Example 6.3 (CDF / Method of Distributions)

$Z = X + Y = g_Z(x,y)$

$g_Y^{-1}(x,z) : Y = Z - X$

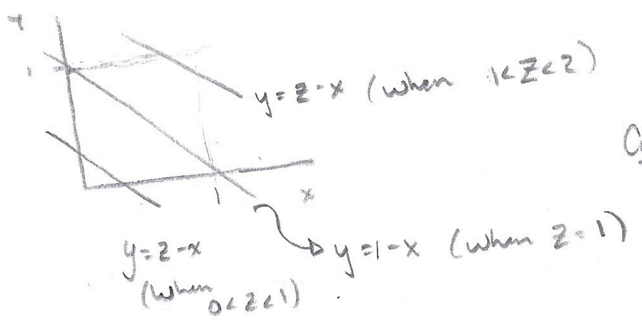
$Y \,\&\, X \sim \text{Uniform}(0,1)$

$\underset{0<x<1}{f_X(x) = 1} \quad \underset{0<y<1}{f_Y(y) = 1} \quad \xrightarrow{\text{ind}} f_{XY}(x,y) = 1$

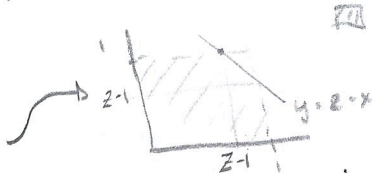
Step 1) Support of Z?

$$\underset{0}{\overset{2}{Z}} = \underset{0}{\overset{1}{X}} + \underset{0}{\overset{1}{Y}}$$

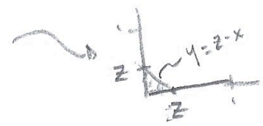
Step 2) graph transformation (wrt $x \,\&\, y$ plane treat Z as a constant)



Cases



$y \le z - x \cap 1 < z < 2$



$y \le z - x \cap 0 < z < 1$

Limits of integration change depending on Z

Step 3)

$$F_Z(z) = P(Z \le z) = P(g_Z(x,y) \le z) = P(X + Y \le z) = P(Y \le z - x)$$

Step 4)

Case 1 : $0 < z < 1$

$$F_Z(z) = \underbrace{\int_{0}^{z} \int_{0}^{z-x} f_{XY}(x,y)\, dy\, dx}_{\text{vert}} = \int_{0}^{z} \int_{0}^{z-x} 1 \, dy\, dx = \int_{0}^{z} \int_{0}^{z-y} 1 \, dx\, dy = \frac{z^2}{2}$$

Case 2 : $1 \le z \le 2$

$$F_Z(z) = \underbrace{\int_{0}^{z-1} \int_{0}^{1} 1 \, dy\, dx + \int_{z-1}^{1} \int_{0}^{z-x} 1 \, dy\, dx}_{\text{vert}} = \underbrace{\int_{0}^{z-1} \int_{0}^{1} 1 \, dx\, dy + \int_{z-1}^{1} \int_{0}^{z-y} 1 \, dx\, dy}_{\text{Hor}} = (z-1) + \left(\frac{2-z}{2}\right) z$$

$$= -\frac{z^2}{2} + 2z - 1$$

$$F_Z(z) = \begin{cases} 0 \,; & -\infty < z < 0 \\ \frac{z^2}{2} \,; & 0 \le z < 1 \\ -\frac{z^2}{2} + 2z - 1 \,; & 1 \le z < 2 \\ 1 \,; & z \ge 2 \end{cases}$$

Step 5)

$$f_Z(z) = \frac{d}{dz}\left(F_Z(z)\right) = \begin{cases} z \,; & 0 \le z < 1 \\ -z + 2 \,; & 1 \le z < 2 \\ 0 \,; & \text{otherwise} \end{cases}$$