Example 6.14 (Δ of variable with jacobian)

$X \sim Exp(\beta=\theta)$
$Y \sim Exp(\beta=\theta)$

$W = \frac{X}{X+Y} \Rightarrow Y = \frac{X}{W} - X \Rightarrow X = \frac{Y}{(\frac{1}{W})-1}$

$V = X+Y \Rightarrow Y = V - X \Rightarrow X = V - Y$

$\frac{X}{W} - X = V - X \rightarrow \boxed{X = VW}$

$V - Y = \frac{Y}{(\frac{1}{W}-1)} \rightarrow \boxed{Y = V - WV}$

Step 1)

$$f_{XY}(x,y) = \frac{e^{-x/\theta}\, e^{-y/\theta}}{\theta^2} \qquad x>0,\ y>0$$

Step 2)

$$f_{V,W}(v,w) = f_{XY}(g_x^{-1}, g_y^{-1}) \left| \det \begin{bmatrix} \frac{\partial}{\partial w} g_x^{-1} & \frac{\partial}{\partial v} g_x^{-1} \\ \frac{\partial}{\partial w} g_y^{-1} & \frac{\partial}{\partial v} g_y^{-1} \end{bmatrix} \right|$$

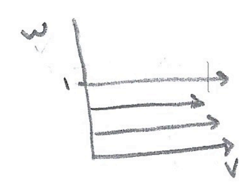
$$= \frac{e^{-(wv)/\theta}\, e^{-(v-wv)/\theta}}{\theta^2} \left| \det \begin{bmatrix} \frac{\partial}{\partial w}(wv) & \frac{\partial}{\partial v}(wv) \\ \frac{\partial}{\partial w}(v-wv) & \frac{\partial}{\partial v}(v-wv) \end{bmatrix} \right|$$

$$= \frac{\exp\left[\frac{-wv-v+wv}{\theta}\right]}{\theta^2} \left| \det \begin{bmatrix} v & w \\ (0-v) & (1-w) \end{bmatrix} \right| = \frac{e^{-v/\theta}}{\theta^2} \left| (v-wv) - (0-wv) \right|$$

$$= \frac{e^{-v/\theta}}{\theta^2} \left| v - wv + wv \right| = \frac{e^{-v/\theta}}{\theta^2} v, \quad \begin{matrix} 0<w<1 \\ v>0 \end{matrix} \overset{\text{ind}}{=} \frac{e^{-v/\theta}\, v^{2-1}}{\Gamma(2)\, \theta^2} \cdot 1$$

Support? $0<x<\infty \Rightarrow 0<wv<\infty \Rightarrow 0<w<\infty \cap 0<v<\infty$
$0<y<\infty \Rightarrow 0<v-wv<\infty \Rightarrow wv<v \Rightarrow w<1$
$\Big\}\ \begin{matrix} 0<w<1 \\ v>0 \end{matrix}$

$V \sim Gamma(\alpha=2, \beta=\theta)$ $\qquad W \sim Uniform(0,1)$



$$f_W(w) = \int_0^\infty \frac{e^{-v/\theta}}{\theta^2} v\, dv = 1 \quad \text{for } 0<w<1$$
v

$$f_V(v) = \int_0^1 v \frac{e^{-v/\theta}}{\theta^2} dw = \frac{v e^{-v/\theta}}{\theta^2} \quad \text{for } v>0$$
w