Example 6.15 (Δ of variable with Jacobian)

$$f_{XY}(x,y) = 2(1-x); \quad 0<x<1, \quad 0<y<1$$

Step 1)

$W = XY \Rightarrow Y = \frac{W}{X} \Rightarrow X = \frac{W}{Y}$

$\boxed{V = X} \Rightarrow \; ? \Rightarrow X = V$

$\downarrow \qquad \downarrow$

$\boxed{V = X} \qquad V = \frac{W}{Y} \Rightarrow \boxed{Y = \frac{W}{V}}$

Step 2)

$$f_{V,W}(v,w) = f_{XY}(g^{-1}_x, g^{-1}_y)\left|\det\begin{bmatrix} \frac{\partial}{\partial w} g^{-1}_x & \frac{\partial}{\partial v} g^{-1}_x \\ \frac{\partial}{\partial w} g^{-1}_y & \frac{\partial}{\partial v} g^{-1}_y \end{bmatrix}\right|$$

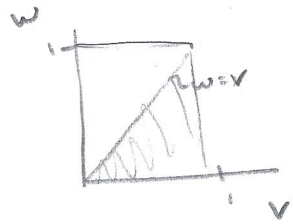
$$= 2(1-v)\left|\det\begin{bmatrix} \frac{\partial}{\partial w}(v) & \frac{\partial}{\partial v}(v) \\ \frac{\partial}{\partial w}(\frac{w}{v}) & \frac{\partial}{\partial v}(\frac{w}{v}) \end{bmatrix}\right| = 2(1-v)\left|\det\begin{bmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{w}{v^2} \end{bmatrix}\right|$$

$$= 2(1-v)\left|0 - \frac{1}{v}\right| = \frac{2(1-v)}{v}$$

So $f_{V,W}(v,w) = 2(1-v)v^{-1}; \quad 0<w<v<1$

Step 3) support of v, w? $\Rightarrow 0<x<1 \Rightarrow 0<v<1$

$\Rightarrow 0<y<1 \Rightarrow 0<\frac{w}{v}<1 \Rightarrow 0<w<v$



$$f_V(v) = \int_w f_{V,W}(v,w)\,dw$$

$$= \int_0^v \frac{2(1-v)}{v}\,dw = \frac{2(1-v)}{v}\Big[v\Big] = 2v^0(1-v)^1 \text{ for } 0<v<1$$

$V \sim Beta(\alpha=1, \beta=2)$

$$f_W(w) = \int_v f_{V,W}(v,w)\,dv = \int_w^1 \frac{2(1-v)}{v}\,dv = 2\left[\int_w^1 \frac{1}{v} - 1\,dv\right] = 2\Big[\ln(v) - v\Big]_{v=w}^{v=1}$$

$$= 2\Big[(\ln(1)-1) - (\ln(w)-w)\Big] = 2\Big[(0-1)-(\ln(w)-w)\Big] = 2\Big[-\ln(w)+w-1\Big]$$

$$= 2w - 2\ln(w) - 2$$

where $0<w<1$