Example 6.2 (CDF / Method of Distributions)

$$f_{XY}(x,y) = \begin{cases} 3x; & 0 \le y \le x \le 1 \\ 0; & \text{otherwise} \end{cases}$$

$Z = g_Z(x,y) = X - Y$

$\Rightarrow g_Y^{-1}(x,z) = Y = X - Z$

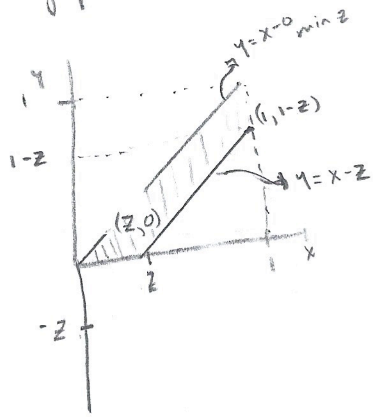
$g_X^{-1}(y,z) = X = Z + Y$

Step 1) Support of Z? what values could it be?

• we know $y \le x$ from joint density of x, y (support)

$$\underset{y}{\overset{1}{X}} - \underset{0}{\overset{x}{Y}} = \underset{0}{\overset{1}{Z}}$$

Step 2) graph transformation function (for CDF method w.r.t. x, y planes treat z as constant)



$y > x - z$ & $0 < y < x < 1$

where $0 < z < 1$

Step 3)

$$F_Z(z) = P(Z \le z) = P(g_Z(x,y) \le z) = P(X - Y \le z) = P(-Y \le z - X) = P(Y \ge X - Z)$$

Step 4)

$$F_Z(z) = \underset{x}{\int_0^z} \underset{y\ \text{vert}}{\int_0^x} 3x \, dy\, dx + \underset{x}{\int_z^1} \underset{y\ \text{vert}}{\int_{x-z}^x} 3x \, dy\, dx = \underset{y}{\int_0^{1-z}} \underset{x\ \text{hor}}{\int_y^{y+z}} 3x \, dx\, dy + \underset{y}{\int_{1-z}^1} \underset{x\ \text{hor}}{\int_y^1} 3x \, dx\, dy = \frac{3z}{2} - \frac{z^3}{2}$$

So $F_Z(z) = \begin{cases} 0, & z \le 0 \\ \frac{3}{2}z - \frac{z^3}{2}, & 0 < z \le 1 \\ 1, & z > 1 \end{cases}$

Step 5) $f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\left(\frac{3}{2}z - \frac{z^3}{2}\right) = \frac{3}{2} - \frac{3z^2}{2}$

$$f_Z(z) = \begin{cases} \frac{3}{2} - \frac{3}{2}z^2; & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \frac{3}{2}(1 - z^2) \quad , \quad 0 < z < 1$$

$$\mu_Z = E[Z] = \int_0^1 z \frac{3}{2}(1-z^2)\, dz = \frac{3}{8}$$

z

$$E[Z^2] = \int_0^1 z^2 \frac{3}{2}(1-z^2)\, dz = \frac{1}{5}$$

z

$$\sigma_Z^2 = VAR[Z] = E[Z^2] - E[Z]^2 = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$$