Example 4: Suppose we obtain a sample $X_1, X_2, \ldots, X_n$, from an Uniform$(0, \theta)$ distributed random variable. Use $\frac{X_{(n)}}{\theta}$ to form a 90% confidence interval for θ.

a) Is $\frac{X_{(n)}}{\theta}$ a pivotal quantity?

1. function of sample? Yes! function of Θ? Yes! Only unknown is Θ? Yes!

2. Dist independent of Θ?

$$f_X(x) = \frac{1}{\Theta - 0} = \frac{1}{\Theta} \quad \text{for } 0 < x < \Theta$$

$$F_X(x) = \int_0^x \frac{1}{\Theta}\, dx = \frac{x}{\Theta}\Big]_{x=0}^{x=x} = \frac{x}{\Theta} \quad \text{for } 0 < x < 1$$

$$f_{X_{(n)}}(x) = n\left[F_X(x)\right]^{n-1} f_X(x) = n\left[\frac{x}{\Theta}\right]^{n-1}\left[\frac{1}{\Theta}\right] = n\frac{x^{n-1}}{\Theta^n}$$

let $= Y = \frac{X_{(n)}}{\Theta} \Rightarrow g^{-1}_{X_{(n)}}(y) = y\Theta$; $\frac{d}{dy} g^{-1}_{X_{(n)}}(y) = \frac{d}{dy}(y\Theta) = \Theta$

change of var Method.

$$f_{\frac{X_{(n)}}{\Theta}}(y) = f_{X_{(n)}}\left(g^{-1}_{X_{(n)}}(y)\right)\left|\frac{d}{dy} g^{-1}_{X_{(n)}}(y)\right| = \frac{n(y\Theta)^{n-1}}{\Theta^n}(\Theta)$$

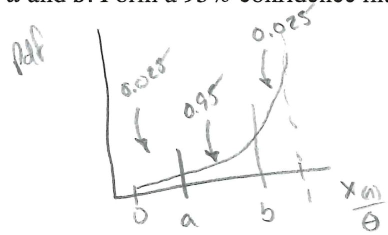
$$= \frac{n\, y^{n-1}\, \Theta^{n-1}\, \Theta}{\Theta^n} = \frac{n\, y^{n-1}\, \Theta^n}{\Theta^n} = n\, y^{n-1} \quad \text{for } 0 < y < 1$$

So $Y = \frac{X_{(n)}}{\Theta}$ $\quad f_Y(y) = n y^{n-1}, \; 0 < y < 1$

free of Θ!

So Yes Y is a pivotal quantity!

b) $P\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = 0.95$, find a and b. Form a 95% confidence interval for θ



$$P\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = 0.95$$

$$\Rightarrow 0.025 = \int_0^a n y^{n-1}\, dy = n\left[\frac{y^n}{n}\right]_{y=0}^{y=a} = \left[a^n - 0^n\right] = a^n = 0.025$$

$$\Rightarrow a = 0.025^{1/n}$$

$$\Rightarrow 0.975 = \int_0^b n y^{n-1}\, dy = b^n = 0.975 \Rightarrow b = 0.975^{1/n}$$

$$\Rightarrow P\left(0.025^{1/n} \le \frac{X_{(n)}}{\theta} \le 0.975^{1/n}\right) = 0.95$$

Say $n = 32$ $\quad X_{(n)} = 4.87$

$$P\left(\frac{X_{(n)}}{0.025^{1/n}} \geq \theta \geq \frac{X_{(n)}}{0.975^{1/n}}\right) = 0.95 = P\left(\frac{X_{(n)}}{0.975^{1/n}} \le \theta \le \frac{X_{(n)}}{0.025^{1/n}}\right)$$

$$P(4.873855 \le \theta \le 5.465040) = 0.95$$

95% confident θ is between 4.873855 & 5.465040