Example 4: If $\hat{\theta} = \hat{S}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$, $\theta = \sigma^2$ when $X_i \sim Norm(\mu, \sigma)$, find $MSE[\hat{\theta}]$.

Recall Example 1 part (3)

$E[\hat{\theta}] = E[\hat{S}^2] = \cdots = \left(\frac{n-1}{n}\right)\sigma^2$ OR $\frac{(n-1)}{(n)}\theta$ Since $\sigma^2 = \theta$

$B[\hat{\theta}] = \frac{(n-1)}{n}\theta - \theta = \theta\left[\frac{n-1}{n} - 1\right] = \theta\left[\frac{-1}{n}\right] = \frac{-\sigma^2}{n}$

typically under estimated "neg"

$V[\hat{S}^2]$:

first consider $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{df=n-1}$ So $Var\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1) \Rightarrow \left[\frac{n-1}{\sigma^2}\right]^2 Var(S^2) = 2(n-1)$

constant

So $V[S^2] = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$

$S^2 = \frac{\Sigma(X_i - \bar{x})^2}{n-1}$ So $\frac{S^2(n-1)}{n} = \frac{\Sigma(X_i - \bar{x})^2}{n} = \hat{S}^2$

So $Var[\hat{S}^2] = Var\left[\frac{(n-1)S^2}{n}\right] = \left(\frac{n-1}{n}\right)^2 Var[S^2] = \frac{(n-1)^2}{n^2}\frac{2\sigma^4}{(n-1)} = \frac{2\sigma^4(n-1)}{n^2}$

Then $MSE[\hat{S}^2] = Var[\hat{S}^2] + B(\hat{S}^2)^2$

$= \frac{2\sigma^4(n-1)}{n} + \left(\frac{-\sigma^2}{n}\right)^2 = \frac{2\sigma^4(n-1) + \sigma^4}{n^2} = \frac{\sigma^4(2(n-1)+1)}{n^2}$

$= \frac{(2n-1)\sigma^4}{n^2}$

So $MSE[\hat{S}^2] = MSE[\hat{\theta}] = \frac{(2n-1)\theta^2}{n^2}$ Since $\theta = \sigma^2$