(A Trivial) Example 1: Let $X_1, X_2, \ldots, X_n$ be Bernoulli random variables with parameter p. What is the method of moments estimator of p?

$X_i \sim Bern(p) \Rightarrow \sum_{i=1}^{n} X_i = X \sim Bin(n, p)$

$E[X_i] = p \qquad \Rightarrow \sum_{i=1}^{n} X_i^2 = X \sim Bin(n,p)$

Set

1st sample moment $\rightarrow$ $p = M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i^1 = \frac{1}{n}\sum X_i = \bar{X} = \frac{X}{n} = \hat{p}_{mm}$

2nd sample moment $\rightarrow$ $M_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} = \frac{X}{n} \Rightarrow \hat{\sigma}^2_{mm} = \frac{X}{n} - \left(\frac{X}{n}\right)^2 = \frac{X}{n}\left(1 - \frac{X}{n}\right) = \hat{p}(1-\hat{p})$

each value is 0 or 1 — 2nd sample moment — 1st sample moment.

Example 2: $X_1, X_2, \ldots, X_n$ be normal random variables with mean μ and variance σ^2. What are the method of moments estimators of the mean μ and variance σ^2?

1st Moment of Normal $\qquad X_i \sim Norm(\mu, \sigma^2)$

$E[X_i] = \mu$

2nd Moment of Normal

$E[X_i^2] = Var[X_i] + E[X_i]^2 = \sigma^2 + \mu^2$

1st sample moment

$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

2nd sample moment

$M_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Set 1st Moment equal to 1st Sample Moment Solve for μ!

$E[X_i] = M_1' \Leftrightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ So $\hat{\mu}_{mm} = \bar{X}$

⋮

$E[X_i^2] = M_2' \Leftrightarrow \sigma^2 + \mu^2 = \frac{1}{n}\sum X_i^2 \Leftrightarrow \sigma^2 = \frac{\sum_{i=1}^{n} X_i^2}{n} - \mu^2$

$\Leftrightarrow \hat{\sigma}^2_{mm} = \frac{\sum X_i^2}{n} - \hat{\mu}_{mm}^2 = \frac{\sum X_i^2}{n} - \bar{X}^2$

$= \frac{1}{n}\left[\sum X_i^2 - n\bar{X}^2\right] \overset{*}{=} \frac{1}{n}\sum (X_i - \bar{X})^2$ * See Intro. Stat. sample prop. & Var Booklet.

$\hat{\sigma}^2_{mm} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{\sum_{i=1}^{n} X_i^2}{n} - \bar{X}^2$