Example 2: Recall that $X_{(n)}$ is a biased estimator for N, when N represents the upper bound of distributed values that is uniform between 0 and N. Is $X_{(n)}$ a consistent estimator for N?

$$f_X(x) = \frac{1}{N}, \quad F_X(x) = \int_0^x \frac{1}{N}\,dx = \frac{x}{N}, \quad 0 < x < N, \quad f_{X_{(n)}}(x) = n\left(\frac{x}{N}\right)^{n-1}\frac{1}{N} = \frac{n}{N^n}x^{n-1}$$

$$E[X_{(n)}] = \int_0^N \frac{x\,x^{n-1}n}{N^n}\,dx = \frac{n}{N^n}\int_0^N x^n\,dx = \frac{Nn}{n+1} \quad \leftarrow \text{baised Estimator for, } N.$$

let $\Theta = N$, $\hat{\Theta} = X_{(n)}$

$$P(|\hat{\Theta}_n - \Theta| < \varepsilon) = P(|X_{(n)} - N| < \varepsilon) = P(-\varepsilon < X_{(n)} - N < \varepsilon)$$

$$= P(N-\varepsilon < X_{(n)} < N+\varepsilon) = \int_{N-\varepsilon}^{N+\varepsilon} \frac{n}{N}x^{n-1}\,dx$$

But max value of support of $X_{(n)}$ is N So Bound of $N+\varepsilon$ impossible

$$= \int_{N-\varepsilon}^{N} \frac{n}{N^n}x^{n-1}\,dx = \frac{n}{N^n}\left[\frac{x^n}{n}\right]_{x=N-\varepsilon}^{N} = \frac{N^n}{N^n} - \frac{(N-\varepsilon)^n}{N^n}$$

$$= 1 - \left(\frac{N-\varepsilon}{N}\right)^n$$

then $\lim_{n\to\infty} P(|X_{(n)} - N| < \varepsilon) = \lim_{n\to\infty}\left[1 - \left(\frac{N-\varepsilon}{N}\right)^n\right]$, clearly $0 \le \frac{N-\varepsilon}{N} < 1$

we know anything between 0 & 1 to power "∞" is 0

$$\lim_{n\to\infty} P(|X_{(n)} - N| < \varepsilon) = 1 - 0 = 1$$

So, $X_{(n)}$, largest number from sample converges in Prob. to, N, Largest possible value. ie As $n\to\infty$, $X_{(n)} = N$

Note: bias estimates can be consistent

Consistency $\not\Rightarrow$ (does Not imply) "Unbiasness"