Example 6.9

$$f_{XY}(x,y) = \begin{cases} 2(1-x) \; ; & 0<x<1,\ 0<y<1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$Z = XY = g_Z(x,y)$$

$$\hookrightarrow g_Y^{-1}(x,z) = z/x$$

Step 1) support of Z?

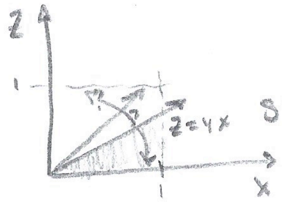
$$\underset{0}{\overset{1}{Z}} = \underset{0}{\overset{1}{X}}\;\underset{0}{\overset{1}{Y}}$$

$$0 < \overbrace{\frac{z}{x}}^{y} < 1$$

$$0 < z < x$$

Step 2) graph z,x plane wrt z,x treat y as constant.

(increasing function)



slope can range depending on y. steep → closer to ∞

shallow but positive closer to 0

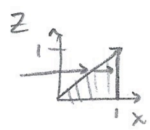
$z < x$

Step 3)

$$f_{XZ}(x,z) = f_{XY}(x, g_Y^{-1}(x,z)) \left| \frac{d}{dz} g_Y^{-1}(x,z) \right|$$

$$= 2(1-x) \left| \frac{d}{dz}\left[z\left(\frac{1}{x}\right)\right] \right| = 2(1-x)\left|\frac{1}{x}\right| = 2\left(\frac{1}{x} - 1\right) = \frac{2}{x} - 2$$

for $0 < z < x < 1$



Step 4)

$$f_Z(z) = \int_x f_{XZ}(x,z)\,dx = \int_z^1 \frac{2}{x} - 2 \, dx = 2\Big[\ln(x) - x\Big]_z^1 = 2\left[(\ln(1) - 1) - (\ln(z) - z)\right]$$

$$= 2\left[(0-1) - (\ln(z) - z)\right]$$

$$= 2(-1 - \ln(z) + z)$$

$$f_Z(z) = \begin{cases} 2(z - \ln(z) - 1) \; ; & 0 \le z \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$$