Example 2: Let $X_1, X_2, \ldots, X_n$ represent a random sample taken from a population of values that is modeled by the following probability distribution function:

$$f(x_i) = \frac{1}{\theta + 2}, \qquad 2 < x_i < \theta + 4$$

a) $E[X_i]$?

$$E[X_i] = E[X] = \int_2^{\theta+4} \frac{x}{\theta+2}\, dx = \frac{1}{\theta+2}\left[\frac{x^2}{2}\right]_{x=2}^{x=\theta+4} = \frac{1}{2(\theta+2)}\left[(\theta+4)^2 - 2^2\right]$$

$$= \frac{1}{2(\theta+2)} (\theta+4)^2 - 4 = \ldots = \frac{\theta}{2} + 3$$

b) What is the bias of $\bar{X}$, as and estimator of θ?

$$E(\hat{\theta}) = E[\bar{X}] = \frac{E[\Sigma X_i]}{n} = \frac{\Sigma[E(X_i)]}{n} = \frac{n}{n} E[X_i] = E[X]$$

$$= \frac{\theta}{2} + 3 = E[\hat{\theta}] \overset{\text{or}}{=} \frac{(\theta + 6)}{2}$$

$$\text{So } B[\hat{\theta}] = E[\hat{\theta}] - \theta = \left(\frac{\theta}{2} + 3\right) - \theta = 3 - \frac{\theta}{2}$$

c) Can you suggest an unbiased estimator that involves $\bar{X}$?

$$E[\bar{X}] = E[\hat{\theta}] \overset{\text{Set}}{=} \bar{x} \Rightarrow \bar{x} = \frac{\theta}{2} + 3 \text{ then let } \hat{\theta}_{un} = \theta \Rightarrow \bar{x} = \frac{\hat{\theta}_{un}}{2} + 3 \Rightarrow 2(\bar{x} - 3) = \hat{\theta}_{un}$$

this will give an unbiased suggestion for an unbiased estimator involving $\bar{X}$ . . .

$$\text{Now } E[\hat{\theta}_{un}] = E[2\bar{x} - 6] = 2E[\bar{x}] - 6 = 2\left[\frac{\theta}{2} + 3\right] - 6$$

$$= \theta + 6 - 6 = \theta$$

So $2\bar{x} - 6$ is an unbiased estimator