Let $M \in \mathsf{Coh}^\beta(X)$ be the object in question, and let its truncated Chern character be $\mathrm{ch}_{\leq 2}(M) = (m_0, m_1H, \frac{m_2}{d}H^2)$, where $d = \deg X$.

Assume there is a short exact sequence $0 \to E \to M \to F \to 0$ which makes M strictly semistable. We can assume that E and F are tilt-semistable using the existence of Harder-Narasimhan or Jordan-Holder filtrations. Then the following conditions must be satisfied:

(a) $\mathrm{ch}_{\leq 2}(M) = \mathrm{ch}_{\leq 2}(E) + \mathrm{ch}_{\leq 2}(F)$;
(b) $\mu_{\alpha,\beta}(E) = \mu_{\alpha,\beta}(M) = \mu_{\alpha,\beta}(F)$;
(c) $\Delta_H(E) \geq 0$ and $\Delta_H(F) \geq 0$;
(d) $\Delta_H(E) \leq \Delta_H(M)$ and $\Delta_H(F) \leq \Delta_H(M)$.

Since $E, F \in \mathsf{Coh}^\beta(X)$, we also must have $\mathrm{ch}^\beta_1(E) \geq 0$ and $\mathrm{ch}^\beta_1(F) \geq 0$. Solving the system of inequalities above gives an even number of solutions of $(m_0, m_1, m_2) \in \mathbb{Z}^{\oplus 3}$; half of them are solutions for the destabilising subobject E, and the other half are the corresponding quotients F.

4.4. **Stability conditions on Kuznetsov components.**

4.4.1. *Double-tilted stability conditions.* Now as in [BLMS17], we pick a weak stability condition $\sigma_{\alpha,\beta}$ and tilt the once-tilted heart $\mathsf{Coh}^\beta(X)$ with respect to the tilt slope $\mu_{\alpha,\beta}$ and some second tilt parameter μ. One gets a torsion pair $(\mathcal{T}^\mu_{\alpha,\beta}, \mathcal{F}^\mu_{\alpha,\beta})$ and another heart $\mathsf{Coh}^\mu_{\alpha,\beta}(X)$ of $\mathrm{D}^b(X)$. Now "rotate" the stability function $Z_{\alpha,\beta}$ by setting

$$Z^\mu_{\alpha,\beta} := \frac{1}{u} Z_{\alpha,\beta}$$

where $u \in \mathbb{C}$ such that $|u| = 1$ and $\mu = -\frac{\operatorname{Re} u}{\operatorname{Im} u}$.

Proposition 4.7 ([BLMS17, Proposition 2.15]). *The pair $(\mathsf{Coh}^\mu_{\alpha,\beta}(X), Z^\mu_{\alpha,\beta})$ defines a weak stability condition on $\mathrm{D}^b(X)$.*

For example, if we choose $\mu = 0$, we have

$$Z^0_{\alpha,\beta}(E) = H^{n-1}\mathrm{ch}^\beta_1(E) + \mathfrak{i}(H^{n-2}\mathrm{ch}^\beta_2(E) - \frac{1}{2}\alpha^2 H^n \mathrm{ch}^\beta_0(E)).$$

Proposition 5.1 in [BLMS17] gives a criterion for checking when weak stability conditions on a triangulated category can be used to induce stability conditions on a subcategory. Each of the criteria of this proposition can be checked for $\mathcal{K}u(X) \subset \mathrm{D}^b(X)$ to give stability conditions on $\mathcal{K}u(X)$.

4.4.2. *Stability conditions on Kuznetsov components.* More precisely, let $\mathcal{A}(\alpha,\beta)=\mathsf{Coh}^{\mu}_{\alpha,\beta}(X)\cap\mathcal{K}u(X)$ and $Z(\alpha,\beta)=Z^{\mu}_{\alpha,\beta}|_{\mathcal{K}u(X)}$. Furthermore, let $0<\epsilon\ll 1$, $\beta=-1+\epsilon$ and $0<\alpha<\epsilon$. Also impose the following condition on the second tilt parameter μ:

$$\mu_{\alpha,\beta}(\mathcal{E}(-H)[1])<\mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1])<\mu<\mu_{\alpha,\beta}(\mathcal{E})<\mu_{\alpha,\beta}(\mathcal{O}_X). \tag{6}$$

Then we get the following theorem.

Theorem 4.8 ([BLMS17, Theorem 6.9])**.** *Let X be a Fano threefold of genus $6,8,10$ or 12, and let ϵ,α,β and μ be as above. Then the pair $\sigma(\alpha,\beta)=(\mathcal{A}(\alpha,\beta),Z(\alpha,\beta))$ defines a Bridgeland stability condition on $\mathcal{K}u(X)$.*

In our paper, we fix $\mu=0$, i.e. $\sigma(\alpha,\beta):=\sigma^0_{\alpha,\beta}|_{\mathcal{K}u(X)}$.

Proposition 4.9. *Let $X:=X_{2g-2}$ be a prime Fano threefold of index one and genus $6\le g\le 12$. Then $\sigma(\alpha,\beta):=(\mathcal{A}^0_{\alpha,\beta}|_{\mathcal{K}u(X)},Z^0_{\alpha,\beta}|_{\mathcal{K}u(X)})$ is a stability condition for (α,β) listed below:*

- *$g=6$: $\beta=-\frac{9}{10}, 0<\alpha<1+\beta$,*
- *$g=7$: $\beta=-\frac{5}{6}$ or $-\frac{71}{84}$, $0<\alpha<1+\beta$,*
- *$g=8$: $\beta=-\frac{22}{25}$ or $-\frac{122}{125}$, $0<\alpha<1+\beta$,*
- *$g=9$: $\beta=-\frac{3}{4}$ or $-\frac{31}{40}$, $0<\alpha<1+\beta$,*
- *$g=10$: $\beta=-\frac{22}{25}$ or $-\frac{10}{11}$, $0<\alpha<1+\beta$,*
- *$g=12$: $\beta=-\frac{21}{25}$ or $-\frac{19}{22}$, $0<\alpha<1+\beta$.*

Moreover, $\sigma(\alpha,\beta)$ is Serre-invariant for these (α,β).

Proof. It is not hard to see that $\mathcal{E}_g,\mathcal{E}_g(-H)[1],\mathcal{O}_X,\mathcal{O}_X(-H)[1]\in\mathsf{Coh}^{\beta}(X)$, and that they satisfy

$$\mu_{\alpha,\beta}(\mathcal{E}(-H)[1])<\mu_{\alpha,\beta}(\mathcal{O}_X(-H)[1])<0<\mu_{\alpha,\beta}(\mathcal{E})<\mu_{\alpha,\beta}(\mathcal{O}_X)$$

for each (α,β) listed above.

First we assume that $g\ge 6$ is even. Then from [PR21, Proposition 3.2] we know that $\sigma(\alpha,\beta)$ is a stability condition for (α,β) as above. The Serre-invariance follows from [PR21, Theorem 3.18].

When $g=7$ or 9, we know that $\mathcal{K}u(X)$ is equivalent to the derived category of a certain curve with positive genus. Thus if one proves that $\sigma(\alpha,\beta)$ is a stability condition, the Serre-invariance follows from [Mac07]. To this end, it is sufficient to show that $\mathcal{E}_g$ and $\mathcal{E}_g(-H)[1]$ are $\sigma_{\alpha,\beta}$-semistable for the (α,β) listed above. From Lemma 4.6 we know that $\mathcal{E}_g$ and $\mathcal{E}_g(-H)[1]$ are both $\sigma_{\alpha,\beta}$-semistable for the β listed above and for $\alpha\gg 0$. The result then follows from Lemma A.2, Lemma A.3, Lemma A.5 and Lemma A.6. □

4.5. **Serre-invariance of stability conditions on Kuznestov components.** Recall the universal covering $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of $\mathrm{GL}^+(2,\mathbb{R})$ acts on the space of stability conditions, see [Bri07, Lemma 8.2].

Definition 4.10. Let σ be a stability condition on the alternative Kuznetsov component $\mathcal{A}_X$. It is called *Serre-invariant* if $S_{\mathcal{A}_X}\cdot\sigma=\sigma\cdot g$ for some $g\in\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

The same definition also applies when we replace $\mathcal{A}_X$ by $\mathcal{K}u(X)$ in the above. We recall several properties of Serre-invariant stability conditions from [PY20, Zha20] below.

Proposition 4.11. *Let σ be a Serre-invariant stability condition on $\mathcal{A}_X$ or $\mathcal{K}u(X)$, where X is a prime Fano threefold of index one and genus $g\ge 6$. Then:*

(i) *the heart of σ has homological dimension is ≤ 2,*
(ii) *Let A be a non-trivial object in the heart of a stability condition σ on $\mathcal{K}u(X_{2g-2})$ for $g\ge 7$. Then*
  - $\mathrm{ext}^1(A,A)\ge 2$ *if $g=6,8,12$ and*
  - $\mathrm{ext}^1(A,A)\ge 1$ *if $g=7,9,10$.*

Proof.

(i) If $g = 7, 9, 10$, we have $\mathcal{K}u(X_{2g-2}) \simeq \mathrm{D}^b(C)$ for certain curves C of positive genus (see [Kuz05], [Kuz09]). Then by [Mac07], the stability condition is given by slope stability on the curve C up to some action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, whose heart is $\mathrm{Coh}(C)$. So the homological dimension is 1. If $g = 12$, then by [DK19, Section 7] the heart of a stability condition on $\mathcal{K}u(X_{22}) \simeq \mathrm{D}^b(K(3))$ is generated by two exceptional objects, so the homological dimension is 1. If $g = 6$ or 8, by [Zha20, Proposition 4.13] and [PY20, Lemma 5.10], the homological dimension is 2.

(ii) Since $\chi(A, A) \geq -1$ for every non-trivial object A in $\mathcal{K}u(X_{10})$, $\mathcal{K}u(X_{14})$ and $\mathcal{K}u(X_{22})$, by (i) we have $\mathrm{ext}^1(A, A) \geq 2$, while $\mathrm{ext}^1(A, A) \geq 1$ since $\chi(A, A) \geq 0$ for every non-trivial object A in $\mathcal{K}u(X_{2g-2})$ for $g = 7, 9$ and 10.

□

To show an object in the Kuznetsov component is stable with respect to a Serre-invariant stability condition, we use the *Weak Mukai Lemma*, written below.

Lemma 4.12 (Weak Mukai Lemma, [PY20, Lemma 5.12], [LZ21, Lemmas 3.15 and 3.16])**.** *Let $X := X_{2g-2}$ be a prime Fano threefold of index one and genus g, and $\sigma = (\mathcal{A}_\sigma, Z_\sigma)$ be a Serre-invariant stability condition on the Kuznetsov component.*

(a) $g = 8$: Let $A \to E \to B$ be an exact triangle in the Kuznetsov component $\mathcal{K}u(X)$ with $\mathrm{Hom}(A, B) = 0$ such that the phases of the σ-semistable factors of A are greater than or equal to the phases of the σ-semistable factors of B. Then $\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leq \mathrm{ext}^1(E, E)$.

(b) $g = 7, 9, 10, 12$: Let $A \to E \to B$ be an exact triangle in Kuznetsov component $\mathcal{K}u(X)$ with $A, B \in \mathcal{A}_\sigma$ and $\mathrm{Hom}(A, B) = 0$. Then $\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leq \mathrm{ext}^1(E, E)$.

(c) $g = 6$: Let $A \to E \to B$ be an exact triangle in the Kuznetsov component $\mathcal{K}u(X)$ with $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, \tau(B)) = 0$. Then $\mathrm{ext}^1(A, A) + \mathrm{ext}^1(B, B) \leq \mathrm{ext}^1(E, E)$.

Finally, we discuss the uniqueness of Serre-invariant stability conditions on Kuznetsov components of certain Fano threefolds, which will be used frequently.

Theorem 4.13 ([JLLZ21, Theorem 4.25], [FP21, Theorem 3.1])**.** *Let $X := X_{4d+2}$ or Y_d for all $d \geq 2$. Then all Serre-invariant stability conditions on $\mathcal{K}u(X)$ are in the same $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit.*

4.6. **Stability of the gluing object.** Let X be an index one prime Fano threefold of genus $6 \leq g \leq 10$. In this subsection, we show that the *gluing object* $i^!\mathcal{E} \in \mathcal{K}u(X)$ is σ-stable with respect to stability conditions on $\mathcal{K}u(X)$ for each Serre-invariant stability condition σ.

Proposition 4.14. *Let X be a prime Fano threefold of index one and genus $10 \geq g \geq 6$, and let $\mathcal{A}_X$ be the alternative Kuznetsov component. Then the* gluing object *$\Xi(i^!\mathcal{E}) \in \mathcal{A}_X$ is σ-stable for each Serre-invariant stability condition σ on $\mathcal{A}_X$.*

Proof.

(i) $g = 6$: $\Xi(i^!\mathcal{E})$ is σ-stable by [JLLZ21, Lemma 5.7].

(ii) $g = 7$: Stability of $\Xi(i^!\mathcal{E})$ follows from [BF14, Lemma 2.9].

(iii) $g = 8$: By Proposition A.7, $\Xi(i^!\mathcal{E})$ is σ-stable by exactly the same argument as in [LZ21, Lemma 7.9].

(iv) $g = 9$: Stability of $\Xi(i^!\mathcal{E})$ follows from [BF13, Proposition 3.10, Remark 3.12].

(v) $g = 10$: Stability of $\Xi(i^!\mathcal{E})$ follows from [Fae13, Definition 2.8, Section II.3.3].

□

Remark 4.15. By Theorem 4.13, this also shows the stability of $i^!\mathcal{E} \in \mathcal{K}u(X)$ in these cases.

5. Embedding Fano threefolds into Bridgeland moduli spaces

In this section, we will embed our Fano threefolds X into certain moduli spaces of stable objects in $\mathcal{K}u(X)$. The numerical class will be that of the projection of a skyscraper sheaf into

$\mathcal{K}u(X)$. In the first subsection, we will give an explicit description of projections of skyscrapers into $\mathcal{K}u(X)$. In the second subsection, we will deal with the stability of these projections with respect to stability conditions on $\mathcal{K}u(X)$. In the third subsection, we will show the existence of a morphism from X to this moduli space, and show that it is in fact an embedding.

5.1. **Projections of skyscraper sheaves into Kuznestov components.**

Proposition 5.1. *Let X be a prime Fano threefold of genus g and $x \in X$ a point. Let $i^* : \mathrm{D}^b(X) \to \mathcal{K}u(X)$ be the projection functor. Then $i^*(\mathcal{O}_x)$ is given by*

(i) $\ker(\mathcal{E}^{\oplus \frac{g}{2}} \twoheadrightarrow I_x)[2]$, *if $g \geq 6$ and g is even such that X is not a special GM threefold;*
(ii) $\ker(\mathcal{E}^{\oplus 5} \twoheadrightarrow I_x)[2]$, *if $g = 7$;*
(iii) $\ker(\mathcal{E}^{\oplus 3} \twoheadrightarrow I_x)[2]$, *if $g = 9$;*
(iv)
 (1) $\ker(\mathcal{E}^{\oplus 3} \twoheadrightarrow I_{\pi^{-1}(\pi(x))})[2] \to i^*(\mathcal{O}_x) \to \mathcal{O}_y[1]$, *if X is a special GM threefold and $\pi(x) \notin \mathcal{B}$, where $\pi : X \to Y_5$ is a double cover with branch locus $\mathcal{B} \subset Y_5$ and $y = \tau(x)$;*
 (2) $\ker(\mathcal{E}^{\oplus 3} \twoheadrightarrow I_{\pi^{-1}(\pi(x))})[2] \to i^*(\mathcal{O}_x) \to \mathcal{O}_x[1]$, *if X is a special GM threefold and $\pi(x) \in \mathcal{B}$.*

Proof. First note that since $H^\bullet(X, \mathcal{O}_x) = k$, we have $\mathbf{L}_{\mathcal{O}_X} \mathcal{O}_x \cong I_x[1]$.

(i) Suppose $g \geq 6$ is even and X is not a special GM threefold. In this case $X \hookrightarrow \mathrm{Gr}(2, \frac{g}{2}+2)$ is an embedding. Applying $\mathrm{Hom}(\mathcal{E}, -)$ to the short exact sequence $0 \to I_x \to \mathcal{O}_X \to \mathcal{O}_x \to 0$, we get an exact sequence
$$0 \to \mathrm{Hom}(\mathcal{E}, I_x) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_x) \to \mathrm{Ext}^1(\mathcal{E}, I_x) \to 0.$$
Note that $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_X) = k^{\frac{g}{2}+2}$ and $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_x) = k^2$. Then $\hom(\mathcal{E}, I_x) \geq \frac{g}{2}$. If $\hom(\mathcal{E}, I_x) \geq \frac{g}{2}+1$, then x would be contained in the zero locus of at least $\frac{g}{2}+1$ linearly independent sections of $\mathcal{E}^\vee$, which is an empty set. Thus $\hom(\mathcal{E}, I_x) = \frac{g}{2}$. Therefore $\mathrm{RHom}^\bullet(\mathcal{E}, I_x) = k^{\frac{g}{2}}$ and we have
$$\mathcal{E}^{\oplus \frac{g}{2}} \to I_x \to \mathbf{L}_\mathcal{E} I_x.$$
We claim that the map $h : \mathcal{E}^{\oplus \frac{g}{2}} \to I_x$ is surjective. Indeed, $\mathrm{im}(h) = I_D$ where D is the zero locus of $\frac{g}{2}$ linearly independent sections of $\mathcal{E}^\vee$ containing the point x. For all cases of g that we consider, this zero locus is $\mathrm{Gr}(2,2)$, which is just a point. Since $X \hookrightarrow \mathrm{Gr}(2, \frac{g}{2}+2)$ is an embedding it follows that $\mathrm{im}(h) = I_x$. So $\mathcal{E}^{\oplus \frac{g}{2}} \to I_x$ is surjective and $\mathbf{L}_\mathcal{E} I_x \cong \ker(\mathcal{E}^{\oplus \frac{g}{2}} \to I_x)[1]$. Hence $i^*(\mathcal{O}_x) = \mathbf{L}_\mathcal{E} I_x[1] = \ker(\mathcal{E}^{\oplus \frac{g}{2}} \to I_x)[2]$.

(ii) If $g = 7$, $i^*(\mathcal{O}_x) \cong \mathbf{L}_{\mathcal{E}_5} I_x[1]$. By similar computations, $\mathrm{RHom}^\bullet(\mathcal{E}_5, I_x) \cong k^5$, so that we have
$$\mathcal{E}^{\oplus 5} \to I_x \to \mathbf{L}_\mathcal{E} I_x.$$
If the map $h : \mathcal{E}^{\oplus 5} \to I_x$ is not surjective, then $\mathrm{im}(h)$ would be an ideal sheaf I_D, where D is zero locus of five linearly independent sections of $\mathcal{E}^\vee$. But in this case $D = \mathrm{Gr}(5,5) \cap X$, and since $X \hookrightarrow \mathrm{OGr}_+(5, 10)$ is cut out by a linear section, we know that $D = \{x\}$. Hence the map is surjective, so $i^*(\mathcal{O}_x) \cong \ker(\mathcal{E}^{\oplus 5} \to I_x)[2]$.

(iii) If $g = 9$, then X is a linear section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$. The map $\mathcal{E}^{\oplus 3} \to I_x$ is surjective by [BF13, Lemma 3.6]. Then $i^*(\mathcal{O}_x) \cong \ker(\mathcal{E}^{\oplus 3} \to I_x)[2]$.

(iv) If $g = 6$ and X is a special GM threefold, since X does not embed into $\mathrm{Gr}(2,5)$, the map $\mathcal{E}^{\oplus 3} \xrightarrow{h} I_x$ is no longer a surjective map. Instead its image is $I_{\pi^{-1}(\pi(x))}$. If $\pi(x) \notin \mathcal{B}$, then $\pi^{-1}(\pi(x)) = x \cup \tau(x)$, otherwise $\pi^{-1}(\pi(x))$ is a non-reduced point with multiplicity two. Then the desired result follows.

□

We define $n_g := \frac{g}{2}$ if g is even, $n_7 = 5$ and $n_9 = 3$.

Remark 5.2. In what follows, we will denote the kernels of $h : \mathcal{E}^{\oplus n_g} \to I_x$ by K_x. For example, if $g \geq 7$ or if X is an ordinary GM threefold, $i^*(\mathcal{O}_x) \cong K_x[2]$. It is an easy computation to show that if $g \geq 6$ and g is even, then

$$\mathrm{ch}(i^* \mathcal{O}_x) = (g-1)v - \frac{g}{2}w = (g-1, -\frac{g}{2}H, \frac{g(g-4)}{4}L, -\frac{1}{24}(g+2)(g-12)P).$$

If $g = 7$, then $\mathrm{ch}(i^* \mathcal{O}_x) = 12v - 10w = (24, -10H, 0, 6P)$ and if $g = 9$, then $\mathrm{ch}(i^* \mathcal{O}_x) = 8v - 3w = (8, -3H, 0, 2P)$.

5.2. **Stability of projections of skyscraper sheaves.** In this subsection, we show that $i^* \mathcal{O}_x$ is semistable with respect to every Serre-invariant stability condition on $\mathcal{K}u(X)$.

Lemma 5.3. *Let $X := X_{2g-2}$ with even genus $g \geq 6$ and $K := K_x = \ker(\mathcal{E}^{\oplus \frac{g}{2}} \to I_x)$. Then K is a μ-stable reflexive sheaf.*

Proof. It is clear that K is reflexive. Assume that K is not μ-semistable, and let $K' \subset K$ be the maximal destabilizing subsheaf. Since $K \subset \mathcal{E}^{\oplus \frac{g}{2}}$ and $\mathcal{E}$ is μ-stable, we know $-\frac{g}{2g-2} < \mu(K') \leq -\frac{1}{2}$. It is not hard to check that the only possible case is $\mu(K') = \mu^+(K) = -\frac{1}{2}$. Now by polystability of $\mathcal{E}^{\oplus \frac{g}{2}}$, K' is contained in a direct summand $\mathcal{E}^{\oplus r}$, where $\mathrm{rk}(K') = 2r < g$. Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, hook] & K'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E}^{\oplus r} \arrow[r] & \mathcal{E}^{\oplus \frac{g}{2}} \arrow[r] & \mathcal{E}^{\oplus \frac{g}{2} - r} \arrow[r] & 0
\end{tikzcd}$$

with the rows being exact, where $K'' := K/K'$. Thus by taking the cokernel, we have a map $\mathcal{E}^{\oplus r}/K' \to \mathcal{E}^{\oplus \frac{g}{2}}/K \cong I_x$ when X is not a special GM threefold, and a map $\mathcal{E}^{\oplus r}/K' \to \mathcal{E}^{\oplus \frac{g}{2}}/K \cong I_Z$ when it is. Note that if $\mathcal{E}^{\oplus r}/K' \neq 0$, then it is a torsion sheaf, and therefore from torsion-freeness of $\mathcal{E}^{\oplus \frac{g}{2}}/K$ we have $\mathrm{Hom}(\mathcal{E}^{\oplus r}/K', \mathcal{E}^{\oplus \frac{g}{2}}/K) = 0$, which gives a contradiction.

Thus we have $\mathcal{E}^{\oplus r}/K' = 0$, i.e. $K' = \mathcal{E}^{\oplus r}$. But this is also impossible, since this implies that $\mathcal{E} \subset K = \ker(\mathcal{E}^{\oplus \frac{g}{2}} \to I_x)$ and this contradicts the construction of the natural map $\mathcal{E}^{\oplus \frac{g}{2}} \to I_x$, which corresponds to the $\frac{g}{2}$ linearly independent sections of $\mathcal{E}^\vee$ whose zero locus contains x. $\square$

Remark 5.4. The stability of K in odd genus can be proved by a similar argument as above, but we will not need this result.

Lemma 5.5. *We have*

$$\mathrm{RHom}^\bullet(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = \begin{cases} k \oplus k^6[-1], & \text{if } g = 6 \text{ and ordinary} \\ k \oplus k^6[-1] \text{ or } k \oplus k^7[-1] \oplus k[-2], & \text{if } g = 6 \text{ and special} \\ k \oplus k^{25}[-1], & \text{if } g = 7 \\ k \oplus k^9[-1], & \text{if } g = 9 \\ k \oplus k^g[-1], & \text{if } g \geq 8 \text{ and } g \text{ is even.} \end{cases}$$

Proof. First we assume that $g \geq 6$ is even. Then we have a triangle $\mathcal{E}^{\oplus \frac{g}{2}} \to I_x \to i^* \mathcal{O}_x[-1]$. We use the standard spectral sequence, see e.g. [Pir20, Lemma 2.27], to do the computations. The first page is

$$E_1^{p,q} = \begin{array}{c|cc} \mathrm{Ext}^3(I_x, \mathcal{E}^{\oplus \frac{g}{2}}) & \mathrm{Ext}^3(\mathcal{E}^{\oplus \frac{g}{2}}, \mathcal{E}^{\oplus \frac{g}{2}}) \oplus \mathrm{Ext}^3(I_x, I_x) & \mathrm{Ext}^3(\mathcal{E}^{\oplus \frac{g}{2}}, I_x) \\ \mathrm{Ext}^2(I_x, \mathcal{E}^{\oplus \frac{g}{2}}) & \mathrm{Ext}^2(\mathcal{E}^{\oplus \frac{g}{2}}, \mathcal{E}^{\oplus \frac{g}{2}}) \oplus \mathrm{Ext}^2(I_x, I_x) & \mathrm{Ext}^2(\mathcal{E}^{\oplus \frac{g}{2}}, I_x) \\ \mathrm{Ext}^1(I_x, \mathcal{E}^{\oplus \frac{g}{2}}) & \mathrm{Ext}^1(\mathcal{E}^{\oplus \frac{g}{2}}, \mathcal{E}^{\oplus \frac{g}{2}}) \oplus \mathrm{Ext}^1(I_x, I_x) & \mathrm{Ext}^1(\mathcal{E}^{\oplus \frac{g}{2}}, I_x) \\ \mathrm{Hom}(I_x, \mathcal{E}^{\oplus \frac{g}{2}}) & \mathrm{Hom}(\mathcal{E}^{\oplus \frac{g}{2}}, \mathcal{E}^{\oplus \frac{g}{2}}) \oplus \mathrm{Hom}(I_x, I_x) & \mathrm{Hom}(\mathcal{E}^{\oplus \frac{g}{2}}, I_x) \\ \hline 0 & 0 & 0 \end{array}$$

It is not hard to show that the dimensions appearing in the first page are of the form

$$\dim E_1^{p,q} = \begin{array}{c|cc} 0 & 0 & 0 \\ g & 3 & 0 \\ 0 & 3 & 0 \\ 0 & \frac{g^2}{4}+1 & \frac{g^2}{4} \\ \hline 0 & 0 & 0 \end{array}$$

The projection $i^* \mathcal{O}_x \in \mathcal{K}u(X)$, so if we apply $\mathrm{Hom}(\mathcal{E}^{\oplus \frac{g}{2}}, -)$ to the triangle $\mathcal{E}^{\oplus \frac{g}{2}} \to I_x \to i^* \mathcal{O}_x[-1]$, we have a natural isomorphism $\mathrm{Hom}(\mathcal{E}^{\oplus \frac{g}{2}}, \mathcal{E}^{\oplus \frac{g}{2}}) \cong \mathrm{Hom}(\mathcal{E}^{\oplus \frac{g}{2}}, I_x)$. This implies the differential $E_1^{0,0} = k^{\frac{g^2}{4}+1} \to E_1^{1,0} = k^{\frac{g^2}{4}}$ is surjective. Thus we have $E_2^{0,0} = k$ and $E_2^{1,0} = 0$.

Next we will compute the differential $d\colon E_1^{-1,2} = k^g \to E_1^{0,2} = k^3$. Since $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$, the map $d : E_1^{-1,2} = \mathrm{Ext}^2(I_x, \mathcal{E}^{\oplus \frac{g}{2}}) \to E_1^{0,2} = \mathrm{Ext}^2(I_x, I_x)$ is the natural map induced by applying $\mathrm{Hom}(I_x, -)$ to the triangle $\mathcal{E}^{\oplus \frac{g}{2}} \to I_x \to i^* \mathcal{O}_x[-1]$. When X is not a special GM threefold, we have $\mathrm{Ext}^2(I_x, i^* \mathcal{O}_x[-1]) = \mathrm{Ext}^3(I_x, K_g)$. Then by Serre duality and Lemma 5.3, we have

$$\mathrm{Ext}^2(I_x, i^* \mathcal{O}_x[-1]) = \mathrm{Ext}^3(I_x, K_g) \cong \mathrm{Hom}(K_g, I_x(-1)) = 0.$$

This means the natural map $d : E_1^{-1.2} \to E_1^{0,2}$ is surjective. Thus the dimensions of the second page are of the form

$$\dim E_2^{p,q} = \begin{array}{c|cc} 0 & 0 & 0 \\ g-3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}$$

This means that when g is even and X is not a special GM threefold, the spectral sequence degenerates at E_2 and the desired result follows.

When X is a special GM threefold, first we assume that $\pi(x)$ is in the branch locus $\mathcal{B}$. Then by Proposition 5.1, we have a triangle $K[1] \to i^* \mathcal{O}_x[-1] \to \mathcal{O}_x$. In this case we have $\mathrm{Ext}^2(I_x, i^* \mathcal{O}_x[-1]) = \mathrm{Ext}^2(I_x, \mathcal{O}_x) = k$, thus the dimensions of the second page are of the form

$$\dim E_2^{p,q} = \begin{array}{c|cc} 0 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}$$

Therefore the spectral sequence degenerates at E_2 and we have $\mathrm{RHom}^\bullet(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = k \oplus k^7[-1] \oplus k[-2]$. When $\pi(x)$ is not in the branch locus, we have $\mathrm{Ext}^2(I_x, i^* \mathcal{O}_x[-1]) = \mathrm{Ext}^2(I_x, \mathcal{O}_{\tau(x)}) = 0$. Then from the arguments above, we obtain $\mathrm{RHom}^\bullet(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = k \oplus k^6[-1]$.

When g is odd, a similar argument shows that $\mathrm{Hom}(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = 1$. Since the homological dimension of $\mathcal{A}_\sigma$ is one, $i^* \mathcal{O}_x$ is the direct sum of shifts of its cohomology objects with respect to $\mathcal{A}_\sigma$. Then from $\hom(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = 1$, we know that $i^* \mathcal{O}_x$ is in the heart up to shifts. Hence we have $\mathrm{Ext}^i(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = 0$ for $i \geq 2$. Finally the dimension of Ext^1 follows from the Euler characteristics. □

Lemma 5.6. *We have*

$$\mathcal{H}^i_{\mathrm{Coh}^0_{\alpha,\beta}(X)}(i^* \mathcal{O}_x) = 0, \quad i \neq -2, -1$$

for (α, β) as in Proposition 4.9.

Proof. By definition of $i^* \mathcal{O}_x[-1]$, we have a triangle $\mathcal{E}^{\oplus n_g} \to I_x \to i^* \mathcal{O}_x[-1]$. Since $\mathcal{E}, I_x \in \mathrm{Coh}^0_{\alpha,\beta}(X)$, the result follows from the long exact sequence of cohomology objects. □

Proposition 5.7. *Let $X := X_{2g-2}$ with genus $g \geq 6$.*

(i) *If $g = 6$, then $i^* \mathcal{O}_x[-1] \in \mathcal{A}(\frac{1}{20}, -\frac{9}{10})$.*

(ii) *If $g=8$, then $i^*\mathcal{O}_x[-1]\in\mathcal{A}(\frac{1}{25},-\frac{22}{25})$.*
(iii) *If $g\neq 6,8$, then $i^*\mathcal{O}_x$ is in the heart of every Serre-invariant stability condition σ on $\mathcal{K}u(X)$ up to some shifts.*

Proof.

(i) First we assume that $g=6$. Let $\sigma:=\sigma(\frac{1}{20},-\frac{9}{10})$. As in [BMMS12, Lemma 4.5], we consider the spectral sequence for objects in $\mathcal{K}u(X)$ whose second page is given by

$$E_2^{p,q}=\bigoplus_i \mathrm{Hom}^p(\mathcal{H}^i_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{i+q}_\sigma(i^*\mathcal{O}_x))\Rightarrow \mathrm{Hom}^{p+q}(i^*\mathcal{O}_x,i^*\mathcal{O}_x)$$

where the cohomology is taken with respect to the heart $\mathcal{A}(\frac{1}{20},-\frac{9}{10})$. Let r be the number of non-zero cohomology objects of $i^*\mathcal{O}_x$ with respect to the heart. Note that since the homological dimension of $\mathcal{A}_\sigma$ is equal to 2, we have that $E_2^{1,q}=E_\infty^{1,q}$. So if we take $q=0$, by Lemma 4.11 we get

$$\mathrm{ext}^1(i^*\mathcal{O}_x,i^*\mathcal{O}_x)\geq\sum_i \mathrm{ext}^1(\mathcal{H}^i_\sigma(i^*\mathcal{O}_x),\mathcal{H}^i_\sigma(i^*\mathcal{O}_x))\geq 2r.$$

If $r\geq 2$, by Lemma 5.6 we know that $r=2$. Recall that in (1) we have $\mathcal{N}(\mathcal{K}u(X))=\langle v,w\rangle$. If we assume that $[\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x[1])]=av+bw$ and $[\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x)[2]]=cv+dw$ for some $a,b,c,d\in\mathbb{Z}$, then we have following equations:

(1) $[\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x[1])]+[\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x)[2]]=[i^*\mathcal{O}_x]$, i.e. $a+b=5, c+d=-3$;
(2) $\mathrm{Im}\,Z^0_{\frac{1}{20},-\frac{9}{10}}(av+bw)\leq 0$, $\mathrm{Im}\,Z^0_{\frac{1}{20},-\frac{9}{10}}(cv+dw)\geq 0$;
(3) $1-\chi(\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x))+1-\chi(\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x))\leq 7$.

Now (1) and (3) imply that the only possible cases are $(a,b,c,d)=(2,-1,3,-2)$, $(a,b,c,d)=(3,-2,2,-1)$, $(a,b,c,d)=(4,-2,1,-1)$ and $(a,b,c,d)=(1,-1,4,-2)$. But it is not hard to check that $\mathrm{Im}\left(Z^0_{\frac{1}{20},-\frac{9}{10}}(v-w)\right)<0$, $\mathrm{Im}\left(Z^0_{\frac{1}{20},-\frac{9}{10}}(2v-w)\right)<0$ and $\mathrm{Im}\left(Z^0_{\frac{1}{20},-\frac{9}{10}}(3v-2w)\right)<0$, which contradicts (2). Therefore we have $r=1$. Now from $\mathrm{Im}\left(Z^0_{\frac{1}{20},-\frac{9}{10}}(i^*\mathcal{O}_x[-1])\right)>0$, we obtain that $i^*\mathcal{O}_x[-1]\in\mathcal{A}(\frac{1}{20},-\frac{9}{10})$.

(ii) Next we assume that $g=8$. Let $\sigma:=\sigma(\frac{1}{25},-\frac{22}{25})$. The method is similar to the case $g=6$. We consider the spectral sequence for objects in $\mathcal{K}u(X)$ whose second page is given by

$$E_2^{p,q}=\bigoplus_i \mathrm{Hom}^p(\mathcal{H}^i_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{i+q}_\sigma(i^*\mathcal{O}_x))\Rightarrow \mathrm{Hom}^{p+q}(i^*\mathcal{O}_x,i^*\mathcal{O}_x)$$

where the cohomology is taken with respect to the heart $\mathcal{A}(\frac{1}{25},-\frac{22}{25})$. Let r be the number of non-zero cohomology objects of $i^*\mathcal{O}_x$ with respect to the heart. Note that since the homological dimension of $\mathcal{A}_\sigma$ is equal to 2, we have that $E_2^{1,q}=E_\infty^{1,q}$. So if we take $q=0$, we get

$$\mathrm{ext}^1(i^*\mathcal{O}_x,i^*\mathcal{O}_x)\geq\sum_i \mathrm{ext}^1(\mathcal{H}^i_\sigma(i^*\mathcal{O}_x),\mathcal{H}^i_\sigma(i^*\mathcal{O}_x))\geq 2r.$$

If $r\geq 2$, by Lemma 5.6 we know that $r=2$. If we assume that $[\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x[1])]=av+bw$ and $[\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x)[2]]=cv+dw$ for some $a,b,c,d\in\mathbb{Z}$, then we have following equations:

(1) $[\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x[1])]+[\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x)[2]]=[i^*\mathcal{O}_x]$, i.e. $a+b=7, c+d=-4$;
(2) $\mathrm{Im}\,Z^0_{\frac{1}{25},-\frac{22}{25}}(av+bw)\leq 0$, $\mathrm{Im}\,Z^0_{\frac{1}{25},-\frac{22}{25}}(cv+dw)\geq 0$;
(3) $1-\chi(\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{-1}_\sigma(i^*\mathcal{O}_x))+1-\chi(\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x),\mathcal{H}^{-2}_\sigma(i^*\mathcal{O}_x))\leq 8$.

Now (1) and (3) imply that the only possible cases are $(a,b,c,d)=(2,-1,5,-3)$, $(a,b,c,d)=(5,-3,2,-1)$, $(a,b,c,d)=(4,-2,3,-2)$ and $(a,b,c,d)=(3,-2,4,-2)$. But it is not hard to check that $\mathrm{Im}\left(Z^0_{\frac{1}{25},-\frac{22}{25}}(5v-3w)\right)<0$, $\mathrm{Im}\left(Z^0_{\frac{1}{25},-\frac{22}{25}}(2v-w)\right)<0$ and $\mathrm{Im}\left(Z^0_{\frac{1}{25},-\frac{22}{25}}(3v-2w)\right)<0$, which contradicts (2). Therefore we have $r=1$. Now from $\mathrm{Im}\left(Z^0_{\frac{1}{20},-\frac{9}{10}}(i^*\mathcal{O}_x[-1])\right)>0$, we obtain that $i^*\mathcal{O}_x[-1]\in\mathcal{A}(\frac{1}{25},-\frac{22}{25})$.

(iii) If $g \neq 6, 8$, then the homological dimension of $\mathcal{A}_\sigma$ is one. Thus $i^* \mathcal{O}_x$ is the direct sum of shifts of its cohomology objects with respect to $\mathcal{A}_\sigma$. Since $\hom(i^* \mathcal{O}_x, i^* \mathcal{O}_x) = 1$, we know that $i^* \mathcal{O}_x$ has only one cohomology object, i.e. $i^* \mathcal{O}_x$ is in the heart $\mathcal{A}_\sigma$ up to a shift.

□

Now we prove the main result of this section. First we need a lemma.

Lemma 5.8. *Let $\mathcal{T} \subset D^b(X)$ be a triangulated subcategory and $\mathcal{A}$ be a heart of $\mathcal{T}$ with homological dimension ≤ 2. Let $A \to E \to B$ be an exact sequence in $\mathcal{A}$. Suppose that* $\mathrm{Hom}(B, A[2]) = 0$ *and* $\mathrm{Hom}(E, E) = k$. *Then we have $\chi(B, A) \leq -1$.*

Proof. First note that $\mathrm{Ext}^1(B, A) \neq 0$, otherwise $E \cong A \oplus B$ which contradicts $\mathrm{Hom}(E, E) = k$. By assumption we have $\mathrm{Hom}(B, A[i]) = 0$ for $i \geq 2$. We also have $\mathrm{Hom}(B, A) = 0$, otherwise we have a non-zero composition of maps $E \twoheadrightarrow B \to A \hookrightarrow E$, which contradicts $\mathrm{Hom}(E, E) = k$. Therefore we obtain $\chi(B, A) = -\operatorname{ext}^1(B, A) \leq -1$. □

Theorem 5.9. *Let X be a prime Fano threefold of genus g where $g \geq 6$, and $x \in X$ a point. Then $i^* \mathcal{O}_x[-1]$ is σ-stable for every Serre-invariant stability condition σ on $\mathcal{K}u(X)$.*

Proof. By Proposition 4.9 and Theorem 4.13, we may assume that $\sigma = \sigma(\alpha_0, \beta_0)$, where

- $g = 6$: $(\alpha_0, \beta_0) = (\frac{1}{20}, -\frac{9}{10})$,
- $g = 7$: $(\alpha_0, \beta_0) = (\frac{1}{12}, -\frac{5}{6})$,
- $g = 8$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{22}{25})$,
- $g = 9$: $(\alpha_0, \beta_0) = (\frac{1}{8}, -\frac{3}{4})$,
- $g = 10$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{22}{25})$,
- $g = 12$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{21}{25})$.

When $g \geq 6$, we know that for (α_0, β_0) as above, $i^* \mathcal{O}_x[-1]$ is in the heart $\mathcal{A}(\alpha_0, \beta_0)$ up to some shifts by Proposition 5.7. If $i^* \mathcal{O}_x$ is not σ-semistable, we can find a triangle in $\mathcal{K}u(X)$

$$A \to i^* \mathcal{O}_x[-1] \to B$$

such that $A[m], B[m], i^* \mathcal{O}_x[-1][m] \in \mathcal{A}(\alpha_0, \beta_0)$ for some integer m with slopes $\mu_\sigma(A) \geq \mu_\sigma^-(A) > \mu_\sigma^+(B) \geq \mu_\sigma(B)$. Indeed, we can take A to be the first HN factor of $i^* \mathcal{O}_x[-1]$. By the Weak Mukai Lemma 4.12, we have

$$\operatorname{ext}^1(A, A) + \operatorname{ext}^1(B, B) \leq \operatorname{ext}^1(i^* \mathcal{O}_x, i^* \mathcal{O}_x).$$

Note that $1 - \chi(A, A) \leq \operatorname{ext}^1(A, A)$. Therefore, we have the following relations:

(1) $[A] + [B] = [i^* \mathcal{O}_x[-1]]$;
(2) $\operatorname{Im} Z^0_{\alpha_0,\beta_0}(A) \cdot \operatorname{Im} Z^0_{\alpha_0,\beta_0}(i^* \mathcal{O}_x[-1]) \geq 0$ and $\operatorname{Im} Z^0_{\alpha_0,\beta_0}(B) \cdot \operatorname{Im} Z^0_{\alpha_0,\beta_0}(i^* \mathcal{O}_x[-1]) \geq 0$;
(3) $\mu^0_{\alpha_0,\beta_0}(A) > \mu^0_{\alpha_0,\beta_0}(B)$;
(4) $1 - \chi(A, A) + 1 - \chi(B, B) \leq \operatorname{ext}^1(i^* \mathcal{O}_x, i^* \mathcal{O}_x)$.

Thus if we assume $[A] = av + bw$ and $[B] = cv + dw$, by the results in Section A.3, we have the following integer solutions for (a, b, c, d):

(i) $g = 6$ and ordinary: $(a, b, c, d) = (-2, 1, -3, 2)$;
(ii) $g = 6$ and special: $(a, b, c, d) = (-2, 1, -3, 2)$ or $(a, b, c, d) = (-4, 2, -1, 1)$;
(iii) $g = 7$: there are no solutions;
(iv) $g = 8$: $(a, b, c, d) = (-2, 1, -5, 3)$ or $(a, b, c, d) = (-4, 2, -3, 2)$;
(v) $g = 9$: there are no solutions;
(vi) $g = 10$: there are no solutions;
(vii) $g = 12$: there are no solutions.

Thus if $g \neq 6, 8$, we know that $i^* \mathcal{O}_x[-1]$ is $\sigma(\alpha_0, \beta_0)$-semistable. Next we consider the cases $g = 6$ and 8:

- Assume that $g = 6$ and X is an ordinary GM threefold. In this case $i^*\mathcal{O}_x[-1] \in \mathcal{A}(\alpha_0, \beta_0)$. The solution $(a, b) = (-2, 1)$ gives a potential destablizing subobject A of $i^*\mathcal{O}_x[-1]$. It is known that $A \cong i^*(E)[1]$ for some $E \in M_{X_{10}}(2, -1, 5)$ by [JLLZ21, Theorem 8.5]. Let $K := \ker(\mathcal{E}^3 \to I_x)$. By Lemma 5.3 and stability, we have $\mathrm{Hom}(E, K) = 0$. Since $K = i^*\mathcal{O}_x[-2]$, by adjointness we have $\mathrm{Hom}(i^*(E)[1], i^*\mathcal{O}_x[-1]) = \mathrm{Hom}(E[1], i^*\mathcal{O}_x[-1]) = \mathrm{Hom}(E, K) = 0$, which gives a contradiction. Therefore, $i^*\mathcal{O}_x[-1]$ is $\sigma(\alpha_0, \beta_0)$-semistable.
- Assume that $g = 6$ and X is a special GM threefold. In this case $i^*\mathcal{O}_x[-1] \in \mathcal{A}(\alpha_0, \beta_0)$. If $(a, b) = (-2, 1)$, then we also have $A \cong i^*(E)[1]$ for some $E \in M_{X_{10}}(2, -1, 5)$. By Lemma 5.3 and stability, we have $\mathrm{Hom}(E, K) = 0$. If we apply $\mathrm{Hom}(E, -)$ to the triangle $K[1] \to i^*\mathcal{O}_x[-1] \to \mathcal{O}_{\tau(x)}$, from $\mathrm{Hom}(E, K) = \mathrm{Hom}(E, \mathcal{O}_{\tau(x)}[-1]) = 0$, we obtain $\mathrm{Hom}(E, i^*\mathcal{O}_x[-2]) = \mathrm{Hom}(A, i^*(\mathcal{O}_x)[-1]) = 0$. This gives a contradiction. If $(a, b) = (-4, 2)$, then we have $\chi(B, A) = 0$, which contradicts Lemma 5.8. Therefore, $i^*\mathcal{O}_x[-1]$ is $\sigma(\alpha_0, \beta_0)$-semistable.
- Assume that $g = 8$. In this case $i^*\mathcal{O}_x[-1] \in \mathcal{A}(\alpha_0, \beta_0)$. If $(a, b) = (-2, 1)$, then by [LZ21, Theorem 1.1] we know that $A \cong i^*(E)[1]$ for some $E \in M_{X_{14}}(2, -1, 6)$. By Lemma 5.3 and stability, we have $\mathrm{Hom}(E, K) = 0$. Since $K = i^*\mathcal{O}_x[-2]$, by adjointness we have $\mathrm{Hom}(i^*(E)[1], i^*\mathcal{O}_x[-1]) = \mathrm{Hom}(E[1], i^*\mathcal{O}_x[-1]) = \mathrm{Hom}(E, K) = 0$, which gives a contradiction. If $(a, b) = (-4, 2)$, then we have $\chi(B, A) = 0$, which contradicts Lemma 5.8. Therefore, $i^*\mathcal{O}_x[-1]$ is $\sigma(\alpha_0, \beta_0)$-semistable.

When $g \neq 7$, $[i^*\mathcal{O}_x[-1]]$ is a primitive class, hence $i^*\mathcal{O}_x[-1]$ is actually σ-stable.

When $g = 7$, the possible JH factors will have numerical class $-6v + 5w$. By [Kuz05], there is an equivalence $\Phi : S^{-1}_{\mathrm{D}^b(X_{12})}(\mathcal{K}u(X_{12})) \to \mathrm{D}^b(\Gamma_7)$. Then we have an equivalence $\Theta = \Phi \circ S^{-1}_{\mathrm{D}^b(X_{12})} : \mathcal{K}u(X_{12}) \to \mathrm{D}^b(\Gamma_7)$, where Γ_7 is a smooth projective curve of genus 7. Moreover, up to some auto-equivalences we have $\Theta(i^!\mathcal{E}_7) \cong \mathcal{O}_{\Gamma_7}$ by [BF13, Lemma 2.9] or [Kuz05, Lemma 5.6]. It is easy to check that if there is a strictly σ-semistable object $i^*\mathcal{O}_x[-1]$, then up to some auto-equivalences, Θ maps the JH filtration of $i^*\mathcal{O}_x[-1]$ to an exact sequence $0 \to L_1 \to E \to L_2 \to 0$ on Γ_7 where L_i are line bundles of degree 6, and $h^0(E) = 5$. Then by Lemma 5.10, the curve Γ_7 admits a line bundle L with $h^0(L) = 2$ and $\deg L = 4$. But this contradicts [Muk01, Theorem 8.1].

Thus the above argument shows that for every $g \geq 6$, every object $i^*\mathcal{O}_x[-1]$ is σ-stable. $\square$

Lemma 5.10. *Let Γ be a smooth projective curve of genus 7. Assume that there is a bundle E on Γ with $h^0(E) = 5$ and an exact sequence*

$$0 \to L_1 \to E \to L_2 \to 0 \tag{7}$$

such that L_i are line bundles with $\deg L_i = 6$ for each i. Then Γ admits a line bundle L with $h^0(L) = 2$ and $\deg L = 4$.

Proof. By Riemann-Roch, we know that $h^0(L_i) = h^1(L_i)$ for each i. By taking the cohomology long exact sequence of (7), there is at least one $j \in \{1, 2\}$ such that $h^0(L_j) = h^1(L_j) \neq 0$. Then applying Clifford's theorem to the line bundle L_j, we have $h^0(L_j) \leq 4$. Thus it is easy to see that there is a $k \in \{1, 2\}$ such that $h^0(L_k) = 4$ or 3.

If $h^0(L_k) = 4$, we take $p \notin \mathrm{Bs}(L_k)$ and $q \notin \mathrm{Bs}(L_k(-p))$. Then $L := L_k(-p-q)$ has $h^0(L) = 2$ and $\deg L = 4$.

If $h^0(L_k) = 3$ such that $\mathrm{Bs}(L_k) \neq \varnothing$, we take $p \in \mathrm{Bs}(L_k)$ and $q \notin \mathrm{Bs}(L_k(-p))$. Then $L := L_k(-p-q)$ has $h^0(L) = 2$ and $\deg L = 4$.

If $h^0(L_k) = 3$ such that $\mathrm{Bs}(L_k) = \varnothing$, we know that L_k is not very ample, otherwise Γ is a plane curve with degree 6, which contradicts $g(\Gamma) = 7$. Thus there exist points $p, q \in \Gamma$ such that $h^0(L_k) - h^0(L_k(-p-q)) < 2$. But from $\mathrm{Bs}(L_k) = \varnothing$, we know that $h^0(L_k) - h^0(L_k(-p-q)) \geq 1$. Thus $L := L_k(-p-q)$ has $h^0(L) = 2$ and $\deg L = 4$. $\square$

5.3. **Embedding X into a Bridgeland moduli space.** We first give an outline for the existence of a morphism $X \to \mathcal{M}_\sigma(\mathcal{K}u(X), [i^*\mathcal{O}_x[-1]])$. We follow exactly the same arguments as in [BMMS12], [APR19] and [Zha20]. First note that the projection functor $i^*: \mathrm{D}^b(X) \to \mathcal{K}u(X)$ is Fourier–Mukai (this follows from [Kuz11, Theorem 7.1] and [BLM+21, Lemma 3.25]), which means that $i^* \cong \Phi_N$ where $N \in \mathrm{D}^b(X \times X)$ is some integral kernel. Let $\mathcal{X}$ be the Hilbert scheme of points on X (note that $\mathcal{X} \cong X$), and let $\mathcal{I}$ be the universal ideal sheaf on $X \times \mathcal{X}$. Define the functor

$$\Phi_N \times \mathrm{id}_{\mathcal{X}} := \Phi_{N \boxtimes \mathcal{O}_{\Delta_{\mathcal{X}}}} : \mathrm{D}^b(X \times \mathcal{X}) \to \mathcal{K}u(X \times \mathcal{X}).$$

The image $\Phi_N \times \mathrm{id}_{\mathcal{X}}(\mathcal{I})$ is a family of objects in $\mathcal{K}u(X)$ parametrised by $\mathcal{X}$, which defines a morphism $p: \mathcal{X} \to \mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$. The following lemma shows that for a point $x \in \mathcal{X}$, the image $p(x)$ is identified with $i^* \mathcal{O}_x[-1]$.

Lemma 5.11. *Let the notation be as above, and let $\iota_x : X \times x \to X \times \mathcal{X}$. Then*

$$\Phi_N(\iota_x^*(\mathcal{I})) \cong \iota_x^*(\Phi_N \times \mathrm{id}_{\mathcal{X}}(\mathcal{I})).$$

Proof. See [Zha20, Lemma 6.1]. □

Proposition 5.12. *For every point $x \in X$, the induced map*

$$\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(i^* \mathcal{O}_x, i^* \mathcal{O}_x)$$

is injective.

Proof. Note that $\mathbf{L}_{\mathcal{O}_X}$ induces an isomorphism $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \cong \mathrm{Ext}^1(I_x, I_x)$. Thus we only need to show that $d_2 : \mathrm{Ext}^1(I_x, I_x) \to \mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}} I_x, \mathbf{L}_{\mathcal{E}} I_x)$ is injective.

We consider the defining triangle of $\mathbf{L}_{\mathcal{E}} I_x$ and apply $\mathrm{Hom}(I_x, -)$ to it. Inside the long exact sequence, we have the sequence

$$\cdots \to \mathrm{Ext}^1(I_x, \mathcal{E}^{\oplus n_g}) \to \mathrm{Ext}^1(I_x, I_x) \xrightarrow{d_2} \mathrm{Ext}^1(I_x, \mathbf{L}_{\mathcal{E}} I_x) = \mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}} I_x, \mathbf{L}_{\mathcal{E}} I_x) \to \cdots.$$

Since $\mathrm{Ext}^1(I_x, \mathcal{E}) = 0$, we can see from the sequence above that the map d_2 is indeed injective. □

Lemma 5.13. *The induced morphism $p: X \cong \mathcal{X} \to \mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$ is injective.*

Proof.

(i) If $g \geq 7$ or if X is an ordinary GM threefold, $i^* \mathcal{O}_x[-2]$ is a reflexive sheaf that is non-locally free only at the point x from Proposition 5.1. Thus $i^* \mathcal{O}_x \cong i^* \mathcal{O}_y$ if and only if $x = y$.

(ii) If X is a special GM threefold, then $\mathcal{H}^0(i^* \mathcal{O}_x[-1]) \cong \mathcal{H}^0(i^* \mathcal{O}_y[-1])$ if and only if $x = y$ by Proposition 5.1.

□

Theorem 5.14. *For any $x \in X$, $p(x)$ is identified with $i^* \mathcal{O}_x[-1]$ in the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$, and there is a closed embedding $p: X \hookrightarrow \mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$ induced by i^*.*

Proof. The discussion at the beginning of Subsection 5.3 together with Theorem 5.9 shows that we have a morphism p. By Lemma 5.11 and 5.13, we know that p is injective. By Proposition 5.12 p is injective at the level of tangent spaces, hence it is an embedding. Note that X and $\mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$ are both proper, hence p is a closed embedding. □

6. Fano threefolds as Brill–Noether loci

In this section, we exhibit X as a Brill–Noether locus inside the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(X), [i^* \mathcal{O}_x[-1]])$. We first fix some notation. Recall that $n_g := \frac{g}{2}$ if g is even, $n_7 = 5$ and $n_9 = 3$. Let $x \in X$ be a point, and let $i_{\mathcal{D}} : \mathcal{K}u(X) \hookrightarrow \mathcal{D} = \langle \mathcal{K}u(X), \mathcal{E} \rangle$ be the inclusion; this has a left adjoint $i_{\mathcal{D}}^*$ and a right adjoint $i_{\mathcal{D}}^!$. In particular, $i_{\mathcal{D}}^* = \mathbf{L}_{\mathcal{E}}$. Therefore we can rewrite the triangle (i) as

$$\mathcal{E}^{\oplus n_g} \to I_x \to i_{\mathcal{D}}^* I_x \to \mathcal{E}^{\oplus n_g}[1].$$

Remark 6.1. We have

$$\mathrm{Ext}^1_{\mathcal{D}}(i^*_{\mathcal{D}}I_x, \mathcal{E}) = \mathrm{Ext}^1_{\mathcal{D}}(i_{\mathcal{D}}i^*_{\mathcal{D}}I_x, \mathcal{E}) \cong \mathrm{Ext}^1_{\mathcal{K}u(X)}(i^*_{\mathcal{D}}I_x, i^!\mathcal{E})$$

because $i^*_{\mathcal{D}}$ and $i^!_{\mathcal{D}}$ are left and right adjoints to $i_{\mathcal{D}}$, respectively. Therefore, $\mathrm{Ext}^1(i^*_{\mathcal{D}}I_x, i^!\mathcal{E}) \cong \mathrm{Hom}(i^*_{\mathcal{D}}I_x, i^!\mathcal{E}[1])$ is precisely the space of morphisms from $i^*\mathcal{O}_x[-1] = i^*_{\mathcal{D}}I_x$ to the shift of the gluing object $i^!\mathcal{E}$ by [1].

We define $(\alpha_6, \beta_6) = (\frac{1}{20}, -\frac{9}{10})$, $(\alpha_7, \beta_7) = (\frac{\sqrt{71}}{84}, -\frac{71}{84})$ $(\alpha_8, \beta_8) = (\frac{2\sqrt{79}}{875}, -\frac{122}{125})$, $(\alpha_9, \beta_9) = (\frac{\sqrt{31}}{40}, -\frac{31}{40})$, $(\alpha_{10}, \beta_{10}) = (\frac{\sqrt{\frac{5}{3}}}{33}, -\frac{10}{11})$, and $(\alpha_{12}, \beta_{12}) = (\frac{1}{22}, -\frac{19}{22})$. From Proposition 4.9 we know that $\sigma(\alpha_g, \beta_g)$ is a Serre-invariant stability condition on $\mathcal{K}u(X)$. In this section we fix $\sigma := \sigma(\alpha_g, \beta_g)$.

Definition 6.2. We define the *Brill–Noether locus* in $\mathcal{M}_\sigma(\mathcal{K}u(X), [i^*\mathcal{O}_x[-1]])$ with respect to the object $i^!(\mathcal{E})$ as

$$\mathcal{BN}_g := \{F \mid \mathrm{ext}^1(F, i^!(\mathcal{E})) = n_g\} \subset \mathcal{M}_\sigma(\mathcal{K}u(X), [i^*\mathcal{O}_x[-1]]).$$

The main theorem we prove in this section is

Theorem 6.3. *With $\sigma = \sigma(\alpha_g, \beta_g)$, we have*

$$\mathcal{BN}_g \cong X.$$

We split the proof of the theorem into a series of propositions and lemmas. First we show that X is embedded in the *Brill-Noether locus* $\mathcal{BN}_g$. Then we show that each object in $\mathcal{BN}_g$ must be $i^*(\mathcal{O}_x[-1])$. The strategy will be to take an object $F \in \mathcal{BN}_g \setminus p(X)$, and show that it cannot exist.

Proposition 6.4. *Let X be a prime Fano threefold of index one and genus $g \geq 6$. Then there is a closed embedding $p : X \hookrightarrow \mathcal{BN}_g$ induced by i^*.*

Proof. By Theorem 5.14, the projection functor i^* already induces a closed embedding

$$p : X \hookrightarrow \mathcal{M}_\sigma(\mathcal{K}u(X), [i^*\mathcal{O}_x[-1]]).$$

It suffices to check that $i^*\mathcal{O}_x[-1]$ satisfies the Brill–Noether condition, i.e. that $\mathrm{ext}^1(i^*\mathcal{O}_x[-1], i^!\mathcal{E}) = n_g$ for $g \geq 6$. Note that $i^*\mathcal{O}_x[-1] \cong \mathbf{L}_{\mathcal{E}}\mathbf{L}_{\mathcal{O}_X}(\mathcal{O}_x)[-1] \cong \mathbf{L}_{\mathcal{E}}I_x \cong i^*_{\mathcal{D}}I_x$. There is an exact triangle

$$\mathrm{RHom}^\bullet(\mathcal{E}, I_x) \otimes \mathcal{E} \to I_x \to \mathbf{L}_{\mathcal{E}}I_x.$$

Then by Proposition 5.1, the triangle above becomes

$$\mathcal{E}^{\oplus n_g} \to I_x \to \mathbf{L}_{\mathcal{E}}I_x.$$

By Remark 6.1, $\mathrm{Ext}^1(i^*\mathcal{O}_x[-1], i^!\mathcal{E}) \cong \mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}}I_x, \mathcal{E})$. Applying $\mathrm{Hom}(-, \mathcal{E})$ to the triangle above, we get a long exact sequence

$$0 \to \mathrm{Hom}(i^*_{\mathcal{D}}I_x, \mathcal{E}) \to \mathrm{Hom}(I_x, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}^{\oplus n_g}) \to \mathrm{Ext}^1(i^*_{\mathcal{D}}I_x, \mathcal{E}) \to \mathrm{Ext}^1(I_x, \mathcal{E}) \to \cdots$$

It is easy to check $\mathrm{Hom}(I_x, \mathcal{E}) = \mathrm{Ext}^1(I_x, \mathcal{E}) = 0$, so $\mathrm{Ext}^1(i^*I_x, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E}^{\oplus n_g}) \cong k^{n_g}$. Then the desired result follows. □

Proposition 6.5. *Let $F \in \mathcal{BN}_g \setminus p(X)$, where $g \geq 6$. Then F is the shift of a vector bundle by [1].*

Proof. We have

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{D}^b(X)}(F, \mathcal{O}_x) &\cong \mathrm{Ext}^{2-i}_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1], F)^\vee \\ &\cong \mathrm{Ext}^{-i}_{\mathcal{K}u(X)}(F[2], S_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1])).\end{aligned}$$

where we have used Serre duality in $\mathrm{D}^b(X)$ and adjunction for the first isomorphism, and Serre duality in $\mathcal{K}u(X)$ for the second. Since F and $i^*\mathcal{O}_x[-1]$ are in the same heart with homological dimension at most two, we have $\mathrm{Ext}^i(F, \mathcal{O}_x) = 0$ for $2 < i$ and $i < 0$. Since $F \notin p(X)$ by assumption, by stability of F and $i^*\mathcal{O}_x[-1]$ we have $\mathrm{Ext}^2(F,\mathcal{O}_x)=\mathrm{Hom}(i^*\mathcal{O}_x[-1],F)^\vee=0$. Thus $\mathrm{Ext}^i(F,\mathcal{O}_x)=0$ for $2\geq i$ and $i<0$.

First we assume that $g\neq 6$. Note that $S_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1])$ is stable in the Kuznetsov component by Serre-invariance of σ. For $g=8$, we have

$$\mathrm{Hom}(F,\mathcal{O}_x)=\mathrm{Ext}^2_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1],F)=0$$

by [PY20, Corollary 5.5, Lemma 5.9], and

$$\mathrm{Hom}(F,\mathcal{O}_x)=\mathrm{Ext}^2_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1],F)=0$$

for $g=7,9,10,12$ since the homological dimensions of the hearts of the stability conditions in these cases are one. This means $\mathrm{RHom}^\bullet(F,\mathcal{O}_x)=\mathrm{Ext}^1(F,\mathcal{O}_x)[-1]$. By [BM99, Proposition 5.4], F is a vector bundle shifted by one.

Now assume that $g=6$. If X is a special GM threefold and $F\in\mathcal{BN}_g\setminus p(X)$, then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}^b(X)}(F,\mathcal{O}_x)&\cong\mathrm{Ext}^2_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1],F)^\vee\\&\cong\mathrm{Hom}(F,\tau(i^*\mathcal{O}_x[-1]))\\&\cong\mathrm{Hom}(F,i^*(\mathcal{O}_y)[-1]),\end{aligned}$$

where $y=\tau(x)$. Then by assumption, $\mathrm{Hom}_{\mathrm{D}^b(X)}(F,\mathcal{O}_x)=0$. The desired result follows from [BM99, Proposition 5.4].

Finally, if X is an ordinary GM threefold and $F\in\mathcal{BN}_g\setminus p(X)$, then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}^b(X)}(F,\mathcal{O}_x)&\cong\mathrm{Ext}^2_{\mathcal{K}u(X)}(i^*\mathcal{O}_x[-1],F)^\vee\\&\cong\mathrm{Hom}(F,\tau(i^*\mathcal{O}_x[-1])).\end{aligned}$$

We claim that $\mathrm{Hom}(F,\tau(i^*\mathcal{O}_x[-1]))=0$. Indeed, if $\mathrm{Hom}(F,\tau(i^*\mathcal{O}_x[-1]))\neq 0$, then $F\cong\tau(i^*\mathcal{O}_x[-1])$ since F and $\tau(i^*\mathcal{O}_x[-1])$ are both σ-stable objects of the same phase in $\mathcal{K}u(X)$. But then this means that $\tau(i^*\mathcal{O}_x[-1])$ would also be in the Brill–Noether locus $\mathcal{BN}_g$, i.e. $\mathrm{Ext}^1(\tau(i^*\mathcal{O}_x[-1]),i^!\mathcal{E})=k^3$, which is impossible by Lemma 6.6 below. Therefore F is the shift of vector bundle by one from the same argument as in the previous cases. □

Lemma 6.6. *We have* $\mathrm{Ext}^1(\tau(i^*\mathcal{O}_x[-1]),i^!\mathcal{E})\neq k^3$.

Proof. As $\mathrm{Ext}^1(\tau(i^*\mathcal{O}_x[-1]),i^!\mathcal{E})\cong\mathrm{Ext}^1(i^*\mathcal{O}_x[-1],\tau^{-1}(i^!\mathcal{E}))$ and $\tau^{-1}\cong i^*\circ(-\otimes\mathcal{O}_X(H))[-1]$, we compute $\tau^{-1}(i^!\mathcal{E})$. By Remark 3.5, the gluing object $i^!\mathcal{E}$ is given by the exact triangle

$$\mathcal{Q}(-H)[1]\to i^!\mathcal{E}\to\mathcal{E}.$$

Then $\tau^{-1}(i^!\mathcal{E})\cong\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee$. Thus $\mathrm{Ext}^1(\tau(i^*\mathcal{O}_x[-1]),i^!\mathcal{E})\cong\mathrm{Ext}^2(i^*\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)\cong\mathrm{Ext}^2(\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)$. On the other hand, there is a triangle

$$\mathcal{E}^{\oplus 2}\to\mathcal{Q}^\vee\to\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee.$$

Applying $\mathrm{Hom}(\mathcal{O}_x,-)$ to the triangle above, we get the following part of the long exact sequence:

$$0\to\mathrm{Ext}^2(\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)\to k^4\to k^3\to\mathrm{Ext}^3(\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)\to 0.$$

Thus $\mathrm{Ext}^1(\tau(i^*\mathcal{O}_x[-1]),i^!\mathcal{E})=k^3$ if and only if $\mathrm{Ext}^3(\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)=k^2$. But $\mathrm{Ext}^3(\mathcal{O}_x,\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee)\cong\mathrm{Hom}(\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee,\mathcal{O}_x)$. Since X is an ordinary GM threefold, $\mathrm{Hom}(\mathcal{E},\mathcal{Q}^\vee)=k^2$ and there is a short exact sequence

$$0\to\mathcal{E}\to\mathcal{Q}^\vee\to I_C\to 0$$

for some conic $C\subset X$ (see [JLLZ21, Proposition 7.1]). Then $\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee\cong\mathbf{L}_{\mathcal{E}}I_C\cong\mathbb{D}(I_C)\otimes\mathcal{O}_X(-H)[1]$ (see [JLLZ21, Lemma 7.8]), where $\mathbb{D}(-)$ is the derived dual functor. Therefore $\mathrm{Hom}(\mathbf{L}_{\mathcal{E}}\mathcal{Q}^\vee,\mathcal{O}_x)\cong\mathrm{Hom}(\mathbb{D}(I_C)\otimes\mathcal{O}_X(-H)[1],\mathcal{O}_x)\cong\mathrm{Hom}(\mathbb{D}(I_C)[1],\mathcal{O}_x)$. Note that $\mathbb{D}(I_C)[1]$ is given by the exact triangle

$$\mathcal{O}_X[1]\to\mathbb{D}(I_C)[1]\to\mathcal{O}_C.$$

Applying $\mathrm{Hom}(-,\mathcal{O}_x)$, we get $\mathrm{Hom}(\mathcal{O}_C,\mathcal{O}_x)\cong\mathrm{Hom}(\mathbb{D}(I_C)[1],\mathcal{O}_x)$. An easy computation shows that $\mathrm{hom}(\mathcal{O}_C,\mathcal{O}_x)$ is either 0 or 1. Then the proof is complete. □

Recall that we define $(\alpha_6,\beta_6)=(\frac{1}{20},-\frac{9}{10})$, $(\alpha_7,\beta_7)=(\frac{\sqrt{71}}{84},-\frac{71}{84})$ $(\alpha_8,\beta_8)=(\frac{2\sqrt{79}}{875},-\frac{122}{125})$, $(\alpha_9,\beta_9)=(\frac{\sqrt{31}}{40},-\frac{31}{40})$, $(\alpha_{10},\beta_{10})=(\frac{\sqrt{\frac{5}{3}}}{33},-\frac{10}{11})$, and $(\alpha_{12},\beta_{12})=(\frac{1}{22},-\frac{19}{22})$. Note that $\mu^0_{\alpha_g,\beta_g}(F)=+\infty$ when $g\neq 6$.

Let $F\in\mathcal{BN}_g$. By assumption, we have a natural map $F\to\mathcal{E}^{\oplus n_g}[1]$. Since F and $\mathcal{E}\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$, the extension C of F and $\mathcal{E}^{\oplus n_g}$

$$\mathcal{E}^{\oplus n_g}\to C\to F$$

is also in the heart $\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$. Note that $\mathrm{ch}(C)=\mathrm{ch}(I_x)$. If we apply $\mathrm{Hom}(-,\mathcal{E})$ to this exact sequence, the natural map $\mathrm{Hom}(\mathcal{E}^{\oplus n_g},\mathcal{E})\to\mathrm{Ext}^1(F,\mathcal{E})$ is bijective by construction. Since $\mathrm{Ext}^1(\mathcal{E}^{\oplus n_g},\mathcal{E})=0$, we have

$$\mathrm{Hom}(C,\mathcal{E})=\mathrm{Ext}^1(C,\mathcal{E})=0. \tag{8}$$

Proposition 6.7. *Let $X:=X_{2g-2}$ and $g\geq 6$. Then $C\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ is $\sigma^0_{\alpha_g,\beta_g}$-semistable.*

Proof. We assume that $C\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ is not $\sigma^0_{\alpha_g,\beta_g}$-semistable. Let B be the minimal destabilizing quotient object of C. Then we have an exact sequence in $\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$

$$0\to A\to C\to B\to 0$$

where $\mu^{0-}_{\alpha_g,\beta_g}(A)>\mu^0_{\alpha_g,\beta_g}(C)>\mu^0_{\alpha_g,\beta_g}(B)$ and B is $\sigma^0_{\alpha_g,\beta_g}$-semistable. Therefore we have a system of inequalities:

- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(A))\geq 0, \mathrm{Im}(Z^0_{\alpha_g,\beta_g}(B))>0$,
- $\mu^0_{\alpha_g,\beta_g}(C)>\mu^0_{\alpha_g,\beta_g}(B)$.

Note that $\mu^0_{\alpha_g,\beta_g}(C)=\mu^0_{\alpha_g,\beta_g}(\mathcal{O}_X)<0$. Since $B\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ is $\sigma^0_{\alpha_g,\beta_g}$-semistable with slope $\mu^0_{\alpha_g,\beta_g}(B)<\mu^0_{\alpha_g,\beta_g}(C)<0$, we know that $B\in\mathsf{Coh}^{\beta_g}(X)$ is $\sigma_{\alpha_g,\beta_g}$-semistable.

Note that $\mathcal{O}_X,\mathcal{O}_X(-1)[2],\mathcal{E},\mathcal{E}(-1)[2]\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ and that they are $\sigma^0_{\alpha_g,\beta_g}$-semistable. Then by Serre duality and the definition of heart we know $\mathrm{Hom}(\mathcal{O}_X,A[n])=\mathrm{Hom}(\mathcal{E},A[n])=0$ for every $n<0$ and $n\geq 2$ and the same holds for B. Also, from $\mu^0_{\alpha_g,\beta_g}(C)=\mu^0_{\alpha_g,\beta_g}(\mathcal{O}_X)>\mu^0_{\alpha_g,\beta_g}(B)$ we have $\mathrm{Hom}(\mathcal{O}_X,B)=0$. Therefore, if we apply $\mathrm{Hom}(\mathcal{O}_X,-)$ and $\mathrm{Hom}(\mathcal{E},-)$ to the triangle $A\to C\to B$, from $\mathrm{RHom}^\bullet(\mathcal{O}_X,C)=0$ and $\mathrm{RHom}^\bullet(\mathcal{E},C)=k^{n_g}$ we obtain $A,B\in\mathcal{O}_X^\perp$, $\mathrm{RHom}^\bullet(\mathcal{E},B)=\mathrm{Hom}(\mathcal{E},B)$, $\mathrm{RHom}^\bullet(\mathcal{E},A)=\mathrm{Hom}(\mathcal{E},A)\oplus\mathrm{Hom}(\mathcal{E},A[1])[-1]$, and a long exact sequence

$$0\to\mathrm{Hom}(\mathcal{E},A)\to\mathrm{Hom}(\mathcal{E},C)=k^{n_g}\to\mathrm{Hom}(\mathcal{E},B)\to\mathrm{Ext}^1(\mathcal{E},A)\to 0.$$

Recall that in (1) we have $\mathcal{N}(\mathcal{K}u(X))=\langle v,w\rangle$. If we assume that $[B]=av+bw+c[\mathcal{E}_g]$ for $a,b,c\in\mathbb{Z}$, from $\chi(\mathcal{E},B)=\hom(\mathcal{E},B)\geq 0$ we have $c=\chi(\mathcal{E},B)\geq 0$.

If we apply the projection functor i^* the the triangle, since $i^*(C)\cong F$, we obtain a triangle

$$i^*(B)[-1]\to i^*(A)\to F. \tag{9}$$

By the definition of projection functor, we have a triangle:

$$i^*(B)[-1]\to\mathcal{E}^{\oplus c}\to B. \tag{10}$$

Claim 1: $i^*(A)\in\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$.

Let $t:=\mathrm{ext}^1(\mathcal{E},A)$. Then $\hom(\mathcal{E},A)=n_g-c+t$. Therefore we have a triangle $\mathcal{E}^{\oplus n_g-c+t}\oplus\mathcal{E}^{\oplus t}[-1]\to A\to i^*(A)$, and a long exact sequence of cohomologies in the heart $\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$:

$$0\to\mathcal{H}^{-1}_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(A))\to\mathcal{E}^{\oplus n_g-c+t}\to A\to\mathcal{H}^0_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(A))\to\mathcal{E}^{\oplus t}\to 0. \tag{11}$$

Since $\mathcal{E}^{\oplus n_g} \subset C$ and $A \subset C$, we know that the natural map $\mathcal{E}^{\oplus n_g - c + t} \to A$ is injective. This implies $\mathcal{H}^0_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(A)) \cong i^*(A) \in \mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ and we have an exact sequence in $\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$:

$$0 \to \mathcal{E}^{\oplus n_g - c + t} \to A \to i^*(A) \to \mathcal{E}^{\oplus t} \to 0. \tag{12}$$

Claim 2: $i^*(B)[-1] \in \mathsf{Coh}(X)$ is torsion free. Thus $a < 0$.

From Proposition 6.5 and the construction of C we have $\mathcal{H}^i(C) = 0$ for $i \neq -1, 0$. Since $B \in \mathsf{Coh}_{\alpha_g,\beta_g}(X)$, we know that $\mathcal{H}^i(B) = 0$ for $i \neq -1, 0$. Thus if we take the cohomology long exact sequence associated to (10) with respect to the standard heart, we have $\mathcal{H}^i(i^*(B)[-1]) = 0$ for $i \notin \{0, 1\}$.

Next note that $i^*(A) \in \mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ from Claim 1, so we obtain $\mathcal{H}^{\geq 1}(i^*(A)) = 0$. Hence if we take the cohomology long exact sequence associated to (9) with respect to the standard heart, since $\mathcal{H}^{-1}(F) \cong G \in \mathsf{Coh}(X)$, we have $\mathcal{H}^1(i^*(B)[-1]) = 0$, i.e. $i^*(B)[-1] \cong \mathcal{H}^0(i^*(B)[-1]) \in \mathsf{Coh}(X)$. Thus we have a long exact sequence in $\mathsf{Coh}(X)$:

$$0 \to \mathcal{H}^{-1}(B) \to i^*(B)[-1] \xrightarrow{\theta} \mathcal{E}^{\oplus c} \to \mathcal{H}^0(B) \to 0.$$

Now we claim that $i^*(B)[-1] \in \mathsf{Coh}(X)$ is a torsion free sheaf. Indeed, if we denote the torsion part of $i^*(B)[-1]$ by $i^*(B)[-1]_{\mathrm{tor}}$, we have $\mathrm{Hom}(i^*(B)[-1]_{\mathrm{tor}}, \mathcal{E}) = 0$ since $\mathcal{E}$ is a bundle. This implies $i^*(B)[-1]_{\mathrm{tor}} \subset \mathcal{H}^{-1}(B) = \ker(\theta)$, which contradicts with $\mu^+(\mathcal{H}^{-1}(B)) \leq \beta_g$. Thus if we assume that $[B] = av + bw + c[\mathcal{E}]$, from $i^*(B)[-1] = -av - bw$, we know that $a \leq 0$. And if $a = 0$, then $i^*(B) = 0$ from torsion-freeness. This means $B \cong \mathcal{E}^{\oplus c}$ and contradicts with (8).

Claim 3: $c > 0$ and $\mu^0_{\alpha_g,\beta_g}(B) \geq \mu^0_{\alpha,\beta}(\mathcal{E})$.

Now we claim that $c = \hom(\mathcal{E}, B) \neq 0$, which implies $\mu^0_{\alpha_g,\beta_g}(B) \geq \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$ by $\sigma^0_{\alpha_g,\beta_g}$-stability of B and $\mathcal{E}$. Indeed, if $c = 0$, then we have $B \in \mathcal{K}u(X)$ and we obtain an exact sequence in the heart $\mathcal{A}(\alpha_g, \beta_g)$

$$i^*(A) \to F \to B.$$

Since F is $\sigma(\alpha_g, \beta_g)$-stable, we have $\mu^0_{\alpha_g,\beta_g}(B) > \mu^0_{\alpha_g,\beta_g}(F)$, which gives a contradiction since $\mu^0_{\alpha_g,\beta_g}(F) > \mu^0_{\alpha_g,\beta_g}(\mathcal{O}_X) > \mu^0_{\alpha_g,\beta_g}(B)$.

Now we are ready to prove our main statement.

Case 1: $g \neq 6$.

We have a triangle $\mathcal{E}^{\oplus c} \xrightarrow{\lambda} B \to i^*(B)$, $\mathcal{H}^{-1}_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(B)) \cong \ker(\lambda)$, and $\mathcal{H}^0_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(B)) \cong \mathrm{cok}(\lambda)$. Note that $\mathcal{H}^{-1}_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(B)), \mathcal{H}^0_{\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)}(i^*(B)) \in \mathcal{A}(\alpha_g, \beta_g)$. Taking the cohomology long exact sequence of (9) with respect to the heart $\mathcal{A}(\alpha_g, \beta_g)$, we have an exact sequence in $\mathcal{A}(\alpha_g, \beta_g)$

$$0 \to \ker(\lambda) \to i^*(A) \to F \to \mathrm{cok}(\lambda) \to 0.$$

From F is $\sigma(\alpha_g, \beta_g)$-stable with $\mu^0_{\alpha_g,\beta_g}(F) = +\infty$, we know either $\mathrm{cok}(\lambda) \cong F$ or $\mathrm{cok}(\lambda) = 0$.

Case 1.1: $\mathrm{cok}(\lambda) \cong F$.

In this case, we have $i^*(A) \cong \ker(\lambda)$ and hence we obtain a triangle

$$i^*(A)[1] \to i^*(B) \to F.$$

Note that $\mu^{0-}_{\alpha_g,\beta_g}(A) > \mu^0_{\alpha_g,\beta_g}(C) > \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$, hence we have $\mathrm{Hom}(A, \mathcal{E}) = 0$. Then if we apply $\mathrm{Hom}(A[1], -)$ to the triangle $\mathcal{E}^c \to B \to i^*(B)$, we obtain $\mathrm{Hom}(A[1], i^*(B)) = \mathrm{Hom}(i^*(A)[1], i^*(B)) = 0$. Thus we have $i^*(B) \cong \mathrm{cok}(\lambda) \cong F$ and $\ker(\lambda) \cong i^*(A) \cong 0$. Therefore, by (12) we know $t = \mathrm{Ext}^1(\mathcal{E}, A) = 0$ and $A \cong \mathcal{E}^{n_g - c}$, which contradicts with $\mu^0_{\alpha_g,\beta_g}(A) > \mu^0_{\alpha_g,\beta_g}(C) = \mu^0_{\alpha_g,\beta_g}(\mathcal{O}_X)$.

Case 1.2: $\mathrm{cok}(\lambda) \cong 0$.

In this case, we have $i^*(B) \cong \ker(\lambda)[1]$ and two exact sequences in $\mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$

$$0 \to \ker(\lambda) \to i^*(A) \to F \to 0$$

and

$$0 \to \ker(\lambda) \xrightarrow{\theta} \mathcal{E}^{\oplus c} \xrightarrow{\lambda} B \to 0. \tag{13}$$

Thus if we apply $\mathrm{Hom}(-, \mathcal{E})$ to the sequence (13), by (8) we obtain that $\mathrm{Hom}(\ker(\lambda), \mathcal{E}) \cong \mathrm{Hom}(\mathcal{E}^{\oplus c}, \mathcal{E}) = k^c$. Then θ is actually the the natural map $\ker(\lambda) \to \mathcal{E} \otimes \mathrm{Hom}(\ker(\lambda), \mathcal{E}) \cong \mathcal{E}^{\oplus c}$. Hence we know that $\mathrm{pr}_i \circ \theta \neq 0$ for any $1 \leq i \leq c$, where $\mathrm{pr}_i : \mathcal{E}^{\oplus c} \to \mathcal{E}$ is the projection map of the i-th component.

By Claim 3, we have $\mu^0_{\alpha_g,\beta_g}(B) \geq \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$. If $\mu^0_{\alpha_g,\beta_g}(B) = \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$, since $\mathcal{E}$ is $\sigma^0_{\alpha_g,\beta_g}$-stable and B is the quotient of $\mathcal{E}^{\oplus c}$, we know that $B \cong \mathcal{E}^{\oplus \frac{\mathrm{rk}\, B}{2}}$, and this contradicts with (8). Hence we have $\mu^0_{\alpha_g,\beta_g}(B) > \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$.

Also, from $\sigma^0_{\alpha_g,\beta_g}$-stability of $\mathcal{E}$, we know that $\mu^0_{\alpha_g,\beta_g}(\ker(\lambda)) \leq \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$ and hence $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(\ker(\lambda))) > 0$. If $\mu^0_{\alpha_g,\beta_g}(\ker(\lambda)) = \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$, then by $\sigma^0_{\alpha_g,\beta_g}$-stability of $\mathcal{E}$ we know that $\ker(\lambda) \cong \mathcal{E}^{\oplus \frac{\mathrm{rk} \ker(\lambda)}{2}}$, which contradicts with $\ker(\lambda) \cong i^*(B)[-1] \in \mathcal{K}u(X)$. Thus the inquality is strict, i.e. $\mu^0_{\alpha_g,\beta_g}(\ker(\lambda)) < \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$.

We have:

- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(A)) \geq 0, \mathrm{Im}(Z^0_{\alpha_g,\beta_g}(B)) > 0$,
- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(\ker(\lambda))) > 0$,
- $\mu^0_{\alpha_g,\beta_g}(\ker(\lambda)) < \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $\mu^0_{\alpha_g,\beta_g}(C) > \mu^0_{\alpha_g,\beta_g}(B) > \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $c > 0, a < 0$.

From Claim 1 we know that $\ker(\lambda) = i^*(B)[-1]$ is a torsion-free sheaf. By Lemma A.10, these inequalities imply $\mu(\ker(\lambda)) \geq \mu(\mathcal{E})$. If $\mu(\ker(\lambda)) = \mu(\mathcal{E})$ and $\ker(\lambda)$ is a μ-semistable sheaf, then by poly-stability of $\mathcal{E}^{\oplus c}$, we know that $\mathrm{im}(\theta)$ is contained in $\mathcal{E}^{\oplus c'}$ for $c' = \frac{\mathrm{rk}(\mathrm{im}(\theta))}{2}$. But we know that $\mathrm{pr}_i \circ \theta \neq 0$ for any $1 \leq i \leq c$, which means that the only possible case is $c = c'$, and either $\mathcal{H}^{-1}(B) \cong 0$ and $B \cong \mathcal{H}^0(B)$ is a torsion sheaf supported in codimension ≥ 2, or $\mathcal{H}^{-1}(B) \neq 0$ is a μ-semistable sheaf with $\mu(\mathcal{H}^{-1}(B)) = \mu(\mathcal{E})$. But the first case contradicts with $\mu^0_{\alpha_g,\beta_g}(B) < 0$ and the second case contradicts with $\mu^+(\mathcal{H}^{-1}(B)) \leq \beta_g$. Therefore, we can assume that $\mu^+(\ker(\lambda)) > \mu(\mathcal{E})$.

Let $K_1 \subset \ker(\lambda)$ be the maximal destabilizing subsheaf of $\ker(\lambda)$. Then we have $\mu(K_1) = \mu^+(\ker(\lambda)) > \mu(\mathcal{E})$. Since $\mu^+(\mathcal{H}^{-1}(B)) \leq \beta_g < \mu(\mathcal{E}) < \mu(K_1)$, we know the composition $K_1 \hookrightarrow \ker(\lambda) \xrightarrow{\theta} \mathcal{E}^{\oplus c}$ is non-trivial, which gives a contradiction since from $\mu(K_1) > \mu(\mathcal{E})$ and stability we have $\mathrm{Hom}(K_1, \mathcal{E}) = 0$.

Therefore, such a minimal destabilizing quotient object B cannot exist, and we can conclude that C is $\sigma^0_{\alpha_g,\beta_g}$-semistable when $g \neq 6$.

Case 2: $g = 6$.

In this case, by the claims above we have a system of inequalities:

- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(A)) \geq 0, \mathrm{Im}(Z^0_{\alpha_g,\beta_g}(B)) > 0$,
- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(i^*(A))) \geq 0$,
- $\mu^0_{\alpha_g,\beta_g}(C) > \mu^0_{\alpha_g,\beta_g}(B) \geq \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $c > 0, a < 0$.

Now since $\beta_g < \mu(\mathcal{E})$, by the same argument as in the previous cases shows that $\mu(i^*(B)[-1]) \leq \mu^+(i^*(B)[-1]) \leq \mu(\mathcal{E})$. By Lemma A.11, we know that when $\mu(i^*(B)[-1]) \leq \mu(\mathcal{E})$, the only possible Chern characters of $i^*(B)[-1]$ are $[i^*(B)[-1]] = v - w$ or $3v - 2w$.

Case 2.1: $[i^*(B)[-1]] = v - w$.

In this case we know $i^*(B)[-1]$ is a torsion-free sheaf with $\mathrm{ch}(i^*(B)[-1]) = \mathrm{ch}(I_C(-H))$, where C is a conic on X. Thus $i^*(B)[-1] \cong I_C(-H)$ for some conic C on X. But then $\mathrm{Ext}^3(\mathcal{O}_X, I_C(-H)) = \mathrm{Hom}(I_C, \mathcal{O}_X) \neq 0$, which contradicts with $i^*(B) \in \mathcal{K}u(X)$.

Case 2.2: $[i^*(B)[-1]] = 3v - 2w$.

In this case $i^*(B)[-1]$ is a torsion-free sheaf with rank 3 and degree -2. We claim that $i^*(B)[-1]$ is actually μ-stable. Indeed, if the minimal destabilizing quotient sheaf of $i^*(B)[-1]$ has rank one, then we have $\mathrm{Hom}(i^*(B)[-1], \mathcal{O}_X(-nH)) \neq 0$ for some $n \geq 1$. But $\mathcal{O}_X(-nH)[1] \in \mathsf{Coh}^{\beta_g}(X)$ is $\sigma_{\alpha_g,\beta_g}$-semistable, hence from $\mu_{\alpha_g,\beta_g}(B) \geq \mu_{\alpha_g,\beta_g}(\mathcal{E}) > \mu_{\alpha_g,\beta_g}(\mathcal{O}_X(-nH)[1])$ we know that $\mathrm{Hom}(B, \mathcal{O}_X(-nH)[1]) = 0$ for every $n \geq 1$. Now applying $\mathrm{Hom}(-, \mathcal{O}_X(-nH))$ to the triangle $i^*(B)[-1] \to \mathcal{E}^c \to B$, and from $\mathcal{E}, B \in \mathsf{Coh}^{\beta_g}(X)$ we have $\mathrm{Hom}(i^*(B)[-1], \mathcal{O}_X(-nH)) = 0$, which gives a contradiction. If the maximal destabilizing subsheaf D_1 of $i^*(B)[-1]$ has rank one, then we know that $\mu(D_1) \geq 0$. By stability we have $\mathrm{Hom}(D_1, \mathcal{E}) = 0$, and since D_1 is μ-stable, we have $D_1 \in \mathsf{Coh}^{\beta_g}(X)$. Thus if we apply $\mathrm{Hom}(D_1, -)$ to the triangle $i^*(B)[-1] \to \mathcal{E}^c \to B$, we obtain $\mathrm{Hom}(D_1, i^*(B)[-1]) = 0$, which gives a contradiction. Therefore, $i^*(B)[-1]$ is μ-stable.

Now by Lemma 4.6 and Lemma A.12, we know that $i^*(B)[-1] \in \mathsf{Coh}^{\beta_g}(X)$ is σ_{α,β_g}-semistable for all $\alpha > 0$. In particular, $i^*(B)[-1]$ is $\sigma_{\alpha_g,\beta_g}$-semistable, hence $i^*(B) \in \mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ is $\sigma^0_{\alpha_g,\beta_g}$-semistable, and is also $\sigma(\alpha_g, \beta_g)$-semistable. But by $\sigma(\alpha_g, \beta_g)$-stability of F and $\mu^0_{\alpha_g,\beta_g}(F) > \mu^0_{\alpha_g,\beta_g}(i^*(B))$, we know that $\mathrm{Hom}(F, i^*(B)) = 0$, which gives a contradiction.

Therefore, such a minimal destabilizing quotient object B cannot exist, and we conclude that C is $\sigma^0_{\alpha_g,\beta_g}$-semistable when $g = 6$. □

6.1. **Proof of Theorem 6.3.** Let $G = F[-1]$ and consider the map of vector bundles $f : G = F[-1] \to \mathcal{E}^{\oplus n_g}$ arising from the map $F \to \mathcal{E}^{\oplus n_g}[1]$.

Lemma 6.8. *Let $f : G = F[-1] \to \mathcal{E}^{\oplus n_g}$ be the natural map in $\mathsf{Coh}(X)$. Then $\mathrm{cone}(f) \cong I_x$ for x a point in X.*

Proof. Let C be the cone of the map f. By Proposition 6.7 we know that $C \in \mathsf{Coh}^0_{\alpha_g,\beta_g}(X)$ is $\sigma^0_{\alpha_g,\beta_g}$-semistable. From $\mu^0_{\alpha_g,\beta_g}(C) < 0$, we know that $C \in \mathsf{Coh}_{\alpha_g,\beta_g}(X)$ is $\sigma_{\alpha_g,\beta_g}$-semistable. Since $\Delta(C) = 0$, C is actually σ_{α,β_g}-semistable for every $\alpha > 0$. Thus C is a torsion free μ-semistable sheaf with $\mathrm{ch}(C) = \mathrm{ch}(I_x)$. From $\mathrm{Pic}(X) \cong \mathbb{Z}$, we conclude that $C \cong I_x$ for a point $x \in X$. □

Now we prove the main theorem of the section.

Proof of Theorem 6.3. By the lemma above, $\mathrm{cok}(f) \cong I_x$ so we have a short exact sequence $0 \to G \to \mathcal{E}^{\oplus n_g} \to I_x \to 0$. Applying $\mathrm{Hom}(\mathcal{O}_x, -)$ to this gives a long exact sequence containing

$$\cdots \to \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{E}^{\oplus n_g}) \to \mathrm{Ext}^1(\mathcal{O}_x, I_x) \to \mathrm{Ext}^2(\mathcal{O}_x, G) \to \cdots .$$

Since $\mathcal{E}$ and G are vector bundles concentrated in degree 0, the first and last terms of the above vanish. However, $\mathrm{Ext}^1(\mathcal{O}_x, I_x) \neq 0$ so we have a contradiction, and the assumption that $F \in \mathcal{BN}_g \setminus p(X)$ is false, i.e. there is no such F. So $\mathcal{BN}_g = p(X)$ as required. □

Theorem 6.3 can be used to prove refined categorical Torelli theorems for all index one Fano threefolds of genus $g \geq 6$.

We first introduce a functor on $\mathcal{K}u(X)$, defined by $T(-) := i^* \circ \mathrm{R}\mathcal{H}om(-, \mathcal{O}_X(-H)[1])$. By [Zha20, Proposition 3.8], this functor $T : \mathcal{K}u(X) \to \mathcal{K}u(X)$ is an anti-equivalence with the property $T \circ T \cong \mathrm{id}_{\mathcal{K}u(X)}$. The involution T induces a linear isometry on $\mathcal{N}(\mathcal{K}u(X))$, which we also denote by T. When $g = 6$, we have $T(v) = -3v + 2w$ and $T(w) = -4v + 3w$. Note that when $g = 6$, there also exists an involution $\mathcal{K}u(X_{10}) \to \mathcal{K}u(X_{10})$, which is $\tau = S_{\mathcal{K}u(X_{10})}[-2]$. The action of τ on $\mathcal{N}(\mathcal{K}u(X_{10}))$ is the identity.

Lemma 6.9. *Let $g \geq 6$ and $\Phi : \mathcal{N}(\mathcal{K}u(X_{2g-2})) = \langle v, w\rangle \xrightarrow{\cong} \mathcal{N}(\mathcal{K}u(X'_{2g-2})) = \langle v', w'\rangle$ be a linear isometry with respect to the Euler form, such that $\Phi([i^!\mathcal{E}]) = [i'^!\mathcal{E}']$. Then $\Phi(v) = v'$ and $\Phi(w) = w'$ if $g \geq 7$. If $g = 6$ we have either $\Phi(v) = v'$ and $\Phi(w) = w'$ or $T \circ \Phi(v) = v'$ and $T \circ \Phi(w) = w'$.*

Proof. Since Φ preserves the Euler form and $\Phi([i^!\mathcal{E}]) = [i'^!\mathcal{E}']$, an elementary computation shows that $\Phi(v) = v'$ and $\Phi(w) = w'$ if $g \geq 7$. If $g = 6$, we have either $\Phi(v) = v'$ and $\Phi(w) = w'$ or $\Phi(v) = -3v' + 2w'$ and $\Phi(w) = -4v' + 3w'$. Then the result follows from $T(v') = -3v' + 2w'$, $T(w') = -4v' + 3w'$ and $T \circ T = \mathrm{id}$. □

Lemma 6.10. *Let $X := X_{10}$. Then $T(i^!\mathcal{E}) \cong \tau(i^!\mathcal{E})$.*

Proof. Recall that $i^!\mathcal{E}$ is the unique object that fits into the triangle

$$\mathcal{Q}(-H)[1] \to i^!\mathcal{E} \to \mathcal{E}.$$

Then note that $i^*(\mathrm{R}\mathcal{H}om(\mathcal{Q}(-H)[1], \mathcal{O}_X(-H)[1])) \cong i^*(\mathcal{Q}^\vee)$ and $i^*(\mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_X(-H)[1])) = i^*(\mathcal{E}[1]) = 0$, so we have $T(i^!\mathcal{E}) \cong i^*(\mathcal{Q}^\vee)$. Then the result follows from $\tau \cong \tau^{-1}$, $i^*(\mathcal{Q}^\vee) = \mathbf{L}_{\mathcal{E}}\,\mathcal{Q}^\vee \cong \tau^{-1}(i^!\mathcal{E})$, and the computations in Lemma 6.6. □

Corollary 6.11 (Refined categorical Torelli)**.** *Let X and X' be smooth index one prime Fano threefolds with genus $g \geq 6$, and suppose there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ such that the gluing object is preserved in the sense that $\Phi(i^!\mathcal{E}) \cong i^!\mathcal{E}'$. Then $X \cong X'$.*

Proof. First we assume that $g \neq 6$. By Theorem 6.3, $X \cong \mathcal{BN}_g$. Let $F \in \mathcal{BN}_g$. Then $\Phi(F) \in \mathcal{BN}'_g$. Indeed, by $\Phi(i^!\mathcal{E}) \cong i^!\mathcal{E}'$ and Lemma 6.9 we know $\mathrm{ch}(\Phi(F)) = \mathrm{ch}(i^*\mathcal{O}_x[-1]))$ when $g \neq 6$. As $\mathrm{Ext}^1(\Phi(F), i^!\mathcal{E}') \cong \mathrm{Ext}^1(\Phi(F), \Phi(i^!\mathcal{E})) \cong \mathrm{Ext}^1(F, i^!\mathcal{E}) \cong k^{n_g}$, it remains to show that $\Phi(F)$ is a σ'-stable object in $\mathcal{K}u(X')$ for every Serre-invariant stability condition σ'. This is from $\Phi(\sigma)$ is Serre-invariant and Theorem 4.13. Thus Φ induces a bijection between the closed points of $\mathcal{BN}_g$ and those of $\mathcal{BN}'_g$. On the other hand, it is clear that the Brill–Noether locus $\mathcal{BN}_g$ admits the universal family coming from X (argued similarly as in [APR19, Section 5.2]). Then the induced morphism $\phi : \mathcal{BN}_g \to \mathcal{BN}'_g$ is an étale morphism and a bijection, thus $\mathcal{BN}_g \cong \mathcal{BN}'_g$, which implies that $X \cong X'$.

In the case of GM threefolds ($g = 6$), we first need to show that X and X' must be ordinary GM threefolds or special GM threefolds simultaneously. Indeed, we may assume X is an ordinary GM threefold and X' is a special GM threefold. Then $\mathrm{Ext}^2(i^!\mathcal{E}, i^!\mathcal{E}) = 0$ and $\mathrm{Ext}^2(i^!\mathcal{E}', i^!\mathcal{E}') = k$ (both facts are shown in [JLLZ21]), which is impossible since Φ is an equivalence. Now by Lemma 6.9, we have either $\mathrm{ch}(\Phi(F)) = \mathrm{ch}(i^*\mathcal{O}_x[-1])$ or $\mathrm{ch}(T \circ \Phi(F)) = \mathrm{ch}(\tau \circ T \circ \Phi(F)) = \mathrm{ch}(i^*\mathcal{O}_x[-1])$. If we are in the former case, then the same argument as in the previous cases shows that $X \cong X'$. Otherwise if $\mathrm{ch}(\tau \circ T \circ \Phi(F)) = \mathrm{ch}(i^*\mathcal{O}_x[-1])$, then since $\tau \circ T(i^!\mathcal{E}) \cong i^!\mathcal{E}$ by Lemma 6.10, we can replace Φ by $\Phi' := \tau \circ T \circ \Phi$. Then we have $\Phi'(i^!\mathcal{E}) = i^!\mathcal{E}$ and $\mathrm{ch}(\Phi'(F)) = \mathrm{ch}(i^*\mathcal{O}_x[-1])$. Thus the result follows from the previous argument. □

7. Reconstruction of Fano threefolds of genus $g < 6$

Proposition 7.1. *Let X be a prime Fano threefold. Then $\sigma'(\alpha, \beta) := \sigma^0_{\alpha,\beta}|_{\mathcal{O}_X^\perp}$ is a stability condition on $\mathcal{O}_X^\perp$ for all $(\alpha, \beta) \in W$, where*

$$W := \{(\alpha, \beta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : -\frac{1}{2} \leq \beta < 0, \alpha < -\beta, or -1 < \beta < -\frac{1}{2}, \alpha \leq 1 + \beta\}.$$

Proof. From [BLMS17, Proposition 2.14] we know $\mathcal{O}_X, \mathcal{O}_X(-1)[1] \in \mathrm{Coh}^\beta(X)$ are $\sigma_{\alpha,\beta}$-stable for all $\alpha > 0$ and $\beta < 0$. Now note that $\mu_{\alpha,\beta}(\mathcal{O}_X(-1)) = \frac{\frac{1}{2}+\beta+\frac{1}{2}\beta^2-\frac{1}{2}\alpha^2}{-1-\beta}$ and $\mu_{\alpha,\beta}(\mathcal{O}_X) = \frac{\beta^2-\alpha^2}{-2\beta}$. Thus the set of solutions of $\mu_{\alpha,\beta}(\mathcal{O}_X(-1)) < 0 < \mu_{\alpha,\beta}(\mathcal{O}_X)$ is exactly W. Then as in [BLMS17, Theorem 6.7], $\sigma'(\alpha, \beta)$ is a stability condition on $\mathcal{K}u(X)$. □

Lemma 7.2. *Let I_x be the ideal sheaf of a point $x \in X$. Then $I_x \in \mathcal{A}'(\alpha, \beta)$ is $\sigma'(\alpha, \beta)$-semistable for every $(\alpha, \beta) \in W$.*

Proof. It is easy to see that $I_x \in \mathcal{O}_X^\perp$. By Lemma 4.6, $I_x \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable for $\beta < 0$ and $\alpha \gg 0$. From $\mathrm{ch}_{\leq 2}(I_x) = \mathrm{ch}_{\leq 2}(\mathcal{O}_X)$ we know there are no walls for I_x. Thus $I_x \in \mathrm{Coh}^0_{\alpha,\beta}(X)$ is $\sigma^0_{\alpha,\beta}$-semistable for every $\alpha > 0$, and hence it is $\sigma'(\alpha,\beta)$-stable. □

Proposition 7.3. *Let $(\alpha,\beta) \in W$ and $F \in \mathcal{A}'(\alpha,\beta)$ be a $\sigma'(\alpha,\beta)$-stable object with $[F] = [I_x]$. Then $F \cong I_x$ for some point $x \in X$.*

Proof. By [BLMS17, Remark 5.12], we know F is $\sigma^0_{\alpha,\beta}$-semistable. Since $\Delta(F) = 0$, F is $\sigma^0_{\alpha,\beta}$-semistable for every $\alpha > 0$. Now by the definition of $\mathrm{Coh}^0_{\alpha,\beta}(X)$ we have a triangle $A[1] \to F \to B$ where $A, B \in \mathsf{Coh}^\beta(X)$, $\mu^+_{\alpha,\beta}(A) \leq 0$, and $\mu^-_{\alpha,\beta}(B) > 0$. Since F is $\sigma^0_{\alpha,\beta}$-semistable with $\mu^0_{\alpha,\beta}(F) < 0$, we know $A = 0$. Thus $F \in \mathrm{Coh}^\beta(X)$ is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$. By Lemma 4.6, we know that F is a μ-stable sheaf, which implies $F \cong I_x$ for some $x \in X$. □

Now we prove the main result of this section.

Theorem 7.4. *Let X be a prime Fano threefold. Then the functor $i^*_{\mathcal{D}}$ induces an isomorphism*

$$X \cong \mathcal{M}_{\sigma'}(\mathcal{O}_X^\perp, [I_x])$$

for $\sigma' \in W$.

Proof. Note that X isomorphic to the moduli space of μ-(semi)stable sheaves with Chern character $1 - P$ by [KPS18, Lemma B.5.6]. Thus using Lemma 7.2 and Proposition 7.3, a similar argument as in [LZ21, Proposition 4.3] shows that $i^*_{\mathcal{D}}$ induces an étale bijective morphism to $\mathcal{M}_{\sigma'}(\mathcal{O}_X^\perp, [I_x])$, which is an isomorphism. □

Remark 7.5. The reason that we cannot deduce a refined categorical Torelli theorem from Theorem 7.4 is that we do not know whether every equivalence $\mathcal{O}_X^\perp \simeq \mathcal{O}_{X'}^\perp$ preserves the stability conditions in W or the class $[I_x]$.

8. Fibers of the period map via the refined categorical Torelli theorem

In the case of Fano threefolds, the Torelli problem asks if the period map $\mathcal{P}_g : \mathcal{M}_g \to \mathcal{A}_g$ is injective, where $\mathcal{M}_g$ is the moduli space of isomorphism classes of index one genus g prime Fano threefolds, and $\mathcal{A}_g$ is the corresponding moduli space of isomorphism classes of principally polarized abelian varieties. In most cases of index one prime Fano threefolds, $\mathcal{P}_g$ is not injective and the fiber has positive dimension. In this section, we compute the fibers of $\mathcal{P}_g$ over $J(X)$ for each genus $g \geq 6$.

Lemma 8.1. *Let X be an index one prime Fano threefold of genus $g \geq 7$. Then its intermediate Jacobian $J(X)$ and Kuznetsov component $\mathcal{K}u(X)$ are mutually determined by each other. More precisely,*

$$J(X) \cong J(X') \iff \mathcal{K}u(X) \simeq \mathcal{K}u(X').$$

Proof.

(i) $g = 7$: We have $J(X) \cong J(X') \iff J(\Gamma_7) \cong J(\Gamma'_7) \iff \Gamma_7 \cong \Gamma'_7 \iff D^b(\Gamma_7) \simeq D^b(\Gamma'_7) \iff \mathcal{K}u(X) \simeq \mathcal{K}u(X')$, where Γ_7 and Γ'_7 are the genus 7 curves associated to X and X', respectively.

(ii) $g = 8$: Assume that $J(X) \cong J(X')$. Then $J(Y) \cong J(Y')$, where Y and Y' are Pfaffian cubics corresponding to X and X', respectively. It follows that $Y \cong Y'$ by the classical Torelli theorem for cubic threefolds. It is known that $\mathcal{K}u(Y) \simeq \mathcal{K}u(X)$ and $\mathcal{K}u(Y') \simeq \mathcal{K}u(X')$ by [Kuz09]. Thus $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. For the other direction, it follows that $\mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$ where Y and Y' are the corresponding Pfaffian cubics of X and X', respectively. Then it follows from [BMMS12] that $Y \cong Y'$, so $J(X) \cong J(Y) \cong J(Y') \cong J(X')$.

(iii) $g = 9$: $J(X) \cong J(X') \iff J(\Gamma_3) \cong J(\Gamma'_3) \iff \Gamma_3 \cong \Gamma'_3 \iff D^b(\Gamma_3) \simeq D^b(\Gamma'_3) \iff \mathcal{K}u(X) \simeq \mathcal{K}u(X')$, where Γ_3 and Γ'_3 are the genus 3 curves associated to X and X', respectively.

(iv) $g = 10$: the argument is the same as for the $g = 9$ and $g = 7$ cases; it suffices to replace Γ_3 by a genus 2 curve Γ_2.
(v) $g = 12$: the statement is trivial since $J(X)$ is trivial.

$\square$

Remark 8.2. If $g = 6$, X is a Gushel–Mukai threefold. It is conjectured that $J(X)$ and $\mathcal{K}u(X)$ are mutually determined by each other. By [JLLZ21, Proposition 11.7], this is equivalent to the *Debarre-Iliev-Manivel conjecture* stated in [DIM12].

Corollary 8.3. *The fiber of period map $\mathcal{P}_g$ over $J(X)$ is parametrised by the family of* gluing objects $i^!\mathcal{E}' \in \mathcal{K}u(X')$ *as X' varies.*

Proof. This follows from Lemma 8.1 and Theorem 1.3. $\square$

Next, we compute the fiber of $\mathcal{P}_g$ over $J(X)$.

8.1. **Genus 6: Gushel–Mukai threefolds.** First, we give another proof of the *categorical Torelli theorem* for general special Gushel–Mukai threefolds via the *refined categorical Torelli Theorem* 1.3. This was proved in [JLLZ21, Theorem 10.9].

Theorem 8.4. *Let X, X' be two general special GM threefolds such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$. Then $X \cong X'$.*

Proof. Let X and X' be two smooth general special GM threefolds. Since there is an equivalence $\Xi : \mathcal{K}u(X) \simeq \mathcal{A}_X$, it is enough to show that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ implies $X \cong X'$. Let $\Phi : \mathcal{A}_X \simeq \mathcal{A}_{X'}$ be an equivalence of the alternative Kuznetsov components. Consider the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, 1-2L)$ of σ-stable objects of class $1-2L$; it is a surface with a unique singular point $[\Xi(i^!\mathcal{E})]$. We claim that $\Phi(\Xi(i^!\mathcal{E})) \cong \Xi(i^!\mathcal{E}') \in \mathcal{A}_{X'}$ up to a shift. First note that the numerical class of $\Phi(\Xi(i^!\mathcal{E}))$ is either $1-2L$ or $2-H+\frac{5}{6}P$ up to sign. It is easy to see $\Phi(\Xi(i^!\mathcal{E}))$ is a stable object in $\mathcal{A}_{X'}$ for every τ-invariant stability condition by the Weak Mukai Lemma 4.12. If $[\Phi(\Xi(i^!\mathcal{E}))] = 2-H+\frac{5}{6}P$, then $\Phi(\Xi(i^!\mathcal{E}))$ is a point in the Bridgeland moduli space $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, 2-H+\frac{5}{6}P) \cong M_{X'}(2,1,5)$ by [JLLZ21, Theorem 11.13], which is a smooth irreducible surface. But $\Phi(\Xi(i^!\mathcal{E}))$ is a singular point since $\mathrm{Ext}^2(\Phi(\Xi(i^!\mathcal{E})), \Phi(\Xi(i^!\mathcal{E}))) \cong \mathrm{Ext}^2(\Xi(i^!\mathcal{E}), \Xi(i^!\mathcal{E})) = k$ since Φ is an equivalence. This leads to a contradiction. Thus $[\Phi(\Xi(i^!\mathcal{E}))] = 1-2L$. But the moduli space $\mathcal{M}_{\sigma'}(\mathcal{A}_{X'}, 1-2L)$ is everywhere smooth apart from at the point $[\Xi(i^!\mathcal{E})']$ by [JLLZ21, Theorem 7.13]. Therefore $\Phi(\Xi(i^!\mathcal{E})) \cong \Xi(i^!\mathcal{E}')$. Then we have an equivalence $\Phi' : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ such that $\Phi'(i^!\mathcal{E}) \cong i^!\mathcal{E}'$. Hence by the *refined categorical Torelli theorem* 1.3, $X \cong X'$. $\square$

Next, we recall a result describing the fiber of the "categorical period map" over the Kuznetsov component of a general ordinary GM threefold, proved in [JLLZ21, Theorem 11.3]. We then study the same problem for *very general* special GM threefolds.

8.1.1. *Ordinary Gushel–Mukai threefolds.*

Theorem 8.5 ([JLLZ21, Theorem 11.3])**.** *Let X be an ordinary GM threefold. Then all GM threefolds X' such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ in the moduli space $\mathcal{M}_3$ of GM threefolds are parametrised by an open subset of the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2,1,5)/\iota'$, where ι, ι' are geometrically meaningful involutions. If in addition, we assume that X is general, then all GM threefolds X' such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ are parametrised by $C_m(X)/\iota \cup M_X(2,1,5)/\iota'$.*

8.1.2. *Special Gushel–Mukai threefolds.*

Lemma 8.6 ([Huy16, Chapter 16])**.** *Let S and S' be two smooth K3 surfaces such that $\mathrm{D}^b(S) \simeq \mathrm{D}^b(S')$. Then their Picard numbers are equal, i.e. $\rho(S) = \rho(S')$.*

Theorem 8.7. *Let X be a very general special GM threefold. Then all GM threefolds X' such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ in the moduli space $\mathcal{M}_3$ of GM threefolds form a subvariety of $\mathcal{C}_m(X)/\iota \bigcup M_X(2,1,5)/\iota'$, where $\mathcal{C}_m(X)$ is the contraction of the Fano surface $\mathcal{C}(X)$ of conics on X along one of its irreducible components $\mathbb{P}^2$, and ι, ι' are geometric involutions.*

Proof. The only possible GM threefold X' with an equivalence $\Phi : \mathcal{K}u(X') \simeq \mathcal{K}u(X)$ is either a special GM threefold such that the Picard number of the branch locus $\mathcal{B}'$ is one, or a non-general ordinary GM threefold. Indeed, X' cannot be a special GM threefold with $\rho(\mathcal{B}') \geq 2$. Otherwise the equivalence Φ would induce an equivalence of equivariant triangulated categories $\Psi : \mathrm{D}^b(\mathcal{B}') \simeq \mathrm{D}^b(\mathcal{B})$, where $\mathcal{B}$ and $\mathcal{B}'$ are smooth K3 surfaces of degree 10. Then by Lemma 8.6, their Picard numbers satisfy $\rho(\mathcal{B}') = \rho(\mathcal{B}) = 1$. In this case, we get $\mathcal{B} \cong \mathcal{B}'$ by [Ogu02, Theorem 1.10] and [HLOY03, Corollary 1.7]. Then we get $X' \cong X$, giving a point in the fiber, which is represented by the unique singular point q on $\mathcal{C}_m(X)$. If X' is an ordinary GM threefold such that the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ is singular but $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ is smooth, then the equivalence Φ identifies the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ with $\mathcal{C}_m(X)$ and $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ with $M_X(2,1,5)$. If X' is an ordinary GM threefold such that the moduli space $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ is smooth but $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ is singular, then the equivalence Φ identifies $\mathcal{M}_\sigma(\mathcal{A}_{X'}, -x)$ with $M_X(2,1,5)$ and $\mathcal{M}_\sigma(\mathcal{A}_{X'}, y-2x)$ with $\mathcal{C}_m(X)$. By Corollary 6.11, X' is uniquely determined by the gluing object $\Xi(i^!\mathcal{E}) \in \mathcal{A}_{X'}$. Thus the set of such ordinary GM threefolds X' is parametrised by $[\Xi(i^!\mathcal{E})] \in \mathcal{C}_m(X) \cup M_X(2,1,5)$. Note that the point $[\Xi(i^!\mathcal{E})]$ is a smooth point by [JLLZ21, Lemma 7.9]. This means the point $[\Xi(\pi(\mathcal{E}))]$ can be any point in $\mathcal{C}_m(X) \cup M_X(2,1,5)$ except the singular point q. Therefore the set of all GM threefolds X' such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ is a subset of $\{q\} \cup (\mathcal{C}_m(X) \setminus \{q\}) \cup M_X(2,1,5) = \mathcal{C}_m(X) \cup M_X(2,1,5)$, up to involutions. □

Remark 8.8. If [GLZ22, Theorem 4.1] holds for very general special GM threefolds X, then we could show that the set of GM threefolds X' with $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ is equal to the union $\mathcal{C}_m(X)/\iota \cup M_X(2,1,5)/\iota'$. Indeed, in the upcoming paper [DK22], the authors show that the contraction of the Hilbert scheme of conics on any special GM threefold X along the $\mathbb{P}^2$ as one of its irreducible components is isomorphic to double dual EPW surface $\widetilde{Y}^{\geq 2}_{A^\perp}$, where $A := A(X) \subset \bigwedge^3 V_6(X)$ is the Lagrangian subspace associated with X. Then the moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -s) \cong \widetilde{Y}^{\geq 2}_{A^\perp}$. If in addition $M_X(2,1,5) \cong \mathcal{M}_\sigma(\mathcal{A}_X, t-2s) \cong \widetilde{Y}^{\geq 2}_{A}$, then by [DK15, Theorem 3.25, Theorem 3.27] every point in $\mathcal{C}_m(X)/\iota \cup M_X(2,1,5)/\iota'$ corresponds to a period partner X' or period dual X'' of X if X is very general. Then by the Duality Conjecture [KP19, Theorem 1.6], $\mathcal{K}u(X') \simeq \mathcal{K}u(X) \simeq \mathcal{K}u(X'')$.

8.2. **Genus 8: degree 14 prime Fano threefolds.** The gluing object $\Xi(i^!\mathcal{E}) \in \mathcal{A}_X$ in the Kuznetsov component $\mathcal{A}_X$ is a σ_1-stable object of class $2(1-2L)$ with respect to every Serre-invariant stability condition on $\mathcal{A}_X$ by Proposition 4.14. Let Y be the Pfaffian cubic associated to X. Let $\Psi : \mathcal{A}_X \simeq \mathcal{K}u(Y)$ be the Kuznetsov-type equivalence from [Kuz09]. It is known that $\Psi(\Xi(i^!\mathcal{E})) \in \mathcal{K}u(Y) \in \mathcal{M}_\sigma(\mathcal{K}u(Y), 2(1-L)) \cong M_Y^{\mathrm{inst}}(2,0,2)$ by [LZ21, Theorem 1.2], where $M_Y^{\mathrm{inst}}(2,0,2)$ is the moduli space of stable instanton sheaves on the cubic threefold Y. Indeed, we can show that $\Psi(\Xi(i^!\mathcal{E})) \in \mathcal{K}u(Y)$ is a rank two stable instanton bundle.

Proposition 8.9. *Let $\Psi : \mathcal{A}_X \simeq \mathcal{K}u(Y)$ be an equivalence of the Kuznetsov component of $X := X_{14}$ and a corresponding Pfaffian cubic Y. Then $\Psi(\Xi(i^!\mathcal{E}))$ is a rank two stable instanton bundle.*

Before the proof, we prove several lemmas. Recall that $\Xi(i^!\mathcal{E}) \in \mathcal{A}_X$ is given by an exact triangle

$$\mathcal{E}[2] \to \Xi(i^!\mathcal{E}) \to \mathcal{Q}^\vee[1]$$

and the equivalence $\mathcal{A}_X \to \mathcal{K}u(Y)$ is given by the Fourier-Mukai functor

$$\Psi := \Psi_{I_W(H_Y)} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

with the kernel being the $\mathcal{O}_{X\times Y}(H_Y)$ twist of the ideal sheaf I_W of an irreducible subvariety of $X \times Y$ of dimension 4 (see [KPS18, Remark B.6.5] and [Kuz04]). Denote by p_X, p_Y the projection maps from $X \times Y$ to X and Y, respectively.

Lemma 8.10. *Let $y \in Y$ be a closed point and $X_y := p_Y^{-1}(y) \cong X$. Then $W_y := W \cap X_y$ is a rational quartic curve on X_y.*

Proof. By definition, we know that W_y is an intersection of a linear section of $\mathrm{Gr}(2,6)$ with a Schubert cell corresponding to $(4,1)$ by standard Schubert calculus (see for example [Ful96]). In particular, we obtain $\deg W_y = 4$. Recall the commutative diagrams defining W below:

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, "\eta"'] \arrow[dr, "q"] & \mathbb{P}_X(\mathcal{U}) \arrow[d, "\phi"] \\ \mathbb{P}_Y(E^\vee) \arrow[r, "\psi"] & Q \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "j"'] \arrow[d, "i"] \arrow[dr, "\lambda"] & \\ X \times Y & \mathbb{P}_Y(E^\vee) \times \mathbb{P}_X(\mathcal{U}) \arrow[l, "p_X \times p_Y"] \arrow[r, "\alpha p_Y \times \mathrm{id}"'] & \mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U}) \end{tikzcd}$$

Then $W_y = (p_Y \circ i)^{-1}(y) = \mathbb{P}^1 \times_Q \mathbb{P}_X(\mathcal{U})$. By [Kuz04, Proposition 2.11], there are the following possible cases of W_y:

(a) $\psi(\mathbb{P}^1) = \mathrm{pt} \in C$. In this case $W_y = \mathbb{P}^1 \times_Q \mathbb{P}_X(\mathcal{U}) \cong \mathbb{P}^1 \times \mathbb{P}^1$;
(b) If $\psi(\mathbb{P}^1) \cap C = \varnothing$, then we have $W_y = \mathbb{P}^1 \times_Q \mathbb{P}_X(\mathcal{U}) \cong \mathbb{P}^1 \times_{Q-C} \phi^{-1}(Q-C) \cong \mathbb{P}^1$. In this case W_y is a smooth rational quartic curve;
(c) $\psi(\mathbb{P}^1) \cap C = \{p_1, ..., p_m\}$. In this case W_y is an intersection of m disjoint lines with a line transversally. In this case W_y is a reducible rational quartic curve.

Since $\deg W_y = 4$ and $X_y \cong X$ is a prime Fano threefold of index 1 and degree 14, case (a) is excluded by the Lefschetz hyperplane section theorem. Thus the only possible cases are (b) and (c), which are both rational quartic curves. □

Lemma 8.11. *Let $C \subset X$ be a rational quartic on X. Then we have:*

(1) $h^2(\mathcal{E} \otimes I_C) = 2$ and $h^i(\mathcal{E} \otimes I_C) = 0$ for $i \neq 2$;
(2) $h^i(\mathcal{E}^\vee \otimes I_C) = 0$ for all i;
(3) $h^i(\mathcal{Q}^\vee \otimes I_C) = 0$ for all i.

Proof.

(1) Applying $\Gamma(X, \mathcal{E} \otimes -)$ to the exact sequence

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and using $H^*(\mathcal{E}) = 0$, we obtain $H^0(\mathcal{E} \otimes I_C) = H^3(\mathcal{E} \otimes I_C) = 0$, and $H^{i+1}(\mathcal{E} \otimes I_C) = H^i(\mathcal{E}|_C)$ for all i. Since $\mathrm{ch}(\mathcal{O}_C) = 4L - P$, we have $\chi(\mathcal{E}|_C) = -2$. By exactly the same argument as in [KPS18, Lemma B.3.3] we show that $h^0(\mathcal{E}|_C) \neq 0$ implies that the span $\langle C \rangle \cong \mathbb{P}^4$ would be contained in X, which is impossible. Thus $H^0(\mathcal{E}|_C) = H^1(\mathcal{E} \otimes I_C) = 0$, hence $h^2(\mathcal{E} \otimes I_C) = h^1(\mathcal{E}|_C) = 2$.

(2) Applying $\Gamma(X, \mathcal{E}^\vee \otimes -)$ to the exact sequence

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and since $h^0(\mathcal{E}^\vee) = 6$ and 0 in other degrees, we obtain $H^3(\mathcal{E}^\vee \otimes I_C) = 0$, and $H^i(\mathcal{E}^\vee|_C) \cong H^{i+1}(\mathcal{E}^\vee \otimes I_C)$ for all $i \geq 1$. Moreover we have an exact sequence

$$0 \to H^0(\mathcal{E}^\vee \otimes I_C) \to H^0(\mathcal{E}^\vee) \to H^0(\mathcal{E}^\vee|_C) \to H^1(\mathcal{E}^\vee \otimes I_C) \to 0.$$

Since $H^1(\mathcal{E}^\vee|_C) = H^0(\mathcal{E}|_C(-1)) \subset H^0(\mathcal{E}|_C) = 0$, we get $H^2(\mathcal{E}^\vee \otimes I_C) = 0$. But $\chi(\mathcal{E}^\vee|_C) = 6$, hence $h^0(\mathcal{E}^\vee) = h^0(\mathcal{E}^\vee|_C) = 6$. Therefore $H^0(\mathcal{E}^\vee \otimes I_C) = H^1(\mathcal{E}^\vee \otimes I_C) = 0$.

(3) Applying $\Gamma(X, - \otimes I_C)$ to exact sequence

$$0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 6} \to \mathcal{E}^\vee \to 0$$

by (2) we obtain $h^i(\mathcal{Q}^\vee \otimes I_C) = 0$ for all i.

□

We define $\mathcal{G} := \Xi(i^!\mathcal{E})$.

Lemma 8.12. $\mathrm{RHom}^\bullet(\mathcal{O}_y, \Psi_{I_{W(H_Y)}}(\mathcal{G})) \cong k^2[-3]$ *for any closed point $y \in Y$, where $\mathcal{G} := \Xi(i^!\mathcal{E})$*

Proof. We know that $\Psi_{I_W(H_Y)}(\mathcal{G}) = (p_Y)_*(p_X^*\mathcal{G}\otimes I_W(H_Y)) = (p_Y)_*(p_X^*\mathcal{G}\otimes I_W\otimes p_Y^*\mathcal{O}_Y(1))$. Then by projection formula we have $\Psi_{I_W(H_Y)}(\mathcal{G}) = (p_Y)_*(p_X^*\mathcal{G}\otimes I_W)(H_Y)$. Let $S := p_X^*\mathcal{G}\otimes I_W$. Then $\Psi(\mathcal{G}) = (p_Y)_*(S)(H_Y)$. Thus

$$\begin{aligned}\mathrm{RHom}^\bullet(\mathcal{O}_y, \Psi_{I_W(H_Y)}(\mathcal{G})) &= \mathrm{RHom}^\bullet(\mathcal{O}_y, (p_Y)_*(S)(H_Y))\\ &= \mathrm{RHom}^\bullet(\mathcal{O}_y, (p_Y)_*(S)) = \mathrm{RHom}^\bullet(p_Y^*\mathcal{O}_y, S)\\ &= \mathrm{RHom}^\bullet(j_{y*}\mathcal{O}_{X_y}, S) = \mathrm{RHom}^\bullet(\mathcal{O}_{X_y}, j_y^!S)\\ &= \mathrm{RHom}^\bullet(\mathcal{O}_{X_y}, \mathcal{G}\otimes I_{W\cap X_y})[-3],\end{aligned}$$

where $j_y : X_y := p_Y^{-1}(y) \cong X \hookrightarrow X\times Y$. Here the first equality is from the projection formula, the second is from $\mathcal{O}_y\otimes\mathcal{O}_Y(-H_Y)\cong\mathcal{O}_y$, and the fourth and fifth are from adjointness. Hence we only need to compute $\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathcal{G}\otimes I_{W_y})$. Applying $-\otimes I_W$ to the triangle

$$\mathcal{E}[2]\to\mathcal{G}\to\mathcal{Q}^\vee[1]$$

and taking cohomology, Lemma 8.10 and Lemma 8.11 give $\hom(\mathcal{O}_X, \mathcal{G}\otimes I_{W_y}) = 2$. □

Proof of Proposition 8.9. It follows from [BM99, Proposition 5.4] that $\Phi_{I_W(H)}(\mathcal{G})$ is a rank 2 vector bundle. Also, $\Psi(\mathcal{G})$ is σ-stable in $\mathcal{K}u(Y)$ with respect to every Serre-invariant stability condition σ by Proposition 4.14. Further note that $\mathrm{ch}(\Psi(\mathcal{G})) = 2(1-L)$. Then by [LZ21, Theorem 7.6], $\Psi(\mathcal{G})$ is an instanton sheaf, hence an instanton bundle. □

Theorem 8.13. *Let X be an index one degree 14 prime Fano threefold. Then all index one prime Fano threefolds X' of degree 14 such that $\mathcal{A}_{X'}\simeq\mathcal{A}_X$ are parametrised by the moduli space of rank 2 instanton bundles of minimal charge over a cubic threefold Y.*

Proof. By Proposition 8.9, all degree 14 index one prime Fano threefolds X' with $\mathcal{A}_{X'}\simeq\mathcal{A}_X$ are parametrised by the family of *gluing objects* $\Psi(\Xi(i^!\mathcal{E}))\in\mathcal{K}u(Y)$, which are rank two instanton bundles on a Pfaffian cubic threefold Y associated to $X := X_{14}$. On the other hand, any pair (Y, E') such that E' is a rank two instanton bundle on Y uniquely determines an index one prime Fano threefold of degree 14 X' such that $\mathcal{A}_{X'}\simeq\mathcal{K}u(Y)\simeq\mathcal{A}_X$ by [Kuz04, Theorem 2.9], [MT01] and [IM99]. Moreover, $\Psi(\Xi(i^!\mathcal{E}'))\cong E'$, where $\mathcal{E}'$ is the tautological subbundle on X'. Then the desired result follows. □

8.3. **Genus 10: degree 18 prime Fano threefolds.** Let X be an index one prime Fano threefold of degree 18. Following [KPS18, Remark B.5.3], we recall some basic properties.

The moduli space of stable sheaves $\mathcal{M} := M_X(3, -H, 9L, -2P)$ consists of only stable vector bundles and $\mathcal{M}$ is isomorphic to a genus 2 curve Γ_2 admitting a universal family $\mathcal{U}$. Consider the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle\mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee\rangle$ where $\mathcal{A}_X = i(\mathrm{D}^b(\Gamma_2))$ and $i : \mathrm{D}^b(\Gamma_2)\to\mathrm{D}^b(X)$[1] is induced by the family $\mathcal{U}$. Next, we recall the explicit formulae for the functors i, i^* and $i^!$. Let p and q be the projection maps $p : X\times\Gamma_2\to X$ and $q : X\times\Gamma_2\to\Gamma_2$:

$$\begin{aligned}&i : \mathrm{D}^b(\Gamma_2)\to\mathrm{D}^b(X), \quad i(-) = \mathrm{R}p_*(q^*(-)\otimes\mathcal{U})\\ &i^! : \mathrm{D}^b(X)\to\mathrm{D}^b(\Gamma_2), \quad i^!(-) = \mathrm{R}q_*(p^*(-)\otimes\mathcal{U}^*(\omega_{\Gamma_2}))[1]\\ &i^* : \mathrm{D}^b(X)\to\mathrm{D}^b(\Gamma_2), \quad i^*(-) = \mathrm{R}q_*(p^*(-)\otimes\mathcal{U}^*(-H_X))[3].\end{aligned}$$

Lemma 8.14. *Let C be a twisted cubic on X. Then we have*

(1) $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$ *and* $\mathrm{Ext}^i(\mathcal{E}, I_C) = 0$ *for* $i\neq 0$,
(2) $h^0(X, \mathcal{E}\otimes\mathcal{O}_C) = 0$ *and* $h^1(X, \mathcal{E}\otimes\mathcal{O}_C) = 1$.

Proof.

(1) Consider the standard exact sequence of the twisted cubic $C\subset X$:

$$0\to I_C\to\mathcal{O}_X\to\mathcal{O}_C\to 0.$$

[1] By abuse of notation, we still use i to denote the functor

Applying $\mathrm{Hom}(\mathcal{E},-)$, we get an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}, I_C) \to H^0(\mathcal{E}^\vee) \to H^0(\mathcal{E}^\vee|_C) \to \mathrm{Ext}^1(\mathcal{E}, I_C) \to 0.$$

Note that $H^0(\mathcal{E}^\vee) = k^7$ and $H^0(\mathcal{E}^\vee|_C) = k^5$, hence $\mathrm{hom}(\mathcal{E}, I_C) \geq 2$. Assume $\mathrm{hom}(\mathcal{E}, I_C) \geq 3$. Then C is contained in the zero locus of at least three independent sections $s \in H^0(\mathcal{E}^\vee)$, i.e., C is contained in linear sections of $\mathrm{Gr}(2,4)$. Note that $\mathrm{Gr}(2,4)$ is a quardic and X does not contain any planes. Therefore C is contained in a conic, which is impossible. Then $\mathrm{Hom}(\mathcal{E}, I_C) = k^2$ and $\mathrm{Ext}^i(\mathcal{E}, I_C) = 0$ for all $i \neq 0$.

(2) A similar argument as in [KPS18, Lemma B.3.3] applies here: $h^0(\mathcal{E} \otimes \mathcal{O}_C) \neq 0$ implies that the linear hull of twisted cubics, which is a $\mathbb{P}^3$, would be contained in X. But this is impossible. Thus $h^0(\mathcal{E} \otimes \mathcal{O}_C) = 0$. On the other hand $\chi(\mathcal{E} \otimes \mathcal{O}_C) = -1$, so $h^1(\mathcal{E} \otimes \mathcal{O}_C) = 1$.

□

Corollary 8.15. *There is a short exact sequence*

$$0 \to \mathcal{E} \to E \to I_C \to 0$$

where $E \cong \mathcal{U}_y$ for some $y \in C$, and E is a vector bundle.

Proof. Note that $\mathrm{Ext}^1(I_C, \mathcal{E}) \cong \mathrm{Ext}^2(\mathcal{E}^\vee, I_C) \cong H^1(X, \mathcal{E} \otimes \mathcal{O}_C) = k$. Then up to isomorphism, the extension of I_C and $\mathcal{E}$ (denoted by E) is unique, and we have a short exact sequence

$$0 \to \mathcal{E} \to E \to I_C \to 0.$$

Note that $\mathcal{E}$ and I_C are both μ-stable with $\mathrm{ch}_{\leq 1}(\mathcal{E}) = (2, -H)$ and $\mathrm{ch}_{\leq 1}(I_C) = (1, 0)$. Then E is also a μ-stable torsion-free sheaf, hence a Gieseker stable torsion-free sheaf with Chern character $3 - H + \frac{1}{2}P$. Thus $E \in M(3, -H, 9L, -2P)$ and $E \cong \mathcal{U}_y$ for some point $y \in C$ since $\mathcal{U}$ is the universal family. □

Lemma 8.16. *Let $E = \mathcal{U}_y$ in $M(3, -H, 9L, -2P)$. Then we have*

- $\mathrm{hom}(\mathcal{E}, E) = 3$ *and* $\mathrm{Ext}^k(\mathcal{E}, E) = 0$ *for* $k \neq 0$,
- $\mathrm{Ext}^k(\mathcal{E}^\vee, E) = \mathrm{Ext}^k(E^\vee, \mathcal{E}) = 0$ *for all* k.

Proof. By Corollary 8.15, we have an exact sequence $0 \to \mathcal{E} \to E \to I_C \to 0$. Applying $\mathrm{Hom}(\mathcal{E},-)$ to it we get an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, E) \to \mathrm{Hom}(\mathcal{E}, I_C) \to 0.$$

Since $\mathcal{E}$ is an exceptional bundle, $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = k$ and thus $\mathrm{Hom}(\mathcal{E}, E) = k^3$. A similar computation gives that $\mathrm{Ext}^j(\mathcal{E}, E) = 0$ for $j \neq 0$. On the other hand, $E \cong \mathcal{U}_y = i(\mathcal{O}_y)$. Thus $\mathrm{RHom}^\bullet(E^\vee, \mathcal{E}) \cong \mathrm{RHom}^\bullet(\mathcal{E}^\vee, i(\mathcal{O}_y)) = 0$ since $i(\mathcal{O}_y) \in \mathcal{A}_X$. □

We prove a proposition, which is an analogue of [BF13, Proposition 3.10].

Proposition 8.17. *Let X be a smooth prime Fano threefold of genus 10. Then the sheaf $\mathcal{V} = q_*(p^*(\mathcal{E}^\vee) \otimes \mathcal{U})$ is a rank 3 vector bundle over Γ_2, and we have a natural isomorphism*

$$\mathcal{V}^* \cong i^*(\mathcal{E}).$$

Proof. In view of Lemma 8.16, we have $\mathrm{R}^k q_*(p^*(\mathcal{E}^\vee \otimes \mathcal{U}) = 0$ for $k \geq 1$, and $\mathcal{V}$ is a locally free sheaf over Γ_2 of rank $h^0(X, \mathcal{E}^\vee \otimes \mathcal{U}_y) \cong \mathrm{Hom}(\mathcal{E}, E) = 3$. By Grothendieck duality, given a sheaf $\mathcal{P}$ on $X \times \Gamma_2$, we have the isomorphism

$$D(\mathrm{R}q_*(\mathcal{P})) \cong \mathrm{R}q_*(\mathcal{O}_X(-H) \otimes D(\mathcal{P}))[3].$$

Setting $\mathcal{P} = p^*(\mathcal{E}^\vee) \otimes \mathcal{U}$, we then have $\mathrm{R}q_*(\mathcal{O}_X(-H) \otimes p^*(\mathcal{E}) \otimes \mathcal{U}^\vee)[3] \cong i^*(\mathcal{E}) \cong D(\mathcal{V}) \cong \mathcal{V}^*$. □

Proposition 8.18. *There are natural isomorphisms $\mathcal{V}^* \cong i^*(\mathcal{E}) \cong i^!(\mathcal{Q}^\vee)[-1]$.*

Proof. Applying the functor i^* to the triangle in Lemma 3.4, we get

$$i^*(\mathcal{E}[1]) \to i^!(\mathcal{Q}^\vee) \to i^*(\mathcal{Q}^\vee).$$

Note that $i^*(\mathcal{Q}^\vee) = 0$ by semiorthogonality in the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$. Then by Proposition 8.17, the result follows. □

Now we prove several lemmas and propositions, which are essentially the same as the ones used in [Fae13].

Lemma 8.19. *Let $\mathcal{U}$ be the Fourier–Mukai kernel of $i : \mathrm{D}^b(\Gamma_2) \to \mathcal{A}_X$ with $\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{E}^\vee \rangle^\perp$ and let $\widetilde{\mathcal{U}}$ be the Fourier–Mukai kernel of $\Psi : \mathrm{D}^b(\Gamma_2) \to \mathcal{B}_X$ with $\mathcal{B}_X := {}^\perp\langle \mathcal{O}_X, \mathcal{E}^\vee \rangle$. Let $E_y := \mathcal{U}_y$ and $F_y := \widetilde{\mathcal{U}}_y$. Then we have the following standard short exact sequences*

$$0 \to E_y^\vee \to (\mathcal{E}^\vee)^3 \to E_y \otimes \mathcal{O}_X(H) \to 0$$

and

$$0 \to E_y \to \mathcal{O}_X^{\oplus 6} \to E_y^\vee \to 0.$$

Proof. By [Fae13], $E_y \cong F_y \otimes \mathcal{O}_X(-H)$, so the result follows from [Fae13, Theorem II.1] □

Corollary 8.20. *We have the isomorphisms $\mathbf{L}_{\mathcal{E}^\vee}(E \otimes \mathcal{O}_X(H)) \cong E^\vee[-1]$, $\mathbf{L}_{\mathcal{O}_X} E^\vee \cong E[-1]$ and $\mathbf{L}_{\mathcal{E}} E \cong F^\vee[-1]$.*

Corollary 8.21. *The rank 3 vector bundle $\mathcal{V}^*$ over the curve Γ_2 has the following property: $\iota^*\mathcal{V} \cong \mathcal{V}^*$, where $\iota\colon C \to C$ is the hyperelliptic involution.*

Proof. By Lemma 3.4, there is a triangle $\mathcal{E}[1] \to ii^!(\mathcal{Q}^\vee) \to \mathcal{Q}^\vee$. Applying the anti-involution τ, we get $\tau(\mathcal{Q}^\vee) \to \tau(ii^!(\mathcal{Q}^\vee)) \to \tau\mathcal{E}[1]$, which is $\mathcal{E}[1] \to \tau(ii^!(\mathcal{Q}^\vee)) \to \mathcal{Q}^\vee$. Thus we have $\tau(ii^!(\mathcal{Q}^\vee) \cong ii^!(\mathcal{Q}^\vee)$. Then the result follows from Proposition 8.22. □

Proposition 8.22. *The functor $\tau : E \mapsto \mathbf{L}_{\mathcal{O}_X} \mathrm{R}\mathcal{H}om(E, \mathcal{O}_X)$ is an anti-autoequivalence of the subcategory $\mathcal{O}_X^\perp$, and $\tau^2(E) \cong E$. Moreover, τ fixes $i(\mathrm{D}^b(\Gamma_2))$ and $\tau(i(E)) \cong i(\mathrm{R}\mathcal{H}om(\iota^*(E), \mathcal{O}_{\Gamma_2}))$, where $i : \mathrm{D}^b(\Gamma_2) \hookrightarrow \mathrm{D}^b(X)$ is the embedding functor.*

Proof. Let $E \in \mathcal{O}_X^\perp$. Then $\mathrm{RHom}^\bullet(\mathcal{O}_X, E) = 0$, so $\mathrm{RHom}^\bullet(E^\vee, \mathcal{O}_X) \cong \mathrm{Hom}^\bullet(\mathcal{O}_X, E) = 0$. Therefore $E^\vee \in {}^\perp \mathcal{O}_X$. Then $\mathbf{L}_{\mathcal{O}_X} E^\vee \in \mathcal{O}_X^\perp$. To see that τ^2 is the identity, note that there is an exact triangle

$$\mathrm{RHom}^\bullet(\mathcal{O}_X, E^\vee) \otimes \mathcal{O}_X \to E^\vee \to \mathbf{L}_{\mathcal{O}_X} E^\vee.$$

Dualizing it, we get

$$(\mathbf{L}_{\mathcal{O}_X} E^\vee)^\vee \to E \to \mathrm{RHom}^\bullet(\mathcal{O}_X, E^\vee)^* \otimes \mathcal{O}_X\,.$$

Applying $\mathbf{L}_{\mathcal{O}_X}$ to the triangle, we get $\tau^2(E) \to \mathbf{L}_{\mathcal{O}_X} E \to 0$, since $\mathbf{L}_{\mathcal{O}_X} \mathcal{O}_X = 0$. Hence $\tau^2(E) \cong E$, since $E \in \mathcal{O}_X^\perp$. Next we show that τ fixes $i(\mathrm{D}^b(\Gamma_2))$. It suffices to check that $\tau(i(k(x)) \cong \tau(\mathcal{U}_x) \in i(\mathrm{D}^b(\Gamma_2))$. By Corollary 8.20, $\tau(\mathcal{U}_x) = \mathbf{L}_{\mathcal{O}_X} E^\vee \cong E[-1] \cong \mathcal{U}_x[-1] \in i(\mathrm{D}^b(\Gamma_2))$. Further note that $i \cong \Psi \circ (\mathcal{O}_X(-H) \otimes -)$. Then

$$\begin{aligned}
\tau(i(E)) &\cong \tau(\Psi(E) \otimes \mathcal{O}_X(-H)) \\
&\cong (\mathcal{O}_X(-H) \otimes -) \circ \tau' \circ \Psi(E) \\
&\cong \Psi(\mathrm{R}\mathcal{H}om(\iota^*(E), \mathcal{O}_{\Gamma_2})) \otimes \mathcal{O}_X(-H) \\
&\cong i(\mathrm{R}\mathcal{H}om(\iota^*(E), \mathcal{O}_{\Gamma_2}))
\end{aligned}$$

by [Fae13, Lemma II.7], where τ' is called the first auto-equivalence of $\mathcal{O}_X(1)^\perp$, defined in [Fae13, Section III.2]. □

Next we state a proposition proved in [Fae13, Theorem 2.10].

Proposition 8.23. *There is a choice of $\xi \in \mathrm{Pic}^2(\Gamma_2)$ such that $\mathcal{V}^*$ is a rank 3 vector bundle over Γ_2 with trivial determinant, and such that $\theta(\mathcal{V})$ lies in the Coble–Dolgachev sextic, where $\theta\colon \mathcal{M}_{\Gamma_2}(3, \mathcal{O}_{\Gamma_2}) \to |3\Theta|$ is the map defined in [Fae13, Section II.1.4].*

Theorem 8.24. *Let X be an index one degree 18 prime Fano threefold. Then the set of all index one degree 18 prime Fano threefolds X' such that $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ is parametrised by the Coble–Dolgachev sextic.*

Proof. By Proposition 8.23, the degree 18 index one prime Fano threefolds X' with $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ are parametrised by the family of *gluing objects* $i^*(\Xi(i^!_{\mathcal{K}u(X)}\mathcal{E})) \in \mathrm{D}^b(\Gamma_2)$, which are self dual rank three vector bundles of trivial determinant by Proposition 8.17, Corollary 8.21 and Theorem 8.23. On the other hand, any pair $(\Gamma_2, \mathcal{V})$ of a genus two curve Γ_2 and a stable vector bundle $\mathcal{V}$ of rank 3 that lies in the Coble–Dolgachev sextic uniquely reconstructs an $X' := X_{18}$ such that $\mathcal{A}_{X'} \simeq \mathrm{D}^b(\Gamma_2) \simeq \mathcal{A}_X$ by [FV22, Section 7]. Therefore the desired result follows. □

8.4. **Genus 12: degree 22 prime Fano threefolds.** In this section, we show that the *gluing objects* $i^!\mathcal{E} \in \mathcal{K}u(X)$ cut out the Hilbert scheme $\Sigma(X)$ of lines from a moduli space $M_X(2,-1,8) \cong \mathbb{P}^2$ of semistable sheaves of rank two as a *Brill–Noether locus*.

Lemma 8.25. *The Brigeland moduli space of σ-stable objects of class $u = 2 - H + 3L + \frac{1}{3}P$ is the moduli space of semistable sheaves $M(2,-1,8)$ for every Serre-invariant stability condition σ on $\mathcal{K}u(X)$. They are both isomorphic to the Fano surface $\mathcal{C}(X)$ of conics on X.*

Proof. Let $E \in M(2,-1,8)$. We show that $\mathrm{RHom}^\bullet(\mathcal{O}_X, E) = 0$ and $\mathrm{RHom}^\bullet(\mathcal{E}, E) = 0$.

(a) If E is locally free, then the first vanishing follows from [BF14, Proposition 3.5]. Note that there is a resolution of E given by the short exact sequence
$$0 \to E \to \mathcal{E}_5 \to \mathcal{E}_3 \to 0,$$
where $\mathcal{E}_5$ and $\mathcal{E}_3$ are rank 5 and rank 3 vector bundles in a full strong exceptional collection of vector bundles $\langle \mathcal{E}, \mathcal{E}_5, \mathcal{E}_3, \mathcal{O}_X\rangle$ on X_{22} (this is actually the foundation of a simple helix in the sense of [Pol11]). It is known that $\mathrm{Hom}(\mathcal{E}, E) = 0$ since $\mathcal{E}$ and $E \in M(2,-1,8)$ are both Gieseker semistable and $p_{\mathcal{E}}(t) > p_E(t)$, where $p(t)$ is the reduced Hilbert polynomial. It is also clear that $\mathrm{Ext}^3(\mathcal{E}, E) \cong \mathrm{Hom}(E, \mathcal{E} \otimes \mathcal{O}_X(-H)) = 0$ by stability. A computation shows that the Euler character $\chi(\mathcal{E}, E) = 0$. Applying $\mathrm{Hom}(\mathcal{E}, -)$ to the short exact sequence above, we get a long exact sequence
$$\cdots \to \mathrm{Ext}^1(\mathcal{E}, E) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}_5) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}_3) \to \mathrm{Ext}^2(\mathcal{E}, E) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}_5) \to \cdots.$$
Note that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}_3) = \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}_5) = 0$. Thus $\mathrm{Ext}^2(\mathcal{E}, E) = 0$. Then $\mathrm{Ext}^1(\mathcal{E}, E) = 0$ since $\chi(\mathcal{E}, E) = 0$. This implies that $E \in \mathcal{K}u(X)$. By [Fae14, Proposition 4.2], $M(2,-1,8) \cong \mathcal{C}(X) \cong \mathbb{P}^2$. In particular, $M(2,-1,8)$ is smooth and hence $\mathrm{Ext}^1(E,E) = 2$. Hence by [LZ21, Lemma 3.14], F is σ-stable in $\mathcal{K}u(X)$ with respect to every Serre-invariant stability condition σ.

(b) If E is not locally free, then by [BF14, Proposition 3.5] it fits into the short exact sequence
$$0 \to E \to \mathcal{E} \to \mathcal{O}_L(-1) \to 0.$$
Applying $\mathrm{Hom}(\mathcal{O}_X, -)$ and $\mathrm{Hom}(\mathcal{E}, -)$ to it, we get $\mathrm{RHom}^\bullet(\mathcal{E}, E) = \mathrm{RHom}^\bullet(\mathcal{O}_X, E) = 0$. Hence $E \in \mathcal{K}u(X)$. The same argument as in (1) shows that E is σ-stable.

Note that $\mathrm{ch}(E) = 2 - H + 3L + \frac{1}{3}P$ and $\chi(\mathrm{ch}(E \otimes \mathcal{O}_L)) = \chi((2 - H + 3L + \frac{1}{3}P)(L + \frac{1}{2}P)) = \chi_0(2L) = 1$. This implies that the moduli space $M(2,-1,8)$ admits a universal family by [HL10, Theorem 4.6.5]. Then the projection functor $i^* := \mathbf{L}_{\mathcal{E}}\,\mathbf{L}_{\mathcal{O}_X}$ induces a morphism $p : M(2,-1,8) \to \mathcal{M}_\sigma(\mathcal{K}u(X), u)$. As in [LZ21, Section 4], it is easy to check that p is an injective, étale and proper morphism, hence a closed and open embedding. On the other hand, because $\chi(u,u) = -1$, by [LZ21, Corollary 4.2] the following moduli spaces are isomorphic: $\mathcal{M}_\sigma(\mathcal{K}u(X), u) \cong \mathcal{M}_{\sigma'}(\mathcal{K}u(Y), v) \cong \mathbb{P}^2$, where Y is del Pezzo threefold of degree 5, $v = 1 - L \in \mathcal{N}(\mathcal{K}u(Y))$ and σ' is any Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then $\mathcal{M}_\sigma(\mathcal{K}u(X), u)$ is irreducible and $\mathcal{M}_\sigma(\mathcal{K}u(X), u) \cong M(2,-1,8)$ as required. □

Proposition 8.26. *The projection $i^*(\mathcal{O}_L(-1))$ is in the Brill–Noether locus*

$$\mathcal{BN} := \{F \in \mathcal{M}_\sigma(\mathcal{K}u(X), -u) \mid \mathrm{Ext}^1(F, i^!\mathcal{E}) = k\}.$$

Moreover, the projection functor i^* induces an embedding of the Hilbert scheme of lines $\Sigma(X)$ on X into $\mathcal{BN}$.

Proof. Since $\mathrm{RHom}^\bullet(\mathcal{O}_X, \mathcal{O}_L(-1)) = 0$, we have $i^*(\mathcal{O}_L(-1)) \cong \mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1)$. It is given by the exact triangle

$$\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{O}_L(-1)) \otimes \mathcal{E} \to \mathcal{O}_L(-1) \to \mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1).$$

Thus $\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1) \cong E[1]$, and by [BF14, Proposition 3.5] we have $E \in M(2, -1, 8)$. Thus $\mathrm{ch}(\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1)) = -u$. Applying $\mathrm{Hom}(-, \mathcal{E})$ to this triangle, we get an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1), \mathcal{E}) \to \mathrm{Hom}(\mathcal{O}_L(-1), \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \\ &\to \mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1), \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{O}_L(-1), \mathcal{E}) \to \cdots. \end{aligned}$$

Then we have $\mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1), \mathcal{E}) \cong \mathrm{Ext}^1(\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1), i^!\mathcal{E}) = k$. It is clear that $\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1) \cong E[1]$ is σ-stable by $E \in M(2, -1, 8)$ and Lemma 8.25. It is also clear that $\Sigma(X)$ admits a universal family. Then by a similar argument as in Lemma 8.25, there is a morphism $\Sigma(X) \xrightarrow{p} \mathcal{M}_\sigma(\mathcal{K}u(X), -u)$. It is easy to see that p is injective since $i^*(\mathcal{O}_L(-1))$ is uniquely determined by the line $L \subset X$ (cf. [BF14, Proposition 3.5]). Applying $\mathrm{Hom}(\mathcal{O}_L(-1), -)$ to the exact triangle defining $\mathbf{L}_{\mathcal{E}}\,\mathcal{O}_L(-1)$, we get an exact sequence

$$0 \to \mathrm{Ext}^1(\mathcal{O}_L(-1), \mathcal{O}_L(-1)) \xrightarrow{dp} \mathrm{Ext}^1(i^*(\mathcal{O}_L(-1)), i^*(\mathcal{O}_L(-1))) \to \cdots.$$

The map dp is exactly the tangent map of $p : \Sigma(X) \to \mathcal{M}_\sigma(\mathcal{K}u(X), -u)$ at the point $[L]$, and it is injective. Since both moduli spaces are proper, p is a closed embedding. □

Theorem 8.27. *The Hilbert scheme of lines of a degree 22 prime Fano threefold X of index one can be exhibited as the Brill–Noether locus of the Bridgeland moduli space of stable objects in $\mathcal{K}u(X)$ with respect to $i^!\mathcal{E} \in \mathcal{K}u(X)$, where $i : \mathcal{K}u(X) \hookrightarrow \mathrm{D}^b(X)$ is the inclusion. In other words, we have $\Sigma(X) \cong \mathcal{BN}$.*

Proof. It remains to show that if $F \in \mathcal{BN}$, then $F \cong i^*(\mathcal{O}_L(-1))$. Assume that $F \in \mathcal{BN}$. Then $F \in \mathcal{M}_\sigma(\mathcal{K}u(X), -u)$. By Lemma 8.25, $F \cong E[1]$ for some $E \in M(2, -1, 8)$. By [Fae14, Proposition 4.2], E is either locally free or non-locally free. If E is locally free, then $\mathrm{Hom}(E, \mathcal{E}) = 0$. Indeed, there is a resolution of E given by the short exact sequence $0 \to E \to \mathcal{E}_5 \to \mathcal{E}_3 \to 0$. Now, applying $\mathrm{Hom}(-, \mathcal{E})$ we get a long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}_3, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}_5, \mathcal{E}) \to \mathrm{Hom}(E, \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E}_3, \mathcal{E}) \to \cdots.$$

But $\mathrm{Hom}(\mathcal{E}_5, \mathcal{E}) = \mathrm{Ext}^1(\mathcal{E}_3, \mathcal{E}) = 0$ since $\langle \mathcal{E}, \mathcal{E}_5, \mathcal{E}_3, \mathcal{O}_X \rangle$ is an exceptional collection. Then $\mathrm{Ext}^1(F, i^!\mathcal{E}) \cong \mathrm{Hom}(E, i^!\mathcal{E}) \cong \mathrm{Hom}_{D^b(X)}(E, \mathcal{E}) = 0$, which is a contradiction. Therefore $F \cong E[1]$ where E fits into the short exact sequence

$$0 \to E \to \mathcal{E} \to \mathcal{O}_L(-1) \to 0,$$

which implies $F \cong i^*(\mathcal{O}_L(-1))$. □

Theorem 8.28. *Let X be a general index one degree 22 prime Fano threefold. Then the set of index one degree 22 prime Fano threefolds X' such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is the moduli space of smooth plane quartic curves.*

Proof. All degree 22 index one prime Fano threefolds X' with $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ are parametrised by the family of *gluing objects* $i^!\mathcal{E} \in \mathcal{K}u(X)$, which gives a Hilbert scheme of lines on X as a plane quartic (by Theorem 8.27). On the other hand, every plane quartic given as a Hilbert scheme of lines on a general X_{22} and a theta characteristic on it determines this Fano threefold up to isomorphism by [Kuz97] and [Muk92]. Then the desired result follows. □

8.5. **Odd genus.** In this section, we include results for the fiber of period map for odd genus index one prime Fano threefolds. For details, we refer to the works [BF13] and [BF14].

8.5.1. *Genus 7: degree 12 prime Fano threefolds.* By Mukai's theorem [Muk01], every genus 7 prime Fano threefold X is reconstructed as *Brill–Noether locus* in a moduli space of rank two vector bundles over a genus 7 curve Γ_7, i.e. $X \cong \{F \in \mathcal{M}_{\Gamma_7}(2, K_{\Gamma_7}) \mid \mathrm{Hom}(\mathcal{O}_{\Gamma_7}, F) = k^5\}$. Thus X is uniquely determined by Γ_7, so every genus 7 prime Fano threefold X' with $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$ is isomorphic to X since $\mathcal{K}u(X') \simeq \mathrm{D}^b(\Gamma'_7)$ and $\mathrm{D}^b(\Gamma'_7) \simeq \mathrm{D}^b(\Gamma_7)$ implies that $\Gamma'_7 \cong \Gamma_7$. Note that the *gluing object* is given by $\mathcal{O}_{\Gamma_7}$ by [BF13, Lemma 2.9] or [Kuz05, Lemma 5.6], so whenever $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$, $\Gamma'_7 \cong \Gamma_7$, such a *gluing object* is fixed.

8.5.2. *Genus 9: degree 16 prime Fano threefolds.* By [BF14, Proposition 3.10] the family of *gluing objects* $i^!\mathcal{E} \in \mathcal{K}u(X)$ is parametrised by a moduli space of rank 2 vector bundles $\mathcal{W}$ over a genus 3 curve Γ_3 such that any section of ruled surface $\mathbb{P}_{\Gamma_3}(\mathcal{W})$ has self intersection at least 3. On the other hand, by Mukai's theorem [Muk01], the pair $(\Gamma_3, \mathcal{W})$ determines a degree 16 prime Fano threefold X' up to isomorphism such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$.

Appendix A. Computations

A.1. **Wall-crossing computations for $g = 7$ and 9.** In this subsection, we compute potential walls for $\mathcal{E}_g$ and $\mathcal{E}_g(-H)[1]$. The lemmas here are used in the proof of Proposition 4.9.

Lemma A.1 ([Li18, Proposition 3.2])**.** *Let $X := X_{12}$ and $F \in \mathrm{D}^b(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some β and $\alpha > 0$.*

(a) If $|\mu(F)| \leq \frac{1}{2\sqrt{2}}$, then $\frac{H\mathrm{ch}_2(F)}{H^3\mathrm{ch}_0(F)} \leq 0$,
(b) If $\frac{1}{2\sqrt{2}} \leq |\mu(F)| \leq 1 - \frac{1}{2\sqrt{2}}$, then $\frac{H\mathrm{ch}_2(F)}{H^3\mathrm{ch}_0(F)} \leq \frac{1}{2}|\mu(F)|^2 - \frac{1}{16}$,
(c) If $1 - \frac{1}{2\sqrt{2}} \leq |\mu(F)| \leq 1 + \frac{1}{2\sqrt{2}}$, then $\frac{H\mathrm{ch}_2(F)}{H^3\mathrm{ch}_0(F)} \leq |\mu(F)| - \frac{1}{2}$,
(d) If $1 + \frac{1}{2\sqrt{2}} \leq |\mu(F)| \leq 2 - \frac{1}{2\sqrt{2}}$, then $\frac{H\mathrm{ch}_2(F)}{H^3\mathrm{ch}_0(F)} \leq \frac{1}{2}|\mu(F)|^2 - \frac{1}{16}$.

Lemma A.2. *Let $X := X_{12}$ and $\beta = -\frac{5}{6}$ or $-\frac{71}{84}$. Let $E \in \mathsf{Coh}^\beta(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some $\alpha > 0$ with $\mathrm{ch}_{\leq 2}(E) = \mathrm{ch}_{\leq 2}(\mathcal{E}_7)$. Then E is $\sigma_{\alpha,\beta}$-semistable for all $\alpha > 0$.*

Proof. We only do computations for $\beta = -\frac{5}{6}$ here. Since $-\frac{71}{84}$ is very close to $-\frac{70}{84} = -\frac{5}{6}$, the argument is almost the same.

We are going to show that there is no wall for E when $\beta = -\frac{5}{6}$. As in Section 4.3, a wall would be given by a short exact sequence

$$0 \to A \to E \to B \to 0$$

in $\mathsf{Coh}^{-\frac{5}{6}}(X)$ such that following conditions hold:

(a) $\mu_{\alpha,-\frac{5}{6}}(A) = \mu_{\alpha,-\frac{5}{6}}(E) = \mu_{\alpha,-\frac{5}{6}}(B)$;
(b) $\Delta(A) \geq 0$ and $\Delta(B) \geq 0$;
(c) $\Delta(A) \leq \Delta(E)$ and $\Delta(B) \leq \Delta(E)$;
(d) $\mathrm{ch}_1^{-\frac{5}{6}}(A) \geq 0$ and $\mathrm{ch}_1^{-\frac{5}{6}}(B) = \mathrm{ch}_1^{-\frac{5}{6}}(E) - \mathrm{ch}_1^{-\frac{5}{6}}(A) \geq 0$.

Note that $\mathrm{ch}_1^{-\frac{5}{6}}(E) > 0$, thus the inequalities in (d) are actually strict. We can assume that $\mathrm{ch}_{\leq 2}(A) = (a, bH, cL)$ and so $\mathrm{ch}_{\leq 2}(B) = (5-a, (-2-b)H, -cL)$ for some $a, b, c \in \mathbb{Z}$. If we divide the discriminant $\Delta(-)$ by $(H^3)^2$, the conditions above can be rewritten as

(a) $\frac{-36a\alpha^2+25a+60b+6c}{12(5a+6b)} = \frac{5-180\alpha^2}{156}$;
(b) $b^2 - \frac{ac}{6} \geq 0$ and $(-2-b)^2 - \frac{(5-a)(-c)}{6} \geq 0$;
(c) $b^2 - \frac{ac}{6} \leq 4$ and $(-2-b)^2 - \frac{(5-a)(-c)}{6} \leq 4$;
(d) $\frac{13}{6} > b + \frac{5}{6}a > 0$.

Since $\alpha^2 > 0$ and $6b + 5a > 0$ by (d), (a) implies

$$(50a + 125b + 13c)(2a + 5b) < 0. \tag{14}$$

Now (a) and (d) imply the following four cases:

(i) $a > 5$, $-\frac{5}{6}a < b < \frac{13-5a}{6}, c > -\frac{25}{13}(2a+5b)$;
(ii) $0 < a \leq 5$, $-\frac{5}{6}a < b < -\frac{2}{5}a, c > -\frac{25}{13}(2a+5b)$;
(iii) $0 < a \leq 5$, $-\frac{2}{5}a < b < \frac{13-5a}{6}, c < -\frac{25}{13}(2a+5b)$;
(iv) $a \leq 0$, $-\frac{5}{6}a < b < \frac{13-5a}{6}, c < -\frac{25}{13}(2a+5b)$.

Combined with (b) and (d), each case (i) to (iv) gives the following:

(i) $5 < a < \frac{169}{5}$, $-\frac{10a}{13} < b \leq -\frac{2}{5}(a+\sqrt{(a-5)a})$, $-\frac{25}{13}(2a+5b) < c \leq \frac{6b^2}{a}$; or $5 < a < \frac{169}{5}$, $-\frac{2}{5}(a+\sqrt{(a-5)a}) < b < \frac{13-5a}{6}$, $-\frac{25}{13}(2a+5b) < c \leq \frac{6(b+2)^2}{a-5}$
(ii) $0 < a \leq 5$, $-\frac{10a}{13} < b < -\frac{2a}{5}$, $-\frac{25}{13}(2a+5b) < c \leq \frac{6b^2}{a}$;
(iii) $0 < a < 5$, $-\frac{2a}{5} < b < -\frac{2(5a-12)}{13}$, $\frac{6(b+2)^2}{a-5} \leq c < -\frac{25}{13}(2a+5b)$;
(iv) $-\frac{144}{5} < a \leq 0$, $-\frac{5a}{6} < b \leq \frac{2}{5}(-a+\sqrt{(a-5)a})$, $\frac{6b^2}{a} \leq c < -\frac{25}{13}(2a+5b)$; or $-\frac{144}{5} < a \leq 0$, $\frac{2}{5}(-a+\sqrt{(a-5)a}) < b < -\frac{2(5a-12)}{13}$, $\frac{6(b+2)^2}{a-5} \leq c < -\frac{25}{13}(2a+5b)$.

Now by a careful computation for each case (i) to (iv), we obtain all possible truncated Chern characters of A and B:

(1) $(-11, 10H, -54L)$ and $(16, -12H, 54L)$;
(2) $(-5, 5H, -29L)$ and $(10, -7H, -29L)$;
(3) $(-4, 4H, -24L)$ and $(9, -6H, 24L)$;
(4) $(-3, 3H, -18L)$ and $(8, -5H, 18L)$;
(5) $(-3, 4H, -27L)$ and $(8, -6H, 27L)$;
(6) $(-2, 2H, -12L)$ and $(7, -4H, 12L)$;
(7) $(-1, H, -6L)$ and $(6, -3H, 6L)$;
(8) $(-1, 2H, -16L)$ and $(6, -4H, 16L)$;
(9) $(0, H, -10L)$ and $(5, -3H, 10L)$;
(10) $(1, 0, -6L)$ and $(4, -2H, 6L)$;
(11) $(1, 0, -5L)$ and $(4, -2H, 5L)$;
(12) $(1, 0, -4L)$ and $(4, -2H, 4L)$;
(13) $(2, -H, 2L)$ and $(3, -H, -2L)$;
(14) $(2, -H, 3L)$ and $(3, -H, -3L)$;
(15) $(2, 0, -8L)$ and $(3, -2H, 8L)$.

Since A and B are both $\sigma_{\alpha,-\frac{5}{6}}$-semistable for some $\alpha > 0$, the cases (13) and (14) are ruled out by using Lemma A.1 on the first character. The other cases are ruled out by using Lemma A.1 on the second character. This implies that there are no walls when $\beta = -\frac{5}{6}$ for E, and E is $\sigma_{\alpha,-\frac{5}{6}}$-semistable for every $\alpha > 0$.

When $\beta = -\frac{71}{84}$, the computation argument is similar, and the solutions for $\mathrm{ch}_{\leq 2}(A)$ and $\mathrm{ch}_{\leq 2}(B)$ are the same as those when $\beta = -\frac{5}{6}$. Thus from the same argument using Lemma A.1, there are no walls when $\beta = -\frac{71}{84}$ for E, and E is $\sigma_{\alpha,-\frac{71}{84}}$-semistable for every $\alpha > 0$. □

Lemma A.3. *Let $X := X_{12}$ and $\beta = -\frac{5}{6}$ or $-\frac{71}{84}$. Let $E \in \mathsf{Coh}^{\beta}(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some $\alpha > 0$ with $\mathrm{ch}_{\leq 2}(E) = \mathrm{ch}_{\leq 2}(\mathcal{E}_7(-H)[1])$. Then E is $\sigma_{\alpha,\beta}$-semistable for all $\alpha > 0$.*

Proof. We assume that there is a wall when $\beta = -\frac{5}{6}$ or $-\frac{71}{84}$ for E, and that it is given by $A \to E \to B$. Then a similar computation as in Lemma A.2 shows that all possible truncated Chern characters of A and B are:

(1) $(-6, 7H, -49L)$ and $(1, 0, -5L)$;
(2) $(-5, 6H, -43L)$ and $(0, H, -11L)$;
(3) $(-4, 5H, -37L)$ and $(-1, 2H, -17L)$;
(4) $(-3, 4H, -32L)$ and $(-2, 3H, -22L)$;
(5) $(-3, 4H, -31L)$ and $(-2, 3H, -23L)$.

Now using Lemma A.1 on the first character in each case, all of the cases (1) to (5) are ruled out. This means that there are no walls for E when $\beta = -\frac{5}{6}$ or $-\frac{71}{84}$, and hence E is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$. □

Lemma A.4 ([Li18, Proposition 3.2])**.** *Let $X := X_{16}$ and $F \in \mathrm{D}^b(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some β and $\alpha > 0$. If $\mu(F) = -\frac{1}{2}$, then $\frac{H\mathrm{ch}_2(F)}{H^3\mathrm{ch}_0(F)} \leq \frac{5}{64}$.*

Lemma A.5. *Let $X := X_{16}$ and $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$. Let $E \in \mathsf{Coh}^\beta(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some $\alpha > 0$ with $\mathrm{ch}_{\leq 2}(E) = \mathrm{ch}_{\leq 2}(\mathcal{E}_9)$. Then E is $\sigma_{\alpha,\beta}$-semistable for all $\alpha > 0$.*

Proof. We assume that there is a wall when $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$ for E, and that it is given by $A \to E \to B$. Then a similar computation as in Lemma A.2 shows that all of the possible truncated Chern characters of A and B are:

(1) $(-1, H, -6L)$ and $(4, -2H, 6L)$;
(2) $(1, 0, -3L)$ and $(2, -H, 3L)$.

Now using Lemma A.4 on the second character in each case, both cases (1) and (2) are ruled out. This means that there are no walls for E on $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$, and hence E is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$. □

Lemma A.6. *Let $X := X_{16}$ and $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$. Let $E \in \mathsf{Coh}^\beta(X)$ be a $\sigma_{\alpha,\beta}$-semistable object for some $\alpha > 0$ with $\mathrm{ch}_{\leq 2}(E) = \mathrm{ch}_{\leq 2}(\mathcal{E}_9(-H)[1])$. Then E is $\sigma_{\alpha,\beta}$-semistable for all $\alpha > 0$.*

Proof. We assume that there is a wall when $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$ for E, and that it is given by $A \to E \to B$. Then a similar computation as in Lemma A.2 shows that there are no such truncated Chern characters of A and B. This means there are no walls for E when $\beta = -\frac{3}{4}$ or $-\frac{31}{40}$, and hence E is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$. □

A.2. **Ext groups of gluing objects.** Let X be an index one prime Fano threefold of even genus $10 \geq g \geq 6$. In this subsection, we compute the self-Ext groups of the *gluing object* $i^!\mathcal{E} \in \mathcal{K}u(X)$.

Proposition A.7. *Let X be prime Fano threefold of index one and even genus $6 \leq g \leq 10$. Then*

(a) $g = 6 : \mathrm{ext}^1(i^!\mathcal{E}, i^!\mathcal{E}) = 2$ *or* 3,
(b) $g = 8 : \mathrm{ext}^1(i^!\mathcal{E}, i^!\mathcal{E}) = 5$,
(c) $g = 10 : \mathrm{ext}^1(i^!\mathcal{E}, i^!\mathcal{E}) = 10$.

Proof. If $g = 6$, the result follows from [JLLZ21, Lemma 5.7]. If $g = 8$, we consider the triangle

$$\mathcal{Q}^\vee[-1] \to \mathcal{E}[1] \to i^!\mathcal{E}[-1]$$

and apply the spectral sequence in [Pir20, Lemma 2.27]. We have:

$$E_1^{p,q} = \begin{cases} \mathrm{Ext}^q(\mathcal{E}[1], \mathcal{Q}^\vee[-1]) = \mathrm{Ext}^{q-2}(\mathcal{E}, \mathcal{Q}^\vee), & p = -1 \\ \mathrm{Ext}^q(\mathcal{E}, \mathcal{E}) \oplus \mathrm{Ext}^q(\mathcal{Q}^\vee, \mathcal{Q}^\vee), & p = 0 \\ \mathrm{Ext}^q(\mathcal{Q}^\vee[-1], \mathcal{E}[1]) = \mathrm{Ext}^{q+2}(\mathcal{Q}^\vee, \mathcal{E}), & p = 1 \\ 0, & p \geq 2, p \leq -2 \end{cases}$$

By convergence of the spectral sequence and Lemma A.8, we have $\mathrm{ext}^1(i^!\mathcal{E}, i^!\mathcal{E}) = 5$ and $\mathrm{ext}^2(i^!\mathcal{E}, i^!\mathcal{E}) = \mathrm{ext}^3(i^!\mathcal{E}, i^!\mathcal{E}) = 0$. Since $i^!\mathcal{E}$ is a (-4)-class, we have $\hom(i^!\mathcal{E}, i^!\mathcal{E}) = 1$. If $g = 10$, $\mathrm{ch}(i^!\mathcal{E}) = -3(1 - 2L)$ and $\chi(i^!\mathcal{E}, i^!\mathcal{E}) = -9$, so $\mathrm{ext}^1(i^!\mathcal{E}, i^!\mathcal{E}) = \hom(i^!\mathcal{E}, i^!\mathcal{E}) - \chi(i^!\mathcal{E}, i^!\mathcal{E}) = 10$ since $\hom(i^!\mathcal{E}, i^!\mathcal{E}) = 1$ by a similar computation as in Lemma A.8. □

Lemma A.8. *Let X be an index one prime Fano threefold of degree 14 and let $\mathcal{E}$ and $\mathcal{Q}$ be the tautological sub and quotient bundles, respectively. Then we have*

(a) $\hom(\mathcal{E}, \mathcal{Q}^\vee) = 5$ *and* $\mathrm{ext}^i(\mathcal{E}, \mathcal{Q}^\vee) = 0$ *for* $i \geq 1$;
(b) $\hom(\mathcal{E}, \mathcal{E}) = \hom(\mathcal{Q}^\vee, \mathcal{Q}^\vee) = 1$ *and* $\mathrm{ext}^i(\mathcal{E}, \mathcal{E}) = \mathrm{ext}^i(\mathcal{Q}^\vee, \mathcal{Q}^\vee) = 0$ *for* $i \geq 1$;
(c) $\mathrm{ext}^2(\mathcal{Q}^\vee, \mathcal{E}) = 1$ *and* $\hom(\mathcal{Q}^\vee, \mathcal{E}) = \mathrm{ext}^1(\mathcal{Q}^\vee, \mathcal{E}) = \mathrm{ext}^3(\mathcal{Q}^\vee, \mathcal{E}) = 0$.

Proof.

(a) Applying $\mathrm{Hom}(\mathcal{E}, -)$ to the standard exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}_X^{\oplus 6} \to \mathcal{E}^\vee \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}, \mathcal{Q}^\vee) \to H^0(\mathcal{E}^\vee)^{\oplus 6} \to H^0(X, \mathcal{E}^\vee \otimes \mathcal{E}^\vee) \to H^1(X, \mathcal{E}^\vee \otimes \mathcal{Q}^\vee) \to 0.$$

We claim that $H^1(X, \mathcal{E}^\vee \otimes \mathcal{Q}^\vee) = 0$ and $H^k(X, \mathcal{E}^\vee \otimes \mathcal{E}^\vee) = 0$ for $k \geq 1$. Since $\chi(\mathcal{E}, \mathcal{E}^\vee) = 31$ and $\mathrm{h}^0(\mathcal{E}^\vee) = 6$, we get $\mathrm{Hom}(\mathcal{E}, \mathcal{Q}^\vee) = \mathbb{C}^5$. The claim on vanishing of $H^1(X, \mathcal{E}^\vee \otimes \mathcal{Q})$ and $H^{k \geq 1}(X, \mathcal{E}^\vee \otimes \mathcal{E}^\vee) = 0$ follows from applying the Borel–Bott–Weil theorem to the cohomologies of the vector bundles $\mathcal{E}^\vee \otimes \mathcal{E}^\vee$ and $\mathcal{E}^\vee \otimes \mathcal{Q}^\vee$ over $\mathrm{Gr}(2, 6)$, via the Koszul complex associated to $X \hookrightarrow \mathrm{Gr}(2, 6)$.

(b) This follows from the exceptionality of $\mathcal{E}$ and $\mathcal{Q}$.

(c) Note that $\mathrm{ext}^i(\mathcal{Q}^\vee, \mathcal{E}) = \mathrm{ext}^{3-i}(\mathcal{E}, \mathcal{Q}^\vee(-1)) = \mathrm{ext}^{3-i}(\mathcal{E}(1), \mathcal{Q}^\vee) = \mathrm{ext}^{3-i}(\mathcal{E}^\vee, \mathcal{Q}^\vee) = \mathrm{ext}^{3-i}(\mathcal{Q}, \mathcal{E})$. Applying $\mathrm{Hom}(\mathcal{Q}, -)$ to the tautological short exact sequence, the statement follows from (2).

□

A.3. **Inequalities in Theorem 5.9.** In this subsection, we compute inequalities used in the proof of Proposition 5.9.

Recall that

- $g = 6$: $(\alpha_0, \beta_0) = (\frac{1}{20}, -\frac{9}{10})$,
- $g = 7$: $(\alpha_0, \beta_0) = (\frac{1}{12}, -\frac{5}{6})$,
- $g = 8$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{22}{25})$,
- $g = 9$: $(\alpha_0, \beta_0) = (\frac{1}{8}, -\frac{3}{4})$,
- $g = 10$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{22}{25})$,
- $g = 12$: $(\alpha_0, \beta_0) = (\frac{1}{25}, -\frac{21}{25})$.

Lemma A.9. *Let $X := X_{2g-2}$. If we assume that $[A] = av + bw$ and $[B] = cv + dw$ for $a, b, c, d \in \mathbb{Z}$, then the solutions of inequalities*

(1) $[A] + [B] = [i^* \mathcal{O}_x[-1]]$;
(2) $\mathrm{Im}\, Z^0_{\alpha_0,\beta_0}(A) \cdot \mathrm{Im}\, Z^0_{\alpha_0,\beta_0}(i^* \mathcal{O}_x[-1]) \geq 0$ *and* $\mathrm{Im}\, Z^0_{\alpha_0,\beta_0}(B) \cdot \mathrm{Im}\, Z^0_{\alpha_0,\beta_0}(i^* \mathcal{O}_x[-1]) \geq 0$;
(3) $\mu^0_{\alpha_0,\beta_0}(A) > \mu^0_{\alpha_0,\beta_0}(B)$;
(4) $1 - \chi(A, A) + 1 - \chi(B, B) \leq \mathrm{ext}^1(i^* \mathcal{O}_x, i^* \mathcal{O}_x)$

are listed below:

(i) $g = 6$ *and ordinary:* $(a, b, c, d) = (-2, 1, -3, 2)$;
(ii) $g = 6$ *and special:* $(a, b, c, d) = (-2, 1, -3, 2)$ *or* $(a, b, c, d) = (-4, 2, -1, 1)$;
(iii) $g = 7$*: there are no solutions;*
(iv) $g = 8$*:* $(a, b, c, d) = (-2, 1, -5, 3)$ *or* $(a, b, c, d) = (-4, 2, -3, 2)$;
(v) $g = 9$*: there are no solutions;*
(vi) $g = 10$*: there are no solutions;*
(vii) $g = 12$*: there are no solutions.*

Proof. Note that for genus $g = 6, 8$ and 12, there are only finitely many $(a, b, c, d) \in \mathbb{Z}^{\oplus 4}$ satisfying the conditions (1) and (4). Thus from a simple computation we obtain solutions to $(i), (ii), (iv)$ and (vii).

- $g = 7$: (1) and (4) give

$$\frac{1}{6}(-6a - \sqrt{33} - 6) \leq b \leq \frac{1}{6}(-6a + \sqrt{33} - 6). \tag{15}$$

From (2) we obtain

$$\frac{1}{12} > \frac{13a + 16b}{48} \geq 0. \tag{16}$$

The combination of (15) and (16) implies $-11 \leq a \leq -1$. Then it is not hard to check that the only possible solution (a,b) for (15) and (16) is $(a,b) = (-6,5)$. But this contradicts (3).

- $g = 9$: (1) and (4) give

$$\frac{1}{4}(-2a - \sqrt{3} - 2) \leq b \leq \frac{1}{4}(-2a + \sqrt{3} - 2). \tag{17}$$

From (2) we obtain

$$\frac{1}{16} > \frac{11a + 32b}{128} \geq 0. \tag{18}$$

The combination of (17) and (18) implies $-7 \leq a \leq -1$. Then it is not hard to check that there are no integer solutions (a,b) for (17) and (18).

- $g = 10$: (1) and (4) give

$$\frac{1}{6}(-4a - \sqrt{7} - 3) \leq b \leq \frac{1}{6}(-4a + \sqrt{7} - 3). \tag{19}$$

From (2) we obtain

$$\frac{167}{1875} > \frac{611a + 1200b}{5625} \geq 0. \tag{20}$$

The combination of (19) and (20) implies $-8 \leq a \leq -1$. Then it is not hard to check that there are no integer solutions (a,b) for (19) and (20).

□

A.4. **Inequalities in Proposition 6.7.** In this subsection, we compute the inequalities used in the proof of Proposition 6.7.

Lemma A.10. *Let the notation and assumptions be as in Case 1.2 of the proof of Proposition 6.7. If we assume $[B] = av + bw + c[\mathcal{E}]$, then the inequalities*

- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(A)) \geq 0, \mathrm{Im}(Z^0_{\alpha_g,\beta_g}(B)) > 0$,
- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(\ker(\lambda))) > 0$,
- $\mu^0_{\alpha_g,\beta_g}(\ker(\lambda)) < \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $\mu^0_{\alpha_g,\beta_g}(C) > \mu^0_{\alpha_g,\beta_g}(B) > \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $c > 0, a < 0$.

imply that $\frac{b}{a} \geq \mu(\mathcal{E})$ for $g \neq 7$ and $\frac{b}{2a} \geq \mu(\mathcal{E})$ when $g = 7$.

Proof. We assume that $\frac{b}{a} < \mu(\mathcal{E}_g)$ for $g \neq 7$ and $\frac{b}{2a} < \mu(\mathcal{E}_g)$ when $g = 7$, and we will show that there are no such integers $a, b, c \in \mathbb{Z}$. Recall that $(\alpha_7, \beta_7) = (\frac{\sqrt{71}}{84}, -\frac{71}{84})$, $(\alpha_8, \beta_8) = (\frac{2\sqrt{79}}{875}, -\frac{122}{125})$, $(\alpha_9, \beta_9) = (\frac{\sqrt{31}}{40}, -\frac{31}{40})$, $(\alpha_{10}, \beta_{10}) = (\frac{\sqrt{\frac{5}{3}}}{33}, -\frac{10}{11})$, and $(\alpha_{12}, \beta_{12}) = (\frac{1}{22}, -\frac{19}{22})$.

First we assume that g is even. In this case $\mu(\mathcal{E}_g) = -\frac{1}{2}$ and $\mathrm{ch}_{\leq 2}(B) = (a+2c, (b-c)H, (\frac{g-4}{2}c - \frac{g}{2}a - \frac{3g-6}{2}b)L)$. We have:

- $0 < \frac{(g-4)c - ga - (3g-6)b}{4g-4} - \beta_g(b-c) + \frac{a+2c}{2}(\beta_g^2 - \alpha_g^2) \leq \frac{\beta_g^2 - \alpha_g^2}{2}$,
- $0 > \frac{-ga-(3g-6)b}{4g-4} - \beta_g b + \frac{a}{2}(\beta_g^2 - \alpha_g^2)$,
- $\frac{\beta_g a - b}{\frac{-ga-(3g-6)b}{4g-4} - \beta_g b + \frac{a}{2}(\beta_g^2 - \alpha_g^2)} < \frac{1+2\beta_g}{\frac{g-4}{4g-4} + \beta_g + \beta_g^2 - \alpha_g^2}$,
- $\frac{2\beta_g}{\beta_g^2 - \alpha_g^2} > \frac{\beta_g(a+2c)-(b-c)}{\frac{(g-4)c-ga-(3g-6)b}{4g-4} - \beta_g(b-c) + \frac{a+2c}{2}(\beta_g^2-\alpha_g^2)} > \frac{1+2\beta_g}{\frac{g-4}{4g-4} + \beta_g + \beta_g^2 - \alpha_g^2}$,
- $a < 0$,
- $-\frac{1}{2}a < b$.

When $g = 8$, we have:

(a) $0<\frac{2332a+4081b+1458c}{12250}\leq\frac{2916}{6125}$,
(b) $b<-\frac{4}{7}a$,
(c) $-\frac{99632}{303389}a<b<-\frac{4}{7}a$,
(d) $-\frac{2989}{1458}>\frac{\frac{-122(a+2c)}{125}-(b-c)}{\frac{2332a+4081b+1485c}{12250}}>-\frac{5831}{729}$,
(e) $c>0, a<0$,
(f) $-\frac{1}{2}a<b$.

Now $(b),(c),(e)$ and (f) imply

$$-\frac{1}{2}a<b<-\frac{4}{7}a. \tag{21}$$

Also (a) is equivalent to

$$-\frac{583(4a+7b)}{1458}<c\leq-\frac{583(4a+7b)-5832}{1458}. \tag{22}$$

Thus (21), (22) and (d) imply

$$\begin{aligned}-\frac{110773008}{11281063}<a\leq-\frac{66479}{9558},-\frac{a}{2}<b<-\frac{4(2654501a-13846626)}{32517071},\\ -\frac{213500a+115559b}{256419}<c\leq-\frac{583(4a+7b)-5832}{1458}\end{aligned} \tag{23}$$

or

$$-\frac{66479}{9558}<a<0,-\frac{a}{2}<b<-\frac{4}{7}a,-\frac{213500a+115559b}{256419}<c\leq-\frac{583(4a+7b)-5832}{1458}. \tag{24}$$

Thus we have $-9\leq a<0$, and it is not hard to see that there are no such integers a,b,c satisfying either (23) or (24).

When $g=10$ or 12, the computation is similar to the $g=8$ case, so we omit the details.

Now we assume that $g=7$, thus $\mathrm{ch}_{\leq 2}(B)=(2a+5c,(b-2c)H,(-5a-6b)L)$. In this case we have:

(a) $0<\frac{290a+348b+71c}{1008}\leq\frac{335}{1008}$,
(b) $b<-\frac{5}{6}a$,
(c) $-\frac{3679}{4926}a<b<-\frac{5}{6}a$,
(d) $-\frac{2244}{71}<\frac{-\frac{71}{84}(2a+5c)-(b-2c)}{\frac{290a+348b+71c}{1008}}<-\frac{12}{5}$,
(e) $c>0, a<0$,
(f) $-\frac{4}{5}a<b$.

Now $(b),(c),(e)$ and (f) imply

$$-\frac{4}{5}a<b<-\frac{5}{6}a. \tag{25}$$

Also (a) is equivalent to

$$-\frac{58(5a+6b)}{71}<c\leq-\frac{58(5a+6b)-335}{71}. \tag{26}$$

Thus (25), (26) and (d) imply

$$-\frac{804}{71}<a<0,-\frac{4}{5}a<b<-\frac{5}{6}a,\frac{-35a-6b}{72}<c\leq-\frac{58(5a+6b)-335}{71} \tag{27}$$

or

$$-\frac{24120}{1309} < a \leq -\frac{804}{71}, -\frac{4}{5}a < b < \frac{4824-3679a}{4926}, \frac{-35a-6b}{72} < c \leq -\frac{58(5a+6b)-335}{71}. \tag{28}$$

It is not hard to see that the only possible solution of (27) and (28) is $(a,b,c) = (-11, 9, 5)$, i.e. $\mathrm{ch}_{\leq 2}(B) = (3, -H, L)$. But since B is $\sigma^0_{\alpha_g,\beta_g}$-semistable, this contradicts Lemma A.1.

Finally we assume that $g = 9$. Then $\mathrm{ch}_{\leq 2}(B) = (a+3c, (b-c)H, (-3a-8b)L)$. In this case we have:

(a) $0 < \frac{33a+88b+31c}{320} \leq \frac{93}{320}$,
(b) $b < -\frac{3}{8}a$,
(c) $-\frac{509}{3703}a < b < -\frac{3}{8}a$,
(d) $-\frac{8}{3} > \frac{-\frac{31}{40}(a+3c)-(b-c)}{\frac{33a+88b+31c}{320}} > -\frac{424}{31}$,
(e) $c > 0, a < 0$,
(f) $-\frac{1}{3}a < b$.

Now $(b), (c), (e)$ and (f) imply

$$-\frac{1}{3}a < b < -\frac{3}{8}a. \tag{29}$$

Also (a) is equivalent to

$$-\frac{11(3a+8b)}{31} < c \leq -\frac{11(3a+8b)-93}{31}. \tag{30}$$

Thus (29), (30) and (d) imply

$$-8 < a < 0, -\frac{1}{3}a < b < -\frac{3}{8}a, \frac{-15a-8b}{32} < c \leq \frac{-33a-88b+93}{31} \tag{31}$$

or

$$-\frac{2976}{265} < a \leq -8, -\frac{1}{3}a < b < \frac{992-197a}{856}, \frac{-15a-8b}{32} < c \leq \frac{-33a-88b+93}{31} \tag{32}$$

Then it is not hard to see that there are no integers a, b, c satisfying either (31) or (32). $\square$

Lemma A.11. *Let the notation and assumptions be as in Case 2 of the proof of Proposition 6.7. If we assume $[B] = av + bw + c[\mathcal{E}]$, then the inequalities*

- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(A)) \geq 0, \mathrm{Im}(Z^0_{\alpha_g,\beta_g}(B)) > 0$,
- $\mathrm{Im}(Z^0_{\alpha_g,\beta_g}(i^*(A))) \geq 0$,
- $\mu^0_{\alpha_g,\beta_g}(C) > \mu^0_{\alpha_g,\beta_g}(B) \geq \mu^0_{\alpha_g,\beta_g}(\mathcal{E})$,
- $c > 0, a < 0$.

imply that $\frac{b}{a} \geq \mu(\mathcal{E})$, or $(a,b) = (-1,1), (-3,2)$.

Proof. We assume that $\frac{b}{a} < \mu(\mathcal{E})$, i.e. $-\frac{1}{2}a < b$. In this case we have $g = 6$ and $\mathrm{ch}_{\leq 2}(B) = (a+2c, (b-c)H, (c-3a-6b)L)$. We have:

(a) $0 < \frac{83a+240b+6c}{800} \leq \frac{323}{800}$,
(b) $b \leq \frac{305-83a}{240}$,
(c) $-\frac{720}{323} > \frac{-\frac{9}{10}(a+2c)-(b-c)}{\frac{83a+240b+6c}{800}} \geq -\frac{320}{3}$,
(d) $c > 0, a < 0$,
(e) $-\frac{1}{2}a < b$.

Now (a), (b), (c), (d) and (e) imply

$$-5 < a < 0, -\frac{1}{2}a < b \leq \frac{305-83a}{240}, -\frac{216a+107b}{253} < c \leq -\frac{83a+240b-323}{6} \tag{33}$$

or

$$-\frac{253}{32} < a \leq -5, -\frac{1}{2}a < b < \frac{253-61a}{186}, -\frac{216a+107b}{253} < c \leq -\frac{83a+240b-323}{6}. \tag{34}$$

It is not hard to see that the only possible values of $a, b \in \mathbb{Z}$ are $(a,b) = (-1,1)$ and $(a,b) = (-3,2)$. □

From a similar computation as in Lemma A.2, we have the following lemma:

Lemma A.12. *Let $X := X_{10}$. Then there are no walls for the class $3v-2w$ on the line $\beta = \beta_6 = -\frac{9}{10}$ with respect to $\sigma_{\alpha,-\frac{9}{10}}$.*

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