Example 4: Let X_1, X_2, X_3, X_4 represent a random sample taken from a population of values that is modeled by the following cumulative and probability distribution functions:

$$F_X(x) = x, \qquad f_X(x) = 1, \qquad 0 < x < 1$$

Find $P\left(\frac{1}{3} < X_{(3)} < \frac{2}{3}\right)$

$$f_{X_{(3)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}\left[F_X(x)\right]^{k-1} f_X(x)\left[1-F_X(x)\right]^{n-k}$$

here $n=4$
$k=3$.

$$= \frac{4!}{2!\,1!}x^2(1)(1-x) = \frac{4\times 3\times \cancel{2!}}{\cancel{2!}}x^2(1-x) = 12\,x^2 - x^3$$

for $0 < x < 1$

$$P\left(\tfrac{1}{3} < X_{(3)} < \tfrac{2}{3}\right) = \int_{1/3}^{2/3} 12x^2 - x^3\,dx = 12\left[\left(\frac{x^3}{3}\right)_{x=1/3}^{2/3} - \left(\frac{x^4}{4}\right)_{x=1/3}^{2/3}\right]$$

$$= 12\left[\left(\frac{(2/3)^3}{3} - \frac{(1/3)^3}{3}\right) - \left(\frac{(2/3)^4}{4} - \frac{(1/3)^4}{4}\right)\right] = 12\left(\frac{7}{81} - \frac{5}{108}\right)$$

$$= 12\left[\frac{13}{324}\right] = \frac{13}{27} = 0.\overline{481}$$