$S^2 = (S^{(1)})^2 + \underline{2S^{(1)}S^{(2)}} + (S^{(2)})^2$

$\circledast$ $S_-^{(1)}S_-^{(2)}$ derived from the $|1\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow - \downarrow\uparrow\rangle)$

Because $|1\ 1\rangle$ and $|1\ -1\rangle$ ~~are~~

$\uparrow\uparrow \qquad \downarrow\downarrow$

$\Rightarrow S_-^{(1)}S_-^{(2)}|\uparrow\downarrow\rangle = \underbrace{(S_-^{(1)}|\uparrow\rangle)(S_-^{(2)}|\downarrow\rangle)}$

$\circledast$ Because χ_1 and χ_2 are independent, the state spin χ_1 has nothing to do with spin state χ_2

$$S^{(1)} S^{(2)} |\uparrow\downarrow\rangle = \left(S_x^{(1)} |\uparrow\rangle\right)\left(S_x^{(2)} |\downarrow\rangle\right)$$
$$+\left(S_y^{(1)} |\uparrow\rangle\right)\left(S_y^{(2)} |\downarrow\rangle\right)$$

(*) axis x, y, z and the main objective is that the core reason for $S_x + S_y + S_z = S$

(*) $S^{(1)} S^{(2)}$ $\qquad S = S_x + S_y + S_z$

$+ S_x^{(1)} S_x^{(2)}$
$+ S_y^{(1)} S_y^{(2)}$
$+ S_z^{(1)} S_z^{(2)}$

$$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\underbrace{S^{(1)} S^{(2)}}$$

$$= 2\left[\left(S_x^{(1)} |\uparrow\rangle\right)\left(S_x^{(2)} |\downarrow\rangle\right) + \left(S_y^{(1)} |\uparrow\rangle\right)\left(S_y^{(2)} |\downarrow\rangle\right) + \left(S_z^{(1)} |\uparrow\rangle\right)\left(S_z^{(2)} |\downarrow\rangle\right)\right]$$

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \qquad S_-^{(1)} S_-^{(2)} |\uparrow\downarrow\rangle \qquad S_y = \frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_y = \frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow S_x = \frac{\hbar}{2}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad = \left(\frac{\hbar}{2} |\downarrow\rangle\right)\left(\hbar/2 |\uparrow\rangle\right) + \left(\frac{i\hbar}{2}(|\downarrow\rangle)\right)\left(-\frac{i\hbar}{2}(|\uparrow\rangle)\right)$$

$$+\left(\left(\frac{\hbar}{2}\right)|\uparrow\rangle\right)\left(-\left(\frac{\hbar}{2}\right)(|\downarrow\rangle)\right)$$

for S_x $\uparrow$ notation converts to down
$\downarrow$ notation converts to up

convert $\uparrow \to \downarrow$

for S_y $\uparrow$ notation converts to down
$\downarrow$ notation converts to up and (−) $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ → convert $\downarrow$ to $\uparrow$ and (−) value or expression

Trick to understand 5:32

$\Rightarrow S_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

0 1 is the first value focused so ↑ becomes ↓

1 0 is the second value focused so ↓ becomes ↑

$S_x \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \begin{matrix} \uparrow \to \downarrow \\ \downarrow \to \uparrow \end{matrix}$

$\Rightarrow S_y$ same thing

$\begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$ same as S_x but negative value present

$\Rightarrow$ same rule but multiply with (-1)

$\Rightarrow S_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

1 0 → doesn't change

0 −1 → remains as it is but add (−) to it

—X—

$\vec{S}^{(1)} \cdot \vec{S}^{(2)} \to S_x^{(2)} + S_y^{(2)} + S_z^{(2)}$

$\hookrightarrow S_x^{(1)} + S_y^{(1)} + S_z^{(1)}$

$\Rightarrow \vec{S}^{(1)} \cdot \vec{S}^{(2)} |\uparrow\downarrow\rangle = \left(\frac{\hbar}{2}|\downarrow\rangle\right)\left(\hbar/_2|\uparrow\rangle\right) + \left(\frac{i\hbar}{2}|\downarrow\rangle\right)\left(-\frac{i\hbar}{2}|\uparrow\rangle\right)$

$+ \left(\frac{\hbar}{2}|\uparrow\rangle\right)\left(-\frac{\hbar}{2}|\downarrow\rangle\right)$

$= \left(\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) + \left(-i^2\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) + \left(\frac{\hbar^2}{4}|\uparrow\downarrow\rangle\right)$

$= \left(\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) + \left(-(-1)\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) + \left(\frac{\hbar^2}{4}|\uparrow\downarrow\rangle\right)$

$= \frac{\hbar^2}{4}|\downarrow\uparrow\rangle + \frac{\hbar^2}{4}|\downarrow\uparrow\rangle$

$\left(+\frac{\hbar^2}{}|\uparrow\downarrow\rangle\right)$

$$\left(\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) + \left(\frac{\hbar^2}{4}|\downarrow\uparrow\rangle\right) - \left(\frac{\hbar^2}{4}|\uparrow\downarrow\rangle\right)\Big\}$$

$$= \left(\frac{2\hbar^2}{4}|\downarrow\uparrow\rangle - \frac{\hbar^2}{4}|\uparrow\downarrow\rangle\right)\Big\}$$

$$= \frac{\hbar^2}{4}\Big\{\big(2|\downarrow\uparrow\rangle\big) - \big(|\uparrow\downarrow\rangle\big)\Big\}\Big\}$$

—X——X——X——X——X——X——

Step wise explanation

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} \, |\downarrow\uparrow\rangle = \frac{\hbar^2}{4}\{2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}$$

$$\left(\vec{S}^{(1)}_x + \vec{S}^{(1)}_y + \vec{S}^{(1)}_z\right) \cdot \left(S^{(2)}_x + S^{(2)}_y + S^{(2)}_z\right)|\downarrow\uparrow\rangle$$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} \, |1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}$$

Ket notation of $|1\,0\rangle$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow + \downarrow\uparrow\rangle\right)$$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$|\uparrow\downarrow\rangle$

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} = \frac{\hbar^2}{4}\{2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\}$$

$$\Rightarrow \vec{S}^{(1)} \cdot \vec{S}^{(2)} |1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\{(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) + (2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\}$$

$$\Rightarrow \vec{S}^{(1)} \cdot \vec{S}^{(2)} |1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\}$$

—x——x—x——x——x——x—x—

PART 2

$$(S^{(1)})^2 = S^2\chi_+ = S^2|\uparrow\rangle$$

$\binom{1}{0}$ so spin up

$$\Rightarrow S_x^2|\uparrow\rangle + S_y^2|\uparrow\rangle + S_z^2|\uparrow\rangle$$

$$= \frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}|\uparrow\rangle$$

$$= 3\times\frac{\hbar^2}{4}|\uparrow\rangle \chi_+$$

$$= \frac{3\hbar^2}{4}|\uparrow\rangle \chi_+$$

$S^2\chi_- :$ same derivation

$$\frac{3\hbar^2}{4}|\downarrow\rangle\chi_-$$

$$\therefore S^2|1\,0\rangle = \left((S_x^{(1)2} + S_y^{(1)2} + S_z^{(1)2}) + (S_x^{(2)2} + S_y^{(2)2} + S_z^{(2)2}) + 2(S^{(1)} \cdot S^{(2)})\right)|1\,0\rangle$$

$$= \left(\frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4}\right) + \left(\frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4}\right) + \left(2\frac{\hbar^2}{4}\right)|1\,0\rangle$$

$\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + \frac{2\hbar^2}{4} |1\ 0\rangle$

$= 2\hbar^2 |1\ 0\rangle$

$\Rightarrow$ proving that $S^2 = 2\hbar^2$ for any ket notation