Progress Report

Lifting Linearization of a UAV

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New Model

$$
\begin{aligned}
\dot{\boldsymbol{p}} &= \boldsymbol{v} \\
\dot{\boldsymbol{v}} &= \boldsymbol{r}_z(\boldsymbol{e})T + \boldsymbol{g} \\
\begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\eta} \end{bmatrix} &= \frac{1}{2}\boldsymbol{J}_E\omega = \frac{1}{2}\begin{bmatrix} \eta\boldsymbol{I} - \boldsymbol{\epsilon}\times \\ -\boldsymbol{\epsilon}^T \end{bmatrix}\boldsymbol{\omega}, \\
\dot{\boldsymbol{\omega}} &= \boldsymbol{w}_1 \\
\dot{T} &= w_2
\end{aligned}
\tag{1}
$$

$$
\boldsymbol{r}_z(\boldsymbol{e}) = \boldsymbol{R}(e)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\left(\epsilon_1\epsilon_3 + \epsilon_2\eta\right) \\ 2\left(\epsilon_2\epsilon_3 - \epsilon_1\eta\right) \\ 1 + 2\left(-\epsilon_1^2 - \epsilon_2^2\right) \end{bmatrix}, \boldsymbol{\epsilon}\times = \begin{bmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{bmatrix}, \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}
$$

- States: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}^T & \boldsymbol{v}^T & \boldsymbol{\epsilon}^T & \eta & \boldsymbol{\omega}^T & T \end{bmatrix}^T_{14\times 1}$
- Inputs: $\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1^T & w_2 \end{bmatrix}^T_{4\times 1}$
- $\dot{p}, \dot{v}, \boldsymbol{\omega}, \boldsymbol{g}$ are measured in Global Frame.
- T is measured in Body Fixed Frame.

Expanded Model

$$\dot{x} = \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\eta} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{T} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ T\left(2\epsilon_1\epsilon_3 + 2\epsilon_2\eta\right) \\ T\left(2\epsilon_2\epsilon_3 - 2\epsilon_1\eta\right) \\ 9.81 - T\left(2\epsilon_1^2 + 2\epsilon_2^2 - 1\right) \\ \frac{\epsilon_3\omega_2}{2} - \frac{\epsilon_2\omega_3}{2} + \frac{\eta\omega_1}{2} \\ \frac{\epsilon_1\omega_3}{2} - \frac{\epsilon_3\omega_1}{2} + \frac{\eta\omega_2}{2} \\ \frac{\epsilon_2\omega_1}{2} - \frac{\epsilon_1\omega_2}{2} + \frac{\eta\omega_3}{2} \\ -\frac{\epsilon_1\omega_1}{2} - \frac{\epsilon_2\omega_2}{2} - \frac{\epsilon_3\omega_3}{2} \\ w_{1,1} \\ w_{1,2} \\ w_{1,3} \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 2\boldsymbol{\eta}_1 + 2\boldsymbol{\eta}_2 \\ 2\boldsymbol{\eta}_3 - 2\boldsymbol{\eta}_4 \\ T - \boldsymbol{\eta}_6 - 2\boldsymbol{\eta}_5 \\ \boldsymbol{\eta}_7 - \boldsymbol{\eta}_8 + \boldsymbol{\eta}_9 \\ \boldsymbol{\eta}_{10} - \boldsymbol{\eta}_{11} + \boldsymbol{\eta}_{12} \\ \boldsymbol{\eta}_{13} - \boldsymbol{\eta}_{14} + \boldsymbol{\eta}_{15} \\ -\boldsymbol{\eta}_{16} - \boldsymbol{\eta}_{17} - \boldsymbol{\eta}_{18} \\ w_{1,1} \\ w_{1,2} \\ w_{1,3} \\ w_2 \end{pmatrix} \tag{3}$$

Auxiliary Parameters

$$\begin{pmatrix} T\epsilon_1\epsilon_3 \\ T\epsilon_2\eta \\ T\epsilon_2\epsilon_3 \\ T\epsilon_1\eta \\ T\epsilon_1 2 \\ 2T\epsilon_2^2 + g \\ \frac{\epsilon_3\omega_2}{2} \\ \frac{\epsilon_2\omega_3}{2} \\ \frac{\eta\omega_1}{2} \\ \frac{\epsilon_1\omega_3}{2} \\ \frac{\epsilon_3\omega_1}{2} \\ \frac{\eta\omega_2}{2} \\ \frac{\epsilon_2\omega_1}{2} \\ \frac{\epsilon_1\omega_2}{2} \\ \frac{\eta\omega_3}{2} \\ \frac{\epsilon_1\omega_1}{2} \\ \frac{\epsilon_2\omega_2}{2} \\ \frac{\epsilon_3\omega_3}{2} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \\ \eta_{10} \\ \eta_{11} \\ \eta_{12} \\ \eta_{13} \\ \eta_{14} \\ \eta_{15} \\ \eta_{16} \\ \eta_{17} \\ \eta_{18} \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} {\epsilon_1}^2 \\ {\epsilon_2}^2 \\ {\epsilon_3}^2 \\ {\omega_1}^2 \\ {\omega_2}^2 \\ {\omega_3}^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \tag{5}$$

$$\epsilon = [0, 0, \sin(5)]$$
$$\eta = \cos(5)$$

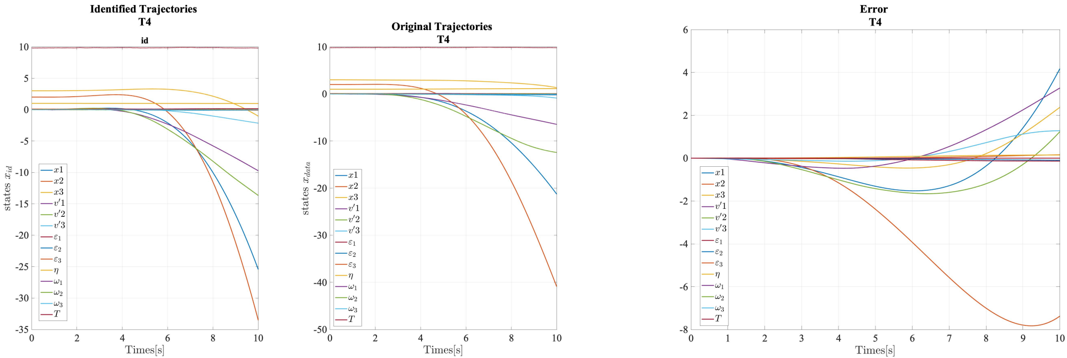

Figure: Linearized vs Real System

T5

$$\epsilon = [0, 0, \sin(10)]$$
$$\eta = \cos(10)$$

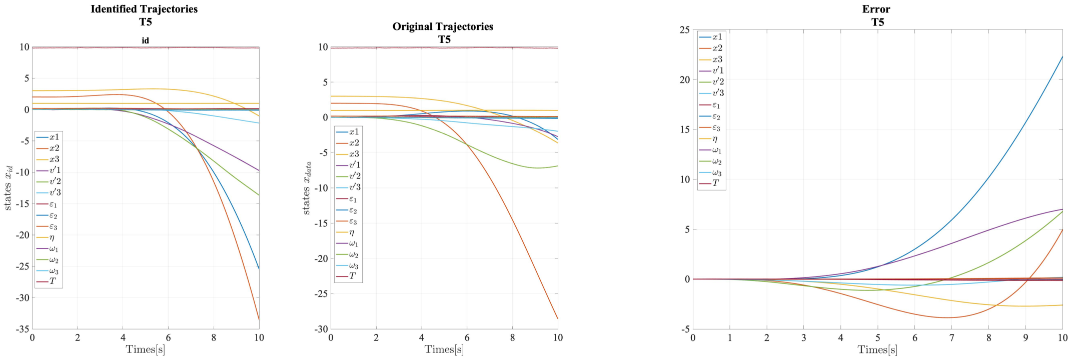

Figure: Linearized vs Real System

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LQR in Matlab

```
%% Controller Settings
Q_x = 2.3e3*[1;1;1];
Q_v = 1e3*[5;5;5];
Q_e = .5e3*[1;1;1;2];
Q_omega =200 * [10;10;10];
Q_dfl = diag([Q_x;Q_v;Q_e;Q_omega]);

R = 1000 * diag([1;1;1;1]);
C_z = [eye(13,13),zeros(13,25)];
K = lqr(A,B,C_z'*Q_dfl*C_z,R);
```

LQR Parameters

Initial State

- Position: $p_0 = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}^T$
- Velocity: $v_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
- Orientation: $\epsilon_0 = \begin{bmatrix} 0 & 0 & \sin 10 \end{bmatrix}^T, \eta_0 = \cos 10$
- Angular Velocity: $\omega_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
- Thrust: $T_0 = 9.81$

Desired State

- Position: $p = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
- Velocity: $v = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
- Orientation: $\epsilon = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \eta = 1$
- Angular Velocity: $\omega = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$
- Thrust: $T = 9.81$

Postion and Velocity

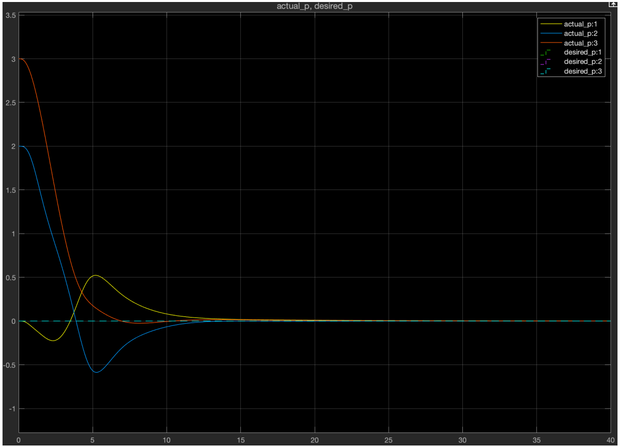

Figure: Position plot

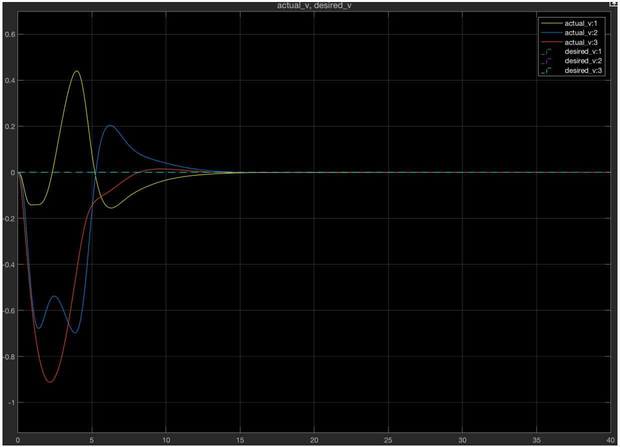

Figure: Velocity plot

Euler Parameters

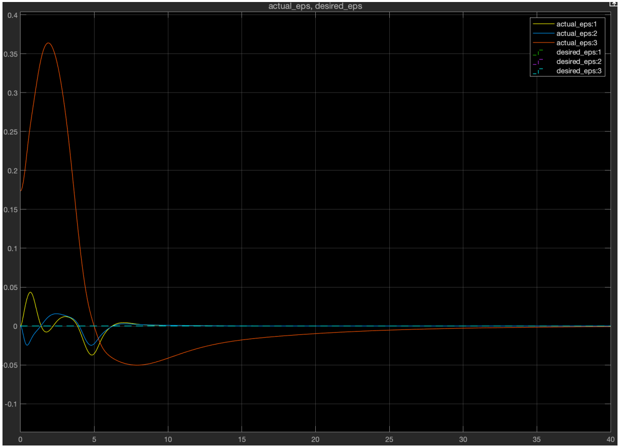

Figure: Epsilon plot

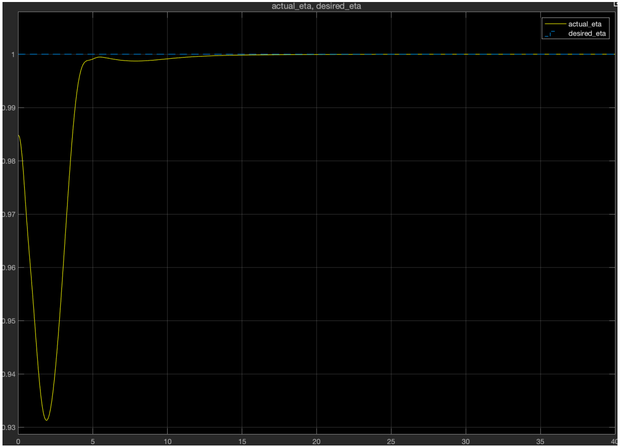

Figure: eta plot

Angular Velocity and Inputs

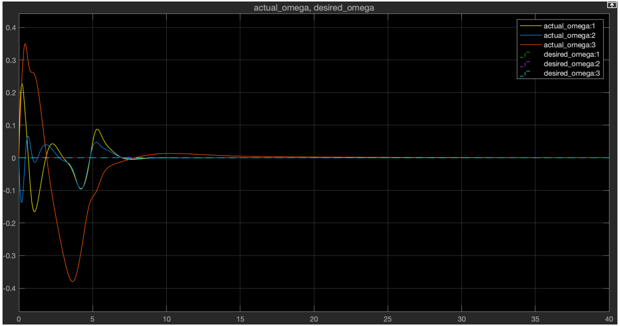

Figure: Angular Velocity plot

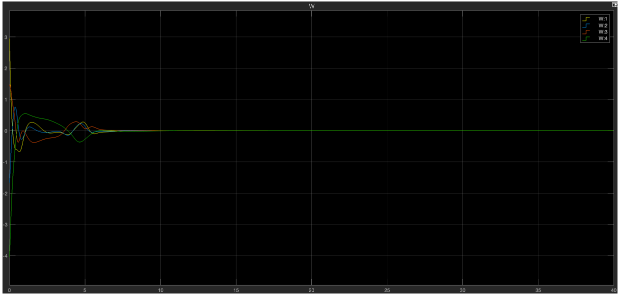

Figure: Inputs plot

Auxiliary Parameters

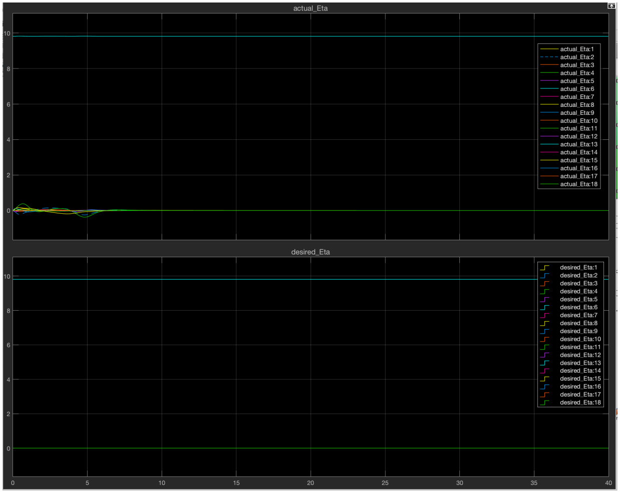

Figure: Etas plot

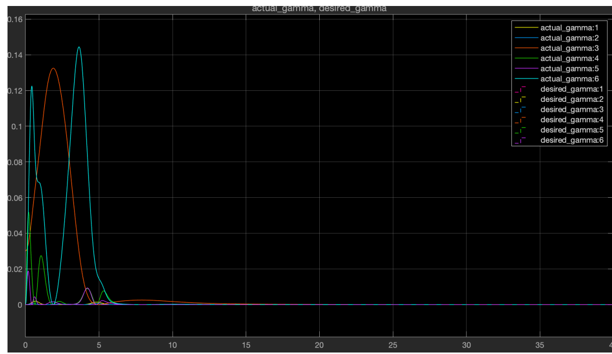

Figure: Gamma plot

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$$\dot{x} = \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\eta} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{T} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ T\left(2\epsilon_1\epsilon_3 + 2\epsilon_2\eta\right) \\ T\left(2\epsilon_2\epsilon_3 - 2\epsilon_1\eta\right) \\ 9.81 - T\left(2\epsilon_1^2 + 2\epsilon_2^2 - 1\right) \\ \frac{\epsilon_3\omega_2}{2} - \frac{\epsilon_2\omega_3}{2} + \frac{\eta\omega_1}{2} \\ \frac{\epsilon_1\omega_3}{2} - \frac{\epsilon_3\omega_1}{2} + \frac{\eta\omega_2}{2} \\ \frac{\epsilon_2\omega_1}{2} - \frac{\epsilon_1\omega_2}{2} + \frac{\eta\omega_3}{2} \\ -\frac{\epsilon_1\omega_1}{2} - \frac{\epsilon_2\omega_2}{2} - \frac{\epsilon_3\omega_3}{2} \\ w_{1,1} \\ w_{1,2} \\ w_{1,3} \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

this equilibrium state is reach when $\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 9.81 \end{bmatrix}^T_{14\times 1}$ So this model can be Linearized in the neighborhood

Linearization of dynamic equation as $\dot{x} = Ax + Bu$, where

$$\begin{aligned} A(x,u) &= \frac{\partial}{\partial x} f(x,u) \\ B(x,u) &= \frac{\partial}{\partial u} f(x,u) \end{aligned} \tag{6}$$

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\,T\epsilon_3 & 2\,T\eta & 2\,T\epsilon_1 & 2\,T\epsilon_2 & 0 & 0 & 0 & 2\,\epsilon_1\,\epsilon_3 + 2\,\epsilon_2\,\eta \\
0 & 0 & 0 & 0 & 0 & 0 & -2\,T\eta & 2\,T\epsilon_3 & 2\,T\epsilon_2 & -2\,T\epsilon_1 & 0 & 0 & 0 & 2\,\epsilon_2\,\epsilon_3 - 2\,\epsilon_1\,\eta \\
0 & 0 & 0 & 0 & 0 & 0 & -4\,T\epsilon_1 & -4\,T\epsilon_2 & 0 & 0 & 0 & 0 & 0 & -2\,{\epsilon_1}^2 - 2\,{\epsilon_2}^2 + 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega_3}{2} & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\eta}{2} & \frac{\epsilon_3}{2} & -\frac{\epsilon_2}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\omega_3}{2} & 0 & -\frac{\omega_1}{2} & \frac{\omega_2}{2} & -\frac{\epsilon_3}{2} & \frac{\eta}{2} & \frac{\epsilon_1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega_2}{2} & \frac{\omega_1}{2} & 0 & \frac{\omega_3}{2} & \frac{\epsilon_2}{2} & -\frac{\epsilon_1}{2} & \frac{\eta}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega_1}{2} & -\frac{\omega_2}{2} & -\frac{\omega_3}{2} & 0 & -\frac{\epsilon_1}{2} & -\frac{\epsilon_2}{2} & -\frac{\epsilon_3}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The rank of the controllability matrix is smaller than the number of states, so I had to remove the uncontrollable states. In this case, only one state was removed, η

Constant disturbance $\left(F_{dis} = [2.50, 1.25, 2.00]^T\right)$ was added to the translational dynamics $(\dot{\boldsymbol{v}})$. The following figures show the outputs:

Postion

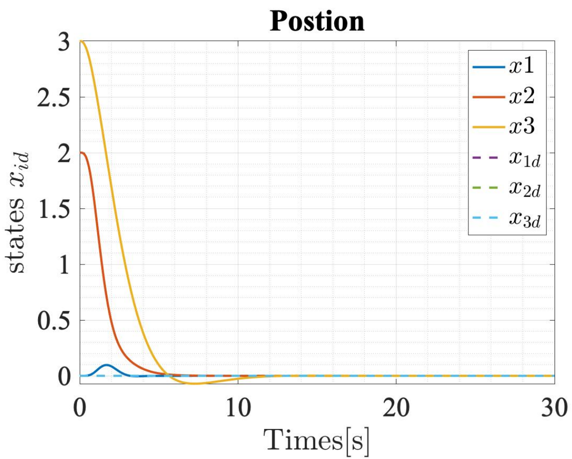

Figure: Postion without disturbance

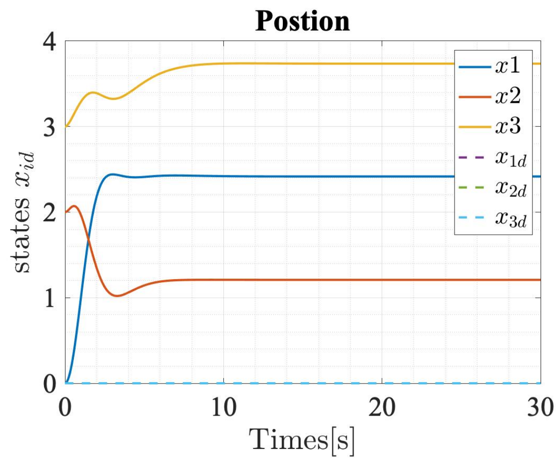

Figure: Postion with disturbance

Velocity

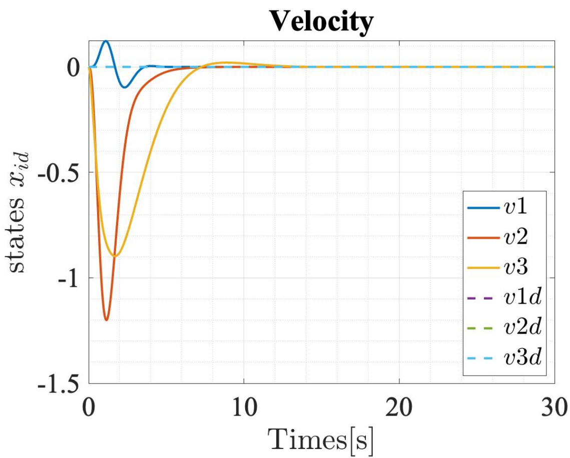

Figure: Velocity without disturbance

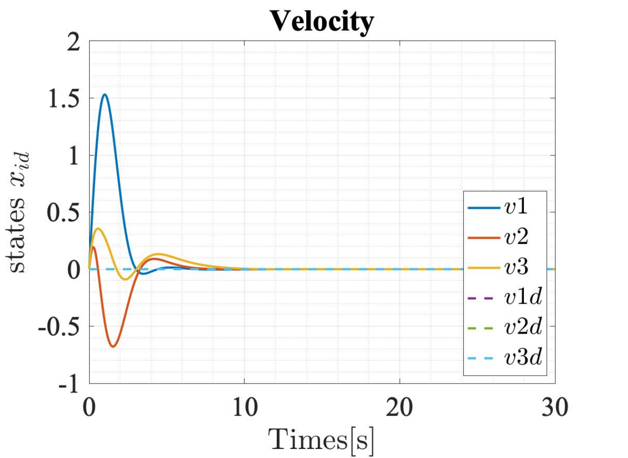

Figure: Velocity with disturbance

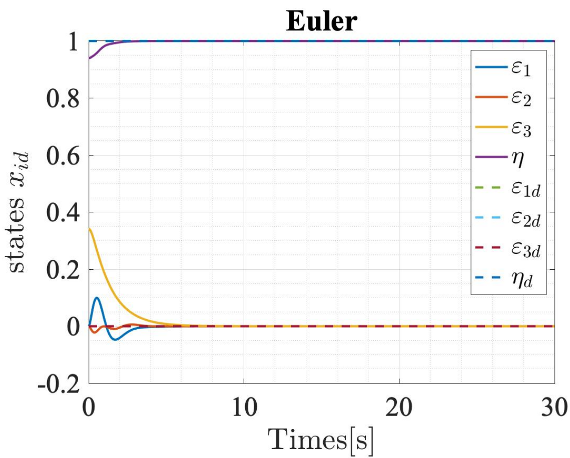

Figure: Euler without disturbance

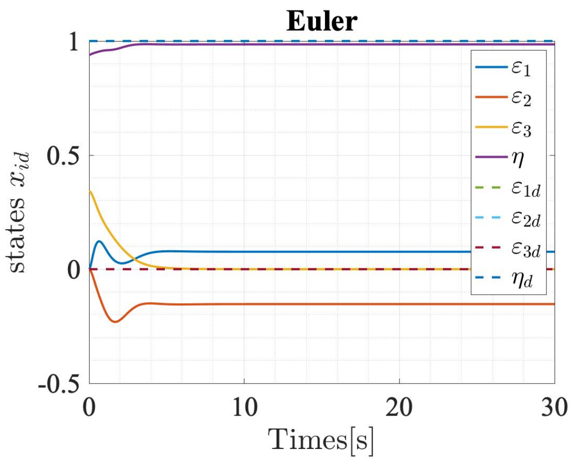

Figure: Euler with disturbance

Angular Velocity

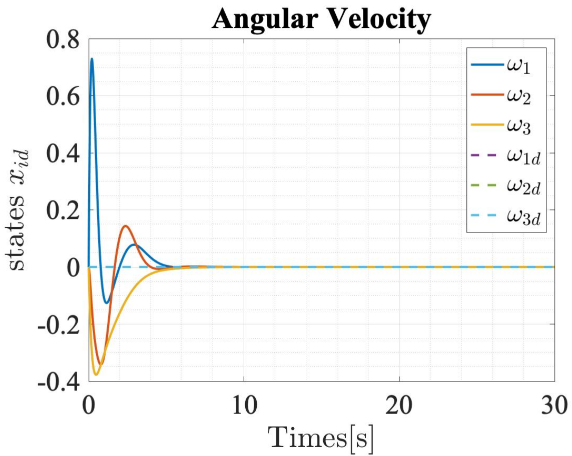

Figure: Angular Velocity without disturbance

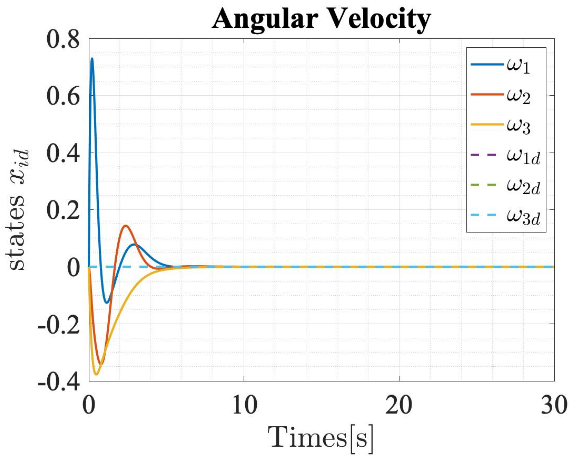

Figure: Angular Velocity with disturbance

Thrust

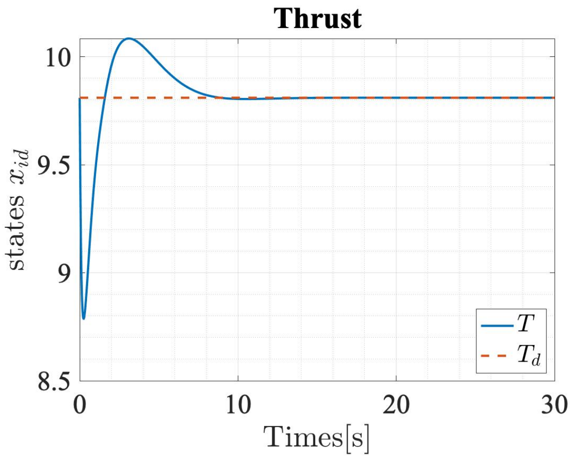

Figure: Thrust without disturbance

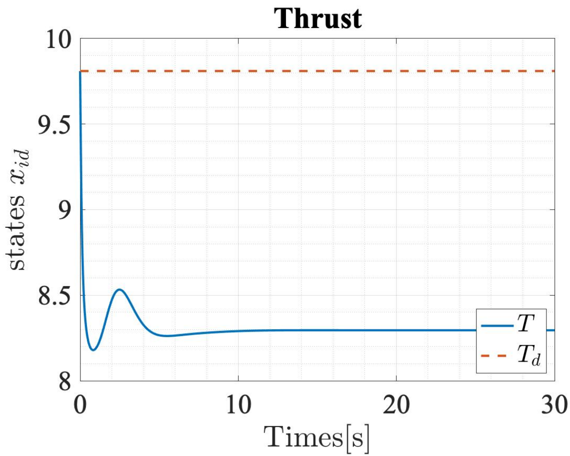

Figure: Thrust with disturbance

Inputs

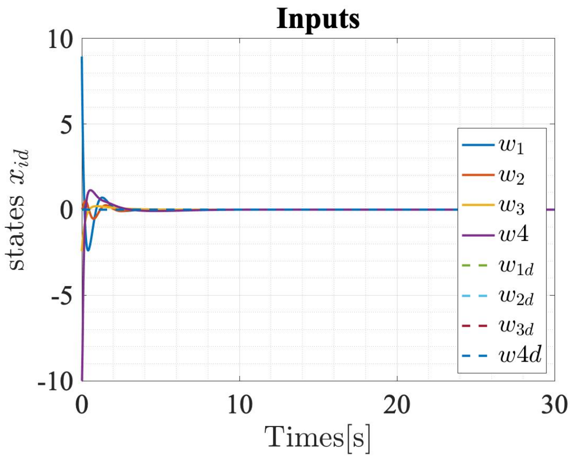

Figure: Inputs without disturbance

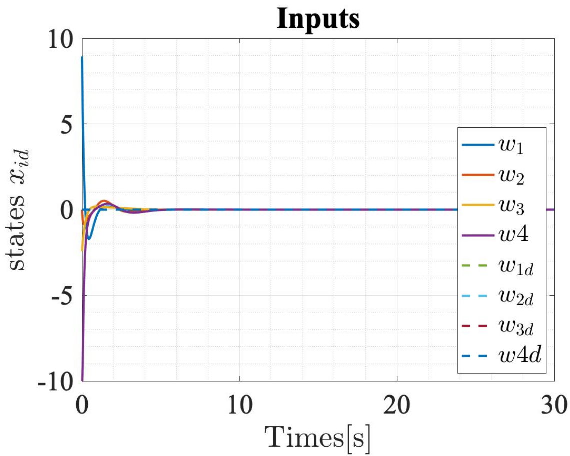

Figure: Inputs with disturbance

Constant disturbance $\left(F_{dis} = [2.50, 1.25, 2.00]^T\right)$ was added to the translational dynamics $(\dot{\boldsymbol{v}})$. The following figures show the outputs:

Postion

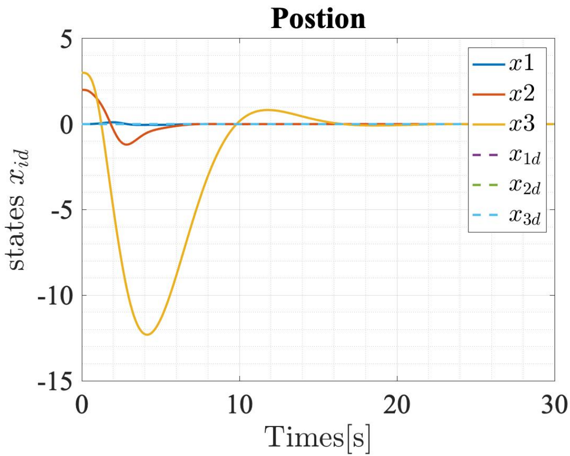

Figure: Postion without disturbance

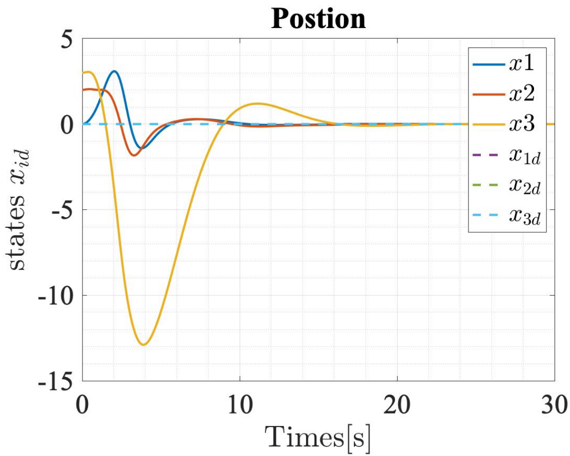

Figure: Postion with disturbance

Velocity

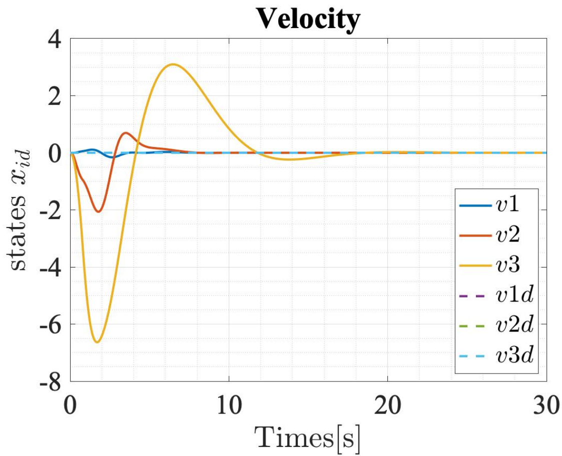

Figure: Velocity without disturbance

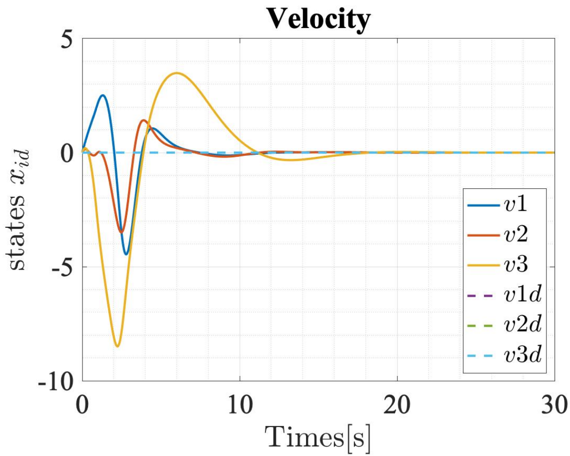

Figure: Velocity with disturbance

Euler

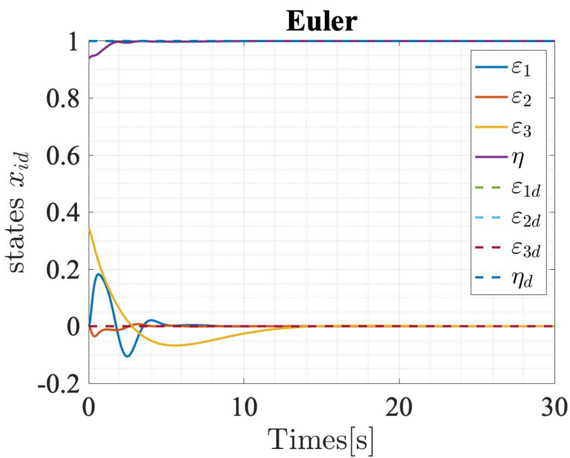

Figure: Euler without disturbance

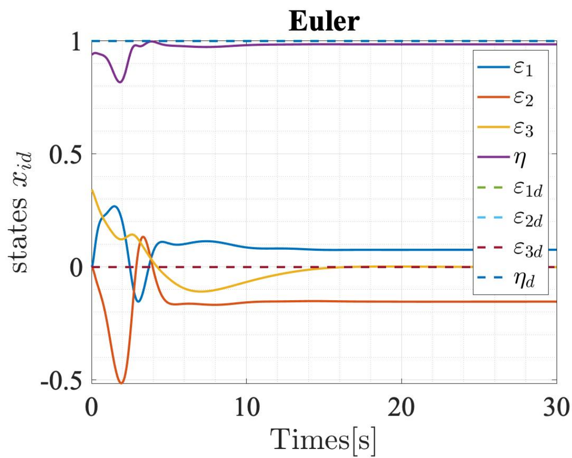

Figure: Euler with disturbance

Angular Velocity

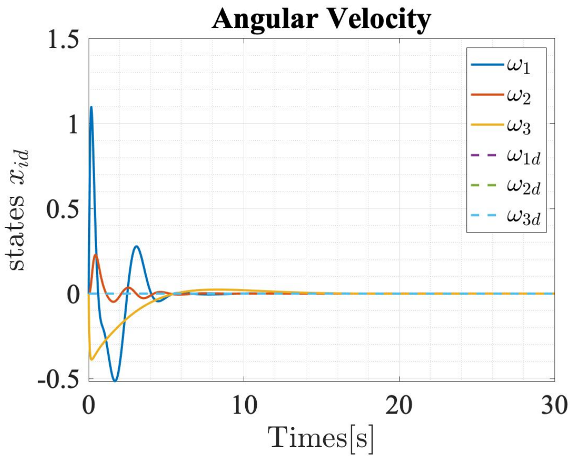

Figure: Angular Velocity without disturbance

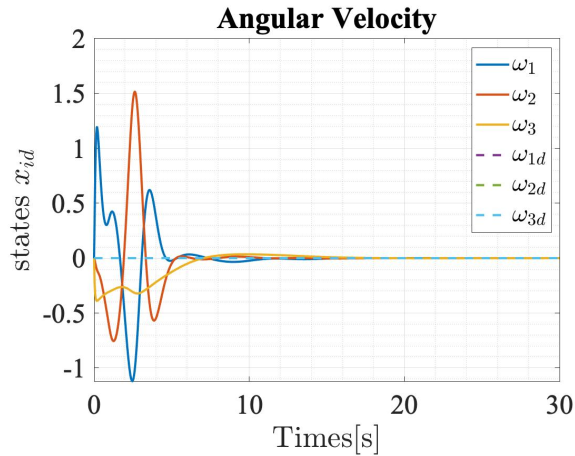

Figure: Angular Velocity with disturbance

Thrust

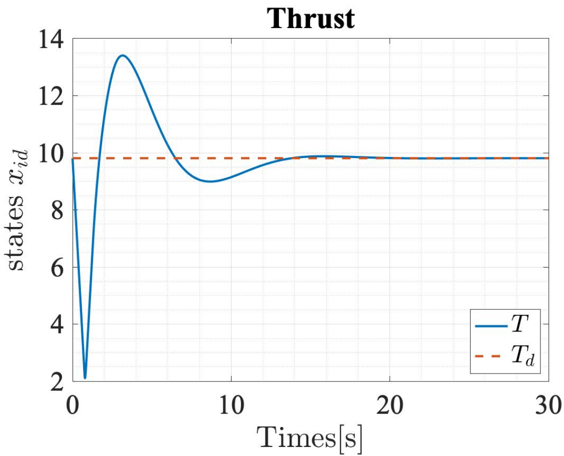

Figure: Thrust without disturbance

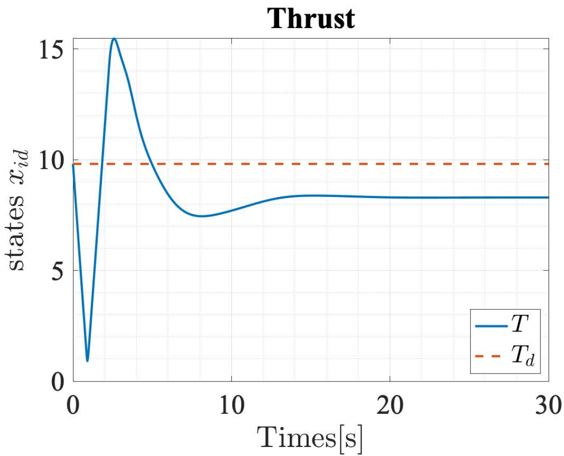

Figure: Thrust with disturbance

Inputs

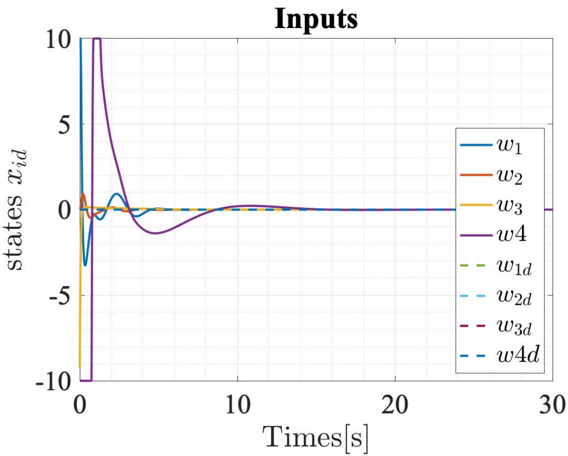

Figure: Inputs without disturbance

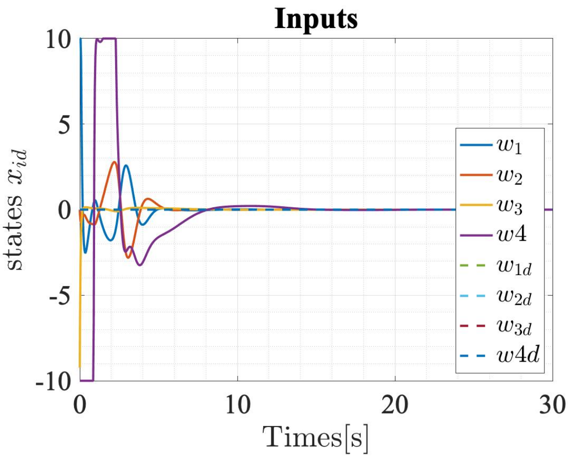

Figure: Inputs with disturbance