Example 3: Let $X_1, \dots, X_n$ represent a random sample taken from a population of values that is modeled by the following cumulative and probability distribution functions:

$$F_X(x) = 1 - e^{-\frac{x}{\theta}}, \qquad f_X(x) = \frac{e^{-\frac{x}{\theta}}}{\theta}, \qquad x > 0$$

Find the relative efficiency of $nX_{(1)}$ and $\bar{X}$. Interpret the meaning.

$$f_{X_{(1)}} = n\left[1-F_X(x)\right]^{n-1} f_X(x) = n\left[1-\left(1-e^{-x/\theta}\right)\right]^{n-1} \frac{e^{-x/\theta}}{\theta} = \frac{n}{\theta}\left[e^{-x/\theta}\right]^{n}$$

$$= \frac{n}{\theta} e^{-\frac{nx}{\theta}} \quad \text{for } x > 0$$

$$E[X_{(1)}] = \frac{n}{\theta}\int_0^\infty x\, e^{-x/(\theta/n)}\, dx = \frac{n}{\theta}\Gamma(2)\left(\frac{\theta}{n}\right)^2 = \frac{\theta}{n}$$

$$E[nX_{(1)}] = n\,E[X_{(1)}] = \theta \Rightarrow nX_{(1)} \text{ is unbiased estimator for } \theta$$

$$E[(nX_{(1)})^2] = n^2 E[X_{(1)}^2] = n^2\int_0^\infty \frac{n}{\theta} x^2 e^{-x/(\theta/n)}\, dx = \frac{n^3}{\theta}\Gamma(3)\left(\frac{\theta}{n}\right)^3 = 2\theta^2$$

$$V[nX_{(1)}] = 2\theta^2 - [\theta]^2 = 2\theta^2 - \theta^2 = \theta^2$$

$$f_{\bar{X}}(x) = ? \qquad M_{\bar{X}}(t) = E[e^{t\bar{x}}] = E[e^{t/n \sum x_i}] = \dots = M_{\sum X_i}(t/n) = \left[1-\theta\frac{t}{n}\right]^{-n}$$

$$\bar{X} \sim \text{Gamma}\left(\alpha = n, \beta = \frac{\theta}{n}\right)$$

$$E[\bar{X}] = \alpha\beta = \frac{n\theta}{n} = \theta, \qquad V[\bar{X}] = \alpha\beta^2 = n\frac{\theta^2}{n^2} = \frac{\theta^2}{n}$$

↳ unbiased!

$$RE[nX_{(1)}, \bar{X}] = \frac{MSE[nX_{(1)}]}{MSE[\bar{X}]} = \frac{V[nX_{(1)}] + B[nX_{(1)}]^2}{V[\bar{X}] + B[\bar{X}]^2} = \frac{\theta^2}{(\theta^2/n)} = n$$

Dependending on sample size n, $(n \geq 1)$, $\bar{X}$ is an n times more efficient estimator for θ, than is, $nX_{(1)}$

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