LECTURE NOTES
381 COMPUTATIONAL FINANCE

Part II

Portfolio Optimisation and Risk Management

Nalan Gülpınar, Berç Rustem
[ng2, br]@doc.ic.ac.uk

Imperial College London,
Department of Computing

180 Queen's Gate, London, SW7 2AZ, UK.

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1 Introduction to Investment Theory

Financial institutions continuously develop and routinely use very advanced analytical, statistical and numerical techniques in order to create and price financial instruments. Advanced quantitative techniques are used in the trading of financial instruments as well as risk management. As a consequence there is a growing demand for knowledge in the area of quantitative finance and the analysis of financial markets.

Computational finance is a research area which applies the most appropriate features of computer science, statistics, mathematics to model and solve problems in finance. Sophisticated and computationally intensive mathematical and statistical techniques for the analysis of real time financial market data, portfolio management and risk management are needs in modern computerised world. This opens up new opportunities as well as challenges in computational finance for portfolio management, hedging, speculative international trading and financial decisions.

1.1 Terminology

Finance

Finance is defined as the commercial or government activity of managing money, debt, credit, and investment.

Investment

In general term, *investment* is defined as the current commitment of resources in order to achieve later benefits. If resources and benefits take the form of the money, investment is basically the present commitment of money for the purpose of receiving more money later. If you invest in some amount of money, you use your initial money in a way that, your capital will increase, by paying it into a bank, or buying shares or property. An investor is a person or an organisation that buys shares or pays money into a bank in order to receive a profit. Investment science is the application of scientific tools, generally mathematical and computational, to investments.

Cash Flows

There is also a broader viewpoint of investment which is based on the idea of flows of expenditures and receipts spanning a period of time. The objective of investment is to make the pattern of flows over time as profitable as possible for the investor. If the expenditures and receipts are denominated in cash, the receipts at any time period are termed cash flow. Therefore, an investment is defined in terms of its resulting cash flow sequence; the amount of money that will flow to and from an investor over time.

Cash flows are either positive or negative to occur at known specific time periods. This is the case when the flows are known deterministically such as bank interests and mortgage payments. For example, if the time period is one year, a stream defined by a series of numbers, $(-1, 1.20)$, corresponds to an initial payment of 1 pound at the beginning of investment period and the receipt of 1.20 a year later. A cash flow of $(-1500, -1000, +3000)$ describes £1500 and £1000 payments at the beginning of investment period and after a year and receipt of £3000 after two years. A cash flow is denoted by $(a_0, a_1, \cdots, a_n)$ at discrete time periods $t = 0, 1, 2, \cdots, n$.

Basic investment problems can be classified as asset pricing, hedging and portfolio selection. We will broadly describe them below.

Asset Pricing

Given an investment with known payoff characteristics (which may be random), *asset pricing* models are concerned with what the reasonable price is or; equivalently what price is consistent with the other securities that are available. For example, consider an investment opportunity that will pay exactly £110 at the end of year. The question to be asked here is "How much is this investment worth today?". If the current interest rate for one year investment is 10%, then the investment should have a price of exactly £100. In this case, £110 paid at the end of year would correspond to a rate of return 10%.

Hedging

Hedging is the process of reducing the financial risks that either arise in the course of normal business operations or are associated with investments. A form of hedging is the insurance where, by paying a fixed amount (a premium), you can protect yourself against certain specified possible losses. There are many examples of financial risks that can be reduced by hedging. Hedging can be carried out in many ways through futures contracts, options, and other special arrangements.

Portfolio Selection Problem

Pure investment refers to the objective of obtaining increased future return for present allocation of the capital. This is the motivation underlying most individual investments. The investment problem arising from this motivation is called the *portfolio selection* problem. The main issue is determine how to compose the portfolio or where to invest the capital so that the profit, the total return obtained from investment is maximised and the risk is minimised.

1.2 The Basic Theory of Interest Rates

In general terms, *interest* is defined as the time value of money. Different markets use different measures. In financial markets, the price for credit is referred to interest rate. It is determined by demand and supply of credit. It is the price today for money that is to be returned at some future date. Fisher (1930) developed a theory on how interest rates can be derived from the consumption and saving decisions of individuals. Interest rate is useful for comparing investments and summarises returns over different time periods. Interest rate basically scales the initial amount.

In this section, we focus on basic elements of interest rate theory and discounting techniques. The techniques are fundamantel for any financial calculation and foundational for subsequent lectures, particularly bond and stock valuations.

1.2.1 Simple Interest

Suppose you invest an amount of money, A and get back the amount of W_1 after a year. There are various ways to describe how A becomes W_1 after a year. In general, if one-period simple interest rate is r_1, then the total amount is obtained as

$$W_1 = A(1 + r_1)$$

This is basically obtained by multiplying the initial investment by $(1 + r_1)$. The initial amount, A, is called *principal*. Let the subsequent one-period rate be r_2. Then after two years, $t = 2$, W_2 is obtained from the initial investment as follows

$$W_2 = A(1 + r_1 + r_2)$$

After $t = n$ years, we have

$$W_n = A(1 + r_1 + r_2 + \cdots + r_n)$$

where $r_1, r_2, \cdots, r_n$ are simple interest rates at years $t = 1, 2, \cdots, n$, respectively. If the interest rates are constant for n years, that is $r_1 = r_2 = \cdots = r_n = r$, then the total amount of money after n years is calculated as

$$W_n = A(1 + nr)$$

Notice that money grows linearly with time; account value at any time is the sum of the principal and the accumulated interest, which is proportional to time.

Example 1.1

If an investor invests £100 in a bank account that pays 8% interest per year, then at the end of one year, he/she gains a total of £108; that is $108 = 100(1 + 0.08)$.

1.2.2 Compound Interest

Most bank accounts and loans employ some form of compounding, producing compound interest. Consider an account that pays interest at a rate of r per year. If interest is compounded yearly, then after one year, the first year's interest is added to the original principal to define larger principal base for the second year. Thus, during second year the account earns interest on the interest.

Yearly compounding

Let one year compound interest rate be r_1. Under yearly compounding, after a year the principal A provides W_1 which is computed as

$$W_1 = A(1 + r_1)$$

If the subsequent one-period compound interest rate is r_2, then the amount hold at $t = 2$ is computed by reinvesting on W_1 as

$$W_2 = A(1 + r_1)(1 + r_2) = W_1(1 + r_2)$$

In other words, after the second year, the money grows by another factor of $(1 + r_2)$ to $(1 + r_1)(1 + r_2)$. After n years, the initial capital becomes

$$W_n = A(1 + r_1)(1 + r_2) \cdots (1 + r_n)$$

If the compound interest rates for n years are constant, $r_1 = r_2 = \cdots = r_n = r$, then after n years, the principal provides W_n

$$W_n = A(1 + r)^n$$

This is the analytic expression of the account growth under compound interest. Notice that the compound interest shows the geometric growth because of its nth-power form. When n increases, the growth due to compounding can be substantial. The simple and compound interest cases are shown in Figure 1 for an interest rate of 10%. Notice that simple interest leads to linear growth over time, whereas compound interest leads to an accelerated increase defined by the geometric growth.

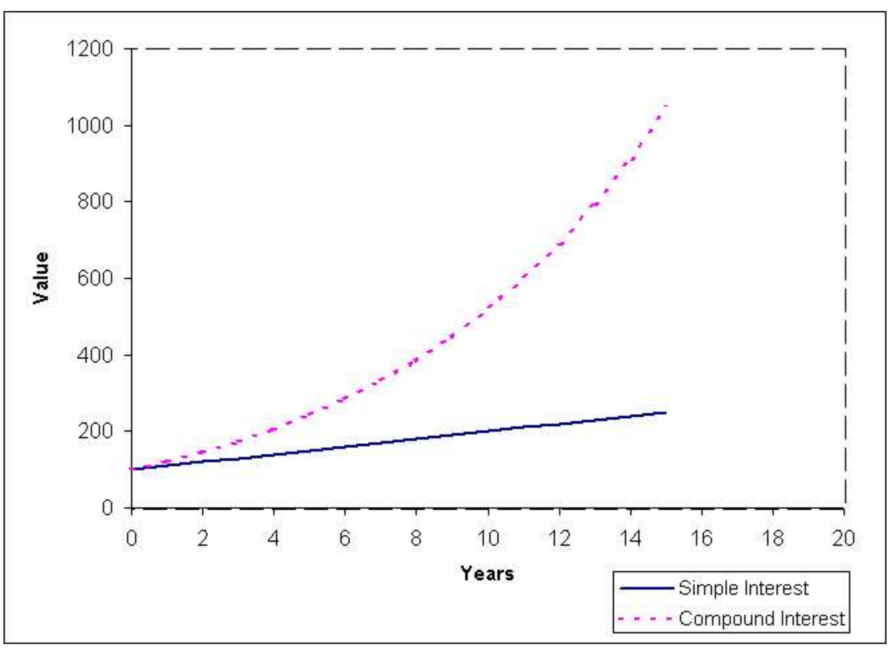

Figure 1: Simple and compound interest rates

Example 1.2

Suppose that you invest £1 in a bank account that pays 8% interest rate per year. What will you have in your account after 5 years under compound and simple interests?

Year	Compound Interest Rate			Simple Interest Rate		
t=1	1.08	=	$1.(1+0.08)^1$	1.08	=	1.(1+0.08)
t=2	1.1664	=	$1.(1+0.08)^2$	1.16	=	1.(1+0.08+0.08)
t=3	1.259712	=	$1.(1+0.08)^3$	1.24	=	1.(1+0.08+0.08+0.08)
t=4	1.36048896	=	$1.(1+0.08)^4$	1.32	=	1.(1+0.08+0.08+0.08+0.08)
t=5	1.4693280768	=	$1.(1+0.08)^5$	1.40	=	1.(1+0.08+0.08+0.08+0.08+0.08)

Example 1.3

Assume that initial amount to invest is $A = £100$ and annual interest rate is constant. What is the compound interest rate and simple interest rate in order to have £150 after 5 years?

Compound Interest Rate			Simple Interest Rate		
W_5	=	$A(1+r)^5$	W_5	=	$A(1+5r)$
150	=	$100(1+r)^5$	150	=	100(1+5r)
1+r	=	$\left[\frac{150}{100}\right]^{\frac{1}{5}}$	1+5r	=	$\frac{150}{100}$
r	=	1.084 -1	5r	=	0.5
r	=	8.4 %	r	=	10 %

Compounding at Various Intervals

Most banks now calculate and pay interest more frequently - quarterly, monthly, or in some cases daily. In this case, the interest rate on a yearly basis is quoted and then the appropriate proportion of that interest rate over each compounding period is applied. For example, quarterly compounding at an interest rate of r per year means that an interest rate of $\frac{r}{4}$ is applied every quarter. Therefore, money left in the bank for one quarter (3 months) will grow by a factor of $1+\frac{r}{4}$ during that quarter. After 1 year money grows by the compound factor of $\left[1+\frac{r}{4}\right]^4$. Notice that for any $r > 0$, $\left[1+\frac{r}{4}\right]^4 > 1+r$ holds.

The effect of compounding on yearly growth is highlighted by stating an *effective interest rate*, which is equivalent to annual interest rate that would produce the same result after 1 year without compounding. For example, an annual rate of 8% compounded quarterly will produce an increase of $\left[1+\frac{0.08}{4}\right]^4 = 1.0824$; hence the effective interest rate is 8.24%. The basic yearly rate, 8%, is called the *nominal rate*.

The general method for frequent compounding is as follows;

1. a year is divided into a fixed number of equally spaced periods–say m periods. For example, for monthly compounding $m = 12$ is set.
2. The interest rate for each of the m periods is thus $\frac{r}{m}$, where r is the nominal annual interest rate.
3. The account grows by $(1+\frac{r}{m})$ during 1 period. After k periods, the growth is $\left[1+\frac{r}{m}\right]^k$.

As a result, the initial amount A becomes $W = A\left[1+\frac{r}{m}\right]^k$ after k periods.

Continuous Compounding

We now divide a year period into smaller periods and apply compounding monthly, weekly, daily or even, minutes and seconds. This leads to continuous compounding where we consider the limit of ordinary compounding as the number of periods m in a year goes to infinity;

$$\lim_{m\to\infty}\left[1+\frac{r}{m}\right]^m = e^r$$

We can also calculate how much an account will have grown after an arbitrary length of time. Let t denote time measured in years. Select a time t and then divide the year into a number m

of small periods. Each period corresponds to a length of $1/m$ and for some k periods $t = \frac{k}{m}$. Using the general formula for compounding, the growth factor for k periods is obtained as

$$\begin{aligned}\left[1+\frac{r}{m}\right]^k &= \left[1+\frac{r}{m}\right]^{mt} \\ \lim_{m\to\infty}\left[1+\frac{r}{m}\right]^k &= \lim_{m\to\infty}\left\{\left[1+\frac{r}{m}\right]^m\right\}^t = e^{rt}\end{aligned}$$

As a result, the investment of A (under the continuous compounding) becomes $W = Ae^{rt}$ after t time periods.

Example 1.4

If the nominal interest rate is 8% per year, then with continuous compounding the growth would be $e^{0.08} = 1.0844$ and hence the effective interest rate is 8.44%. Recall that quarterly compounding produces an effective rate of 8.24%.

Example 1.5

See the following table for the effective interest rates for the different time periods in a year.

Time periods in a year		**Interest rate per period(%)**	**APR (%)**	**Value after a year**			**Effective Interest rate (%)**
annual	1	6	6	1.06^1	=	1.06	6.000
semiannual	2	3	6	1.03^2	=	1.0609	6.090
quarter	4	1.5	6	1.015^4	=	1.06136	6.136
month	12	0.5	6	1.005^{12}	=	1.06168	6.168
week	52	0.1154	6	1.001154^{52}	=	1.06180	6.180
day	365	0.0164	6	1.000164^{365}	=	1.06183	6.183

1.3 Future Value

In the previous section, we only calculate the future value of an investment. Future value is defined as what we have if we invest the cash for some period. In other words; it is basically the value tomorrow of money today. Suppose that you put £1000 into a bank account at a compound interest rate of 6% per year. How does this compare to receiving a payment of £1000 one year from now? You can invest £1000 at 6% so at the end of one year, the sum in your savings account has grown to £1060: you own £60 more a year from now if you invest £1000 today rather than a year hence. If we compare £1000 today with a saving of £1000 two years from now, we will have $\pounds 1123.60 = \pounds 1060 \times 1.06$. Note that we already have £1060 at the end of the first year. We call the amount $\pounds 1123.60$ the *future value* (FV) of £1000 at 6%, 2 years from now.

This process can be extended in order to find out what the future value of A today at $r\%$ is at the end of n years as

$$FV = A(1+r)^n$$

For the cash stream $(a_0, a_1, \cdots, a_n)$, the future value is computed by the formula

$$\begin{aligned} FV &= FV(a_0) + FV(a_1) + \cdots + FV(a_n) \\ &= a_0(1+r)^n + a_1(1+r)^{n-1} + a_2(1+r)^{n-2} + \cdots + a_{n-1}(1+r) + a_n\end{aligned}$$

At the end of n periods, the initial investment will grow to $a_0(1+r)^n$, where the interest rate is r per year. The cash received after the first period a_1 will be in the account only for $n-1$ periods; therefore, it will only grow to $a_1(1+r)^{n-1}$. At the last time period, we get only a_n cash flow and it will not get any interest. If the period is less than a year then this will be divided by the number of periods per year.

Example 1.6

Suppose you get two payments; £5000 today and £5000 exactly one year from now. Put these payments into a saving account and earn interest at a rate of 5%. What is the balance in your savings account exactly 5 years from now?

See the following table for interest and balance values.

Year	Cash Inflow (£)	Interest (£)	Balance (£)
0	5,000	0.00	5,000.00
1	5,000	250.00	10,250.00
2	0	512.50	10,762.50
3	0	538.13	11,300.63
4	0	565.03	11,865.66
5	0	593.28	12,458.94

The future value of the cash flow is also computed as follows;

$$\begin{aligned} FV &= 5000(1+0.05)^5 + 5000(1+0.05)^4 \\ &= \pounds 12,458.94 \end{aligned}$$

1.4 Present Value (Discounting)

The investment today leads to an increased value in the future as a result of interest. This concept can be reversed in time to calculate the value that should be assigned now, in the present, to money that is to be received at a later time. This is the essence of *present value* (PV), which is the value today of a pound tomorrow.

A pound received in the future has less value than a pound received today. The question is that how much you have to put into your bank account today, so that in one year from now, the balance is exactly W, if you accrue interest at a rate of r on the balance. Let's consider two situations which are identical: i) after one year you will receive £110 or, ii) £100 now and deposit it in a bank account for a year at 10% interest. These situations can be restated as "£110 received in a year is equivalent to the receipt of £100 now when the interest rate is 10%" or "£110 to be received in a year has a present value of £100". This is formulated as

$$PV = \frac{W}{(1+r)} = \frac{110}{(1+0.10)}$$

A similar transformation applies to future obligations such as the repayment of debt. In order to calculate the present value of this obligation, you determine how much money you would need now to cover the future obligation. The process of evaluating future obligations as an equivalent present value is also referred to *discounting*. The present value of the future monetary amount is less than the face value of that amount, so the future value must be discounted to obtain the present value. The factor by which the future value must be discounted is called *discount factor*. Therefore, the discount factor for a year investment for the example above is

$$d_1 = \frac{1}{(1+r)}$$

If the bank quotes rates with compounding, then such a compound interest rate must be used in the calculation of the present value. Suppose that the annual interest rate r is compounded at the end of each of m equal periods per year and W will be received at the end of the kth period. The discount factor is computed as

$$d_k = \frac{1}{\left[1+\frac{r}{m}\right]^k}$$

The present value of a payment of W to be received kth periods in the future is

$$PV_k = d_k W$$

Present Value of Cash Flow Stream

Assets (investments) typically generate multiple future cash flows–a stream of cash flows. We convert each component of the cash flow stream of the asset into an equivalent amount today by using the discount factor for the time of that cash flow. Adding these equivalent amount gives the preesent value of the cash flow stream. In notation, the present value of the cash flow stream $(a_0, a_1, \cdots, a_n)$ is calculated as:

$$\begin{aligned} PV &= PV(a_0) + PV(a_1) + PV(a_2) + \cdots + PV(a_n) \\ &= a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n} \end{aligned}$$

Notice that the present value of initial investment is itself since no discounting is necessary. However, for the present value of a_k, the flow must be discounted k periods.

Present Value of Frequent Compounding

Let r be nominal interest rate which is compounded at m equally spaced periods per year. For n periods, consider the cash flow stream $(a_0, a_1, \cdots, a_n)$ which is paid initially and at the end of each period. Then the present value of the cash flow is calculated by the following formula

$$\begin{aligned} PV &= a_0 + \frac{a_1}{\left[1+\frac{r}{m}\right]} + \frac{a_2}{\left[1+\frac{r}{m}\right]^2} + \cdots + \frac{a_n}{\left[1+\frac{r}{m}\right]^n} \\ &= \sum_{k=0}^{n} \frac{a_k}{\left[1+\frac{r}{m}\right]^k} \end{aligned}$$

Present Value of Continuous Compounding

Suppose that the nominal interest rate r is compounded continuously and cash flow $(a_0, a_1, \cdots, a_n)$ occurs at times $t = 0, 1, \cdots, n$. Hence, the cash flow at time t is a_t. The present value of the cash flow under the continuous compounding is formulated as

$$PV = \sum_{t=0}^{n} a_t e^{-rt}$$

Example 1.7

Assume that you have just bought a new computer for £3,000. The payment terms are two years the same as the cash. If you can earn 8% on your money, how much money should you set aside today in order to make the payment when due in two years?

$$PV = \frac{3000}{(1+0.08)^2} = £2,572.02$$

Example 1.8

Consider the cash flow stream $(-2, 1, 1, 1)$. Calculate the present and future values using an interest rate of 10%.

$$\begin{aligned} PV &= -2 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \frac{1}{(1.1)^3} = 0.487 \\ FV &= -2 \times (1.1)^3 + 1 \times (1.1)^2 + 1 \times (1.1) + 1 = 0.648 \end{aligned}$$

As it can be seen from this example the following relation holds;

$$PV = \frac{FV}{(1.1)^3} = \frac{0.648}{1.331} = 0.487$$

Example 1.9

Show that the relationship $PV = \frac{FV}{(1+r)^n}$ between present value and future value of a cash flow holds.

Net Present Value (NPV)

The time value of money has a nice application in investment decisions of firms. In order to determine the exact cost or benefit of an investment decision we subtract the cost of investment. This yields the net present value of the investment as

$$NPV = -Cost + PV \tag{1}$$

In deciding whether or not to undertake an investment NPV is used in the following way: invest in any project with a positive net present value. This is termed the net present value rule.

Example 1.10

At these days buying a two bedroom flat in London costs £150,000 on average. Experts predict that a year from now it will cost £175,000. Thus you would be investing £150,000 now in the expectation of the expected £175,000 a year. You have to make a decision on whether you should buy a flat or invest £150,000 on safe government securities with 6

You should buy a flat if the present value of the expected £175,000 payoff is greater than the investment of £150,000. Therefore, you need to ask, "what is the value today of £175,000 to be received a year from now, and is that present value greater than £150,000?". The PV and NPV are calculated as

$$\begin{aligned} PV &= \frac{175,000}{1+0.06} = £165,094 \\ NPV &= £165,094 - £150,000 = £15,094 \end{aligned} \tag{2}$$

There is another way to evaluate investment projects. It involves calculating rates of return on projects. This rule says that investment is worth under-taking if its rate of return exceeds the cost of capital. This is termed the rate of return rule. The rate of return on investment in the residential property is simply the profit as a proportion of the initial outlay.

$$\begin{aligned} Rate\ of\ return &= \frac{Profit}{Investment} \\ &= \frac{175,000 - 150,000}{150,000} \\ &= 0.167\ or\ 16.7\% \end{aligned}$$

Since the 16.7% return on the residential building exceeds the 6% opportunity cost, you should go ahead with the investment.

Example 1.11

The following cash flow is obtained from the construction and sale of an office building

Year 0	**Year 1**	**Year 2**
-150,000	-100,000	+300,000

Given 7% interest rate, create a present value worksheet and show the net present value, NPV.

t	a_t	d_t	PV_t
0	-150,000	1.0	-150,000
1	-100,000	$\frac{1}{1+0.07} = 0.935$	-93,500
2	+300,000	$\frac{1}{(1+0.07)^2} = 0.873$	+261,900
			NPV = 18,400

1.5 Annuity Valuation

An annuity is a cash flow stream which is equally spaced and equal amount. For example, £250,000 mortgage at 9% per year, which is paid off with a 180 month annuity of £2,535.67. Now we will show how to calculate the present value of annuity and how to determine the size of annuity. Let $d = \frac{1}{1+r}$ be discount factor; that is the present value of £1 at the end of one period. Consider a cash flow $a_1 = a_2 = \cdots = a_n = a$ payments per year $t = 1, \cdots, n$, respectively, and r is the annual interest rate.

The present value of n period annuity is computed by

$$PV_A = \frac{a.d(1-d^n)}{1-d} \tag{3}$$

This can be easily shown by using the formula of the present value of the cash flow.

$$\begin{aligned} PV_A &= \sum_{t=1}^{n} \frac{a}{(1+r)^t} &(4)\\ &= \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + \cdots + \frac{a}{(1+r)^n} \end{aligned}$$

If we multiply both side of the last equation with $\frac{1}{(1+r)}$ we obtain

$$\frac{1}{(1+r)} PV_A = \frac{a}{(1+r)^2} + \frac{a}{(1+r)^3} + \cdots + \frac{a}{(1+r)^{n+1}} \tag{5}$$

By subtracting (5) from (4),

$$\begin{aligned} \left(1 - \frac{1}{(1+r)}\right) PV_A &= \frac{a}{(1+r)} - \frac{a}{(1+r)^{n+1}} &(6)\\ \frac{r}{(1+r)} PV_A &= \frac{a}{(1+r)} - \frac{a}{(1+r)^{n+1}} &(7)\\ PV_A &= \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right) &(8) \end{aligned}$$

which is equivalent to right side of the formula (3) for $d = \frac{1}{1+r}$. When we consider m periods per year, the interest rate corresponding to per period is $i = \frac{r}{m}$ and the discount factor is $d = \frac{1}{1+\frac{r}{m}} = (1+i)^{-1}$. This simplifies Equation (3) as

$$PV_A = \frac{a.(1-(1+i)^{-n})}{i} \tag{9}$$

Notice that Equations (3) and (9) are equivalent. Suppose that the kth payoff is $a(1+g)^k$ (instead of a constant cash flow) where g is the growth rate of the payoff per year. The present value of growing annuity is calculated as

$$PV_{GA} = \frac{a}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right]$$

Example 1.12

Suppose that you borrow £250,000 mortgage and repay over 15 years. The interest rate is 9% annually and payments are made monthly. The effective periodic rate of interest is 9%/12 = 0.75% per month. What is the monthly payment which is needed to pay off the mortgage?

Given $n = 180,\ m = 12,\ T = 15,\ r = 9\%,\ PV_A = £250,000$, we can calculate the discount factor

$$\begin{aligned} d &= \frac{1}{\left[1+\frac{r}{m}\right]}\\ &= \frac{1}{\left[1+\frac{0.09}{12}\right]} = 0.9925558\\ PV_A &= \frac{a.d(1-d^n)}{1-d}\\ 250,000 &= \frac{a\left(0.9925558(1-0.9925558^{180})\right)}{(1-0.9925558)}\\ &= a(98.59319)\\ a &= \frac{250,000}{98.59319} = £2,535.67 \end{aligned}$$

The present value of perpetuity: Perpetuities are assets that generate the same cash flow forever. In other words, the annuity is called a *perpetuity* when the number of payments becomes infinite. Perpetuity pays a coupon at the end of each year and never matures. The British consol bond is an example of a perpetuity.

Letting n go to infinity, the present value of annuity gives the value of a perpetuity as

$$PV_P = \frac{a}{r} \tag{10}$$

In words, the present value of a sure stream of payments of a, discounted at a rate of r, is equal to $\frac{a}{r}$. An alternative proof is based on the present value of the cash flow stream of

$$PV_P = \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + \cdots + \frac{a}{(1+r)^n} + \cdots$$

The formula for an infinite geometric progression $1 + x + x^2 + x^3 + \cdots$ is $\frac{1}{1-x}$. In our context $x = \frac{1}{1+r}$ and we need to note that the first term is $\frac{1}{1+r}$ not 1. Thus,

$$\begin{aligned}
PV_P &= a\left[\frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots\right] \\
&= \frac{a}{1+r}\left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots\right] \\
&= \frac{a}{1+r}\left[\frac{1}{1-x}\right] = \frac{a}{1+r}\left[\frac{1}{1-\frac{1}{1+r}}\right] \\
&= \left[\frac{a}{1+r}\right]\left[\frac{1+r}{r}\right] = \frac{a}{r}
\end{aligned}$$

For m periods per year (10) becomes

$$PV_P = \frac{a}{\left[\frac{r}{m}\right]} = \frac{a}{i} \tag{11}$$

The present value of growing perpetuity: In the same way the present value of the growing perpetuity can be found as

$$PV_{GP} = \frac{a}{r-g}$$

Let the growth rate be g. Then the present value of cash flow $a, a(1+g), \cdots, a(1+g)^n, \cdots$ is

$$PV_{GP} = \frac{a}{(1+r)} + \frac{a(1+g)}{(1+r)^2} + \frac{a(1+g)^2}{(1+r)^3} + \cdots + \frac{a(1+g)^{n-1}}{(1+r)^n} + \cdots$$

The common factor is $x = \frac{1+g}{1+r}$ and

$$\begin{aligned}
PV_{GP} &= \frac{a}{1+r}\left[1 + \frac{1+g}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \cdots\right] \\
&= \frac{a}{1+r}\left[\frac{1}{1-x}\right] = \frac{a}{1+r}\left[\frac{1}{1-\frac{1+g}{1+r}}\right] \\
&= \left[\frac{a}{1+r}\right]\left[\frac{1+r}{r-g}\right] = \frac{a}{r-g}
\end{aligned}$$

2 Bonds and Stocks

2.1 Valuing Bonds

2.1.1 Introduction to Bonds

Bonds are securities that establish a creditor relationship between purchaser and the issuer. The issuer receives a certain amount of money in return for the bond and is obligated to repay the principal at the end of lifetime of the bond to the purchaser. Bonds typically require coupon or interest payments which are determined as part of the contracts. Therefore, bonds are also called fixed income securities. The interest payment is called coupon payment. The standard coupon payments are made at regular intervals and represent the interest on bonds. The final interest payment and the principal amount are paid at a specific date of maturity.

Think of a bond in the context of a mortgage, we usually need to borrow when we buy a house. Similar to your mortgage with the bank, bonds are issued by a borrower (issuer) to a lender (investor). When you buy a bond and loan your money to the borrower there is also a pre-specified period of time that the loan has to be repaid; this is called *maturity date*. The *par value or face value* is an amount paid to bond holder at maturity. You are not loaning your money free though, so the borrower must also pay you a premium or "coupon" at a pre-determined interest rate in exchange for using your money. The interest payments are usually made every 6 months until maturity. There are some exceptions to this such as zero coupon bonds which instead give you a large lump-sum payment once the bond has reached maturity.

It is important to know that interest payments are not the only way you can profit from bonds. Publicly traded bonds often fluctuate in price, similar to stocks, therefore it is possible to have a capital gain (or loss) once you sell the bond or once it matures. As discussed in the next section, bond prices and its yield (or overall return) are inversely related, that is, as bond prices rise, yields shrink. Should bond prices fall then the yield will increase.

What types of bonds are there?

Bonds are issued by many different entities, including corporations, governments and government agencies. Some of bonds are described below;

Corporate Bond is issued by a corporation, usually through the public securities markets, which trades publicly just like stocks. They offer a higher yield because they carry a higher default risk than government bonds. Three major types of corporate bonds are mortgage bonds, debentures and convertible bonds. Mortgage bonds are secured by real property such as real estate or buildings. Debentures are unsecured debt and backed only by the name and goodwill of the corporation. Convertible bonds can be exchanged for the stock in the corporation.

Municipal Bond is issued by a municipality or state. The advantage of "muni's" is that the returns are tax-free and no tax is paid on the interest you earn. Since it is tax-free, the yield is usually lower than for a taxable bond because the "tax savings" are priced into the bond. Muni's are very popular with people in higher tax brackets.

Treasury Bond is issued by a national government. In most cases, they are considered safe investments because the taxing authority of the government backs them. In the US, interest on Treasury bonds is not subject to state income tax. T-bonds have maturity greater than 10 years, while notes and bills have lower maturity. They pay coupons twice per year, with the principal paid in full at maturity.

Treasury Notes are similar to the treasury bonds except a treasury note is issued for a shorter time period, typically 1 to 10 years. They pay coupons twice per year, with the principal paid in full at maturity.

Treasury Bills are held for a shorter time period, maturity vary from 3, 6, or 9 months (sometimes up to 12 months). There is no coupon payment and return is paid at maturity. Therefore, bills are priced at a discount. T-bills are zero coupon bonds, so the only cash flow is the face value received at maturity.

How Do You Read a Bond Table?

An example of a bond table is presented in Figure 2. The meaning of each column in the table is described as follows.

	Coupon	Mat. date	Bid $	Yld%
Corporate				
AGT Lt	8.800	Sep 22/25	100.46	8.75
Air Ca	6.750	Feb 02/04	94.00	9.09
AssCap	5.400	Sep 04/01	100.01	5.38
Avco	5.750	Jun 02/03	100.25	5.63
Bell	6.250	Dec 01/03	101.59	5.63
Bell	6.500	May 09/05	102.01	5.95
BMO	7.000	Jan 28/10	106.55	6.04
BNS	5.400	Apr 01/03	100.31	5.24
BNS	6.250	Jul 16/07	101.56	5.95
CardTr	5.510	Jun 21/03	100.52	5.27
Cdn Pa	5.850	Mar 30/09	93.93	6.83
Clearn	0.000	May 15/08	88.50	8.61
CnCrTr	5.625	Mar 24/05	99.78	5.68
Coke	5.650	Mar 17/04	99.59	5.80

Figure 2: An example of bond table

Column 1: Issuer is the company, state (or province), or country that is issuing the bond.

Column 2: Coupon refers to the fixed interest rate that the issuer pays to the lender. The coupon rate varies by bond.

Column 3: Maturity Date is the date when the borrower will pay the lenders (investors) their principal back. Typically only the last two digits of the year are quoted, 25 means 2025, 04 is 2004, etc.

Column 4: Bid Price is the price in which someone is willing to pay for the bond. It is quoted in relation to 100, no matter what the par value is. Think of the bond price as a percentage, a bond with a bid of 93 means it is trading at 93% of its par value.

Column 5: Yield indicates annual return until the bond matures. Yield is calculated by the amount of interest paid on a bond divided by the price. It is a measure of the income generated by a bond.

2.1.2 Bond Prices

We use the following notation:

Notation	Explanation
P	market price of the bond
F	principal payment (face or par value)
M	annual coupon rate of the bond: yearly coupon payment
m	the number of coupon payments per year
c	periodic coupon rate ($c = M/m$)
R	APR (Annual Percentage Rate) for today's cash flows
i	effective periodic interest rate ($i = R/m$)
t	the number of years to maturity
λ	yield to maturity
n	the total number of periods ($n = mt$)

The coupon amount is described as a percentage of the face value. For example, consider a 20% coupon bond with par value of £100. The bond has three years from now to maturity and pays the coupon semi-annually. The annual percentage rate is 13%. The bond will have a coupon of £20 in a year and £10 after six months. The notation is, therefore,

$$F = £100,\ M = £20,\ c = £10,\ m = 2,\ t = 3,\ n = 6, R = 13\%,\ i = 6.5\%.$$

The price of the bond is simply the sum of the present values of all future payments.

$$\begin{aligned} P &= PV[F] + \sum_{k=1}^{n} PV_k \left[\frac{M}{m}\right] \\ &= \frac{F}{\left[1+\frac{R}{m}\right]^n} + \sum_{k=1}^{n} \frac{\frac{M}{m}}{\left[1+\frac{R}{m}\right]^k} \end{aligned}$$

The price of a zero coupon bond is

$$P = \frac{F}{\left[1+\frac{R}{m}\right]^n}$$

Notice the immediate consequences of this formula: higher interest rates imply lower zero coupon bond prices.

Example 2.1

Reconsider the three-year coupon bond given above. In order to find price of the bond, we apply the general formula

$$\begin{aligned} P &= \frac{100}{(1+0.065)^6} + \sum_{n=1}^{6} \frac{10}{(1+0.065)^n} \\ &= \frac{100}{1.065^6} + \frac{10}{1.065} + \frac{10}{1.065^2} + \frac{10}{1.065^3} + \frac{10}{1.065^4} + \frac{10}{1.065^5} + \frac{10}{1.065^6} \\ &= 68.53 + 9.39 + 8.82 + 8.28 + 7.77 + 7.30 + 6.85 \\ &= £116.95 \end{aligned}$$

Bond Price Dynamics

It is important to show effects of changes in the interest rate on the present value of the cash flows obtained from a bond investment. Recall that we have expressed the present value in terms of the interest rate and the cash flows. Therefore, the direction of change is determined by the first derivative of the price function. Consider the price function of a zero coupon bond, P, given

$$P = \left[1+\frac{R}{m}\right]^{-n} .F$$

The first derivative of the price equation above with respect to interest rate is

$$\begin{aligned} \frac{\partial P}{\partial R} &= \frac{-n}{m}\left[1+\frac{R}{m}\right]^{-n-1} .F \\ &= -t\left[1+\frac{R}{m}\right]^{-n-1} .F \end{aligned}$$

which gives the change of the price (in currency) of the bond in response to a change in the level of the interest rate. Notice that the sign of the derivative is negative, which means that the price of the zero coupon bond or the present value of the cash flow decreases with an increase in the interest rate. Note that the first derivative of the price function is a first order (or linear) approximation to the slope of the function. This approximation is accurate for small changes in the interest rate. Hence, the price response ΔP to a change in the interest rates by ΔR is approximately

$$\Delta P = -t(1+i)^{-n-1}.F\Delta R$$

In other words, it is an approximation for the absolute price change in pounds ΔP in response to a shift in the interest rate by ΔR. The percentage response of the bond price is obtained by dividing the derivative by the value of the bond as follows;

$$\begin{aligned}\frac{\partial P}{\partial R}\frac{1}{P} &= \frac{-t(1+i)^{-n-1}.F}{(1+i)^{-n}.F} \\ &= \frac{-t}{1+i}\end{aligned}$$

The percentage price change of the bond in response to a change by ΔR is

$$\frac{\Delta P}{P} = \frac{-t}{1+i}\Delta R$$

As it can be easily seen from this expression, the percentage price change of a zero coupon bond is proportional to the maturity of the bond.

Example 2.2

Consider a zero coupon bond with a term to maturity of 5 years, a face value of £1000. The interest rate is 8%. The change of price of the bond in response to change in the level of interest rate is computed as

$$\begin{aligned}\frac{\partial P}{\partial R} &= -1000 \times 5 \times (1+0.08)^{-6} \times \frac{1}{100} \\ &= 31.5085\end{aligned}$$

It means that when interest rate changes by one percentage point, then the bond price changes by £31.51. See the following table for the comparison of this with the exact numbers obtained from different interest rates of $R = 8\%, 9\%, 7\%$.

Interest Rate	Bond Value	Change (£)	Change (%)
8%	680.5832		
9%	649.9314	-30.6518	-4.5038
7%	712.9862	+32.4030	+4.7611
Average		31.5274	4.6324

Hence, the error for the case of an interest rate movement of one percentage point is small.

It said that a coupon bond sell

- *at face value* if the coupon rate is equal to the market interest rate
- *at a discount* if the coupon rate is below the market interest rate
- *at a premium* if the coupon rate is above the market interest rate

Bonds Volatility

The absolute value of percentage price change is also called *volatility.* The factors affecting bond volatility are *level of yield, time to maturity*, and *coupon rate.*

2.1.3 Yield to Maturity

For bond valuation, the future payments are discounted at the same interest rate R. Now, we reverse the present value procedure. Given the bond market price, we solve for the interest rate that equates the present value of the cash flows to its price. The solution is the interest rate at which the present value of the stream of payments (consisting of the coupon payments and the final face value redemption payment) is exactly equal to the current price. This value is termed the *yield to maturity* (YTM). Yield is the interest rate implied by the payment structure.

Suppose that a bond with a face value F makes m coupon payments of $\frac{M}{m}$ per period and there are n periods remaining. The coupon payments sum to M within a year. The current price of the bond is P. Then the yield to maturity is the value of λ such that the following equality holds

$$P \quad = \quad \frac{F}{\left[1+\frac{\lambda}{m}\right]^{n}} + \sum_{k=1}^{n} \frac{\frac{M}{m}}{\left[1+\frac{\lambda}{m}\right]^{k}}$$

Note that the first term is the present value of the face value payment. The kth term in the summation is the present value of the kth coupon payment $\frac{M}{m}$. The sum of the present values is set to equal to the bond price. As a result, the yield is determined at that particular discount rate that makes the present value of all future payments to the bondholder equal to the current market price.

YTM includes all the interest payments you will receive up to maturity and also assumes that you reinvest the interest payment at the same rate as the current yield on the bond. YTM also accounts for the difference between the par value and the current price of the bond. Whenever the price of the bond is equal to its par value or principal value, then the yield must be equal to the coupon rate. If the current price of the bond is higher than par value, we say the bond is trading at a premium. Then the market has used a discount rate lower than the coupon rate. Conversely, if the bond price is below its par value, we say it is trading at a discount, and the yield must exceed the coupon rate.

There is no easy way to calculate the yield to maturity for coupon-paying bond. A computer program solves the equation numerically using iterative methods.

For a zero coupon bond the bond price is $P = F\left[1+\frac{\lambda}{m}\right]^{-n}$. and the yield is found

$$\lambda = m\left[\left\{\frac{F}{P}\right\}^{\frac{1}{n}} - 1\right]$$

With continuous compounding the bond price is computed as $P = e^{-\lambda t}F$. Hence, rearranging and taking logarithms, the yield is obtained as $\lambda = \frac{1}{t} ln\left[\frac{F}{P}\right]$

Example 2.3

Consider a zero coupon bond with par value £100. It matures in six years from now and is trading £55. What is the yield under annual, semiannual and continuous compoundings?

Annual compounding: $\lambda = \frac{100}{55}^{\frac{1}{6}} - 1 = 10.48\%$

Semiannual compounding: $\lambda = 2(\frac{100}{55}^{\frac{1}{12}} - 1) = 10.22\%$.

Continuous compounding: $\lambda = \frac{1}{6} ln(\frac{100}{55}) = 9.96\%$

Notice that the more periods per year, the lower the yield we obtain and the yield with continuous compounding is always the lowest.

Price-Yield Curve

Price-yield curves show how yield and price are related. Consider bonds with 5%, 10%, 15% coupon rates and 10 years to maturity. The bond 10% coupon rate means 10% of the face value is paid each year. Price-yield curves of the bonds are plotted in Figure 3 where the price is shown as a function of YTM expressed in percentage terms. Yield to maturity varies between 0 and 25%. It can be observed that

- the price-yield curve has a negative slope; that is, price and yield have an inverse relation. If yield goes up, price goes down.
- when $\lambda = 0$, the bond is priced as if it offered no interest.
- the price of the bond must tend to zero as the yield increases - large yields imply heavy discounting, so even the nearest coupon payment has little present value. Overall the shape of the curve is convex.

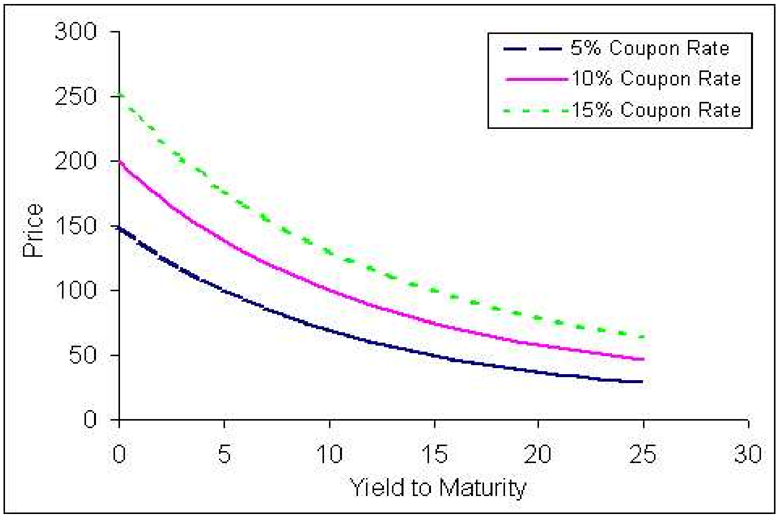

Figure 3: Price-yield curves

- when the value of the bond is equal to the par value, it means that the yield is exactly equal to the coupon rate. The bond is called a par bond.
- the price-yield curve rises as the coupon rate increases.

If the yield is high, then it gives the low volatility. This also implies that volatility is asymmetric respect to direction of yield change (for large change). For a given yield and maturity, the lower the coupon rate implies the greater the price volatility.

Example 2.4

This example demonstrates some of problems in using the yield to maturity. Consider two types of bonds A and B. They both cost £1000, have 3 years maturity, and compounded annually. The cash flows and yields are presented in the following table.

	Bond A	**Bond B**
Year 1	145	430
Year 2	145	430
Year 3	1145	430
Yield (%)	14.5	13

From this table it might appear as if Bond A is better since it has a higher yield. However, the yield to maturity rule might not lead us the higher returns since there is an assumption of equal annual rates of return in computing the yield to maturity.

When we consider the different interest rates over periods 1, 2, and 3, this might not be the case. Suppose $i_1 = 10\%$, $i_2 = 20\%$, $i_3 = 15\%$ interest rates prevailing over periods 1, 2, and 3, respectively. The present value of two cash flows for the bonds A and B are

$$\begin{aligned} P(A) &= \frac{145}{1.1} + \frac{145}{1.1 \times 1.2} + \frac{1145}{1.1 \times 1.2 \times 1.15} \\ &= £996 \\ P(B) &= \frac{430}{1.1} + \frac{430}{1.1 \times 1.2} + \frac{350}{1.1 \times 1.2 \times 1.15} \\ &= £1000 \end{aligned}$$

From these calculations, the present value of bond B is greater than the present value of bond A. As a result, the yield to maturity rule does not always work as a guide to higher returns, as we vary the pattern of i_1, i_2, i_3.

Time to Maturity

Now let's consider the influence of the time to maturity on price of the bond. Bonds with long maturities have steeper price-yield curves than bonds with short maturities. Hence the prices of long bonds are more sensitive to interest rate changes than those of short bonds. Figure 4 presents the price-yield curves for three different bonds. Each bond has 10% coupon rate with

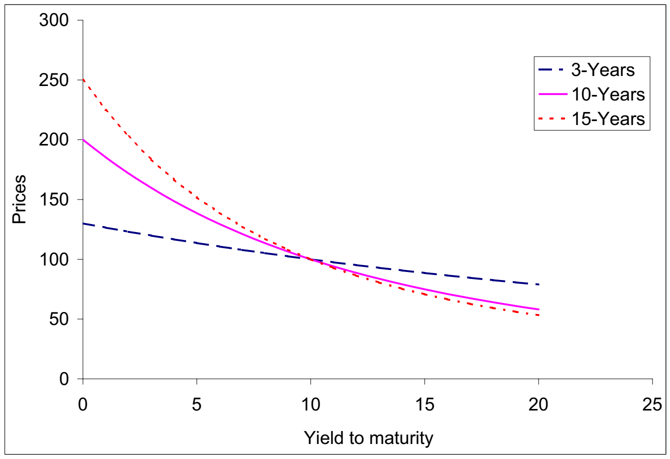

Figure 4: Price-yield curves and maturity.

par value of £100, and 3 years, 10 years, and 15 years to maturity. The bonds are at par when the yield is 10%; hence the curves all pass through the common par point. However, the curves pivot upward around that point by various amounts, depending on the maturity. As maturity is increased, the price-yield curve becomes steeper. This indicates that longer maturities imply greater sensitivity of price to yield. As it will be discussed in the next section, duration is high if maturity is long; so higher maturity implies higher volatility.

Bond holders are subject to yield risk as well as the interest rate risk. If yields change, bond prices also change. This is immediate risk, affecting the near-term value of the bond. You may continue to hold the bond and thereby continue to receive promised coupon payments and the face value at maturity. The cash flow stream is not affected from interest rates; however, if you plan to sell the bond before maturity, the price will be governed by the price-yield curve.

Example 2.5

Show the price-yield relation in tabular form for bonds with a 8% coupon rate, 1, 3, 5, and 10 years to maturity and 3%, 5%, 8%, and 10% yield values, respectively.

Time to	Yield			
Maturity	3%	5%	8%	10%
1 year	104.85	102.85	100	98.18
3 years	114.14	108.16	100	95.02
5 years	122.89	112.98	100	92.41
10 years	142.65	123.16	100	87.71

For example, the price of bound with yield to maturity of $\lambda = 5\%$, and 3 years to maturity is calculated as

$$\begin{aligned} P &= \frac{100}{(1+0.05)^3} + \frac{8}{(1+0.05)} + \frac{8}{(1+0.05)^2} + \frac{8}{(1+0.05)^3} \\ P &= £108.1697 \end{aligned}$$

Notice that the bond with 10 years to maturity is much more sensitive to yield changes than the one with 1 year maturity. For the 8% coupon bond at yield $\lambda = 8\%$, the price will be equal to the par value of $F = £100$.

2.1.4 Duration

Maturity itself does not give a complete quantitative measure of interest rate sensitivity. Another measure of time length termed *duration* gives a direct measure of interest rate sensitivity. It is calculated as a weighted average of the times that cash flow payments are made. The weighting coefficients are the present values of the individual cash flows.

Suppose that cash flows $a_0, a_1, \cdots, a_n$ are received at times $t_0, t_1, t_2, \cdots, t_n$, respectively. The duration of the stream, which has units of time, can be formalised as

$$D = \frac{PV(a_0)t_0 + PV(a_1)t_1 + \cdots + PV(a_n)t_n}{PV}$$

where $PV(a_k)$ is the present value of cash flow a_k at k and PV is the total present value of the cash flow. The expression for D is indeed a weighted average of the cash flow times. Hence D itself has units of time. When the cash flows are all nonnegative, then it is clear that $t_0 \leq D \leq t_n$. Duration is a time intermediate between the first and the last cash flows. It is a fact that a zero-coupon bond, which makes only a final payment at maturity, has a duration equal to its maturity date, $D = T$. Nonzero-coupon bonds have durations strictly less than their maturity dates. This shows that duration can be viewed as a generalised maturity measure. If maturity is long, then duration is high. So higher maturity implies higher volatility.

Macaulay Duration

The definition of duration given above is a bit vague about how the present value is calculated, what interest rate to use. For a bond it is natural to base those calculations on the bond's yield. If the yields is used, the general duration formula (D) becomes the *Macaulay duration*, (D_{mac}). When time periods are given in terms of years as $t = 1, 2, \cdots, T$ and the corresponding cash flow a_t, the Macaulay duration is defined as

$$\begin{aligned} D_{mac} &= \frac{1 \times \frac{a_1}{(1+\lambda)} + 2 \times \frac{a_2}{(1+\lambda)^2} + \cdots + T \times \frac{a_T}{(1+\lambda)^T}}{\frac{a_1}{(1+\lambda)} + \frac{a_2}{(1+\lambda)^2} + \cdots + \frac{a_T}{(1+\lambda)^T}} \\ &= \frac{\sum_{t=1}^{T} t \times \frac{a_t}{(1+\lambda)^t}}{\sum_{t=1}^{T} \frac{a_t}{(1+\lambda)^t}} = \frac{\sum_{t=1}^{T} t \times PV(a_t)}{P} \end{aligned}$$

Suppose that a financial instrument makes payments m times per year, with the payment in period k being a_k and there are n periods remaining. The Macaulay duration becomes

$$D_{mac} = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{a_k}{(1+\frac{\lambda}{m})^k}}{PV}$$

where λ is the yield to maturity and $PV = \sum_{k=1}^{n} \frac{a_k}{\left[1+\frac{\lambda}{m}\right]^k}$. Note that the factor $\frac{k}{m}$ in the numerator of the formula for D_{mac} is time, measured in years.

Example 2.6

A bond has a maturity of 5 years with a coupon payments of £50 and principal repayment of £500 on redemption. The current price of the bond is £539.93. Find its yield to maturity and duration.

The cash flow is $(50, 50, 50, 50, 550)$ for $t = 1, 2, 3, 4, 5$ and the price $P = £539.93$. Recall that yield to maturity is the rate of interest that equates the PV of future cash flows to the price and computed as

$$539.93 = \frac{50}{(1+\lambda)} + \frac{50}{(1+\lambda)^2} + \frac{50}{(1+\lambda)^3} + \frac{50}{(1+\lambda)^4} + \frac{50+500}{(1+\lambda)^5}$$

By solving the equation, $\lambda = 8\%$ is found.

t	a_t	$PV(a_t)$	$t \times PV(a_t)$
1	50	46.30	46.30
2	50	42.87	85.73
3	50	39.69	119.08
4	50	36.75	147.01
5	550	374.32	1871.62
P		539.93	
$\sum_{t=1}^{5} t \times PV(a_t) =$			2269.73

Then Macaulay duration is $D_{mac} = \frac{2269.73}{539.93} = 4.20$ years. This is the average time to receipt of the cash flows.

Modified Duration

Duration is useful because it measures directly the sensitivity of price to changes in yield. This also follows a simple expression for the derivative of the present value of expression. In the case where payments are made m times per year and the yield is based on those same periods, we have $PV(a_k) = \frac{a_k}{\left[1+\frac{\lambda}{m}\right]^k}$. The first order derivative of $PV(a_k)$ with respect to λ is

$$\begin{aligned} \frac{dPV(a_k)}{d\lambda} &= -\frac{\frac{k}{m}a_k}{\left[1+\frac{\lambda}{m}\right]^{k+1}} \\ &= -\frac{\frac{k}{m}}{\left[1+\frac{\lambda}{m}\right]}PV(a_k) \end{aligned}$$

Similiarly, the first order derivative of bond price $P = \sum_{k=1}^{n} PV(a_k)$ with respect to λ is calculated as

$$\begin{aligned} \frac{dP}{d\lambda} &= \sum_{k=1}^{n} \frac{dPV(a_k)}{d\lambda} \\ &= -\sum_{k=1}^{n} \frac{\frac{k}{m}}{1+\frac{\lambda}{m}} PV(a_k) \\ &= -\frac{1}{1+\frac{\lambda}{m}} D_{mac} P \\ &\approx -D_M P \end{aligned}$$

where

$$D_M = \frac{D_{mac}}{1+\frac{\lambda}{m}}$$

D_M is called the *Modified duration.* It is the usual duration modified by the extra term in the denominator. This measures the relative change in a bond's price directly as λ changes. By using the approximation $\frac{dP}{d\lambda} = \frac{\Delta P}{\Delta\lambda}$, the change in price due to a small change in yield (or vice versa) can be estimated as

$$\Delta P \approx -D_M P \Delta\lambda \tag{12}$$

Explicit values for the impact of yield variations can be obtained by using $\Delta P = P_N - P_0$ and $\Delta\lambda = \lambda_N - \lambda_0$ (where P_N and P_0 are the new and old prices and λ_N and λ_0 are the new and old values of the yields) as

$$P_N \quad \approx \quad P_0 - D_M P_0(\lambda_N - \lambda_0)$$

Example 2.7

Consider the bond given in Example 2.6. What is the new price of the bond if yield increases from 8% to 8.1%? Use the Modified Duration.

The Modified duration is

$$D_M = \frac{4.20}{1+0.08} = 3.89 \text{ years}$$

The change on price is obtained as

$$\begin{aligned} \Delta P &\approx -D_M P \Delta\lambda \\ &\approx -3.89 \times 539.93 \times (0.081 - 0.08) \\ &\approx -2.1003277 \end{aligned}$$

and the new price is calculated as

$$\begin{aligned} P_N - P_O &\approx -2.1003277 \\ P_N &\approx \text{£}537.83 \end{aligned}$$

As a result, price decreases from £539.93 to £537.83 when the yield increases from 8% to 8.1%.

2.1.5 Convexity

Modified duration measures the relative slope of the price-yield curve at a given point. This leads to a linear approximation to price-yield curve. However, a better approximation can be obtained by including a second order (quadratic) term which is based on convexity, C. Convexity is defined as

$$C \quad = \quad \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2}$$

This is the relative curvature at a given point on the price-yield curve. Given a cash flow a_t, for $t = 1, 2, \cdots, n$, the convexity is calculated as

$$\begin{aligned} C &= \frac{1}{P}\sum_{k=1}^{n} \frac{d^2 PV(a_k)}{d\lambda^2} \\ &= \frac{1}{P(1+\lambda)^2}\sum_{k=1}^{n} \frac{k(k+1)a_k}{(1+\lambda)^k} \qquad (13) \\ &= \frac{1}{P(1+\lambda)^2}\sum_{k=1}^{n} k(k+1)PV(a_k) \end{aligned}$$

Suppose that a financial instrument makes payments m times per year, with the payment in period k being a_k and there are n periods remaining. Then Equation (13) becomes

$$C \quad = \quad \frac{1}{P\left[1+\frac{\lambda}{m}\right]^2}\sum_{k=1}^{n} \frac{k(k+1)}{m^2}\frac{a_k}{\left[1+\frac{\lambda}{m}\right]^k}$$

This reflects the sensitivity of modified duration to changes in yield. Assume that price P and the corresponding yield λ are given and D_M, C are calculated. The second order approximation to the price-yield curve is

$$\Delta P \quad \approx \quad -D_M P_0 \Delta\lambda + \frac{1}{2}P_0 C(\Delta\lambda)^2 \qquad (14)$$

Let $\Delta\lambda$ be change on yield from λ_0 to λ_N and ΔP be the corresponding change on price of bond from P_0 to P_N. The new price of bond is found as

$$P_N \quad \approx \quad P_0 - D_M P_0(\lambda_N - \lambda_O) + \frac{1}{2}P_0 C(\lambda_N - \lambda_O)^2$$

2.1.6 The Term Structure of Interest Rates

The term structure of interest rates is the name given to the pattern of interest rates available on instruments of similar credit risk but with different terms to maturity. There are three standard theories for the term structure, each of which provides some important insight. We outline them briefly below.

Expectation Theory (Fisher) Long-term rates should reflect expected future short-term rates. It implies that the implied forward rate for a given period is equal to the expected futures zero rate for that period. Theory starts with assuming that market participants which trade on default free bonds are risk neutral, i.e. are indifferent to risk. They have no preference for a maturity over another and seek bonds with highest expected return. Markets are assumed to operate efficiently. Arbitrage across maturities forces interest rates on different assets to conform to market expectations of the appropriate rate for each instrument.

Liquidity Preference Theory (Keynes and Hicks) This theory suggests that market investors prefer liquidity and they demand compensation in term of higher yield, for moving to longer maturity. In this theory market participant as assumed risk-averse and in general long term bonds are less liquid because higher bid-ask spread and are more volatile. Borrowers on the other hand prefer to borrow at a fixed rate for a long period of time. Financial intermediaries do not want to finance long-term fixed rate loans with short-term deposits because of interest rate risk. To match depositors and borrowers than financial intermediaries would rise long-term interest rates. So long-term interest rates are determined by market expectation plus a liquidity premium. The theory implies that implied forward rates are upward-biased estimates of future zero rates.

Preferred Habitat Theory (Modigliani and Sutch) It says that different investor have preferences for different maturities because of the nature of their liabilities or their risk aversion. Participant are reluctant to switch across maturities. So securities of different maturities are imperfect substitute for one another and intertermporal arbitrage does not work. Thus the short-term rate is determined by demand and supply in the short-term market, the medium-term rate is determined by demand and supply in the medium-term market, and the long-term rate is determined by demand and supply in the long-term market. The theory implies that forward rates are biased estimates of future zero rates; this can be upward or downward biased.

2.1.7 Spot Rates

Spot rates are the basic interest rates defining term structure. The spot rate s_t is expressed on annualised basis, charged for money held from the present time ($t = 0$) until time t. Both interest and the original principal are paid at time t. Let s_t denote t-year spot rate. This is the rate paid for money held for t-years. For example, s_2 represents the rate that is paid for money held 2-years; held 1 year. 2-years; compounded yearly,

Suppose that an investor puts A amount of money to a bank with spot rate of s_t for t years. Let' see how the account grows under different compoundings.

Under yearly compounding convention: The spot rate s_t is defined such that $(1 + s_t)^t$ is the factor by which a deposit held t years will grow. Then A becomes $A(1 + s_t)^t$

Under a convention of compounding m periods per year: The factor is $(1 + \frac{s_t}{m})^{mt}$ and the account grows to $A(1 + \frac{s_t}{m})^{mt}$.

Under continuous compounding convention: The growth factor is $e^{s_t t}$ and the capital provides $Ae^{s_t t}$ after t years.

Assume that the 1-year spot rate s_1 is known. In order to determine 2-year spot rate, we can solve the following the equation for s_2,

$$P = \frac{M}{1 + s_1} + \frac{M + F}{(1 + s_2)^2}$$

where the bond has coupon payments of amount M at end of both years with a face value of F and its current price P. This basically means that the price should equal to discounted value of the cash flow stream. Carrying out forward in this way we can determine $s_3, s_4, \cdots$ step by step. Once the spot rates have been determined the corresponding discount factors d_k at time period k can be defined for various compounding conventions as follows;

For yearly compounding, $d_k = \frac{1}{(1+s_k)^k}$

For compounding m periods per year, $d_k = \frac{1}{(1+\frac{s_k}{m})^{mk}}$

For continuous compounding, $d_k = e^{-s_t t}$

These discount factors transform future cash flows directly into an equivalent present value. In other words, the future cash flows are multiplied by the factors to obtain an equivalent present value as

$$PV = a_0 + d_1 a_1 + \cdots + d_n a_n$$

Example 2.8

Consider two bonds A and B which mature in a year and two years with coupon payments of 10% and 8%, respectively. The face value is £100 and prices of bonds are £98 and £96. What are 1-year and 2-year spot rates s_1, s_2?

$$\begin{aligned}
P(A) &= \frac{M+F}{(1+s_1)} \\
98 &= \frac{100+10}{(1+s_1)} \Rightarrow s_1 = 12.24\% \\
P(B) &= \frac{M}{(1+s_1)} + \frac{M+F}{(1+s_2)^2} \\
96 &= \frac{8}{(1+0.1224)} + \frac{108}{(1+s_2)^2} \Rightarrow s_2 = 10.24\%
\end{aligned}$$

2.1.8 Forward Rates

Forward rate is defined as an interest rate for money to be borrowed between two dates in the future under terms agreed upon today. Suppose that you invest one pound in with spot rates of s_1 and s_2 for two years. There are two possibilities:

1. leave one pound in two-year account which money grows with factor of $(1+s_2)^2$, or
2. leave one pound in one-year account which money grows with factor of $(1+s_1)$, then lend your money for another year with interest rate f.

This loan gains an interest at a pre-arranged rate f as

$$(1+s_1)(1+f)^{2-1} = (1+s_2)^2 \Longrightarrow f = \frac{(1+s_2)^2}{(1+s_1)} - 1$$

In general term, the forward rate, denoted by $f_{i,j}$ between times i and j such that $i < j$, is the rate of interest charged for borrowing money at time i which is to be repaid with interest at time j. Therefore the following equation holds

$$(1+s_j)^j = (1+s_i)^i (1+f_{i,j})^{j-i}$$

where s_i and s_j are spot rates at i and j, respectively. The left side represents the factor by which money grows if it is directly invested for j years with spot rate s_j. The right side is the factor by which money grows if it is invested first for i years with spot rate of s_i and then in a forward contract (arranged today) between years i and j ($i < j$) with rate of f_{ij}. The term $(1+f_{i,j})$ is raised to the $(j-i)$th power because the forward rate is expressed in yearly terms.

Under various compounding conventions the forward rates can be determined as follows;

For yearly compounding, forward rate $f_{i,j}$ satisfies $(1+s_j)^j = (1+s_i)^i(1+f_{i,j})^{j-i}$ and is calculated as

$$f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{\frac{1}{(j-i)}} - 1$$

For compounding m periods per year, forward rate satisfies $(1+\frac{s_j}{m})^j = (1+\frac{s_i}{m})^i(1+\frac{f_{i,j}}{m})^{j-i}$ and is obtained as

$$f_{i,j} = m\left[\frac{(1+\frac{s_j}{m})^j}{(1+\frac{s_i}{m})^i}\right]^{\frac{1}{(j-i)}} - m$$

For continuous compounding, forward rate $f_{i,j}$ satisfies $e^{s_j j} = e^{s_i i}e^{f_{i,j}(j-i)}$ and is calculated as

$$f_{i,j} = \frac{s_j j - s_i i}{j-i}$$

Example 2.9 *Taken from Luenberger's Book Chp 4 - Exercises 1*

The spot rates for 1 and 2 years are $s_1 = 6.3\%$ and $s_2 = 6.9\%$. What is the forward rate $f_{1,2}$?

$$\begin{aligned} f_{1,2} &= \frac{(1+s_2)^2}{1+s_1} - 1 \\ &= \frac{(1.069)^2}{1.063} - 1 = 7.5\% \end{aligned}$$

2.2 Valuing Stocks

2.2.1 Introduction to Stocks

Stocks are sometimes called as shares, securities or equity. In general term, a stock is an ownership in part of a company. For every stock you own in a company you own a small piece of office furniture, company cars, and even that lunch the boss paid with the company credit card. More importantly though, you are entitled to a portion of the company's profits and any voting rights attached to the stock. The profits are typically paid out in dividends. The more shares you own, the larger portion of the company (and profits) you own. In addition to owning part of a corporation, owning stocks allows you to utilise the power of compounding, that is, to earn a return on top of returns. Compounding is part of the reason that over the past several decades stocks have outperformed practically every other investment tool.

You might think why a company would want to issue stocks and share the profits with thousands of people when they could keep profits to themselves. The reason companies issue stock is to raise money (called equity financing). By selling some ownership in the company (in the form of stocks), they get money that can be used for expansion, upgrading equipment, marketing, etc.

There are two main types of stocks:

Common Stock is just that, "common". The majority of stocks trading today are in this form. Common stock represents ownership in a company and a portion of profits (dividends). Investors also have voting rights (one vote per share) to elect the board members who oversee the major decisions made by management. In the long term, common stocks, by means of capital growth, yield higher rewards than other forms of investment securities. This higher return comes at a cost as common stocks entail the most risk. Should a company go bankrupt and liquidate, the common shareholders will not receive money until the creditors, bond holders, and preferred shareholders are paid.

Preferred Stock represents some degree of ownership in a company but usually don't have voting rights (this may vary depending on the company). On preferred shares investors are guaranteed a fixed (or sometimes variable) dividend forever, meanwhile common stocks have variable dividends. However, one advantage is in the event of liquidation; they are paid off before the common shareholder (but still after debt holders). Preferred stock may also be callable, meaning that the company has the option to purchase the shares from shareholders at anytime, and usually for a premium.

Investors purchase stocks for their returns which come in the form of capital gains (the appreciation in value over time) and dividends (paid periodically by most companies). There are three ways of transacting in stocks;

Buy: The stock will appreciate in value over time, or require the stock for its characteristics as part of portfolio. It is also said that we are **long** in the stock.

Sell: The stock will depreciate in value over time or we require funds for another purpose.

Short sale: It is also said that we are **short** in the stock. Here, we do not own the stock; but we borrow it from another investor, sell it to a third party and receive the proceeds. We are obligated to pass on to the lender of the stock any dividends declared on the stock and also to pay to the lender the market price of the stock if we decide to sell. When the short sale is the case, it is expected that the stock will decline in value in order to enable us to buy it back at a low price later on to make up out obligations to the lender. We are expecting a bearish market for the stock.

How To Read A Stock Table

An example of a stock table is presented in Figure 5. The meaning of each column in the table is introduced as follows;

52W high	52W low	Stock	Ticker	Div	Yield %	P/E	Vol 00s	High	Low	Close	Net chg
s45.39	19.75	ResMed	RMD			52.5	3831	42.00	39.51	41.50	-1.90
11.63	3.55	Revlon A	REV				162	6.09	5.90	6.09	+0.12
77.25	55.13	RioTinto	RTP	2.30	3.2		168	72.75	71.84	72.74	+0.03
31.31	16.63	RitchieBr	RBA			20.9	15	24.49	24.29	24.49	-0.01
8.44	**1.75**	**RiteAid**	**RAD**				**31028**	**4.50**	**4.20**	**4.31**	**+0.21**
s38.63	18.81	RobtHalf	RHI			26.5	6517	27.15	26.50	26.50	+0.14
51.25	27.69	Rockwell	ROK	1.02	2.1	14.5	6412	47.99	47.00	47.54	+0.24
Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10	Column 11	Column 12

Figure 5: An example of stock table

Columns 1 and 2: 52-Week Hi and Low are the highest and lowest prices that a stock has traded at over the previous 52-weeks (1 year). This typically does not include the previous day's trading.

Column 3: Company Name (and Type of Stock) lists the name of the company. If there are no special symbols or letters following the name, it is common stock. Different symbols imply different classes of shares. For example, "pf" means the shares are preferred stock.

Column 4: Ticker Symbol is the unique alphabetic name which identifies the stock on the exchange's ticker. The ticker tape will quote the latest prices alongside this symbol. If you are looking for stock quotes online, you always search for a company by the ticker symbol.

Column 5: Dividend Per Share indicates the annual dividend payment per share. If this space is blank, the company does not currently pay out dividends.

Column 6: Dividend Yield is the percentage return for the dividend, which is calculated as annual dividends per share divided by price per share.

Column 7: Price/Earnings Ratio is calculated by dividing the current stock price by earnings per share from the last four quarters.

Column 8: Trading Volume shows the total number of shares traded for the day, listed in hundreds. To get the actual number traded, add "00" to the end of the number listed.

Column 9 and 10: Day High - Low indicate the price range the stock has traded at throughout the day's trading. In other words, these are the maximum and the minimum price people have paid for the stock.

Column 11: Close is the last trading price recorded when the market closed on the day. If the closing price is up or down more than 5% than the previous day's close, the entire listing for that stock is bold-faced. Keep in mind you are not guaranteed to get this price if you buy the stock the next day. Because a stock price is constantly changing (even after an exchange is closed for the day) the close merely serves as an indicator of past performance.

Column 12: Net Change is the dollar value change in the stock price from the previous day's closing price. When you hear about a stock being: "up for the day" it means the net change was positive.

2.2.2 Valuation of Stocks

Stock prices are changed everyday by "the market". Buyers and sellers cause prices to change as they decide how valuable each stock is. Basically, share prices change because of supply and demand. If more people want to buy a stock than sell it, then the price moves up. Conversely, if more people want to sell a stock, there would be more supply (sellers) than demand (buyers); therefore, the price would start to fall.

Stock represents an ownership in a company. Therefore, the price of a stock shows what investors feel the company is worth. Stock prices can change at any rate, some have dramatic swings in one day while others stay the same for a long time. There are hundreds of variables which drive stock prices, the most important of which is earnings. Think of earnings as the profit of a company, the money left after all expenses have been paid, this is what shareholders desire.

Securities issued by the firm have no set maturity date or dividend rate; however, the par value is set by the firm. Unlike bonds, there are many approaches to the valuation of stocks. The reason is that cash flows associated with stocks are extremely difficult to estimate. In this section we will consider "discounted dividend model" where the stock price is the present value of future cash flows to be received by the investor. The price the investor is willing to pay for a share of stock depends upon magnitude, timing of expected future dividends and the risk of the stock.

If the investor buys a stock, then he is entitled to receive all future dividends and can sell the stock in the future. The cash flow to holders of common stock consists of dividends plus a future sale price. Consider the following cash flow which investors expect to receive.

Time	0	1	2	$\cdots$	T	$\cdots$
Cash Flow		D_1	D_2	$\cdots$	$D_T + P_T$	$\cdots$

where the current dividend is denoted by D_0. The stock's discount rate r_e is the rate of return that investors expect to earn on securities with similar risk. Shareholders require a rate of return for buying a share. They buy for P_0 and sell after one year for P_1 and receive dividend D_1.

$$P_0 = \frac{D_1 + P_1}{1 + r_e}$$

The next buyer also sells after one year:

$$P_1 = \frac{D_2 + P_2}{1 + r_e} \Longrightarrow P_0 = \frac{D_1}{1 + r_e} + \frac{D_2 + P_2}{(1 + r_e)^2}$$

At period T, we have $P_{T-1} = \frac{D_T + P_T}{1 + r_e}$. Continuing the same process (using recursive substitution), the current price of a stock can be written as

$$\begin{aligned} P_0 &= \frac{D_1}{1+r_e} + \frac{D_2}{(1+r_e)^2} + \frac{D_3}{(1+r_e)^3} + \cdots + \frac{D_t}{(1+r_e)^t} + \cdots \\ P_0 &= \sum_{t=1}^{\infty} \frac{D_t}{(1+r_e)^t} \qquad (15) \end{aligned}$$

This formula shows that the current value of stock is the present value of all future cash flows, i.e., dividends and expected selling price. Note that the expression $\frac{P_T}{(1+r_e)^T}$ can be neglected for a large time horizon since $(1+r_e)^T$ becomes very large as T becomes very large. The disadvantages of this approach are that the value of T is determined by the investor, the dividends are uncertain and we need to estimate the expected selling price in order to calculate the current price.

Required Returns

Consider $P_0 = \frac{D_1+P_1}{1+r_e}$. We can reexpress it in terms of returns as

$$r_e = \frac{D_1}{P_0} + \frac{P_1 - P_0}{P_0} \tag{16}$$

The first part on the right hand side is a financial ratio widely used by practitioners and called as the *dividend yield*. However, in practice, D_1 is not known since it is an expected value about a future dividend payment. Practitioners commonly refer to the dividend yield as D_0/P_0. Because of this difference we will refer to D_0/P_0 as the historic or trailing dividend yield, and to D_1/P_0 as the prospective dividend yield. The second part on the right hand side of (16) is the capital gain, which is the percentage of the current stock price. As a result, the return on equity is the sum of prospective dividend yield and expected capital gain.

2.2.3 Stock's Value Estimation

In order to make use of expression (15) we will make some assumptions about future dividends and consider the special cases. This helps to estimate the value of a stock under different assumptions.

The Zero Dividend Growth Model: The simplest assumption about dividends is that they stay constant over time with zero growth, so that $D_1 = D_2 = \cdots = \bar{D}$. Then the formula (15) becomes

$$P_0 = \frac{\bar{D}}{r_e} \quad \Longrightarrow r_e = \frac{\bar{D}}{P_0} \tag{17}$$

As a result, if the dividend is expected to stay constant over time (the cash flow is always the same), then shares can be valued like perpetual bonds and the expected return on equity is equal to the dividend yield. This model does not reflect reality because of constancy of dividends. More general assumption is the constant dividend growth.

The Constant Dividend Growth Model: An amount that grows at a constant rate forever is called a *growing perpetuity*. The stock with a constant dividend growth is actually a growing perpetuity. Let g be a constant rate which dividends grow forever. Therefore, dividends are computed as

$$\begin{aligned}
D_2 &= D_1(1+g) \\
D_3 &= D_2(1+g) = D_1(1+g)^2 \\
D_4 &= D_3(1+g) = D_1(1+g)^3 \\
&\cdots \\
D_T &= D_{T-1}(1+g) = D_1(1+g)^{T-1}
\end{aligned}$$

When we substitute them in Equation (15), the current price of the stock is obtained as

$$P_0 = \frac{D_1}{1+r_e} + \frac{D_1(1+g)}{(1+r_e)^2} + \frac{D_1(1+g)^2}{(1+r_e)^3} + \cdots + \frac{D_t(1+g)^{T-1}}{(1+r_e)^T} + \cdots$$

Assume that g is smaller than r_e. In this case the general formula (15) becomes

$$P_0 = \frac{D_1}{r_e - g} \tag{18}$$

Notice that it gives the equation (17) when $g = 0$. From (18), we can obtain the expected return on equity as

$$r_e = \frac{D_1}{P_0} + g \tag{19}$$

which is the sum of prospective dividend yield and growth rate. Using (16) with (19), the growth rate is obtained as $g = \frac{P_1 - P_0}{P_0}$ and therefore, $P_1 = (1+g)P_0$. Hence, if we assume that the company is in steady state where dividends are expected to grow at a constant rate g, we also expect that the stock price grows at the same constant rate g.

Example 2.10

If the dividend at $t = 1$ is £2 and the expected growth rate is 5%, what is the dividiend D_5 at $t = 5$?

$$D_5 = D_1(1+g)^4 = 2 \times (1+0.05)^4 = 2 \times 1.276 = £2.55$$

Example 2.11

Next year dividends per share for company X is expected to be £0.95. The dividends are expected to grow at 14% per year in the future. What should be the current price if the required rate of return is 16% per year?

$$\begin{aligned} P_0 &= \frac{0.95}{(1+0.16)} + \frac{0.95 \times (1+0.14)}{(1+0.16)^2} + \frac{0.95 \times (1+0.14)^2}{(1+0.16)^3} + \cdots \\ &= \frac{0.95}{0.16 - 0.14} = 47.5 \text{ growing perpetuity} \end{aligned}$$

Non-constant Dividend Growth Model: The model with a constant dividend growth rate is not a suitable for the companies which are not in a steady state. them. For simplicity, we can view g as a kind of average. In this case, we need to extend the constant dividend growth model and define sub-periods with different growth rates. Then we can estimate the value of the stock by considering each sub-period with individual discounting in two steps:

1. Find present value, $PV(NCD)$, of non-constant growth dividends, $D_k(NC)$ at each sub-period $1 \le k \le n$ as

$$PV(NCD) = \sum_{k=1}^{n} \frac{D_k(NC)}{(1+r_e)^k} \tag{20}$$

2. Find present value of constant growth dividends $PV(P_t)$ where

$$P_t = \frac{D_{t+1}}{r_e - g} \tag{21}$$

Then the present value of the stock is basically sum of present values of all dividends as

$$P_0 = PV(NCD) + PV(P_t)$$

Example 2.12

The next three dividends for Company Y are expected to be £0.50, 1.00, 1.50. Then the dividends are expected to grow at a constant rate of 5% forever. If the required return is 10%, what is the value of the stock?

For the non-constant dividend growth

$$\begin{aligned} PV(NCD) &= \sum_{k=1}^{3} \frac{D_k(NC)}{(1+r_e)^k} \\ &= \frac{0.50}{(1+0.10)} + \frac{1.00}{(1+0.10)^2} + \frac{1.50}{(1+0.10)^3} \\ &= 0.454 + 0.826 + 1.127 = 2.407 \end{aligned}$$

For the constant dividend growth based on the third year dividend D_3,

$$\begin{aligned} P_3 &= \frac{D_3(1+g)}{r_e - g} \\ &= \frac{1.5(1+0.05)}{0.10-0.05} = 31.50 \\ PV(P_3) &= \frac{P_3}{(1+r_e)^3} = \frac{31.50}{1.331} = 23.67 \end{aligned}$$

Then $P_0 = PV(NCD) + PV(P_3) = 2.407 + 23.67 = 26.07$ is the present value of the stock.

Dividend Forecast

The general stock valuation formula is a function of all future dividends. A forecasting method for dividends $D_1, D_2, \cdots, D_\infty$ which extends into all periods is needed.

If we forecast the values of $D_1, D_2, \cdots, D_T$ for a finite time horizon T, then the price of stock becomes

$$P_0 = \sum_{i=1}^{T} \frac{D_i}{(1+r_e)^i} + \frac{P_T}{(1+r_e)^T} \qquad (22)$$

Then apply the constant growth formula for the horizon value P_T by substituting for the dividends after period T,

$$P_T = \frac{D_{T+1}}{r_e - g} = \frac{(1+g)D_T}{r_e - g}$$

then the general stock valuation formula given in Equation (22) becomes

$$P_0 = \sum_{i=1}^{T} \frac{D_i}{(1+r_e)^i} + \frac{(1+g)D_T}{(1+r_e)^T(r_e - g)}$$

If we forecast the dividends one period into the future, then a special version of (22) for $T = 1$ is obtained. In this case, we obtain the standard equation for the dividend growth model as

$$P_0 = \frac{D_1}{r_e - g}$$

If we do not have a forecast for this year's dividend, then the historic dividend value D_0 can be used so that $D_1 = D_0(1+g)$. The stock price is obtained as

$$P_0 = \frac{(1+g)D_0}{r_e - g}$$

Example 2.13

Consider a company pays a dividend of £0.75 per share. Demand for this company's product is growing at 2% per year and inflation averages 2.5% per year. The company expects its profits and dividends to grow at about 4.55% per year ($1.02 \times 1.025 = 1.0455$). Stockholders require a 10% rate of return. What is the market price of this company's stock?

The dividend next period is $0.75 \times 1.0455 = 0.0784$. Using the formula for a growing perpetuity $P = \frac{D_1}{(r_e - g)}$, we obtain

$$P = \frac{0.75 \times 1.0455}{0.10 - 0.0455} = £14.39$$

2.2.4 Earnings and Sales Based Valuation Models

Price-Earnings (or P/E) ratios are frequently used to price equities. This model claims that stocks have a fair or normal price-to-earnings per share rate. If this ratio is known, then the fair value of the stock can be determined with the P/E multiple. Let E_1 be the earnings per share. The earning yield is defined as E_1/P_0. Dividends and earnings are related via the company's pay out policy. This can be expressed in the pay out ratio p as the ratio of dividends per share and earnings per share as

$$p = \frac{D_1}{E_1} \Longrightarrow D_1 = pE_1$$

The required return on equity is related to earnings yield. The relationship obtained by substituting $D_1 = pE_1$ in (19) can be formalised as;

$$r_e = \frac{E_1}{P_0} \times p + g \Longrightarrow \frac{P_0}{E_1} = \frac{p}{r_e - g}$$

Therefore, companies should have the same P/E ratio if they have the same pay out ratio (similar technology, efficiency), discount rate (business risk, financial risk), growth rate (business prospects, market share development). Analysts often report the historical (or trailing) price-earnings ratio, P_0/E_0 where E_0 is the current earnings. The relation between the historical and forward-looking P/E can be seen from the following equation

$$\frac{P_0}{E_1} = \frac{P_0}{E_0}\frac{1}{(1+g)}$$

where $E_1 = E_0(1+g)$. This shows that the historical P/E overstates the forward-looking P/E. When it is assumed that dividends and earnings grow at the same constant rate g from now on, (that is $D_1 = (1+g)D_0$, $E_1 = (1+g)E_0$), the required return and P/E ratio are obtained as

$$\begin{aligned} r_e &= g + (1+g)\frac{D_0}{P_0} \qquad (23) \\ \frac{P_0}{E_0} &= (1+g)\frac{P_0}{E_1} = \frac{p(1+g)}{r_e - g} \end{aligned}$$

Rearranging (23), we can find $g = \frac{r_e - \frac{D_0}{P_0}}{1+\frac{D_0}{P_0}}$ where $\frac{D_0}{P_0}$ is referred as the historical dividend yield.

3 Single–Period Markowitz Model

Having invested a capital at the beginning of investment horizon, payoff is attained at the end of the investment period. An investment in a zero-coupon bond that will be held to maturity and an investment in a physical project that will not provide payment until it is completed are two examples of single–period real life applications. It is worthwhile to mention that some common investments are not tied to a single–period, since they can be liquidated at will and may return dividends periodically. Nevertheless, as a simplification, such investments are often analysed on a single period basis. Typically, when making an investment, the initial capital is known, but the amount to be returned is uncertain. Uncertainty can be treated by the mathematical methods such as mean-variance analysis, utility function analysis and arbitrage (or comparison) analysis. In this section, we focus on the mean-variance analysis which uses probability theory and leads to convenient mathematical expressions and procedures.

3.1 Notation and Terminology

Random variables

Suppose that y is a random quantity and can take on any one of a finite number of specific values, say $y_1, \ldots, y_m$. Assume that associated with each possible y_j for $j = 1, \cdots, m$, there is a probability p_j that represents the relative chance of an occurrence of y_j. They are determined such that the following relations hold.

$$\sum_{j=1}^{m} p_j = 1 \;\text{ and }\; p_j \geq 0 \text{ for } j = 1, \cdots, m$$

Each p_j can be thought of as the relative frequency with which y_j would occur if an experiment of observing y_j were repeated infinitely often. The quantity y, characterised in this way before its value known, is called a random variable.

Expected Value

The expected value (mean value or mean) of a random variable y is the average value obtained by regarding the probabilities as frequencies. For the case of a finite number of possibilities, it is defined as

$$E(y) = \sum_{j=1}^{m} p_j y_j \tag{24}$$

The expected value of any variable z is denoted by either $E(z)$ or $\bar{z}$.

Variance

Variance is the measure of the degree of possible deviation from the mean. Given the expected value, $\bar{y}$, of a random variable y, the variance of y is the expected value of the squared variable $(y - \bar{y})^2$, which is the measure of how much y tends to vary from its expected value. It can be mathematically formulated as

$$\begin{aligned} var(y) &= E\left[(y-\bar{y})^2\right] \\ &= E(y^2) - 2E(y)\bar{y} + \bar{y}^2 \\ &= E(y^2) - \bar{y}^2 \end{aligned} \tag{25}$$

Covariance

If two or more random variables are considered, their mutual dependence can be summarised by their covariance. Let y_1 and y_2 be two random variables with expected values $\bar{y_1}$ and $\bar{y_2}$. The covariance of these variables (denoted by $cov(y_1, y_2)$ or σ_{12}) is defined as

$$\begin{aligned} \sigma_{12} &= E\left[(y_1 - \overline{y_1})(y_2 - \overline{y_2})\right] \\ &= E(y_1.y_2) - \overline{y_1}\,\overline{y_2} \end{aligned} \tag{26}$$

Notice that $\sigma_{12} = \sigma_{21}$. If two random variables are independent, then they are called uncorrelated and $\sigma_{12} = 0$. If $\sigma_{12} > 0$, then two variables are said to be positively correlated. In this case, if one variable is above its mean, the other one is likely to be above its mean as well. If $\sigma_{12} < 0$, the two variables are said to be negatively correlated.

Asset return

An investment instrument that can be bought and sold is called an *asset.* The amount of money to be obtained when selling an asset is uncertain at the time of purchase. In this case, the return is random and the uncertainty can be described in probabilistic terms. Suppose that you purchase an asset at time zero and one year later you sell it. The *total return* on this investment is defined as

$$\text{total return} = \frac{\text{amount received}}{\text{amount invested}} \tag{27}$$

Let x_0 and x_1 be amounts of money invested and received after one year, respectively. The total return R is formalised as

$$R = \frac{x_1}{x_0} \tag{28}$$

The *rate of return* is defined as

$$\text{rate of return} = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}}$$

and formulated

$$r = \frac{x_1 - x_0}{x_0} \tag{29}$$

Notice that total return and rate of return are related by $R = 1 + r$ and the equation (29) can be rewritten as

$$x_1 = (1 + r)x_0$$

This shows that the rate of return acts much like an interest rate.

Example 3.1

Consider two assets whose return series r_1, r_2 are given as

Time Period	Return Series r_1	r_2
1	6	8
2	4	2
3	7	11
4	3	-1
5	8	12
6	2	-2
7	11	13
8	-1	-3

and plotted in Figure 6. Although both assets have the same mean return value of $\bar{r_1} = \bar{r_2} = 5$, their standard deviations are ${\sigma_1}^2 = 3.535534$ and ${\sigma_2}^2 = 6.284903$.

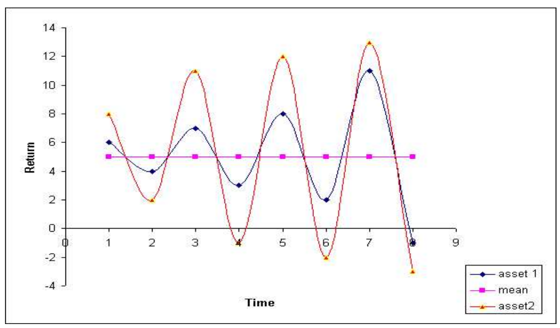

Figure 6: The return series around the mean.

Portfolio return

Consider n risky assets in order to form a portfolio by apportioning an amount x_0 among n assets. We can select amounts x_{0i} for $i = 1, \ldots, n$ such that

$$x_0 = \sum_{i=1}^{n} x_{0i} \tag{30}$$

where x_{0i} represents an amount invested in the ith asset. If we are allowed to sell an asset short, then some of the x_{0i}'s can be negative; otherwise we restrict the x_{0i}'s to be nonnegative. The amounts invested can be expressed as fractions of the total investment. Thus, we can write

$$x_{0i} = w_i x_0 \qquad i = 1, \ldots, n \tag{31}$$

where w_i is the weight or fraction of asset i in the portfolio. It is clear that $\displaystyle\sum_{i=1}^{n} w_i = 1$ and some w_i's may be negative if short selling is allowed. Let R_i denote the total return of asset i. The amount of money gained at the end of the period by the ith asset is

$$R_i x_{0i} = R_i w_i x_0$$

The total amount received by this portfolio at the end of period is, therefore, $\sum_{i=1}^{n} R_i w_i x_0$. Hence, the overall total return of the portfolio is calculated as follows;

$$R = \frac{\sum_{i=1}^{n} R_i w_i x_0}{x_0} = \sum_{i=1}^{n} R_i w_i \tag{32}$$

Equivalently, we have $r = \sum_{i=1}^{n} w_i r_i$, since $\sum_{i=1}^{n} w_i = 1$. As a result we can state the following lemma.

Both the total return and the rate of return of a portfolio of assets are equal to the weighted sum of the corresponding individual asset returns, with the weight of an asset being its relative weight in the portfolio; that is,

$$R = \sum_{i=1}^{n} w_i R_i, \qquad r = \sum_{i=1}^{n} w_i r_i \tag{33}$$

Expected return of portfolio

Suppose that there are n assets with random rates of return $r_1, r_2, \cdots, r_n$. Their expected values are denoted by $E(r_i)$ or $\bar{r_i}$ for $i = 1, \cdots, n$. We form a portfolio of n assets using weights w_i. The rate of return of the portfolio in terms of return of individual returns is given by

$$r = \sum_{i=1}^{n} w_i r_i$$

The mean return of a portfolio is calculated by taking the weighted sum of the individual expected rates of return as

$$\begin{aligned} E(r) &= E(r_1)w_1 + \cdots + E(r_n)w_n \\ &= \sum_{i=1}^{n} w_i E(r_i) \end{aligned}$$

Example 3.2

Consider a portfolio of a risk-free asset and two risky stocks. Suppose that there exists three equally likely states presented in the scenario tree presented in Figure 7. The root node represents today and the future uncertainty is discretised by three events (states). The asset returns for each state s are given in the following table.

	Returns (%)		
States	**T-Bill**	**Stock** A	**Stock** B
s	$i = 1$	$i = 2$	$i = 3$
s_1- boom	5	16	3
s_2- normal	5	10	9
s_3- recession	5	1	15

What is the expected return of the portfolio formed?

The probabilities of each event is $p_1 = p_2 = p_3 = \frac{1}{3}$. The expected return of the risk free asset is $E[r_1] = 5\%$. For stocks A and B,

$$\begin{aligned} E[r_2] &= \frac{1}{3}[16 + 10 + 1] = 9\% \\ E[r_3] &= \frac{1}{3}[3 + 9 + 15] = 9\% \end{aligned}$$

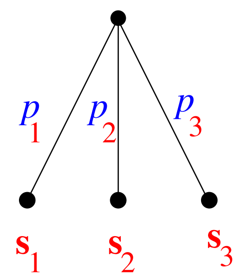

Figure 7: Scenario tree consists of three states

The expected return of the portfolio is

$$E[r] = 5 \times w_1 + (w_2 + w_3) \times 9$$

For equally weighted portfolio, $w_1 = w_2 = w_3 = \frac{1}{3}$ and the expected return of the portfolio is

$$E[r_{ew}] = \frac{1}{3}\,[5 + 2 \times 9] = 7.666$$

Risk

Everyone exposes themselves to risk whether it is investing, driving, or just walking down the street. Your personality and lifestyle play a big deal on how much risk you are comfortably able to take on. If you invest in stocks and have trouble sleeping at nights because of your investments you are probably taking on too much risk. According to Investopedia dictionary, "risk" is defined as the chance that an investment's actual return will be different than expected". This includes the possibility of losing some or all of the original investment. Those who work hard for every penny earned have a harder time parting with money. These people are considered to be more risk averse. On the other end of the spectrum, day traders feel if they aren't making dozens of trades a day there is a problem, these people are risk loving. When investing in stocks, bonds, or any investment instrument there is a lot more risk than you'd think. Two basic types of risk are given here.

Systematic Risk influences a large number of assets. An example is political events. It is virtually impossible to protect yourself against this type of risk.

Unsystematic Risk is sometimes referred to as "specific risk" that affects very small number of assets. An example is news that affects a specific stock such as a sudden strike by employees. Diversification is the only way to protect yourself from unsystematic risk.

There are several types of risk that a smart investor should consider and pay careful attention to. Deciding your potential return while respecting risk is the age old decision that investors must make. More specific types of risk, particularly when we talk about stocks and bonds are summarised as follows;

Credit Risk is a risk that a company or individual will be unable to pay the contractual interest or principal on its debt obligations. It is sometimes called default risk.

Country Risk is a risk that a country won't be able to honor its financial commitments. When a country defaults it can harm the performance of all other financial instruments in that country as well as other countries it has relations with.

Foreign Exchange Risk appears when investing in foreign countries. The fact that currency exchange rates can change the price of the asset as well must be considered.

Interest Rate Risk is a risk that interest rates will rise during the term of your investment. A rising interest rate hurts the performance of stocks and bonds.

Political Risk is a financial risk that a government in a country will suddenly change its policies. This is a major reason that second and third world countries lack foreign investment.

Market Risk is the one we are all familiar with. It's the day to day fluctuations in a stocks price and referred to volatility.

Portfolio Risk

There are alternative risk measures such as value at risk (VaR), downside risk, utility functions. In the mean-variance framework risk is measured as the variance of the portfolio return.

Let σ_i^2 and σ^2 denote the variances of the return of asset i and portfolio, respectively. The covariance of return of asset i with asset j is σ_{ij}. The variance of rate of return of the portfolio is calculated as follows;

$$\begin{aligned}
\sigma^2 &= E\left[(r-\bar{r})^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r_i}\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r_i})\right)\left(\sum_{j=1}^{n} w_j (r_j - \bar{r_j})\right)\right] \\
&= E\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r_i})(r_j - \bar{r_j})\right] \\
&= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}
\end{aligned} \tag{34}$$

Example 3.3

Consider the portfolio given in Example 3.2. What is variance of the portfolio?

The variances of assets $i = 1, 2, 3$ are computed as follows;

$$\begin{aligned}
\sigma_1^2 = Var[r_1] &= 0 \\
\sigma_2^2 = Var[r_2] &= \frac{1}{3}\left[(16-9)^2 + (10-9)^2 + (1-9)^2\right] = \frac{114}{3} \\
\sigma_3^2 = Var[r_3] &= \frac{1}{3}\left[(3-9)^2 + (9-9)^2 + (15-9)^2\right] = \frac{72}{3}
\end{aligned}$$

and covariances are

$$\begin{aligned}
\sigma_{23} = Cov[r_2, r_3] &= \frac{1}{3}\left[(7)(-6) + (1)(0) + (-8)(6)\right] = \frac{-90}{3} \\
\sigma_{21} = Cov[r_2, r_1] &= 0 \\
\sigma_{31} = Cov[r_3, r_1] &= 0
\end{aligned}$$

The portfolio risk is calculated as

$$\begin{aligned}
\sigma^2 &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + w_1w_2\sigma_{12} + w_1w_3\sigma_{13} \\
&\quad + w_2w_1\sigma_{21} + w_2w_3\sigma_{23} + w_3w_1\sigma_{31} + w_3w_2\sigma_{32} \\
&= w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_2w_3\sigma_{23} \\
&= \frac{1}{3}\left[w_2^2 114 + w_3^2 72 - w_2w_3 180\right]
\end{aligned}$$

For equally weighted portfolio $w_1 = w_2 = w_3 = \frac{1}{3}$, then $\sigma_{ew}^2 = \frac{6}{27}$.

3.2 Portfolio Optimisation

Portfolio Theory deals with the effects of investor decisions on security prices; specifically the relationship that should exist between the returns and risk. Harry Markowitz (1990 Nobel Prize winner in Economic Sciences) formalised an integrated theory of diversification, portfolio risks, efficient (and inefficient) portfolios in 1952 *Journal of Finance* article titled "Portfolio Selection". The theory has revolutionised the money management industry with the goals of quantifying the investment risk and expected return of a portfolio.

3.2.1 Asset Allocation

Asset allocation is an investment portfolio technique that divides assets among major categories such as bonds, stocks, real estate, or cash, usually to balance risk and to create diversification. Each asset class will generally have different levels of return and risk so they will also behave differently. At the time one asset is increasing in value, another may be decreasing or not increasing as much and vice versa. The underlying principle of asset allocation is that the older you get, the less amount of risk you should be exposed to. The more that you depend on your retirement savings for income, the more conservatively you should invest because asset preservation is a key. Determining the right mix of investments in your portfolio is very important. The optimal decision of what percentage of your portfolio you should put into stocks, mutual funds, and low risk instruments like bonds and treasuries is not an easy task.

3.2.2 The Optimal Portfolio

A rational framework for investment decisions is provided by the maximisation of return for an acceptable level of risk. A fundamental example is the single-period Markowitz model in which expected portfolio return is maximised and risk measured by the variance of portfolio return is minimised. The optimal portfolio was formally used in 1952 by Harry Markowitz. He showed that it is possible to have different portfolios varying levels of risk and return. Each investor must decide how much risk they can handle and allocate (or diversify) their portfolio according to this decision. Figure 8 presents a graphical example of how the optimal portfolio

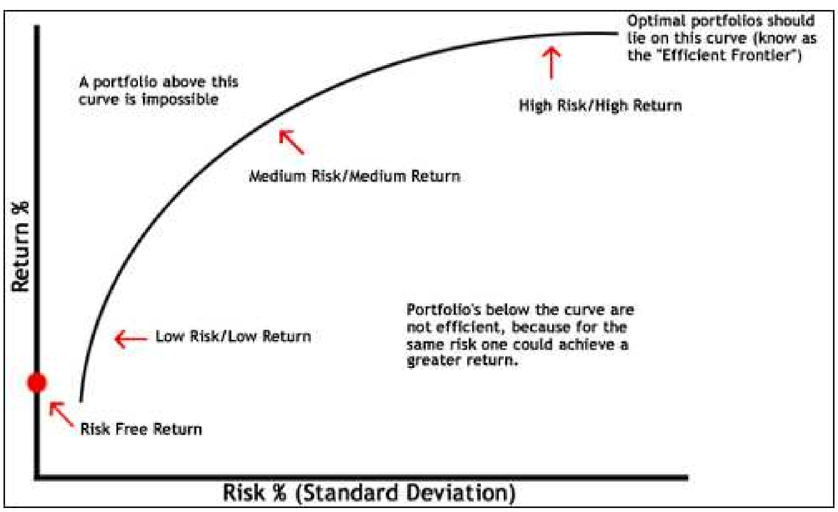

Figure 8: An optimal portfolio

works. The "optimal risky portfolio" is usually determined to be somewhere in the middle of the curve, this is because as you go higher up the curve you take on proportionately more risk for a lower incremental return and so you're assuming a larger amount of risk for a smaller increase in returns. On the other end, low risk/low return portfolios are pointless because you can achieve a similar return by investing in risk-free assets like government securities.

3.2.3 Short sale

It is possible to sell an asset that you do not own through the process of short selling, or shorting, the asset. In order to do this, you borrow the asset from someone who owns it, such as a brokerage firm. Then you sell the borrowed asset to someone else, receiving an amount x_0. At a later date, you repay your loan by purchasing the asset for, say x_1, and return the asset to your lender. If the later amount x_1 is lower than the original amount x_0, you have profit of $x_0 - x_1$. Hence the short selling is profitable if the asset price declines.

Short selling is considered quite risky by many investors. The reason is that potential for loss is unlimited. If the asset value increases the loss is $x_1 - x_0$; since x_1 can increase arbitrarily, so can the loss. For this reason, short selling is prohibited within certain financial institutions, and it is purposely avoided as a policy by many individuals and institutions. However, it is not universally forbidden, and there is, in fact a considerable level of short selling of stock market securities.

Example 3.4

Assume that company A has a poor outlook next month. The stock is now trading at £65, but you see it trading much lower than this price in the future. You decide to take risk and trade on this stock. Two things can happen; stock price can go up or down. If it goes down the stock price is predicted as £40; otherwise, it is £85. The process of short selling under the two states is summarised in the following table.

	Predicted stock price: £40		**Predicted stock price: £85**	
Action taken	**Price**	**Cost**	**Price**	**Cost**
Borrowed 100 shares of A and sell	65	6500	65	6500
Bought back 100 shares of A	40	-4000	85	-8500
Profit		2500		-2000
	MAKE MONEY		**LOSE MONEY**	

3.2.4 Diversification

Diversification is a risk management technique that mixes a wide variety of investments within a portfolio in order to minimise the impact that any one security will have on the overall performance of the portfolio. What do you need to have a well diversified portfolio? There are 3 main aspects you should have to ensure for the best diversification:

- portfolio should be spread among many different investment vehicles such as cash, stocks, bonds, mutual funds, and perhaps even some real estate.
- securities should vary in risk. You're not restricted to picking only blue chip stocks, the opposite is actually true. Picking different investments with different rates of return will ensure that large gains offset losses in other areas.
- securities should vary by size and industry, minimising unsystematic risk to small groups of companies.

Most investment professionals agree that while it does not guarantee against a loss, diversification is the most important component for helping you to reach your long-range financial goals while minimising your risk. But, remember that no matter how much diversification you do, it can never reduce risk down to zero.

3.2.5 Risk and Return Trade off

The risk-return tradeoff could easily be called the "sleep at night" test. Deciding what amount of risk you can take on while still being able to get a comfortable rest at night without worrying yourself to death is a most important decision. The risk/return tradeoff is the balance, an investor must decide on between the desire for the lowest possible risk and the highest possible returns. Remember to keep in mind that low levels of uncertainty (low risk) are associated with low potential returns and high levels of uncertainty (high risk) are associated with high potential

Figure 9: The risk-return trade off

returns. This is demonstrated graphically in Figure 9. A higher standard deviation means a higher risk, which for the most part means a higher potential return.

How do you know what risk is most appropriate for you? This isn't an easy question to answer. The amount of risk you can comfortably undertake differs from person to person. Generally, if you are having anxiety attacks every time when the market moves up or down, then perhaps you should consider reducing your risk.

3.3 The Markowitz Model

The portfolio selection model of Markowitz laid the foundations of modern portfolio theory. Markowitz showed how rational investors can construct optimal portfolios under conditions of uncertainty. The mean and variance of a portfolios return represent the benefit and risk associated with the investment.

Consider a portfolio of n assets defined in terms of a set of weights w_i for $i = 1, \cdots, n$, which sum to unity. Given an expected rate of return $\bar{r}$, the optimal portfolio is defined in terms of the solution of quadratic or linear programming problem.

Maximum Expected Wealth Problem

This problem maximises the expected portfolio return. It is a risk-neutral approach which does not take risk attitudes into account. This can be achieved by the following linear programming problem

$$\max_{w} \sum_{i=1}^{n} w_i r_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} w_i &= 1 \\ w_i &\geq 0 \qquad i = 1, \cdots, n \end{aligned} \tag{35}$$

The linear constraints in the model describe the allocation of initial investment and bounds on decision variables. Notice that the non-negativity of weights are to prevent short selling.

Minimum Variance Portfolio

The minimum variance portfolio is obtained by solving the following quadratic programming problem which only takes risk into account.

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} w_i &= 1 \\ w_i &\geq 0 \qquad i = 1, \cdots, n \end{aligned} \tag{36}$$

Mean Variance Model

The Markowitz problem provides the foundation for single-period investment theory. The mean-variance problem describes basically the trade off between expected rate of return and variance of rate of return in a portfolio. This approach attempts to inject risk aversion into the optimisation model. It incorporates the quadratic variance term. Therefore, it naturally leads to diversification by taking risk attitudes into account. The mean-variance problem can be stated as follows;

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} w_i r_i &= \bar{r} \\ \sum_{i=1}^{n} w_i &= 1 \\ w_i &\geq 0 \qquad i = 1, \cdots, n \end{aligned} \tag{37}$$

The first constraint presents the expected portfolio return fixed at a certain value. The other constraints are the same as those given in the maximum wealth problem. Notice that the mean-variance model becomes the minimum variance problem when the expected wealth constraint is ignored. Financial reality dictates that the highest performing portfolio strategy is also the most risky efficient strategy available. In order to obtain other points on the Markowitz efficient frontier, it is necessary to consider risk (variance) in conjunction with the mean return. In this case the required expected return can be provided as constant $\bar{r}$.

Markowitz Solution without Short-sale

Consider the mean-variance optimisation problem presented above. Introducing dual variables $\mu, \bar{\mu}, \bar{\bar{\mu}}$ corresponding to the constraints of quadratic programming problem, respectively, the Lagrangian function can be stated as

$$\mathcal{L}(w, \mu, \bar{\mu}, \bar{\bar{\mu}}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} + \mu \left[\sum_{i=1}^{n} w_i r_i - \bar{r} \right] + \bar{\mu} \left[\sum_{i=1}^{n} w_i - 1 \right] - \sum_{i=1}^{n} \bar{\bar{\mu}}_i w_i$$

The KKT conditions are presented below. The solution of linear equation system provides the optimal investment strategy.

$$\begin{aligned} \sum_{j=1}^{n} w_j \sigma_{ij} + \mu r_i + \bar{\mu} - \bar{\bar{\mu}}_i &= 0 \quad i = 1, 2, \cdots, n \\ \sum_{i=1}^{n} w_i r_i &= \bar{r} \\ \sum_{i=1}^{n} w_i &= 1 \\ w_i \bar{\bar{\mu}}_i &= 0, \quad i = 1, 2, \cdots, n \\ w_i, \bar{\bar{\mu}}_i &\geq 0, \quad i = 1, 2, \cdots, n \end{aligned}$$

3.3.1 Efficient Frontier

Varying the desired level of return and repeatedly solving the quadratic programming problem identifies the minimum variance portfolio for each value of $\bar{r}$. These are the efficient portfolios that compose the efficient set. Plotting corresponding values of the objective function and $\bar{r}$ (variance and return, respectively) traces the Markowitz efficient frontier in the mean-variance space. An example of the efficient frontier is presented in Figure 10. How much more or less

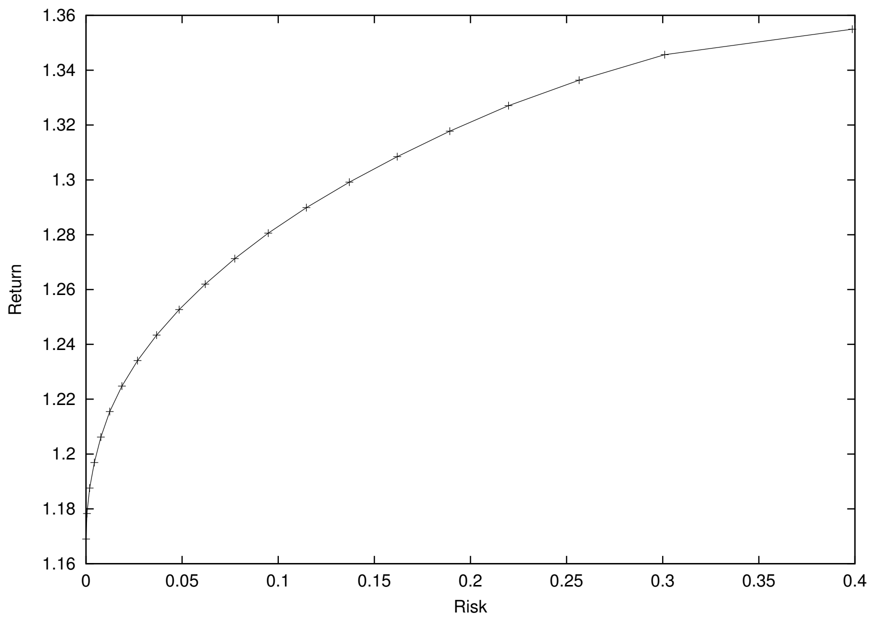

Figure 10: An example of the efficient frontier

volatility you are willing to bear in your portfolio is done by choosing any other point that falls on the "Efficient Frontier", since this will give you the maximum return for the amount of risk you wish to take on. The set of points (that corresponds to portfolios) is called *feasible set* or *feasible region*. Suppose that investor's choice of portfolio is restricted to the feasible points on a given horizontal line in mean-variance plane. All portfolios on this line have the same mean rate of return, but different variances. Most investors prefer the portfolio corresponding to the leftmost point on the line; that is the point with the smallest variance with the given mean. An investor who agrees with this viewpoint is said to be *risk averse*. In this case, he or she seeks to minimise risk as measured by variance. An investor who would select a point other than the one of minimum standard deviation is said to be *risk preferring*.

Optimising your portfolio is not something you can calculate in your head. In fact, there are computer programs dedicated to estimating 100's (and sometimes 1000's) of estimates for expected returns in determining optimal portfolios for each given amount of risk.

4 Asset Pricing Models

Two main problem types dominate the investment science.

1. Determining the best course of action in an investment situation: problems of this type include how to determine the best portfolio, how to devise the optimal strategy for managing an investment, how to select from a group of potential investment projects, and so forth. Mean-variance model is an example of this type presented in the previous section.

2. Determining the correct, arbitrage-free, fair, or equilibrium price of an asset: we also saw some examples of this type such as the formula for the correct price of a bond in terms of the term structure of interest rates.

We will carry on the pricing issue in this chapter, especially the correct price of risky asset within the framework of the mean-variance setting. This is the *Capital Asset Pricing Model*

(CAPM) which was developed by Sharpe, Lintner and Mossin. An alternative asset pricing model, we consider here is the *Arbitrage Pricing Model* (APT), is based on factor models. The APT model was first developed by Stephen Ross, then have been created for applications in most cash and derivative markets. Most of the criticism of the CAPM comes from two sources. The first is the assumption that investors are mean-variance optimisers. The second comes from single–period assumption.

4.1 The Capital Asset Pricing Model (CAPM)

In the mean-variance framework, the investor chooses portfolios on the efficient frontier. In practice, deciding whether or not a given portfolio is on the efficient frontier is difficult. For one thing, there is no guarantee that a portfolio that was efficient ex ante will be efficient ex post. Furthermore, the statistical considerations regarding time period over which to estimate and which assets to include are non-trivial. Also we have not mentioned the implication of mean-variance optimisation on asset prices. The CAPM describes mean-variance portfolios and provides asset pricing formula. In order to derive the CAPM we will make the following assumptions.

- All investors are mean-variance optimisers and a single–period world is considered.
- All investors use the same joint probability distribution for asset returns; that is everyone assigns to the returns of assets the same mean values, the same variances, and the same covariances.
- There is no transaction costs or taxes.
- Investors can borrow or lend at the risk free rate and all assets are traded.
- All investors are price takers, i.e. their purchases and sales do not influence the price of an asset.

4.1.1 Basic Concepts

Market Portfolio According to the one-fund theorem, everyone will purchase a single fund of risky asset, and may borrow or lend at the risk-free rate. In addition, since everyone uses the same means, variances, and covariances, everyone will use the same risky fund. The mix of the risky asset with risk-free asset will likely vary across individuals according to their individual tastes for risk. Some will seek to avoid risk and have a high percentage of the risk free asset in their portfolios; those who are more aggressive, will have a high percentage of the risky asset. However, every individual will form a portfolio that is a mix of the risk-free asset and the single risky one fund. Hence the one fund in the theorem is really the only fund that is used.

If everyone purchases the same risky asset, what must that fund be? The answer to this question is the key insight to underlying the CAPM. The answer to this question is that this fund must equal to the market portfolio which is the summation of all assets. A weight w_i of asset i is defined as the proportion of portfolio capital that is allocated to that asset. Hence the weight of an asset in the market portfolio is equal to the proportion of that asset's total capital value to the total market capital value. These weights are also referred as *capitalisation weights*.

In the idealised world, every investor is a mean-variance investor. In other words, every investor is facing the same mean variance efficient frontier. This means that all investors hold the same proportion of each asset, including the risk free asset and have the same estimates, and everyone buys the same portfolio (which must be equal to the market portfolio). Therefore, if we aggregate all investors' portfolios we have a portfolio that is on the frontier, but the aggregation of all portfolios is the entire market. Hence we have shown that *the market is mean-variance efficient*. The implication is that investors do not need to perform the optimisation. The efficient portfolios are the ones connecting the risk free rate to the market portfolio.

Market Equilibrium The return on an asset depends on both the initial and final prices. The investors solve the mean-variance portfolio problem using their common estimates, and they place orders in the market to acquire their portfolios. If the orders placed do not match what is available, the prices must change. The prices of assets under heavy demand will increase; the prices of assets under light demand will decrease. Of course, the price changes affect the estimates of asset returns directly, and hence investors will recalculate their optimal portfolios. This process continues until demand exactly matches supply; that is it carries on until there is an equilibrium.

The Capital Market Line The single efficient fund of risky assets is the market portfolio and we can label this fund on the $\bar{r} - \sigma$ diagram with M for market. The efficient set therefore consists of the straight line, emanating from the risk free point and passing through the market portfolio. This line is called the Capital Market Line (CML) and presented in Figure 11. It is also called as a pricing line, since prices should adjust so that efficient

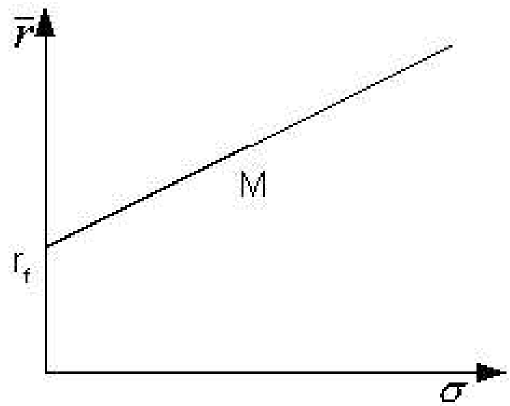

Figure 11: The Capital Market Line

assets fall on this line. In mathematical terms, the capital market line states that

$$\bar{r} = r_f + \frac{\bar{r_M} - r_f}{\sigma_M}\sigma$$

where $\bar{r_M}$ and σ_M are the expected value and standard deviation of the market rate of return and $\bar{r}$ and σ are the expected value and standard deviation of the rate of return of an arbitrary efficient asset. This equation describes all portfolios on capital market line which are combinations of riskless asset and portfolio M where r_f is the intercept and $\frac{\bar{r_M} - r_f}{\sigma_M}$ is the slope of the line. The slope is also called the price of the risk and describes how much the expected rate of return of a portfolio must increase if its standard deviation increases by one unit. As a consequence all portfolios that contain some proportion of portfolio M and risk free asset will be perfectly correlated with it. An investor can attain positions between r_f and M by investing some of his/her money in portfolio M and the rest of it in the risk free bond. For these positions he/she buys the risk free asset. On the other hand, the positions on the line beyond point M can be obtained by selling the risk free bond and using the proceeds of sale to buy portfolio M.

4.1.2 The Pricing Model

The capital market line relates the expected rate of return of an efficient portfolio to its standard deviation but it does not show how the expected rate of return of an individual asset relates to its individual risk. This relation is expressed by the CAPM.

If the market portfolio M is efficient, then the expected return $E[r_i]$ of an asset i satisfies

$$E[r_i] - r_f = \beta_i(E[r_M] - r_f)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Proof:

Consider the portfolio consists of a portion λ invested in asset i and a portion $1-\lambda$ invested on the market portfolio M. Here $\lambda < 0$ is allowed, which corresponds to borrowing at the risk-free rate. In order to derive the CAPM we consider the portfolio with return

$$r_\lambda = \lambda r_i + (1-\lambda) r_M \tag{38}$$

where r_i is the return of asset i and r_M is the return on the market. Taking expectation of two sides of Equation (38) gives

$$\bar{r_\lambda} = \lambda \bar{r_i} + (1-\lambda)\bar{r_M}$$

and the standard deviation of the rate of return is

$$\sigma_\lambda^2 = \lambda^2\sigma_i^2 + 2\lambda(1-\lambda)Cov[r_i, r_M] + (1-\lambda)^2\sigma_M^2$$

where $E[r_*] = \bar{r_*}$ and σ_*^2 are the expected return and the variance of r_*. When $\lambda = 0$, it corresponds to the market itself, so r_λ is on CML; in other words, it is efficient. Observe that for any λ the point $(\sigma_\lambda, \mu_\lambda)$ must lie to the right of the CML; otherwise it would be a portfolio with the same variance but higher expected return than an efficient portfolio, thus violating the definition of efficiency. As λ varies, the values of $\bar{r_\lambda}$ and σ_λ trace the curve.

The tangency condition can be translated into the condition that the slope of the curve is equal to the slope of the capital market line at point M. In order to show this, we first need to set up the following derivations;

$$\begin{aligned} \frac{d\bar{r_\lambda}}{d\lambda} &= \bar{r_i} - \bar{r_M} \\ \frac{d\sigma_\lambda}{d\lambda} &= \frac{\lambda\sigma_i^2 + (1-2\lambda)\sigma_{iM} + (\lambda-1)\sigma_M^2}{\sigma_\lambda} \end{aligned}$$

Thus

$$\left.\frac{d\sigma_\lambda}{d\lambda}\right|_{\lambda=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$$

In addition, using the relation

$$\frac{d\bar{r_\lambda}}{d\sigma_\lambda} = \frac{d\bar{r}_\lambda/d\lambda}{d\sigma_\lambda/d\lambda}$$

we obtain the slope as

$$\left.\frac{d\bar{r_\lambda}}{d\sigma_\lambda}\right|_{\lambda=0} = \frac{(\bar{r_i} - \bar{r_M})\,\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

which must be equal to the slope of the capital market line. Hence we find

$$\frac{(\bar{r_i} - \bar{r_M})\,\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r_M} - r_f}{\sigma_M}$$

or the stated formula for the CAPM as

$$\bar{r_i} = r_f + \frac{\bar{r_M} - r_f}{\sigma_M^2}\sigma_{iM} = r_f + \beta_i(\bar{r_M} - r_f) \tag{39}$$

■

The value $\bar{r_i} - r_f$ is called the *expected excess rate of return of asset i*. This is basically the amount by which the rate of return is expected to exceed the risk-free rate. Likewise $\bar{r_M} - r_f$ is the *expected excess rates of the return of market portfolio*. According to this, CAPM refers that the expected excess rate of the return of an asset is proportional to the expected excess rates of the return of market portfolio and proportionality factor is β.

The value β_i is referred to the beta of an asset and measures the riskiness of each asset with respect to the market portfolio M. High beta assets earn higher average return in equilibrium because of the factor of $\beta_i(\bar{r_M} - r_f)$. The beta of the market portfolio is then

$$\beta_M = \frac{Cov(r_M, r_M)}{Var(r_M)} = 1$$

This is used as a reference point which the risk of other assets can be measured. The average risk of all assets is the beta of the market, which is one. Assets or portfolios that have a beta greater than one have above average risk, tending to move more than market. For example, if the riskless rate of interest (T-bill rate) is 5% per year and the market rises by 10%, assets with a beta of 2 will tend to increase by 15%. If however, the market falls by 10%, assets with a beta of 2 will tend to fall by 25% on average. Conversely, assets with betas less than one are of below average risk and tend to move less than the market portfolio. Assets that have betas less than zero tend to move in opposite direction to the market. These assets are known as hedge assets.

Beta of Portfolio

It is easy to calculate the overall beta of a portfolio β_p in terms of the betas of the individual assets in the portfolio. Suppose that the portfolio has n assets with the weights $w_1, \cdots, w_n$. If the rate of return of the portfolio is $r = \sum_{i=1}^{n} w_i r_i$ and the $cov(r, r_M) = \sum_{i=1}^{n} w_i cov(r_i, r_M)$, then $\beta_p = \sum_{i=1}^{n} w_i \beta_i$. The CAPM can be expressed in graphical form by regarding the formula as a linear relationship which is called security market line. The CAPM in the covariance form and in beta form is expressed in Figure 12. The market portfolio is at the point σ_M^2 in the first graph

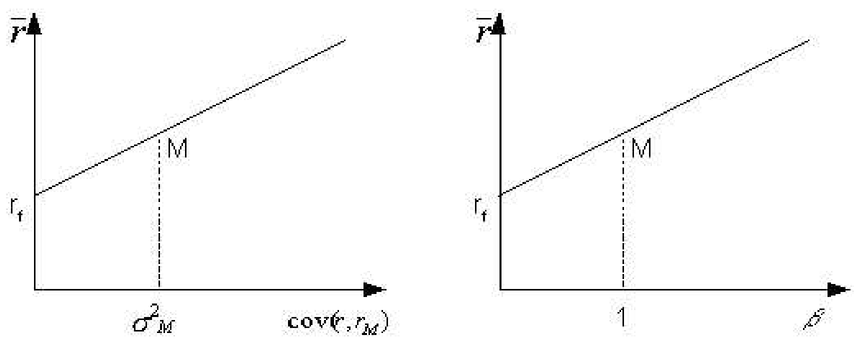

Figure 12: The Security Market Line

and at $\beta = 1$ at the second graph. The security market line expresses the risk-reward structure of assets according to CAPM. In other words, it describes the risk associated with each asset in the portfolio as a function of its covariance with market or equivalently a function of its beta.

4.1.3 Systematic and Specific Risk

The CAPM divides the risk of holding assets into systematic and specific risk. Systematic risk is the risk of holding the market portfolio. Specific risk is the risk, which is unique to an individual asset. The total risk is the sum of these two risks.

Consider the random rate of return of an asset i as $r_i = r_f + \beta_i(r_M - r_f) + e_i$. If we take expected value of it, we find that $E[e_i] = 0$ (according to CAPM). Taking the correlation with r_M and using the definition of β_i, we obtain $cov(e_i, r_M) = 0$ as well. We can therefore obtain σ_i^2 as the sum of the two terms

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2$$

The first term $\beta_i^2\sigma_M^2$ is called the systematic risk which is associated with the market as a whole. As the market moves, each individual asset is more or less affected. To the extent that any asset is affected by such general market moves, that asset entails systematic risk. The second term $\sigma_{e_i}^2$ is called non-systematic (idiosyncratic, or specific risk). This represents the component of an asset's volatility, which is uncorrelated with general market moves. Therefore, the specific risk can be reduced by diversification. The systematic risk measured by beta is most important since it directly combines with the other assets' systematic risk.

4.1.4 CAPM as a Pricing Formula

The CAPM is a pricing model. However, the standard CAPM formula does not hold the prices explicitly, only the expected rates of return. Now we will obtain the CAPM in terms of the asset prices. Suppose that an asset is purchased at price P and later sold at price S. The rate of return is

$$r = \frac{S-P}{P}$$

Here P is known, but S is uncertain (random). If we substitute it in the CAPM formula, we obtain

$$\begin{aligned} \frac{\bar{S}-P}{P} &= r_f + \beta(\bar{r}_M - r_f) \\ P &= \frac{\bar{S}}{1 + r_f + \beta(\bar{r}_M - r_f)} \end{aligned}$$

Notice that this can also be seen as a discounting formula. In the deterministic case, it is appropriate to discount the future payment at the interest rate r_f, using a factor of $\frac{1}{1+r_f}$. In the random case, the appropriate interest rate is $r_f + \beta(\bar{r}_M - r_f)$, which can be regarded as a risk-adjusted interest rate.

4.2 Factor Models

The main idea of factor models is that the riskiness of an asset can be explained by number of factors. The simplest form of factor models is a single factor which is based on one factor. However, multi-factor models are based on more than one factor.

4.2.1 Single Factor Models

Consider n assets indexed by i with rates of return r_i for $i = 1, \cdots, n$ and a single factor f which is a random quantity such as inflation, interest rate so on. Assume that the rate of return can be expressed in terms of the factor as

$$r_i = a_i + b_i f + e_i \qquad i = 1, \cdots, n \tag{40}$$

Notice that the rates of return and the single factor are linearly related. In Equation (40), a_i and b_i are fixed constants. The quantity e_i is random and represents the errors. We assume that errors

- have zero mean; that is $E[e_i] = 0$, for $i = 1, \cdots, n$
- are uncorrelated with the factor f; that is $E[fe_i] = 0$ for $i = 1, \cdots, n$
- are uncorrelated each other; that is $E[e_i e_j] = 0$ for $i \neq j$

Equation (40) also defines a linear fit to data. Imagine that several independent observations are made of both the rate of return and factor. When the data is plotted the lines are likely to be scattered. A straight line defined by the single factor equation is fitted through these points in such a way that the average value of the error is zero. The error is measured by the vertical distance from a point to the line. The fitting process is carried out for each asset separately to obtain the parameters as follows. The mean return and the variance of assets $i = 1, \cdots, n$ are calculated as

$$E[r_i] = \bar{r_i} = a_i + b_i \bar{f} \tag{41}$$

and

$$Var[r_i] \;=\; \sigma_i^2 = b_i^2\sigma_f^2 + \sigma_{e_i}^2 \tag{42}$$

In addition, the covariance between asset i and j is

$$\sigma_{ij} = b_i b_j \sigma_f^2 \qquad\qquad i \neq j$$

Therefore, the parameter b_i for asset i is calculated as

$$b_i \;=\; \frac{cov(r_i, f)}{\sigma_f^2} \tag{43}$$

Substituting (43) in (41), parameter a_i can be obtained. The estimated parameters are valid only for the current data set. Different values of b_i and a_i are obtained for the different sets of historical observations of r_i and f. Since a_i is the intercept of the line for asset i with the vertical axis, it is sometimes called *intercepts*. The parameter b_i measures the sensitivity of the return to the factor; therefore, it is named *factor loading*. If a portfolio is formed with weights of w_i for $i = 1, \cdots, n$ such that $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} r_p &= \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i b_i f + \sum_{i=1}^{n} w_i e_i \\ &= a + bf + e \end{aligned} \tag{44}$$

where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i, \quad e = \sum_{i=1}^{n} w_i e_i$$

The expected value and variance of e are

$$\begin{aligned} E[e] &= \sum_{i=1}^{n} w_i E[e_i] = 0 \\ Var[e] &= E[e^2] \\ &= \left[\left(\sum_{i=1}^{n} w_i e_i\right)\left(\sum_{i=1}^{n} w_i e_i\right)\right] \\ &= E\left[\sum_{i=1}^{n} w_i^2 e_i^2\right] \\ &= \sum_{i=1}^{n} w_i^2 Var[e_i] \end{aligned} \tag{45}$$

If the variance of each asset is bounded by some constant, say σ, and if we put $w_i = \frac{1}{n}$, then we get

$$Var[e] = \frac{\sigma^2}{n} \to 0 \ \text{ as } \ n \to \infty$$

In a well diversified portfolio, the error term in the factor equation is small.

4.2.2 Multi-factor Models

The single factor model can be extended to include more than one factor. Let's consider two factors f_1 (being a broad index of the market return) and f_2 (being an index of change). The model for the rate of return of asset i has the form

$$r_i = a_i + b_{1i} f_1 + b_{2i} f_2 + e_i$$

The constant a_i is called the intercept, and b_{1i}, b_{2i} are the factor loadings. The factors f_1 and f_2 and the error e_i are random variables. It is assumed that the expected value of error is zero and the error is uncorrelated with two factors as well as with errors of other assets. However, it is not assumed that two factors are uncorrelated with each other. These factors are observable variables and their statistical properties can be studied independently of the asset returns. We easily derive the following values for the expected rates of return and the covariances

$$\bar{r_i} = a_i + b_{1i}\bar{f_1} + b_{2i}\bar{f_2} \tag{46}$$

$$cov(r_i, r_j) = \begin{cases} b_{1i}b_{1j}\sigma_{f_1}^2 + [b_{1i}b_{2j} + b_{2i}b_{1j}]\, cov(f_1, f_2) + b_{2i}b_{2j}\sigma_{f_2}^2, & i \neq j \\ b_{1i}^2\sigma_{f_1}^2 + 2b_{1i}b_{2i}cov(f_1, f_2) + b_{2i}\sigma_{f_2}^2 + \sigma_{e_i}^2, & i = j \end{cases} \tag{47}$$

The factor loadings b_{1i} and b_{2i} can be obtained by forming the covariance of r_i with f_1 and f_2, as follows;

$$\begin{aligned} cov(r_i, f_1) &= b_{1i}\sigma_{f_1}^2 + b_{2i}\sigma_{f_1,f_2} \\ cov(r_i, f_2) &= b_{1i}\sigma_{f_1,f_2} + b_{2i}\sigma_{f_2}^2 \end{aligned} \tag{48}$$

Solution of this equation system gives the values of unknowns b_{1i} and b_{2i}. When the factor loadings are substituted in Equation (46) the intercepts a_i can be found.

4.2.3 Selection of Factors

The selection of appropriate factors for a factor model is part science and part art. However, we can categories the factors in the three groups;

External factors It is very common that factors are chosen to be variables that are external to the securities being explicitly considered in the model, such as gross national product (GNP), consumer price index (CPI), unemployment rate.

Extracted factors It is possible to extract factors from the known information about security returns. For example, the rate of return on the market portfolio is the most frequently used factor. It is constructed directly from the returns of the individual securities. The rate of return of one security can also be used as a factor for others. The method of principal components is used to extract factors. The method uses the covariance matrix of the returns to find the combinations of securities that have a large variances.

Firm characteristics Firms are characterised financially by a number of firm–specific values, such as price–earnings ratio, the dividend–pay out ratio, and earnings forecast.

4.2.4 The CAPM as a Factor Model

The CAPM can be derived as a special case of a single-factor model. For stock returns a single factor model is used in order to express the CAPM. The factor is the market rate of return $f = r_M$.

$$\begin{aligned} r_i &= a_i + b_i r_M + e_i \\ r_i - r_f &= a_i - (1 - b_i)r_f + b_i(r_M - r_f) + e_i \\ &= \alpha_i + b_i(r_M - r_f) + e_i \\ E[r_i] &= r_f + \alpha_i + b_i\,(E[r_M] - r_f) \end{aligned}$$

Hence the CAPM predicts that $\alpha_i = 0$. On the other hand, we have

$$\begin{aligned} Cov[r_i - r_f, r_M] &= Cov[\alpha_i + b_i(r_M - r_f) + e_i, r_M] \\ Cov[r_i, r_M] &= b_i Var[r_M] \\ b_i &= \frac{Cov[r_i, r_M]}{Var[r_M]} \end{aligned}$$

and the intercept is obtained as in the CAPM. Notice that $b_i = \beta_i$. An example of CAPM as a factor model is presented in Figure 13. While y-axis represents the returns on the asset Lloyds, the market returns are on the x-axis.

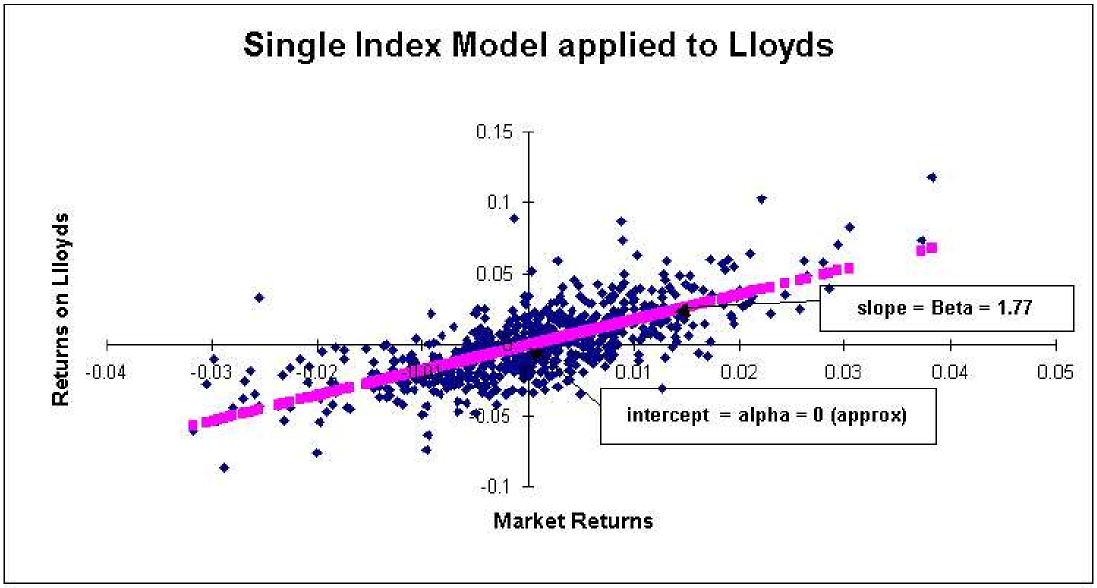

Figure 13: An Example of Capital Asset Pricing Model as a Factor Model

4.3 The Arbitrage Pricing Theory

The factor model framework leads to an alternative approach to asset pricing, called Arbitrage Pricing Theory (APT). This theory does not require the assumption that investors evaluate portfolios on the basis of means and variances; only that when returns are certain, investors prefer greater return to lesser return. In this sense, the theory is much more satisfying than the CAPM theory, which relies on both the mean-variance framework and a strong version of equilibrium, which assumes that everyone uses the mean-variance framework. Instead the APT assumes that the universe of assets being considered is large (we consider an infinite number of assets) and these securities differ from each other in nontrivial ways. The APT uses an arbitrage argument to force the relationship between a_i and b_i so that a_i can be eliminated from the pricing formula.

Example 4.1

Consider three securities A, B and C whose prices and payoff values in two states are presented in the following table. How can you produce an arbitrage opportunity involving three securities?

Security	**Price (£)**	**Payoff in State 1 (£)**	**Payoff in State 2 (£)**
A	70	50	100
B	60	30	120
C	80	38	112

Let w_A and w_B be the proportions of security A and B in the portfolio. We can combine securities A and B in such a way that they replicate the payoffs of security C in either state. The portfolio has the payoff $50w_A + 30w_B$ in state 1 and $100w_A + 120w_B$ in state 2. Portfolio consisting of securities A and B will reproduce the payoff of C regardless of the state occurs one year from now. Therefore, we can write

$$50w_A + 30w_B = 38 \text{ and } 100w_A + 120w_B = 112 \tag{49}$$

An arbitrage opportunity exists if the cost of the portfolio is different than the cost of security C. Solving the equation system (49) we obtain weights $w_A = 0.6$, $w_B = 0.4$. Then the cost of the portfolio is $0.4 \times £70 + 0.6 \times £60 = £64$. However, the price of security C is £80. The synthetic security is cheap relative to security C. A risk-less arbitrage profit can be obtained by *buying A and B* in these proportions and *shorting security C*.

Suppose that you have £1m capital to construct the arbitrage portfolio. Therefore, we allocate the capital between asset A and B as £400k and £600k, respectively. In other words, we buy £400k/£70 = 5714 shares of A and £600k/£60 = 10,000 shares of B. In addition, we short-sale £1m in asset C; that is (£1m/£80) = 12,500 shares of C. The outcome of forming an arbitrage portfolio of £1m is summarised in the following table.

Security	**Investment**	**State 1**	**State 2**
A	-400000	$5714 \times 50 = 285700$	$5714 \times 100 = 574100$
B	-600000	$10000 \times 30 = 300000$	$10000 \times 120 = 1200000$
C	1000000	$12500 \times 38 = -475000$	$12500 \times 112 = -1400000$
Total	0.00	110,700	371,400

4.3.1 The Simple Version of APT

Consider a single–factor model. Assume that the factor model holds exactly (that is there is no e_i term);

$$r_i = a_i + b_i f$$

The uncertainty comes from the factor f. The APT says that a_i and b_i are related if there is no arbitrage. Choose another asset j, which is different from the asset i, such that $b_i \neq b_j$ and form a portfolio with w of asset i and $(1-w)$ of asset j.

$$r_p = wa_i + (1-w)a_j + [wb_i + (1-w)b_j]f$$

If we choose $w = \frac{b_j}{b_j - b_i}$ so that $wb_i + (1-w)b_j = 0$, then

$$r_p = \frac{a_i b_j}{b_j - b_i} + \frac{a_j b_i}{b_i - b_j} \tag{50}$$

and since r_p has no exposure to f, it is risk-less. Therefore, r_p must be equal to the risk-less rate r_f. The special portfolio is risk free because the equation for r_i contains no random. If there is a separate risk-free asset with a rate of return r_f, it is clear that the portfolio constructed in (50) must have the same rate; otherwise there would be an arbitrage opportunity. Even if there is no explicit risk-free asset, all portfolios constructed this way, with no dependence on f must have the same rate of return. We denote this rate by α_0, ($\alpha_0 = r_f$, if there is a risk free asset).

Setting the right-hand side of (50) equal to α_0, we find

$$\begin{aligned}
\alpha_0 &= \frac{a_i b_j}{b_j - b_i} + \frac{a_j b_i}{b_i - b_j} \\
\alpha_0 (b_j - b_i) &= a_i b_j - a_j b_i \\
\frac{a_j - \alpha_0}{b_j} &= \frac{a_i - \alpha_0}{b_i} \qquad (51)
\end{aligned}$$

The relationship must hold for all assets, therefore $\frac{a_i - \alpha_0}{b_i}$ must be a constant c which is not depending on i. Hence we obtain the following

$$\frac{a_i - \alpha_0}{b_i} = c$$

for all i and a constant c. This shows explicitly that the values of a_i and b_i are not independent; that is indeed

$$a_i = \alpha_0 + b_i c$$

We can use this information to write a formula for the expected rate of return of asset i.

$$\begin{aligned}
E[r_i] &= a_i + b_i E[f] \\
&= \alpha_0 + b_i c + b_i E[f] \\
&= \alpha_0 + b_i E[f + c]
\end{aligned}$$

If $E[f] + c = \alpha_1$, then we can get

$$E[r_i] = \alpha_0 + b_i \alpha_1 \tag{52}$$

Notice that once the constants α_0 and α_1 are known, the expected return of an asset i is determined entirely by the factor loading b_i since a_i must follow b_i.

The pricing formula given in (52) looks similar to the CAPM. If the factor f is chosen to be the rate of return on the market r_M, then we can set $\alpha_0 = r_f$ and $\alpha_1 = E[r_M] - r_f$. Therefore, the APT is identical to the CAPM when $b_i = \beta_i$. Therefore the CAPM is a consequence of the APT if an exact single factor model holds. This result can also be extended for multi factor models as follows:

Suppose that there are n assets whose rates of return are governed by $m < n$ factors according to the equation

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$

for $i = 1, \cdots, n$. Then there are constants $\alpha_0, \alpha_1, \cdots, \alpha_m$ such that

$$E[r_i] = \alpha_0 + \sum_{j=1}^{m} b_{ij} \alpha_j$$

for $i = 1, \cdots, n$.

4.3.2 The APT Model for Well-Diversified Portfolios

Consider more realistic factor models which have error terms as well as factor terms. Suppose that there are n assets and the rate of return on asset i can be represented by

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j + e_i$$

where $E[e_i] = 0$ and $E[e_i^2] = \sigma_{e_i}^2$. Also assume that e_i is uncorrelated with the factors and with the error terms of other assets. Let us form a portfolio using the weights $w_1, \cdots, w_n$ with $\sum_{i=1}^{n} w_i = 1$. The rate of the return of the portfolio is

$$r = a + \sum_{j=1}^{m} b_j f_j + e_i$$

where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b_j = \sum_{i=1}^{n} w_i b_{ij}, \quad \sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

Suppose that for each i there holds $\sigma_{e_i}^2 \leq S^2$ for a constant S. In addition, the portfolio is well-diversified, that is for each i there holds $w_i \leq \frac{W}{n}$ for a constant $W \approx 1$. Therefore, no asset is heavily weighted in the portfolio. The following inequality holds

$$\sigma_e^2 \leq \frac{1}{n^2} \sum_{i=1}^{n} W^2 S^2 \leq \frac{1}{2} W^2 S^2$$

If $n \to \infty$ and $\sigma_{e_i}^2 \leq S^2$ hold, then $\sigma_e^2 \to 0$. In other words, the error term associated with well-diversified portfolio of an infinite number of assets has a variance of zero. If we apply the simple APT, we obtain the following;

Suppose that there are n assets whose rates of return are governed by $m < n$ factors according to the equation

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$

for $i = 1, \cdots, n$. Then there are constants $\alpha_0, \alpha_1, \cdots, \alpha_m$ such that for any well-diversified portfolio having an expected rate of return is

$$E[r_i] = \alpha_0 + \sum_{j=1}^{m} b_{ij} \alpha_j \text{ for } i = 1, \cdots, n$$

This is again basically a relation describing that a_i is not independent of b_{ij}'s. The risk free term must be related to the factor loadings. This is true even when there are error terms, provided there is a large number of assets so that error terms can be effectively diversified away.

4.3.3 APT and CAPM

The multi-factor model underlying APT can be applied to the CAPM framework to derive a relation between the two theories. For simplicity, we consider two-factor model with the factors f_1 and f_2. Then the rate of return for asset i is

$$r_i = a_i + b_{i1}f_1 + b_{i2}f_2 + e_i$$

The covariance of this asset with the market portfolio is

$$Cov(r_M, r_i) = b_{i1}Cov(r_M, f_1) + b_{i2}Cov(r_M, f_2) + Cov(r_M, e_i)$$

If the market represents a well-diversified portfolio, the term $Cov(r_M, e_i)$ is ignored since there is no error term in this portfolio. The beta of the asset i is

$$\beta_i = b_{i1}\beta_{f_1} + b_{i2}\beta_{f_2}$$

where

$$\beta_{f_1} = \frac{\sigma_{M,f_1}}{\sigma_M^2}, \;\; \beta_{f_2} = \frac{\sigma_{M,f_2}}{\sigma_M^2}$$

Hence the overall beta of asset can be obtained from underlying factor betas that do not depend on the particular asset. The weights of these factor betas in the overall asset beta is equal to the factor loadings. Notice that different assets have different betas since they have different loadings.