Polyhedral Combinatorics (Summer 2025)

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July 6, 2025

Homework 1

Problem 1. Let $d \geq 1$ be an integer. An undirected graph is called d-regular if every node has degree d. Use Corollary 1.5 to show that every d-regular bipartite graph has a perfect matching.

Solution. Let $G = (V, E)$ be the given bipartite graph. Then the perfect matching polytope, according to Corollary 1.5 (Birkhoff-von Neumann theorem), is given by

$$\mathcal{P}_{\text{perfmatch}}(G) = \{x \in \mathbb{R}^E_{\geq 0} : \sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in V\}.$$

Note that it is not empty as it contains $x_0 := \dfrac{1}{d}\mathbb{1}$. Since $\mathcal{P}_{\text{perfmatch}}$ is a face of

$$\mathcal{P}_{\text{match}} = \{x \in \mathbb{R}^E_{\geq 0} : \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V\}$$

and any vertex of $\mathcal{P}_{\text{match}}$ is integer (Lemma 1.2) then any vertex of $\mathcal{P}_{\text{perfmatch}}$ corresponds to a perfect matching of G. $\square$

Problem 2. Let $C_n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$ be the ℓ_1-ball, also known as the *cross polytope*.

a) What are the vertices of C_n?

b) Show that

$$C_n = \left\{ x \in \mathbb{R}^n : \sum_{i \in I} x_i - \sum_{i \notin I} x_i \leq 1 \text{ for all } I \subseteq [n] \right\}.$$

Solution.

a) The vertex set of C_n is exactly $V := \{\pm e_i, i \in [n]\}$ where $\{e_i\}_{i \in [n]}$ is the standard basis. First, $V \subset C_n$ so $\operatorname{conv} V \subseteq C_n$. On the other hand, let $x \in C_n \setminus V$. Let $I \subseteq [n]$ be the indices of nonzero coordinates of x. Then there exists a small enough vector $\varepsilon \in \mathbb{R}^n$ with $\varepsilon_i = 0$ for all $i \notin I$ such that $x + \varepsilon$ and $x - \varepsilon$ both lie in C_n. However x is their nontrivial convex combination so cannot be a vertex.

b) Let C_n' denote the RHS of b). If $x \in C_n$ then $\forall I \subseteq [n]$

$$\sum_{i\in I} x_i - \sum_{i\notin I} x_i \le \sum_{i\in[n]} |x_i| \le 1$$

so $x \in C_n'$. Now let $x \in C_n'$. Choosing I in the definition of C_n' to be the index set of x's nonnegative coordinates we obtain $\sum_{i\in[n]} |x_i| \le 1$ so $x \in C_n$.

Problem 3. A vertex cover of an undirected graph $G = (V, E)$ is a set of nodes $U \subseteq V$ such that every edge is incident to a node in U. The vertex cover polytope of G is the convex hull of characteristic vectors of vertex covers of G. Give an inequality description of the vertex cover polytope of a bipartite graph.

Solution. We prove that the desired polytope

$$\mathcal{P}_{\text{vertcover}} := \text{conv}\{\chi(U) \in \{0,1\}^V : U \subseteq V, e \cap U \neq \varnothing \quad \forall e \in E\}$$

is

$$\mathcal{Q} := \{x \in [0,1]^V : x_u + x_v \ge 1 \quad \forall \{u,v\} \in E\}.$$

Obviously $\mathcal{P}_{\text{vertcover}} \subseteq \mathcal{Q}$. To prove the opposite inclusion we prove that all the vertices of $\mathcal{Q}$ are integer (then they will lie in $\mathcal{P}_{\text{vertcover}}$).

Suppose that $\mathcal{Q}$ has a non-integer vertex u. Let $F = \{v \in V : 0 < x_v < 1\}$ be the index set of its fractional coordinates.

Case 1: F contains vertices forming a cycle $v_1, \ldots, v_k, v_1$. Since G is bipartite, $2 \mid k$. Thus we can wiggle x at these vertices (coordinates) alternately by $\pm\varepsilon$ small enough; the obtained point u^+ with changed coordinates $u_{v_1}+\varepsilon, u_{v_2}-\varepsilon, u_{v_3}+\varepsilon, \ldots, u_{v_k}-\varepsilon$ will still be in $\mathcal{Q}$. Similarly obtained point u^- with changed coordinates $u_{v_1}-\varepsilon, u_{v_2}+\varepsilon, u_{v_3}-\varepsilon, \ldots, u_{v_k}+\varepsilon$ will be in $\mathcal{Q}$. But then u will be a nontrivial convex combination of $u^\pm$ so cannot be a vertex.

Case 2: F has no vertices forming a cycle. Then the induced subgraph $G[F]$ is a forest, so has a tree, which in its turn has two leaves. Consider the path joining these leaves and operate as in the previous case. (When wiggling, a problem might arise only in the two leaves; but notice that if any of them cannot be wiggled then it is not in F.)

Problem 4. König's theorem states that, in a bipartite graph, the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover. Give a proof of this fact using linear programming duality.

Show that the statement is not true for general undirected graphs.

Solution. Let $G = (V, E)$ be a bipartite graph. The two polytopes we are interested in are

$$\mathcal{P}_{\text{match}}(G) := \{x \in [0,1]^E : \sum_{e\in\delta(v)} x_e \le 1 \quad \forall v \in V\},$$

$$\mathcal{P}_{\text{vertcover}}(G) := \{y \in [0,1]^V : \sum_{v\in e} x_v \ge 1 \quad \forall e \in E\}.$$

Now if $H \in \mathbb{R}^{V\times E}$ is the node-edge matrix of G, i.e. $H_{ve} = \mathbb{I}\{v \in e\}$ then the cardinality of the maximal matching is

$$\max\{\mathbb{1}^T x : Hx \le \mathbb{1}, x \in \mathbb{R}^E_{\ge 0}\}$$

and the cardinality of the minimal vertex cover is

$$\min\{\mathbb{1}^T y : H^T y \geq \mathbb{1}, x \in \mathbb{R}^V_{\geq 0}\}.$$

By linear programming duality the first one does not exceed the second one. Since both linear programs have optimal solutions, by strong duality theorem they coincide.

To see that the claim fails for non-bipartite graphs consider a triangle. Its maximal matching is of size 1, while its minimal vertex cover is of size 2.

Homework 2

Problem 5. Show that optimizing over the intersection of three matroids is NP-hard.

Hint: You may use the fact that the following problem (directed Hamiltonian path) is NP-hard: Given a directed graph D with nodes s, t, is there an s-t-path in D that contains all nodes of D?

Solution. Let $D = (V, A)$ be a directed graph with different nodes $s, t \in V$. Let M_1 be the graphic matroid of the underlying undirected multigraph and M_2 be the matroid on ground set A where a set F is independent if $|\delta^{\text{in}}(v) \cap F| \leq 1$ for all $v \in V \setminus \{s\}$. Similarly define M_3, but F is independent if $|\delta^{\text{out}}(v) \cap F| \leq 1$ for all $v \in V \setminus \{t\}$. We want to optimize over the face of $M_1 \cap M_2 \cap M_3$ for which $|\delta^{\text{in}}(v) \cap F| = 1$ for all $v \in V \setminus \{s\}$ and $|\delta^{\text{out}}(v) \cap F| = 1$ for all $v \in V \setminus \{t\}$, which is exactly the problem in the hint.

Problem 6. Show that the incidence matrix of the union of two laminar set families is totally unimodular (TU) (see Lemma 4.2).

Solution. First we prove that the incidence matrix of a laminar family is TU. Indeed, consider any square submatrix A of it.

- First we may rearrange the rows of A, possibly changing $\operatorname{sgn}\det A$, so that the first several rows correspond to some monotonously decreasing subsets $S_1 \subseteq \cdots \subseteq S_{i_1}$, the second block of several rows corresponds to some other monotonously decreasing subsets $S_{i_1+1} \subseteq \cdots \subseteq S_{i_2}$ and so on.

- Now, for the j-th block of rows do the following. Subtract the last row of that block from the rows in that block above it. Then subtract the row above the last row from the rows in that block above it, and so on. Now we have a $0-1$ matrix with only one 1 in each column.

Hence the determinant of resulting matrix is either 0 (if there is a zero row) or 1.

Now consider the incidence matrix A of the union of two laminar families. Again, permute the rows so that the first several rows correspond to the first laminar family, and the remaining last rows correspond to the other laminar family. Then in each of these two blocks to the operations described above. Then we are left with a matrix A' with at most two 1s in each column.

Take any subset I of rows. Partition $I = I_1 \sqcup I_2$ so that I_i corresponds to the i-th laminar family. Let S_i be the sum of the rows I_i. Then $S_1 - S_2$ is a $0, \pm 1$ vector. Hence by the theorem of Ghouila-Houri A' is TU.

Problem 7. Prove the following (see Proposition 6.2).

(a) A connected graph has a T-join if and only if $|T|$ is even.

(b) A set J is a T-join if and only if it is the edge-disjoint union of cycles and $\frac{1}{2}|T|$ paths connecting disjoint pairs of nodes in T.

(c) If J is a T-join, then J' is a T-join if and only if $J \Delta J'$ is an edge-disjoint union of cycles.

Solution. (a) Suppose that the graph $G = (V, E)$ has a T-join $J \subseteq E$. Double-count the number of pairs $(v, e) \in V \times J$ where $v \in e$: $\sum_{v \in T} \deg v + \sum_{u \in V \setminus T} \deg u = 2|J|$. Since $\deg u$ is even for all $u \in V \setminus T$ and $\deg v$ is odd for $v \in T$ we get that $|T|$ is even.

Now the other direction. Let $G = (V, E)$ be a connected graph and $T \subseteq V$ with $|T|$ even. Let $T = \{v_1, \ldots, v_{2n}\}$. Since G is connected we can join v_1 and v_2 with a path p_1. Also join v_3 and v_4 with a path p_2. If p_1 and p_2 have any common edge (u, w) then we can flip the paths so that they are disjoint, i.e. change p_1 and p_2 WLOG to $v_1 u v_3$ and $v_2 w v_4$. Continue the procedure and we will be left with n disjoint paths. Their edges form a T-join.

(b) An edge-disjoint union of cycles and $\frac{1}{2}|T|$ paths is clearly a T-join. To prove the converse take any vertex v_0 and go to its neighbor v_1, then to the neighbor of v_1 and so on. Thus we will build a cycle or a path. Delete its edges and continue. (Note that both endpoints of a path will be from T.)

(c) J' is a T-join iff the set of odd degree vertices of the graph (V, J') is T, which happens iff the degrees of vertices in T have even degree in $J \Delta J'$, that is iff $J \Delta J'$ is an edge-disjoint union of cycles.

Homework 3

Problem 8. Given an undirected graph with *nonnegative* edge weights, we have shown that a minimum-weight T-join can be found in polynomial time. Show that we can also find a minimum-weight T-join in polynomial time for (possibly negative) arbitrary weights (see Theorem 9.1).

Solution. Let $F = \{e \in E : w(e) < 0\}$ and U be the set of nodes that are incident to an odd number of edges in F. Set $T' = T \Delta U$.

Let G' be the graph G but with negated weights $w' = |w|$ on F. Let J' be a T'-join. Then $J = J' \Delta F$ is a T-join. We have

$$\begin{aligned} w(J) &= w(J' \Delta F) = w(J' \setminus F) + w(F \setminus J') \\ &= w(J' \setminus F) - w(J' \cap F) + w(J' \cap F) + w(F \setminus J') \\ &= w(J' \setminus F) + w'(J' \cap F) + w(F) \\ &= w'(J') + w(F). \end{aligned}$$

Thus minimizing $w(J)$ in G is equivalent to minimizing $w'(J')$ in G' in force of the bijection $J \leftrightarrow J'$ on the sets of T- and T'-joins.

Problem 9. We have shown that the problem of finding a maximum-weight perfect matching can be solved in polynomial time. Show that the problem of finding a maximum-weight matching can be solved in polynomial time.

Solution. Duplicate the graph and join each vertex with its duplicate and give these edges zero weights.

For any matching in the initial graph we can complete it to a perfect matching in the new graph by adding the edges joining duplicated vertices. And vice versa, any perfect matching in the new graph can be reduced to a matching in the initial graph by removing the edges joining duplicated vertices.

Thus we can find a maximum-weight matching in the initial graph by finding a maximum-weight perfect matching in the new graph in polynomial time.

Problem 10. Show that if $\mathcal{L}$ is a laminar family of nonempty subsets of a set $S \neq \varnothing$, then $|\mathcal{L}| \leq 2|S| - 1$ (see Lemma 9.7).

Solution. Induct on $|S|$. Let $S_1, \ldots, S_n$ be the largest disjoint sets of $\mathcal{L}$ with their respective laminar subfamilies $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

$$|\mathcal{L}| = \sum_{i \in [n]} |\mathcal{L}_i| \leq \sum_{i \in [n]} (2|S_i| - 1) \leq 2|S| - n \leq 2|S| - 1.$$

Problem 11. Recall that the parity polytope is the convex hull of all $x \in \{0,1\}^n$ with $\mathbf{1}^T x$ even. In the lecture, we have seen an extended formulation for the parity polytope that has size $O(n^2)$. Give an extended formulation of size $O(n)$.

Solution. Define the digraph $D = (V, A)$ with nodes $V := \{0, 1, \ldots, n\} \times \{0,1\} \setminus \{(0,1),(n,1)\}$ and arcs $A := \{((i-1,\alpha),(i,\beta)) \in V \times V : i \in [n], \alpha, \beta \in \{0,1\}\}$. Call the nodes $s := (0,0)$ the source and $t := (n,0)$ the sink. Clearly, the set $P^{s-t}_{\text{flow}}(D)$ of $s-t$-flows $y \in \mathbb{R}^A_+$ of flow-value 1 in any such acyclic digraph is described by the linear constraints

$$\begin{aligned} y(\delta^{\text{out}}(v)) - y(\delta^{\text{in}}(v)) &= 0 && \forall v \in V \setminus \{s,t\} \\ y(\delta^{\text{out}}(s)) &= 1 && \\ y(\delta^{\text{in}}(t)) &= 1 && \\ y_a &\geq 0 && \text{for all } a \in A. \end{aligned}$$

Due to the total unimodularity of D's incidence matrix, $P^{s-t}_{\text{flow}}(D)$ is an integral polytope. The following map projects it onto the parity polytope:

$$\pi : \mathbb{R}^A \to \mathbb{R}^n \quad \text{with} \quad \pi(y_i) := \begin{cases} y_{((0,0),(1,1))} & \text{for } i = 1 \\ y_{((i-1,0),(i,1))} + y_{((i-1,1),(i,0))} & \text{for } 2 \leq i \leq n-1 \\ y_{((n-1,1),(n,0))} & \text{for } i = n. \end{cases}$$

It is not hard to see that the integer points in $P^{s-t}_{\text{flow}}(D)$ are precisely the binary vectors with an even number of 1's, since an $s-t$-path in D must use an even number of diagonal arcs (show on graph than in each layer one node is 0 and the other 1, going from s to t on an example). The claim follows because $P^{s-t}_{\text{flow}}(D)$ is integral and because π maps integral points to integral points.

Homework 4

Problem 12. Determine the extension complexity of the n-dimensional cross-polytope (and prove your claims).

Solution. We prove that the answer is $2n$. A size-$2n$ extension was described in the lecture. We show that no smaller one exists by finding a fooling set in a slack matrix.

We will use the description of cross-polytope with the 2^n inequalities $a^T x \le 1$ where a runs through all $\{0,1\}^n$. We will index a slack matrix with $a \in \{0,1\}^n$ for rows and with $\pm e_i$ for the columns, where e_i is the i-th basis vector. For example, a slack matrix of octahedron is, with its fooling set boxed,

$$\begin{pmatrix} 0 & 0 & 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & \boxed{2} & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & \boxed{2} & 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & \boxed{2} & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 & 0 & \boxed{2} \\ \boxed{2} & 2 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

The fooling set is defined as

$$((-1,-1,-1), e_1), \quad ((1,-1,-1), e_2), \quad ((1,1,-1), e_3),$$
$$((1,1,1), -e_1), \quad ((-1,1,1), -e_2), \quad ((-1,-1,1), -e_3).$$

Now we check that this is indeed a fooling set. In the first line, for any indices $i < j$ the dot product of e_i with the $(\dots,1,-1,\dots)$ vector of j is 1, so its slack is zero. Same for the second line. Same for two elements from different lines.

Problem 13. (a) Prove Lemma 14.3: Let S be a nonnegative matrix with m rows, such that the supports of any two rows are distinct. Show that $\mathrm{rc}(S) \ge \log_2(m)$.

(b) Prove Proposition 14.4: If P is a convex n-gon, then $\mathrm{xc}(P) \ge \log_2(n)$.

Solution. (a) Let S be of size $m \times n$. WLOG S is a 0-1 matrix.

Denote by $\mathrm{gen}(S)$ the minimal set of vectors in $\{0,1\}^n$ such that any row of S can be represented as the union of vectors from $\mathrm{gen}(S)$.

Note that $\mathrm{rc}(S) = \mathrm{gen}(S)$. Indeed, for any $e \in \mathrm{gen}(S)$ define the rectangle R_e as the union of all subrows e in S. Then these rectangles cover S so $\mathrm{rc}(S) \le \mathrm{gen}(S)$. On the other hand, for a given rectangle covering, for any $R = I \times J, I \subseteq [m], J \subseteq [n]$ of its rectangles define $e_R \in \{0,1\}^n$ as the characteristic vector of J. Then e_R's generate the rows of S so $\mathrm{gen}(S) = \mathrm{rc}(S)$.

Now it remains to check that $2^{|\mathrm{gen}(S)|} \ge m$ which is obvious.

(b) Note that any regular n-gon has a slack matrix of the form

$$\begin{pmatrix} + & + & + & 0 & 0 \\ 0 & + & + & + & 0 \\ 0 & 0 & + & + & + \\ + & 0 & 0 & + & + \\ + & + & 0 & 0 & + \end{pmatrix}.$$

The supports of any two rows of it are distinct so point (a) applies.

Problem 14. Show that the rectangle covering number of the perfect matching polytope of K_n is $O(n^4)$.

Solution. We consider K_{2n}. The slack matrix S of its perfect matching polytope

$$\{x \in \mathbb{R}^E_{\geq 0} : \ x(\delta(v)) = 1 \ \ \forall v \in V, \ \ x(\delta(S)) \geq 1 \ \ \forall S \subseteq V \text{ with } |S| \text{ odd}\}$$

has rows indexed by sets $U \subseteq V$ with $|U|$ odd and columns indexed by perfect matchings. We disregard the linearly many equality constraints. The slack matrix's value at U, M is

$$S_{U,M} = |\delta(U) \cap M| - 1.$$

Define its rectangle covering as follows. For fixed disjoint edges e_1, e_2 put

$$\begin{aligned}
\mathcal{M}_{e_1,e_2} &= \{M \text{ matching} : e_1, e_2 \in M\}, \\
\mathcal{U}_{e_1,e_2} &= \{U \subseteq V : |U| \text{ odd}, e_1, e_2 \in \delta(U)\}, \\
\mathcal{R}_{e_1,e_2} &= \mathcal{U}_{e_1,e_2} \times \mathcal{M}_{e_1,e_2}.
\end{aligned}$$

For any $(U, M) \in \mathcal{R}_{e_1,e_2}$ we have $|\delta(U) \cap M| \geq 2$ which implies that it is a valid rectangle. On the other hand,

$$\bigcup_{\text{disjoint } e_1,e_2} \mathcal{R}_{e_1,e_2} = \operatorname{supp}(S)$$

since any entry in $\operatorname{supp}(S)$ will contain two edges which are shared. Thus we can cover $\operatorname{supp}(S)$ by $O(n^4)$ rectangles.

Problem 15. Prove Proposition 16.2: For every $n \in \mathbb{Z}_{\geq 1}$, $P_{\text{corr}}(n)$ is linearly isomorphic to $P_{\text{cut}}(K_{n+1})$.

Solution. We define the map

$$f : \mathbb{R}^{n^2} \to \mathbb{R}^{\binom{n+1}{2}} : \qquad f(P_{\text{corr}}(n)) = P_{\text{cut}}(K_{n+1})$$

as follows. For every vertex $X := xx^T$ of $P_{\text{corr}}(n)$ extract the vector x from its diagonal. Then define the set $S_x := \{i \in [n] : x_i = 1\} \subseteq [n+1]$, and thus get the cut vector $\delta(S_x)$ in K_{n+1}. Moreover, the components of $\delta(S_x)$ can be derived as linear combinations of the entries of X:

$$\begin{aligned}
\delta_{i,j} &= \mathbb{I}\{(i \in S_x \wedge j \notin S_x) \vee (i \notin S_x \wedge j \in S_x)\} \\
&= x_i(1 - x_j) + x_j(1 - x_i) \\
&= X_{ii} + X_{jj} - 2X_{ij}, \\
\delta_{i,n+1} &= X_{ii}.
\end{aligned}$$

This defines a linear map with $f(xx^T) = \delta(S_x)$ and thus $f(P_{\text{corr}}(n)) = P_{\text{cut}}(K_{n+1})$. The inverse map is

$$\begin{aligned}
X_{ii} &= \delta_{i,n+1}, \\
X_{ij} = X_{ji} &= \frac{1}{2}(\delta_{i,n+1} + \delta_{j,n+1} - \delta_{ij})
\end{aligned}$$

so f is an isomorphism.