Finding the Most Powerful Test: the Neyman-Pearson Lemma

The goodness of a test is measured by α and β, the probabilities of type I and type II errors, respectively. Typically, the value of α is chosen in advance and determines the location of the rejection region. A related but more useful concept for evaluating the performance of a test is called the power of the test. Basically, the power of a test is the probability that the test will lead to rejection of the null hypothesis.

If a random sample is taken from a distribution with parameter θ, a hypothesis is said to be a simple hypothesis if that hypothesis uniquely specifies the distribution of the population from which the sample is taken. Any hypothesis that is not a simple hypothesis is called a composite hypothesis.

The hypothesis $H_a: \lambda = 2$ is therefore an example of a simple hypothesis. In contrast, the hypothesis $H_a^* : \lambda > 2$ is a composite hypothesis because under H_a^* the density function is not *uniquely* determined. The form of the density is known, but the parameter λ could be 3 or 15 or any value greater than 2.

> **The Neyman-Pearson Lemma:** Suppose that we wish to test the simple null hypothesis $H_o: \theta = \theta_o$ versus the simple alternative hypothesis $H_a: \theta = \theta_a$, based on a random sample from a distribution with parameter θ. Let $L(\theta)$ denote the likelihood of the sample when the value of the parameter is θ. Then, for a given α, the test that maximizes the power at θ_a has a rejection region, RR, determined by
>
> $$\frac{L(\theta_o)}{L(\theta_a)} < k$$
>
> The value of k (critical region) is chosen so that the test has the desired value for α. Such a test is a most powerful α-level test for H_o versus H_a.

Example1: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with parameter β. Is the hypothesis H: $\beta = 3$ a simple or a composite hypothesis?

$H_0: \beta = 3$ vs. $H_0: \beta \geq 3$

$\Rightarrow f_X(x) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 < x < \infty$

CAN specify Dist → Simple

infinitly many possibilities . . . → Composite

Example 2: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with mean μ and unknown variance σ^2. Is the hypothesis H: $\mu = 12$ a simple or a composite hypothesis?

Cannot specify Dist of Hypothesis (σ?) ?

$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$

$\Rightarrow$ Composite.

Example 3: Suppose X is a single observation from a population with probabilitiy density function given by:

$$f_X(x) = \theta x^{\theta-1}, \qquad 0 < x < 1$$

Find the test with the best critical region, that is, find the most powerful test, with significance level $\alpha = 0.05$, for testing the simple null hypothesis $H_o: \theta = 3$ against the simple alternative hypothesis $H_a: \theta = 2$.

$H_0: \theta = 3$

$H_a: \theta = 2.$

$$L(\theta) = \prod_{i=1}^{1} \theta x_i^{\theta-1} = \theta x_1^{\theta-1}$$

$L(\theta_0) = 3x_1^2 \Rightarrow$ 

$L(\theta_a) = 2x_1^1 \Rightarrow$ 

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{3x_1^2}{2x_1^1} = \tfrac{3}{2}(x_1) < k.$$

Reject when:

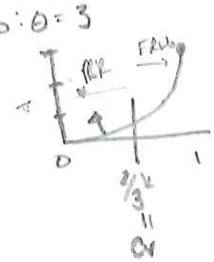
$\Rightarrow \boxed{x_1} < \boxed{\frac{k\,2}{3}}$ Let $CV = \frac{2k}{3}$.

TEST STAT critical value.

critical value for Most powerful test from one-single obs

$\alpha = P[RH_0 \mid H_0 \text{ true}] \Rightarrow \alpha = P\left[x_1 < \frac{2k}{3} \,\middle|\, \theta = 3\right].$

$H_0: \theta = 3$



Say $\alpha = 0.05$.

$$0.05 = \int_0^{CV} 3x^2\,dx = \frac{3}{3}x^3 \Big|_{x=0}^{x=CV}$$

$\Rightarrow 0.05 = CV^3 \Rightarrow CV = 0.05^{1/3} \doteq$

$CV \doteq 0.3684$

Dist X? $f_X(x) = 3x^2$, $0<x<1$ $\quad X \sim Beta(\alpha=3, \beta=1)$.

Say a Sample $x_1 = 0.7$.

$H_0: \theta = 3$

$H_a: \theta = 2$.

most powerful test ... RH_0 when $x_1 < 0.3684$ $(\alpha = 0.05)$.

Here $x_1 = 0.7 \Rightarrow \cancel{R}H_0$ (under the most powerful test)

16 obs. what could you do? Sum? median? mean? smallest...?

Example 4: Suppose $X_1, X_2, \ldots, X_{16}$ is a random sample from a normal distribution with mean μ and known variance $\sigma^2 = 16$. Find the most powerful test, with significance level $\alpha = 0.05$, for testing the simple null hypothesis $H_o: \theta = 10$ against the simple alternative hypothesis $H_a: \theta = 15$.

$H_0: \mu = \theta = 10$ $\quad\quad$ $H_0: \mu = \theta = 10$

$H_a: \mu = \theta = 15$ $\quad\quad$ $H_a: \mu > 10$

$$L(\theta) = \prod_{i=1}^{16}\left[\frac{1}{\sqrt{2\pi}(4)}\right]\exp\left[\frac{-(x_i-\theta)^2}{2(16)}\right] = \left(\frac{1}{\sqrt{2\pi}\,4}\right)^{16}\exp\left[\frac{-\sum_{i=1}^{16}(x_i-\theta)^2}{2(16)}\right]$$

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{\left(\frac{1}{\sqrt{2\pi}(4)}\right)^{16}}{\left(\frac{1}{\sqrt{2\pi}(4)}\right)^{16}}\;\frac{\exp\left[\frac{-\sum_{i=1}^{16}(x_i-10)^2}{32}\right]}{\exp\left[\frac{-\sum_{i=1}^{16}(x_i-15)^2}{32}\right]} = \exp\left[\frac{\sum_{i=1}^{16}(x_i-15)^2}{32} - \frac{\sum_{i=1}^{16}(x_i-10)^2}{32}\right]$$

$$= \exp\left[\frac{\left(\cancel{\sum x_i^2} - 30\sum x_i + 16(15)^2\right) - \left(\sum x_i^2 - 20\sum x_i + 16(10)^2\right)}{32}\right]$$

$$= \exp\left[\frac{-10\sum_{1}^{16} x_i + 2000}{32}\right] = \exp\left[\frac{-10}{32}\sum x_i + \frac{2000}{32}\right] < k.$$

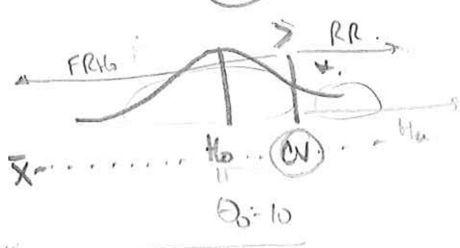
$$\Rightarrow \frac{-10}{32}\sum x_i + \frac{2000}{32} < \ln(k) \quad \Rightarrow \frac{-10}{32}\sum x_i < \ln(k) - \frac{2000}{32}$$

$$\Rightarrow \sum x_i \overset{\text{Mult By 20}}{>} \underbrace{\left(\ln(k) - \frac{2000}{32}\right)\frac{32}{-10}}_{\substack{\text{Constant.}\\ \text{CV}}}$$

$$\text{OR } \bar{x} = \frac{\boxed{\sum x_i}}{16} \boxed{>} \underbrace{\left(\ln(k) - \frac{2000}{32}\right)\frac{32}{-10(16)}}_{\substack{\text{Constant.}\\ \text{CV}}}$$

Recall $x \sim Norm(\mu, \sigma = 4)$

What is Dist of $\sum x_i$ or $\bar{x} \sim norm(\overset{10}{\mu}, \sigma_{\bar{x}} = \overset{\sigma/\sqrt{n}}{4/\sqrt{16}} = 1)$



$H_0: \mu = 10$
$H_a: \mu = 15$

$$\text{rnorm}(n=16, \overbrace{\text{mean} = \text{runif}(1, 10, 20)}^{10,\ ?,\ 20}, \text{sd}=4)$$

if $\alpha = 0.05$ most powerful test says... RR > CV

$$CV \Rightarrow \bar{X} \sim \text{Norm}(\overbrace{\mu = 10}^{H_0}, \sigma = 4/\sqrt{16})$$

$$CV = \text{qnorm}(0.95, 10, 1) = 11.64485$$

Example 5: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli distribution with probability p. Find the most powerful test, with significance level $\alpha = 0.05$, for testing the simple null hypothesis $H_o: p = 0.6$ against the simple alternative hypothesis $H_a: p = 0.4$.

$H_0: p = 0.6$

$H_a: p = 0.4$.

$$L(\theta) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{x_1} p^{x_2} \cdots p^{x_n} (1-p)^{1-x_1}(1-p)^{1-x_2} \cdots (1-p)^{1-x_n}$$

$$= p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{0.6^{\sum x_i}(0.4)^{n-\sum x_i}}{0.4^{\sum x_i}(0.6)^{n-\sum x_i}} = \left(\frac{0.6}{0.4}\right)^{\sum x_i}\left(\frac{0.4}{0.6}\right)^{n-\sum x_i} < k.$$

$$\Rightarrow \sum x_i \ln\left(\frac{0.6}{0.4}\right) + (n-\sum x_i)\ln\left(\frac{0.4}{0.6}\right) < \ln(k)$$

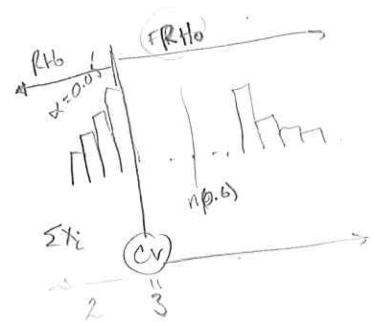
$$\Rightarrow \sum x_i \ln\left(\frac{0.6}{0.4}\right) + n\ln\left(\frac{0.4}{0.6}\right) - \sum x_i\left(\frac{0.4}{0.6}\right) < \ln(k)$$

$$\Rightarrow \sum x_i \underbrace{\left[\ln\left(\frac{0.6}{0.4}\right) - \ln\left(\frac{0.4}{0.6}\right)\right]}_{>0} < \ln(k) - n\ln\left(\frac{0.4}{0.6}\right)$$

$$\boxed{\sum x_i} \boxed{<} \underbrace{\frac{\left[\ln(k) - n\ln\left(\frac{0.4}{0.6}\right)\right]}{\ln\left(\frac{0.6}{0.4}\right) - \ln\left(\frac{0.4}{0.6}\right)}}_{k^* \quad CV}$$

What is the Dist of $\sum X_i$? when $X \sim Bern(p=0.6)$. And X_i independent.

$$\sum X_i \sim Bin(n=n, p=0.6).$$



RHo.

$H_0: p = 0.6$.

$H_a: p < 0.6$

when n Big $\frac{\sum X_i}{n} = \hat{p} \sim Norm\left(0.6, \sigma = \sqrt{\frac{0.24}{n}}\right)$

Ho: $p=0.6$.

Ha: $p=0.4$. $\alpha=0.05$, $n=10$

CV ⇒ qbinom (0.05, n, 0.6)

Most Powerful test RR if $\Sigma X \leq CV$ RHo. otherwise FRHo.

$CV \cong 3$ $\alpha \cong 0.0122945536$.

Example 6: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean β. Find the most powerful test, with significance level $\alpha = 0.05$, for testing the simple null hypothesis $H_o: \beta = \beta_o$ against the simple alternative hypothesis $H_a: \beta = \beta_a$, *where* $\beta_a > \beta_o$.

$\beta_o \ngtr \beta_a$.

Big 4 Small 3

$H_0: \beta = \beta_0$

$H_a: \beta = \beta_a$.

$$L(\beta) = L(\theta) = \prod_{i=1}^{n} \frac{1}{\beta} \exp\left[-x_i/\beta\right] = \frac{1}{\beta^n} \exp\left[\frac{-\sum x_i}{\beta}\right]$$

$$\frac{L(\theta_o)}{L(\theta_a)} = \frac{\frac{1}{\beta_o^n} \exp\left[\frac{-\sum x_i}{\beta_o}\right]}{\frac{1}{\beta_a^n} \exp\left[\frac{-\sum x_i}{\beta_a}\right]} < k.$$

$$\Rightarrow \frac{\beta_a^n}{\beta_o^n} \frac{\exp\left[\frac{-\sum x_i}{\beta_o}\right]}{\exp\left[\frac{-\sum x_i}{\beta_a}\right]} < k \Rightarrow n\ln(\beta_a) - n\ln(\beta_o) + \frac{-\sum x_i}{\beta_o} - \frac{-\sum x_i}{\beta_a} < \ln(k)$$

$$\Rightarrow \frac{\sum x_i}{\beta_a} - \frac{\sum x_i}{\beta_o} < \ln(k) - n\ln(\beta_a) + n\ln(\beta_o)$$

$$\Rightarrow \sum x_i \underbrace{\left(\frac{1}{\beta_a} - \frac{1}{\beta_o}\right)}_{<0} < \ln(k) - n\ln(\beta_a) + n\ln(\beta_o)$$

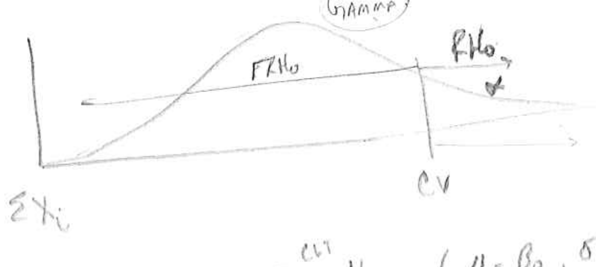
$$\Rightarrow \boxed{\sum x_i} \boxed{>} \underbrace{\frac{\left[\ln(k) - n\ln(\beta_a) + n\ln(\beta_o)\right]}{\left(\frac{1}{\beta_a} - \frac{1}{\beta_o}\right)}}_{\text{let CV.}}$$

when $X_i \sim Exp[\beta_o]$

$\left(\sum X_i \sim \ldots \quad \text{mgf} \left[[1-\beta_o t]^{-1}\right]^n = [1-\beta_o t]^{-n} \quad Gamma(\nu = n, \beta = \beta_o)\right)$

$\left(\bar{X} \sim \ldots \left[1 - \frac{\beta_o}{n} t\right]^{-n}\right)$

$\bar{X} \sim Gamma\left(\nu = n, \beta = \frac{\beta_o}{n}\right)$



$\approx$ when n Big $\frac{\sum X_i}{n} = \bar{X} \overset{CLT}{\sim} Norm\left(\mu = \beta_o, \sigma = \frac{\beta_o}{\sqrt{n}}\right)$

$H_0: \beta = 4$

$H_a: \beta > 4$ $n = 11$;

Cv of most powerful test.

RR if $\sum y_i >$ qgamma$(0.95, n, 1/\beta)$

Example 7: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a uniform distribution $[0, \theta]$. Find the most powerful test, with significance level $\alpha = 0.05$, for testing the simple null hypothesis $H_o: \theta = 4$ against the simple alternative hypothesis $H_a: \theta < 4$.

$f_X(x) = \frac{1}{\theta}$, $0 < x < \theta$

$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$

$$\boxed{\begin{array}{l} H_0: \theta = 4 \\ H_a: \theta < 4 \end{array}}$$

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{\left(\frac{1}{4^n}\right)\text{ when } 0 \le x \le 4 \text{ only}}{\left(\frac{1}{\theta_a^n}\right)\text{ when } 0 \le x \le \theta_a < 4} = \frac{\theta_a^n}{4^n} < k. \quad \text{when } 0 \le x \le \theta_a < 4$$

let $I_{[a,b]}(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{I_{[0,4]}(x_{(n)})}{I_{[0,\theta_a]}(x_{(n)})} \cdot \frac{\theta_a^n}{4^n} < k \Rightarrow \boxed{\frac{I_{[0,4]}(x_{(n)})}{I_{[0,\theta_a]}(x_{(n)})}} \boxed{<} \frac{k \cdot 4^n}{\theta_a^n}$$

Consider $\frac{I_{[0,4]}(x_{(n)})}{I_{[0,4)}(x_{(n)})}$

$\frac{0}{0}$: → $X_{(n)} > 4$ → impossible under either Hypothesis ⇒ rewrite hypothesis.

$\frac{1}{1}$: → $X_{(n)} < 4$ ⇒ Possible either RHo or FRHo based on cv which is determined on α and $f_{X_{(n)}}(x)$

$\frac{1}{0}$: → $X_{(n)} = 4$ ⇒ Possible means you would fail to reject Ho clearly Ha is incorrect.

Recall $f_{X_{(n)}}(x) = \frac{n x^{n-1}}{\theta^n}$ for $0 < x < \theta$

$$0.05 = \int_0^{cv} \frac{n x^{n-1}}{4^n}\, dx \Rightarrow 0.05 = \frac{n}{4^n}\left[\frac{x^n}{n}\right]\Bigg|_{x=0}^{x=cv}$$

$$0.05 = \frac{cv^n}{4}$$

$$\Rightarrow \left[4^n(0.05)\right]^{1/n} = cv$$

$n = 5$ $H_0: \theta = 4$ $H_a: \theta < 4$ $cv = 2.197$ RR: if $X_{(n)} < cv$ RHo.