Example 4: Consider a random sample of values taken from a population that follows a normal distribution with a mean of 0 and an unknown variance σ^2. Is $\frac{\sum_{i=1}^{n}(X_i)^2}{n}$ a consistent estimator for σ^2?

$\lim_{n\to\infty} P\left(\left|\frac{\Sigma X_i^2}{n} - \sigma^2\right| < \varepsilon\right) = ?$ Need dist of ΣX_i^2 ... if Standard Normal $\Rightarrow \chi^2$ but it's Not standard normal ... Could find it ...

OR Check if $\frac{\Sigma X_i^2}{n}$ is unbiased then show $V\left[\frac{\Sigma X_i^2}{n}\right] \to 0$ as $n \to \infty$

$E\left[\frac{\Sigma X_i^2}{n}\right] = \frac{1}{n} n E[X_i^2] = E[X_i^2] \leftarrow$ 2nd moment

$V[X_i] = E[X_i^2] - E[X_i]^2 \Rightarrow \sigma^2 = E[X_i^2] - \mu^2 \Rightarrow \sigma^2 = E[X_i^2] - 0^2$

$\Rightarrow E[X_i^2] = \sigma^2$ So $E\left[\frac{\Sigma X_i^2}{n}\right] = \sigma^2$ So This is unbiased estimate

$V\left[\frac{\Sigma X_i^2}{n}\right] = \frac{1}{n^2} n V[X_i^2] = \frac{1}{n}\left[E[X_i^{2^2}] - (E[X_i^2])^2\right] = \frac{1}{n}\left[E[X_i^4] - E(X_i^2)^2\right]$

$= \frac{1}{n}\left[E[X_i^4] - \sigma^{2^2}\right] = \frac{1}{n}\left[E[X_i^4] - \sigma^4\right]$

we need $E[X_i^4] \leftarrow$ 4th moment

Mgf of $X \Rightarrow$ Norm$(\mu=0, \sigma^2) \Rightarrow \exp\left[\frac{\sigma^2}{2}t^2\right]$

$E[X_i^4] = \frac{d^4}{dt^4}\left(\exp\left[\frac{\sigma^2}{2}t^2\right]\right)\Big|_{t=0} = \cdots = \left[3\sigma^4 e^{\frac{1}{2}\sigma^2 t^2} + 6\sigma^6 t^2 e^{\frac{1}{2}\sigma^2 t^2} + \sigma^8 t^4 e^{\frac{1}{2}\sigma^2 t^2}\right]_{t=0}$

$= 3\sigma^4 = E[X_i^4]$

So $V\left[\frac{\Sigma X_i^2}{n}\right] = \frac{1}{n}\left[3\sigma^4 - \sigma^4\right] = \frac{2\sigma^4}{n}$

$\lim_{n\to\infty}\left(\frac{2\sigma^4}{n}\right) = 0$ So $\frac{\Sigma X_i^2}{n} \xrightarrow{P} \sigma^2$ (Consistent Estimator)