heisenberg uncertainty derivation

Derivation

$$\langle \phi_m | \phi_n \rangle = \delta_{mn} \rightarrow \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$\langle \phi_m | \phi_n \rangle \geq 0$ [maximum is one]

or, $\langle \phi | \phi \rangle \geq 0$

proof $\sigma_x \sigma_p \geq \frac{\hbar}{2}$ | $\Delta x \Delta p \geq \hbar/2$ uncertainity

example

$\hat{A}$ & $\hat{B}$:- Hermitian operators

$\hat{A}\psi = a\psi$; $\hat{B}\psi = b\psi$

$(\hat{A} - a)\psi = 0$; $(\hat{B} - b)\psi = 0$

$\sigma_A^2 = \langle (\hat{A} - a)^2 \rangle$, $\sigma_B^2 = \langle (\hat{B} - b)^2 \rangle$

Bra-Ket

$\sigma_A^2 = \langle \psi | (\hat{A} - a)^2 | \psi \rangle$, $\sigma_B^2 = \langle \psi | (\hat{B} - b)^2 | \psi \rangle$ notel

$(\hat{A} - a) + i\lambda(\hat{B} - b)$; ψ ; $\tilde{\psi}$

$(\hat{A} - a) + i\lambda)(\hat{B} - b) | \psi \rangle = |\tilde{\psi}|$

$\langle \tilde{\psi} | \tilde{\psi} \rangle \geq 0$

I call it big ~~hermitian~~ conjugate

$$\langle\tilde{\Psi}| = \langle\Psi|\left((\hat{A}-a)+i\lambda(\hat{B}-b)\right)^{\dagger}$$

$$= \langle\Psi|\left((\hat{A}-a)-i\lambda(\hat{B}-b)\right)$$

Changing from ket-notation to bra-notation

$\langle\tilde{\Psi}|\tilde{\Psi}\rangle$

$$\underbrace{\left[\langle\tilde{\Psi}|\left((\hat{A}-a)-i\lambda(\hat{B}-b)\right)\right]}_{\text{Bra}}\underbrace{\left[\left(\hat{A}-a\right)+i\lambda\left(\hat{B}-b\right)\right]}_{\text{Ket}} \geq 0$$

Bra — Ket

notation

$$\left[(\hat{A}-a)(\hat{A}-a)\right] + \cancel{\left[(\hat{A}-a)(i\lambda(\hat{B}-b))\right]}$$

$$\cancel{\hat{B}}\left[i\lambda(\hat{B}-B)\right]\left[-i\lambda(\hat{B}-B)\right]$$

$$= \left[(\hat{A}-a),(\hat{B}-b)\right]$$

$$\Rightarrow \left[(\hat{A}-a), (\hat{B}-b)\right] = (\hat{A}-a)(\hat{B}-b) - (\hat{B}-b)(\hat{A}-a)$$

$$= (\hat{A}\hat{B} - b\hat{A} - a\hat{B} + ab - \hat{B}\hat{A} + a\hat{B} + b\hat{A} - ab)$$

$$= [\hat{A}, \hat{B}]$$

[hermitian operator A , hermition operator B]

$$\left[\therefore \langle\psi|(\hat{A}-a)^2|\psi\rangle + \lambda^2\langle\psi|(\hat{B}-b)^2|\psi\rangle + i\lambda\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right] \geq 0$$

$$\underbrace{\sigma_A^2 + \lambda^2\sigma_B^2 + i\lambda[\hat{A},\hat{B}]}_{\blacksquare} \geq 0 \qquad \boxed{i^2 = -1}$$

$$\blacksquare = 0 \qquad \blacksquare \geq 0$$

$\lambda =$ [most minimum value] $\qquad \lambda =$ [some maximum value]

$$\Rightarrow f(\lambda) = \frac{df}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left[\sigma_A^2 + \lambda^2\sigma_B^2 + i\lambda[\hat{A},\hat{B}]\right] = 0$$

$$\left[\sigma + 2\lambda\sigma_B^2 + i[\hat{A}\,\hat{B}]\right] = 0$$

replace the lambda in

$$\Leftarrow \lambda = -i\frac{[\hat{A},\hat{B}]}{2\sigma_B^2} \qquad \boxed{\lambda = \frac{1}{2i}\frac{[\hat{A},\hat{B}]}{\sigma_B^2}}$$

$$\sigma_A^2 \sigma_B^2 [\hat{A},\hat{B}] \geq 0 \Rightarrow \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}[\hat{A},\hat{B}]\right)^2$$

$$\Rightarrow \sigma_A \sigma_B \geq \frac{[\hat{A},\hat{B}]}{2i} \qquad [\hat{x},\hat{p}] = i\hbar$$

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

By errnfest theorm
one can proov for each
sigma notation