Proof of CLT for Numeric Variables

Maclaurin Series

$$f(x) = f(0) + x\frac{d}{dx}(f(x)) + \frac{x^2}{2!}\frac{d^2}{dx^2}(f(x)) + \cdots = \sum_{i=0}^{\infty}\frac{d^i}{dx^i}(f(x))\frac{x^i}{i!}$$

$$E[e^{tx}] = \overset{\text{mgf}}{M_x(t)} = M_x(0) + \frac{d}{dx}(M_x(0))\,x + \frac{d^2}{dx^2}(M_x(0))\frac{x^2}{2!} + \frac{d^3}{dx^3}(M_x(0))\frac{x^3}{3!} + \cdots$$

where $\frac{d}{dx}[M_x(0)] = \frac{d}{dt}[M_x(t)]\Big|_{t=0}$

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let X_i be the i^{th} random selection from a Distribution X where $\overbrace{\mu_{x_i} = E[X_i] = 0}^{\text{if not 0 shift it to be 0!}}$

and $Var[X_i] = \sigma_{x_i}^2 = E[X_i^2] - E[X_i]^2 = \sigma_x^2$ Here $E[X_i] = 0$

$\Rightarrow E[X_i^2] - 0^2 = E[X_i^2] = \sigma_x^2$

So $M_x(t) = M_x(0) + t\,\underbrace{M_x'(0)}_{E[X]} + \frac{t^2}{2!}\underbrace{M_x''(0)}_{E[X^2]} + \underbrace{\sum_{i=3}^{\infty}\left[\frac{d^i}{dt^i}(M_x(t))\Big|_{t=0}\left(\frac{t^i}{i!}\right)\right]}_{\text{let } e_x}$

Consider $M_x(0) = E[e^{x\cdot 0}] = E[e^0] = E[1] = 1$

So

$$M_x(t) = 1 + t\,\underbrace{E[X]}_{0} + \frac{t^2}{2!}\sigma_x^2 + e_x = 1 + \frac{t^2}{2!}\sigma_x^2 + e_x$$

CLT for Numeric claims

$$\frac{\sum_{i=1}^{n} X_i - n\mu_x}{\sqrt{n\sigma_x^2}} = Z \quad \text{where } \mu_x = E[X_i] = 0 \Rightarrow Z = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n\sigma_x^2}}$$

as $n \to \infty$

let $S = \sum_{i=1}^{n} X_i$ So $Z = \dfrac{S}{\sqrt{n\sigma_x^2}}$ or $\sqrt{n}\sigma_x Z = S$

let $t^* = \dfrac{t}{\sqrt{n}\sigma_x}$

$$M_Z(t) = E[e^{tZ}] = E\left[e^{t\frac{S}{\sqrt{n\sigma_x^2}}}\right]$$

$$= E\left[e^{t^*S}\right] = E\left[e^{t^*\Sigma X_i}\right] = E\left[e^{t^*(X_1 + X_2 + \cdots + X_n)}\right]$$

$$= E\left[e^{t^*X_1 + t^*X_2 + \cdots + t^*X_n}\right] = E\left[e^{t^*X_1} e^{t^*X_2} \cdots e^{t^*X_n}\right] = E\left[\prod_{i=1}^{n} e^{t^*X_i}\right]$$

each X_i comes from Dist. X so,

$$E[e^{t^*X_i}] = E[e^{t^*X}] = M_{X_i}(t^*) = M_X(t^*) = 1 + \frac{t^{*2}}{2!}\sigma_x^2 + e_x$$

So,

$$M_Z(t) = E\left[e^{t^*X_1} \cdots e^{t^*X_n}\right] = E\left[\overbrace{e^{t^*X} e^{t^*X} \cdots e^{t^*X}}^{n \text{ times}}\right] = E\left[e^{nt^*X}\right]$$

$$= \left(E\left[e^{t^*X}\right]\right)^n = \left[M_X(t^*)\right]^n \quad \text{let } t^* = t/\sqrt{n}\sigma_x \quad = \left[M_X\left(\frac{t}{\sqrt{n}\sigma_x}\right)\right]^n$$

$$= \left[1 + \left(\frac{t}{\sqrt{n}\sigma_x}\right)^2 \frac{1}{2!}\sigma_x^2 + e_{x^*}\right]^n = \left[1 + \frac{t^2}{2n\sigma_x^2}\sigma_x^2 + \frac{n}{n}e_{x^*}\right]^n$$

$$= \left[1 + \frac{(t^2/2) + n\,e_{x^*}}{n}\right]^n$$

Consider: $e_x = \sum_{i=3}^{\infty}\left[\frac{t^i}{i!}E[X^i]\right] = \frac{t^3}{3!}E[X^3] + \frac{t^4}{4!}E[X^4] + \frac{t^5}{5!}E[X^5] + \ldots$

let $t^* = \dfrac{t}{\sqrt{n}\sigma_x}$

$$e_{x^*} = \sum_{i=3}^{\infty}\left[\left[\frac{t}{\sqrt{n}\sigma_x}\right]^i \frac{1}{i!}E[X^i]\right] = \frac{t^3}{n^{3/2}\sigma_x 3!}E[X^3] + \frac{t^4}{n^{4/2}\sigma_x 4!}E[X^4] + \ldots$$

$$n \cdot e_{x^*} = \sum_{i=3}^{\infty} \left[\frac{n \quad t^i}{n^{i/2} \sigma_x \, i!} E[X^i] \right] = \sum_{i=3}^{\infty} \left[\frac{t^i}{n^{\frac{2-i}{2}} \sigma_x \cdot i!} E[X^i] \right]$$

$$= \frac{t^3}{n^{1/2} \sigma_x \, 3!} E[X^3] + \frac{t^4}{n^{2/2} \sigma_x \cdot 4!} E[X^4] + \frac{t^5}{n^{3/2} \sigma_x \, 5!} E[X^5] + \dots$$

$$\lim_{n\to\infty} \left[n \cdot e_{x^*} \right] = \lim_{n\to\infty} \left[\overbrace{\frac{t^3}{\cancel{n^{1/2}} \sigma_x \, 3!}}^{\to 0} E[X^3] + \overbrace{\frac{t^4}{\cancel{n} \sigma_x \, 3!}}^{\to 0} E[X^4] + \overbrace{\cancel{\dots}}^{\to 0} \right]$$

$$= 0$$

So as $n \to \infty$ ...

$$\lim_{n\to\infty} M_Z(t) = \lim_{n\to\infty} \left[1 + \frac{(t^2/2) + \overbrace{\cancel{\lim_{n\to\infty} (n \cdot e_{x^*})}}^{\to 0}}{n} \right]^n = \lim_{n\to\infty} \left[1 + \frac{t^2/2}{n} \right]^n$$

Recall $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$

So $\lim_{n\to\infty} M_Z(t) = e^{t^2/2} = e^{\mu_Z t + \sigma_Z^2 \frac{t^2}{2}}$ when $\mu_Z = 0$; $\sigma_Z^2 = 1$

This is Mgf of Normal when $\mu = 0$ $\sigma = 1$

Standard Normal!!

So $\dfrac{\sum_{i=1}^{n} X_i - n\mu_x}{\sqrt{n\sigma_x^2}} \sim N(0,1)$ and $\dfrac{\frac{\sum_{i=1}^{n} X_i}{n} - \mu_x}{\left(\sqrt{n\sigma_x^2}/n\right)} = \dfrac{\bar{X} - \mu_x}{(\sigma_x/\sqrt{n})} \sim N(0,1)$