Example 4: A random variable $X_i \sim Gamma(\alpha, \beta)$. If $\alpha = 2$, find an MLE for β.

$$f(x_i | \alpha, \beta) = \frac{x_i^{\alpha-1} e^{-x_i/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \quad \text{where} \quad x_i > 0$$

$$f(x_i | \alpha=2, \beta) = \frac{x_i e^{-x_i/\beta}}{\beta^2}$$

$$L(\beta) = \prod_{i=1}^{n}\left(\frac{x_i e^{-x_i/\beta}}{\beta^2}\right) = \frac{\left(\prod_{i=1}^{n} x_i\right) e^{-\Sigma x_i/\beta}}{\beta^{2n}}$$

$$\ln(L(\beta)) = \ln\left(\prod x_i\right) - \frac{\Sigma x_i}{\beta} - 2n \cdot \ln(\beta)$$

$$\frac{\partial \ln(L(\beta))}{\partial \beta} = \frac{\Sigma x_i}{\beta^2} - \frac{2n}{\beta} \overset{\text{set}}{=} 0 \quad \Rightarrow \frac{\Sigma x_i}{\beta^2} = \frac{2n}{\beta}$$

$$\Rightarrow \frac{\Sigma x_i}{2n} = \beta = \frac{\bar{X}}{2}$$

$$\frac{\partial^2}{\partial \beta^2} \ln(L(\beta)) = -\frac{2\Sigma(x_i)}{\beta^3} + \frac{2n}{\beta^2}$$

let $\beta = \frac{\Sigma x_i}{2n} = \frac{\bar{X}}{2} \Rightarrow -\frac{2\Sigma x_i}{\left(\frac{\Sigma x_i}{2n}\right)^3} + \frac{2n}{\left(\frac{\Sigma x_i}{2n}\right)^2}$

$= \frac{-8n}{\bar{X}^2} < 0$ So by 2nd derivative test. $\frac{\bar{X}}{2}$ is a max!

$$MLE(\beta) = \frac{\bar{X}}{2} = \frac{\Sigma x_i}{2n}$$