

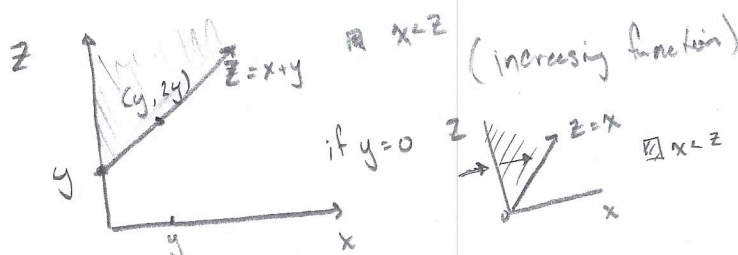
Example 6.8 (Δ of variable / Method of transformations)

$$f_{x,y}(x,y) = \begin{cases} e^{-(x+y)} & ; x > 0; y > 0 \\ 0 & ; \text{otherwise} \end{cases} \quad \begin{array}{l} \text{product of 2, Exp}(\beta=1) \\ x \sim \text{Exp}(\beta=1) \\ y \sim \text{Exp}(\beta=1) \end{array}$$

$$g_z(x,y) = z = x+y \Rightarrow g_y^{-1}(x,z) = y = z-x$$

Step 1) support of z ? $\begin{matrix} \infty \\ \uparrow \\ z \\ \downarrow \\ 0 \end{matrix} = \begin{matrix} \infty \\ \uparrow \\ x \\ \downarrow \\ 0 \end{matrix} + \begin{matrix} \infty \\ \uparrow \\ y \\ \downarrow \\ 0 \end{matrix}$ $0 < z < \infty$

Step 2) graph x, z plane wrt z, x or y, z



Step 3)

$$f_{x,z}(x,z) = f_{x,y}(x, g_y^{-1}(x,z)) \left| \frac{d}{dz} g_y^{-1}(x,z) \right| = e^{-(x+(z-x))} \left| \frac{d}{dz} (z-x) \right|$$

$$= e^{-z} |1-0| = e^{-z}; \quad 0 < x < z < \infty$$

Step 4)

$$f_z(z) = \int_x f_{x,z}(x,z) dx = \int_0^z e^{-z} dx = z e^{-z}; \quad 0 < z < \infty$$

$$f_z(z) = \begin{cases} z e^{-z} & , z > 0 \\ 0 & , \text{otherwise} \end{cases} \quad z \sim \text{Gamma}(\alpha=2, \beta=1)$$