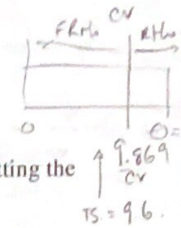
Example 3: A random sample $X_1, X_2, \cdots, X_8$ is taken from a population of values that is modeled by the uniform distribution from 0 to θ. From this sample, one wishes to test

$$H_0: \theta = 10 \qquad H_A: \theta > 10$$

upper tailed



Use $X_{(n)}$ as a test statistic,

(a) derive a decision rule, based on the previously derived probability density function of $X_{(n)}$, setting the probability of committing a Type I error to be 0.10.

$$f_{X_{(n)}} = n\left[F_X(x)\right]^{n-1} f_X(x) = 8\left[\frac{x}{\theta}\right]^7 \frac{1}{\theta} = \frac{8}{\theta^8}x^7 \quad \text{for } 0 \le x_{(n)} \le \theta$$

$$P[\text{Type I}] = \alpha = P[X_{(n)} > CV \mid \theta = 10] = P[\text{RHo} \mid \text{Ho true}]$$

$$\Rightarrow 0.1 = 1 - \int_0^{CV} \frac{8}{10^8}x^7\,dx \Rightarrow 0.9 = \frac{8}{10^8}\left[\frac{x^8}{8}\right]_{x=0}^{CV} \Rightarrow 0.9 = \frac{CV^8}{10^8}$$

$$\Rightarrow \left[10^8(0.9)\right]^{1/8} = CV = 90000000^{1/8} \cong 9.869162814$$

if $X_{(n)} = X_{(8)} > 9.869162814$ Then RHo ↓ we Think $\theta > 10$. otherwise FRHo

(b) The sample of n = 8 was taken, the largest value in the sample was observed to be 9.6. Does this support the null hypothesis? Provide the P-value and interpret its meaning.

This Appears to suppor the The null hypothesis!

Based on This Sample we are 90% Confident (@ 10% level of significance) it appears The Largest Number is Not Sig. different from 10.

$$\text{p-value} = P[X_{(n)} > \underset{TS}{9.6} \mid \theta = 10] = 1 - \int_0^{9.6} \frac{8x^7}{10^8}\,dx = 1 - \frac{1100753141176}{1525878906250} \cong 0.2786104210$$

FRHo

if θ is 10, The prob. of obtaining another sample (size = 8) That has a larger (alt sign. >) sample max than 9.6 (TS) is about 0.2786

This is quite high for most acceptable α's higher than $\alpha = 0.1$ (part a)

This implies The prob. of another sample being more "unexpected" under The Assumption of Ho is "high" meaning we are sig. close to what we expected to see under assumption of Ho $\Rightarrow$ FRHo.

(c) What is the probability of concluding, $\theta = 10$ when in fact $\boxed{\theta = 11}$?

$H_0: \theta = 10$
$H_a: \theta = 11$

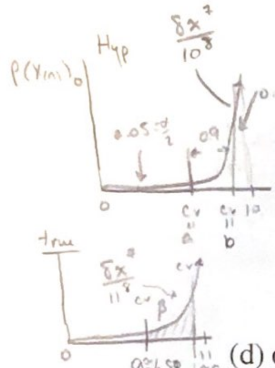
or

$H_0: \theta = 10$
$H_a: \theta > 10$ when $\theta = 11$

$P[\text{FRHo} \mid \text{Ho false}] = \beta = P[\underset{cv}{X_{(n)} \leq 9.86} \mid \underset{\text{Ho is false}}{\theta = 11}] = \int_0^{9.86} \frac{8x^7}{(11)^8}\,dx$

$= \frac{8(9.86)^8}{8 \cdot 11^8} = \frac{(9.86)^8}{11^8} \cong 0.4167 = P[\text{Type II}]$

$P[\text{Not making type II}] = 1 - \frac{(9.86)^8}{11^8} = 0.583 = \text{Power of Test}.$



a. What if the $H_A: \theta \neq 10$ → two tailed test

CV's $\Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.5$

$0.05 = \int_0^a \frac{8x^7}{10^8}\,dx \Rightarrow \cdots \Rightarrow a = [0.05(10^8)]^{1/8} \cong 6.876560219$

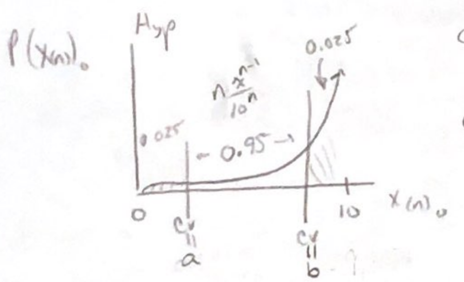
$0.95 = \int_0^b \frac{8x^7}{10^8}\,dx \Rightarrow b = [0.95(10^8)]^{1/8} \cong 9.93608849$

$\beta = P[\text{Type II}] = P[\text{FRHo} \mid \text{Ho false}] \cong \int_{6.876560219}^{9.93608849} \frac{8x^7}{11^8}\,dx \cong 0.4198566420$

(d) Consider $H0: \theta = 10$ versus $HA: \theta \neq 10$. If one wanted to see if $P(TypeI) = P(TypeII) = 0.05$, how large a sample must be taken? $(\theta = 11)$?

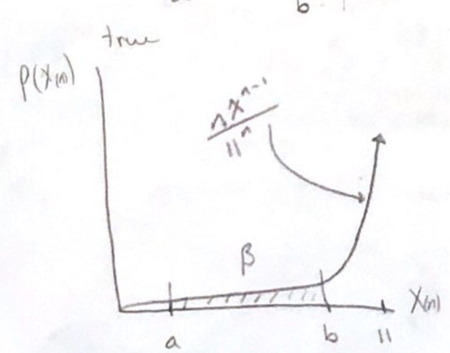
$\alpha = 0.05 \quad \frac{\alpha}{2} = 0.025 \quad \beta = 0.05 \qquad H_a: \theta \neq 10$ two tailed..

CV's: ?



$0.025 = \int_0^a \frac{nx^{n-1}}{10^n}\,dx \Rightarrow [(0.025)10^n]^{1/n} = a = 0.025^{1/n}(10) = 10\sqrt[n]{0.025}$

$0.975 = \int_0^b \frac{nx^{n-1}}{10^n}\,dx = [(0.975)10^n]^{1/n} = b = 0.975^{\frac{1}{n}}(10) = 10\sqrt[n]{0.975}$



$\overset{set}{=} 0.05 = \int_{10\sqrt[n]{0.025}}^{10\sqrt[n]{0.975}} \frac{nx^{n-1}}{11^n}\,dx \Rightarrow 0.05 = \left[\frac{x^n}{11^n}\right]_{x=0.025^{1/n}(10)}^{0.975^{1/n}(10)}$

$\Rightarrow 0.05 = \left[\frac{[0.975^{1/n}(10)]^n}{11^n} - \frac{[0.025^{1/n}(10)]^n}{11^n}\right]$

$= \frac{0.975 \cdot 10^n - 0.025 \cdot 10^n}{11^n} = \left(\frac{10}{11}\right)^n [0.975 - 0.025] = 0.95\left(\frac{10}{11}\right)^n$

$\Rightarrow 0.95\left(\frac{10}{11}\right)^n \overset{set}{=} 0.05 \Rightarrow \frac{0.05}{0.95} = \left(\frac{10}{11}\right)^n \Rightarrow \log\left(\frac{0.05}{0.95}\right) = n \log\left(\frac{10}{11}\right)$

$\Rightarrow n = \frac{\log\left(\frac{0.05}{0.95}\right)}{\log(10/11)} = 30.893 \Rightarrow n \geq 31$ ensure $\alpha \leq 0.05$ & $\beta \leq 0.05$