Angular momentum

n :- energy quantum number f

$\Rightarrow \vec{L} = \vec{r} \times \vec{p}$ [momentum of a rotating body]

$\Rightarrow p = mv$ — rotational analogue of mass

$\vec{L} = \vec{I}\vec{\omega}$ $\rightarrow$ angular velocity

$\downarrow$

Inertia

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{i} - (xp_z - zp_x)\hat{j} + (xp_y - yp_x)\hat{z}$$

$\Rightarrow \dfrac{dp}{dt} = \vec{F}$ $\qquad \vec{\tau} = \dfrac{d\vec{L}}{dt}$

$\therefore \vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$

$L_x = yp_z - zp_y$ $\qquad L_z = xp_y - yp_x$

$L_y = zp_x - xp_z$

$x \to y \to z$

$p_x \to p_y \to p_z$

$L_x \to L_y \to L_z$

$p_x = -i\hbar \frac{\partial}{\partial x}$

$p_y = -i\hbar \frac{\partial}{\partial y}$

$p_z = -i\hbar \frac{\partial}{\partial z}$

Commutetions : $[A, B] = \underbrace{\hat{A}\hat{B} - \hat{B}\hat{A}}$ $= 0$: Set of eignen values

$\neq 0$

not commuting

order doesn't

$L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

ARE HARDER TO HIDE THAN I THOUGHT

— X — X — X — X — X —

$\vec{r} = x, y, z$
$P \to P_x, P_y, P_z$ } respectively

WANNA BE YOURS

WANNA BE YOURS

$[r_i, P_j] = i\hbar\,\delta_{ij}$

$\hat{i} = x$

$[x, P_x] = i\hbar\,\boxed{\delta_{ij}} = 1$

$[x, P_x] = i\hbar$

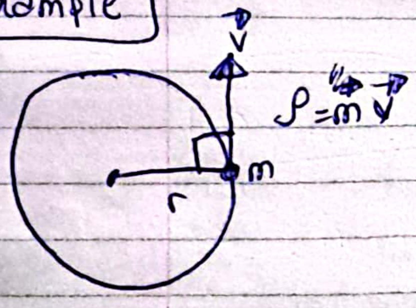
↳ why, because they are not commuting

$[y, P_x] = [x, P_y] = 0$ [Because it commutes]

— x — x — x — x — x — x — x — x —

$L = rp\sin(\theta)$

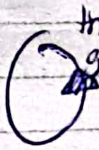
example

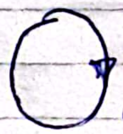


At maximam Angular momentum
$L = rp\sin(\theta)$
$L = rp\sin(90)$
$\boxed{L = rp}$ maxima

WANNA BE YOURS

1) Right hand capsule rule





$L \begin{cases} r = 0 \\ p = 0 \\ \sin(\theta) = 0 \end{cases}$ $[\theta]$ = parallel

[Angular momentum]

$[L_x, L_y] = [(yp_z - zp_y), (zp_x - xp_z)] = [yp_z, zp_x] + [zp_y, xp_z]$ $[zp_z][p_y, x]$ $- [zp_y, zp_x] - [yp_z, xp_z]$

1.1, 1.2, 2.1, 2.2

0 0

$[z, z] = 0$
$[p_y, p_x] = 0$ } commuting

Question to ask?

Why would $[(yp_z), (xp_z)]$ commute even though p_z and p_z different doesn't correlate with y & x ?

$[(yp_z), (zp_x)] = (yp_z zp_x - zp_x yp_z)$

different

different

$[(yp_z zp_x - zp_x yp_z) + (z(p_y)xp_z - (x)p_z zp_y)]$

~~$(yp_x)(p_z z) - (zp_z)(p_x y)$~~

$[(yp_x)(p_z z)] + [zp_z][p_y x]$

p_x, y $\quad [z, x]$
p_y, x
p_z, y

$z p_y \, x p_z \quad - x p_z \, z$

$(z p_z)[p_y, x]$

~~$(y p_x)[p_z, z] + [z p_y][p_y, x]$~~

~~$(y p_x)[i\hbar] + [(z p_y)(0)]$~~ $[(p_z x)[z, p_y]$

~~$(y p_x)[i\hbar]$~~

$[y p_z, z p_x] + [z p_y, x p_z]$

$[(y p_x)[p_z, z]] + [(p_y x)[z, p_z]]$

$[(y p_x) - i\hbar] + (p_y x)[i\hbar]$

~~$(y p_x) - i\hbar$~~

$-i\hbar(y p_x) + i\hbar(p_y x)$
$i\hbar(-y p_x + p_y x)$
$i\hbar(p_y x - y p_x)$

$i\hbar(x p_y - y p_x)$
$i\hbar(L_z)$

$\therefore [L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_z$

— X — X — X — X —

<u>HUP in terms of Angular momentum</u>

MAKE PROUD

$[x, p_x] = i\hbar \quad , \sigma_x \sigma_{p_x} \geq \hbar/2$

$\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} |L_z|$ [Because they do not commute so HUP exists]

$L^2 : P^2$

[Because L: Angular momentum depends on the linear momentum, ~~law of~~ the linear momentum itself fails in terms of the vectorial nature of momentum thus failing the law of conservation of momentum which then focuses on law of conservation of energy P^2 meaning angular momentum should also be squared]

—X————X————X————X————X————X————

$[L^2, L_x]$

$$[L^2, L_x] = [(L_x^2) + (L_y^2) + (L_z)^2, (L_x)] \quad 0$$

$$= [\cancel{[(L_x)^2, (L_x)]}, [(L_y)^2, (L_x)], [(L_z)^2, (L_x)]]$$

I know

$\vec{L} = L_x + L_y + L_z$

$\therefore L^2 = (L_x)^2 + (L_y)^2 + (L_z)^2$

$$= [(L_x)^2, (L_x)] + [(L_y)^2, (L_x)] + [(L_z)^2, (L_x)]$$

$$= [(L_y L_y), (L_x)] + [(L_z L_z), (L_x)]$$

$$[(AB), (C)] = A[B, C] + [A, C]B$$

$$[(L_y L_y), (L_x)] = [L_y[L_y, L_x] + [L_y, L_x]L_y] +$$

$$+ [(L_z L_z), (L_x)] = [L_z[L_z, L_x] + [L_z, L_x]L_z]$$

$$[L_y[L_y, L_x] + [L_y, L_x]L_y] + [L_z[L_z, L_x] + [L_z, L_x]L_z]$$

$$[L_y[-i\hbar L_z] + [-i\hbar L_z]L_y] + [L_z[i\hbar L_y] + i\hbar L_z]$$

$$[-i\hbar L_y L_z - i\hbar L_z L_y] + [i\hbar L_z L_y + i\hbar L_y L_y]$$

$[L^2, L_x] = 0$ prooved

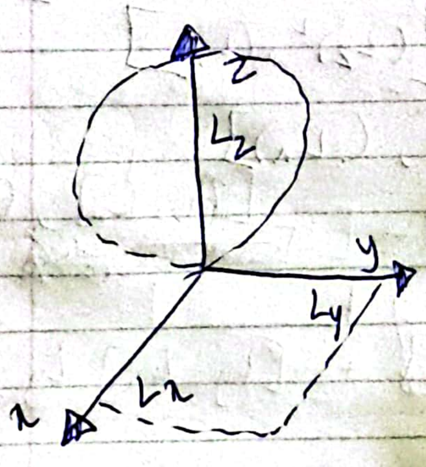
$[L^2, L_y] = 0$ $[L^2, L_z] = 0$

component form

$$\overset{1}{L_x^2} + \overset{2}{L_y^2} + \overset{3}{L_z^2}, \; \overset{1}{L_x} \; \overset{2}{L_y} \; \overset{3}{L_z} = 0$$

1 commutes
2 comutes
3 commutes

—x—x—x—x—x—x—x—



$$[\overbrace{L_z, L_x}^{i\hbar L_y}] \pm [\overbrace{i(L_z, L_y)}^{-i\hbar L_x}]$$

$$i\hbar L_y \pm [-i(i\hbar L_x)]$$

$L_+ = L_x + iL_y$; L_z

$L_- = L_x - iL_y$; L_z

$[L_z, L_\pm] : [L^2, L_\pm]$

$L_z, [L_x \pm iL_y]$

Eigen Values and Eigen functions

$[L_x, L_y, L_z, L_\pm, L^2;]$

They don't commute with each other so they don't have the same set of eigen functions

$L^2: L_x^2, L_y^2, L_z^2$ — L_x, L_y, L_z commutes

$L_\pm: L_x \pm i L_y$

L_z: separate component

$L^2; L_z$:

eigen function: f: $L^2 f = \lambda f$; $L_z f = \mu F$

$[L^2, L]$ they commute so same set of eigen functions f

example $L_\pm (L^2 F) = L_\pm(\lambda f) = \lambda L_\pm f$ [order doesn't matter]

$\boxed{\text{Step 1}}$ $L^2(L_\pm f)$: If L^2 and $L_\pm$ commutes with f then $L_\pm$ and f will commute with L^2

$\boxed{\text{Step 2}}$ $L_z(L_\pm f) = L_z L_\pm f$ [since they don't commute Order cannot be changed]

L_z and $L_\pm$ don't commute

[Changing position of values when multiplying doesn't change anything] $= L_\pm L_z f + \boxed{[L_z L_\pm f - L_\pm L_z f]}$

$= (L_z L_\pm - L_\pm L_z) f + L_\pm L_z f$

$([L_z, L_\pm] + L_\pm \mu) f = [L_z, L_\pm] f + L_\pm(\mu f) =$

$= (\pm \hbar L_\pm + \mu L_\pm) f$

$(\mu \pm \hbar)(L_\pm f) = L_z(L_\pm f) =$ [$L_\pm f$ is an eigen function of with an eigen value $\mu \pm \hbar$]

SHO

- ⇒ L_+ raising operator because the values alters from μ to $\mu + \hbar$
- ⇒ L_- lowering operator changes $\hbar$ values to $\mu - \hbar$

Step 3 $\quad l\hbar \to$ max $\quad f_t$

Ladder operators of angular states

$\mu + 2\hbar \longleftarrow L_+^2 f$

[raising eigen value] $\mu + \hbar \longleftarrow L_+ f$ [raising eigen function]

[eign value] μ ——— f [eigen function of L_z]

$\mu - \hbar \longleftarrow L_- f$

[lowering eigen value] $\mu - 2\hbar \longleftarrow L_-^2 f$

$-1(-1) = 1$

$\bar{l}\hbar \longleftarrow f_b$

minimum

$L_+ f_t = 0$

raising operator more than max value does not have a max value so reaches 0

$L_- f_b = 0$

same for bottom state

$L_z f_t = l\hbar f_t$

$L^2 f_t = \lambda f_t$

—— x —— x —— x —— x —— x —— x ——

Derivation of $L_\pm$ and $L_\mp$ in terms of L^2, L_z

$$L_\pm L_\mp = L^2 - L_z \pm \hbar L_z$$

$$= (L_x \pm i L_y)(L_x \mp i L_y)$$

$$= L_x^2 + L_y^2 \mp i L_x L_y \pm i L_y L_x$$

$$= L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) \to [L_x, L_y]$$

not commuting

$$= L_x^2 + L_y^2 \mp i(i\hbar L_z) \quad \mp \hbar L_z$$

$$= L_x^2 + L_y^2 \pm \hbar L_z$$

(2) (-1) times

1

excluded

$= L_x^2 + L_y^2 = L^2 - L_z^2$

$L_x + L_y^2 + L_z^2 = L^2$

proved $= L^2 - L_z^2 \pm \hbar L_z$

$L_\mp = L^2 - L_z^2 \pm$

proov $L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$

$L_\pm L_\mp = ($

Anything further

$$\boxed{\text{Find } \lambda}$$

$L^2 f_t = (L_\pm L_\mp + L_z^2 \mp \hbar L_z) f_t$

step 1) $\boxed{+}$: $L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t$

$\Rightarrow L^2 f_t = L_- L_+ f_t + L_z^2 f_t + \hbar L_z f_t$

$L_+ f_t = 0$

$L_- (L_+ f_t) = (L_-) 0 = 0$

$= L_z^2 f_t + \hbar L_z f_t$

$\Rightarrow \ell^2 \hbar^2 f_t + \hbar^2 \ell f_t$

$L^2 f_t = \ell(\ell+1) \hbar^2 f_t$

$\lambda = \ell(\ell+1)\hbar^2$

$\boxed{-}$ most min value

$L^2 f_b = L_+ L_- f_b + L_z^2 f_b \mp L_z \hbar f_b$

$L^2 f_b = 0 + \bar{\ell}\hbar^2 f_b - \bar{\ell}\hbar(\hbar f_b)$

$= 0 + \bar{\ell}\hbar^2 f_b - \bar{\ell}\hbar^2 f_b$

$L^2 f_b = \bar{\ell}(\bar{\ell}-1)\hbar^2 f_b$

$\lambda = \bar{\ell}(\bar{\ell}-1)\hbar^2$

$\lambda = \ell(\ell+1) = \bar{\ell}(\bar{\ell}+1)$

for $\bar{\ell} = -\ell$: $-\ell(\ell-1)$

$= \ell^2 + \ell = \ell(\ell+1)$

$\Rightarrow \bar{\ell} = -\ell$

$L^2 f = \lambda f$ & $L_z f = \mu f$

$\lambda = \ell(\ell+1)\hbar^2 \quad \mu = m_\ell \hbar \quad m_\ell = -\ell, -\ell+1, 0, \ell, \ell+1, \ldots$

N: discrete values

for each ℓ: $m_\ell = 2\ell + 1$