Discussion point

→ How would you explain the result for parallel lines?

To illustrate the result for perpendicular lines, try Activity 5.1 on squared paper.

ACTIVITY 5.1

(i) Draw two congruent right-angled triangles in the positions shown in Figure 5.2. p and q can take any value.

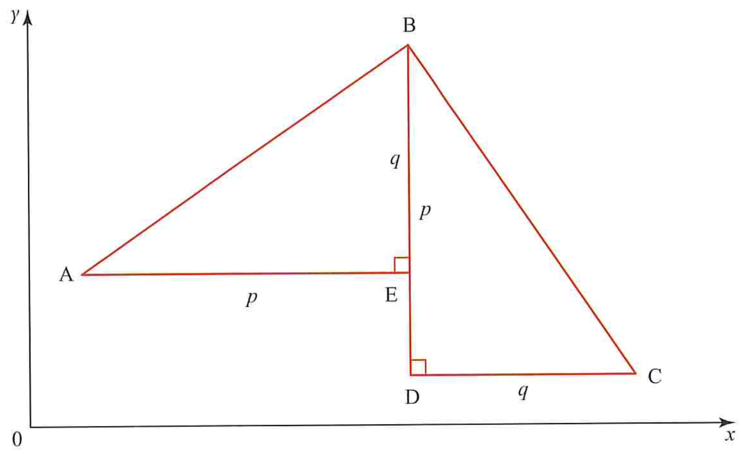


Figure 5.2

(ii) Explain why $\angle ABC = 90°$.

(iii) Calculate the gradient of AB (m_1) and the gradient of BC (m_2).

(iv) Show that $m_1 m_2 = -1$

❗ The gradient of a line perpendicular to a line of gradient m is given by $-\frac{1}{m}$.
Don't forget to change the sign when taking the reciprocal.

2 The distance between two points

Prior knowledge

Students are expected to use Pythagoras' theorem to calculate the distance between two points with known coordinates.

Look at Figure 5.3. P is (3, 1) and Q is (6, 5).

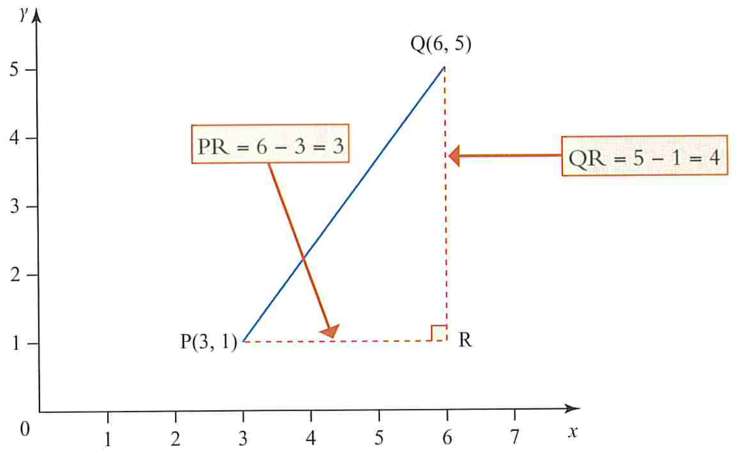


Figure 5.3

$$PQ = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Generalising this, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then

length $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

3 The midpoint of a line joining two points

Look at the line joining the points P(1, 2) and Q(7, 4) in Figure 5.4. The point M is the midpoint of PQ.

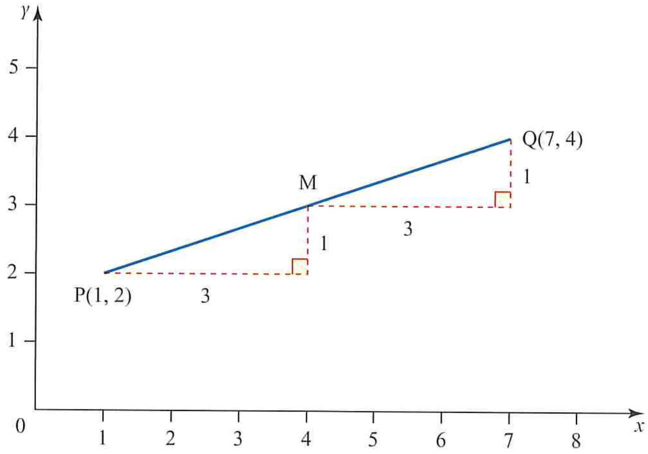


Figure 5.4

The coordinates of M are the means (averages) of the coordinates of P and Q.

$$\tfrac{1}{2}(1 + 7) = 4 \text{ and } \tfrac{1}{2}(2 + 4) = 3$$

M is $(4, 3)$.

Again, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then the coordinates of the midpoint of PQ are given by

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 5.1

A and B are the points (–4, 2) and (2, 5). Work out

(i) the gradient of AB

(ii) the gradient of the line perpendicular to AB

(iii) the length of AB

(iv) the coordinates of the midpoint of AB.

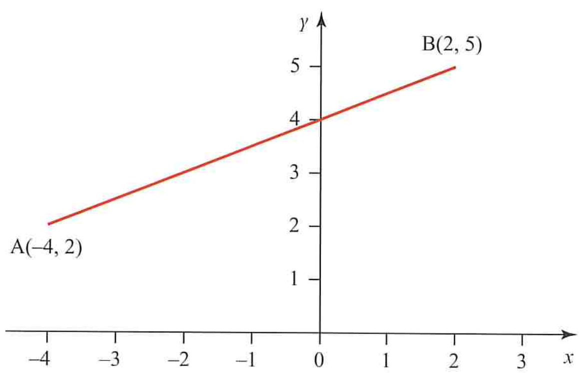


Figure 5.5

Solution

(i) Taking $(-4, 2)$ as (x_1, y_1) and $(2, 5)$ as (x_2, y_2)

$$\text{gradient} = \frac{5-2}{2-(-4)} = \frac{3}{6} = \frac{1}{2}$$

(ii) $m_1 = \frac{1}{2}$ and $m_1 m_2 = -1$

$$\Rightarrow \quad \frac{1}{2}m_2 = -1$$

$$\Rightarrow \quad m_2 = -2$$

The line perpendicular to AB has gradient -2

(iii)
$$\begin{aligned} \text{length} &= \sqrt{(2-(-4))^2 + (5-2)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} \\ &= 6.71 \text{ (3 s.f.)} \end{aligned}$$

(iv)
$$\begin{aligned} \text{midpoint} &= \left(\frac{-4+2}{2}, \frac{2+5}{2}\right) \\ &= (-1, 3.5) \end{aligned}$$

Example 5.2

P is the point (a, b) and Q is the point $(3a, 5b)$.

Write expressions, in terms of a and b, for

(i) the gradient of PQ

(ii) the length of PQ

(iii) the midpoint of PQ.

Solution

Taking (a, b) as (x_1, y_1) and $(3a, 5b)$ as (x_2, y_2)

(i)
$$\begin{aligned} \text{gradient} &= \frac{5b-b}{3a-a} \\ &= \frac{4b}{2a} = \frac{2b}{a} \end{aligned}$$

(ii)
$$\begin{aligned} \text{length} &= \sqrt{(3a-a)^2 + (5b-b)^2} \\ &= \sqrt{4a^2 + 16b^2} \end{aligned}$$

(iii)
$$\begin{aligned} \text{midpoint} &= \left(\frac{a+3a}{2}, \frac{b+5b}{2}\right) \\ &= (2a, 3b) \end{aligned}$$

Discussion point

→ How can the length in part (ii) be simplified further?

Example 5.3

A, B and C are the points (1, 2), (5, b) and (6, 2). $\angle ABC = 90°$.

(i) Work out two possible values of b.

(ii) Show all four points on a sketch and describe the shape of the figure you have drawn.

Solution

(i) Gradient of AB $= \dfrac{b-2}{5-1} = \dfrac{b-2}{4}$

Gradient of BC $= \dfrac{2-b}{6-5} = 2-b$

$$\angle ABC = 90° \Rightarrow \text{AB and BC are perpendicular}$$

$$\Rightarrow \left(\frac{b-2}{4}\right) \times (2-b) = -1$$

$$\Rightarrow (b-2)(2-b) = -4$$

$$\Rightarrow 2b - b^2 - 4 + 2b = -4$$

$$\Rightarrow 4b - b^2 = 0$$

$$\Rightarrow b(4-b) = 0$$

So $b = 0$ or $b = 4$.

(ii)

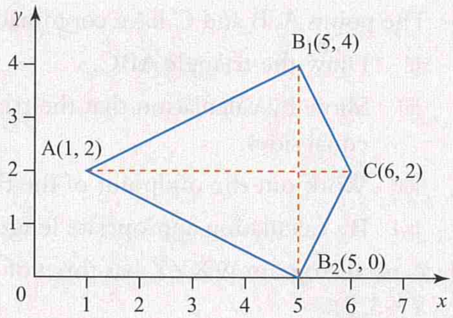


Figure 5.6

AB_1CB_2 is a quadrilateral with diagonals that are perpendicular, since AC is parallel to the x-axis and B_1B_2 is parallel to the y-axis.

This makes AB_1CB_2 a kite.

Exercise 5A

① For each of the following pairs of points A and B, calculate

(a) the gradient of the line perpendicular to AB

(b) the length of AB

(c) the coordinates of the midpoint of AB.

(i)	A(4, 3)	B(8, 11)
(ii)	A(3, 4)	B(0, 13)
(iii)	A(5, 3)	B(10, −8)

(iv) A(−6, −14) B(1, 7)
(v) A(6, 0) B(8, 15)
(vi) A(−2, −4) B(3, 9)
(vii) A(−3, −6) B(2, −7)
(viii) A(4, 7) B(7, −4)

② A(0, 5), B(4, 1) and C(2, 7) are the vertices of a triangle. Show that the triangle is right-angled

(i) by finding the gradients of the sides
(ii) by finding the lengths of the sides.

③ A(3, 6), B(7, 4) and C(1, 2) are the vertices of a triangle. Show that ABC is a right-angled isosceles triangle.

④ A(3, 5), B(3, 11) and C(6, 2) are vertices of a triangle.

(i) Work out the perimeter of the triangle.
(ii) Using AB as the base, work out the area of the triangle.

⑤ A quadrilateral PQRS has vertices at P(−2, −5), Q(11, − 7), R(9, 6) and S(−4, 8).

(i) Work out the lengths of the four sides of PQRS.
(ii) Work out the midpoints of the diagonals PR and QS.
(iii) Without drawing a diagram, show why PQRS cannot be a square. What is it?

⑥ The points A, B and C have coordinates (2, 3), (6, 12) and (11, 7) respectively.

(i) Draw the triangle ABC.
(ii) Show by calculation that the triangle is isosceles and name the two equal sides.
(iii) Work out the midpoint of the third side.
(iv) By calculating appropriate lengths, work out the area of triangle ABC.

⑦ A parallelogram WXYZ has three of its vertices at W(2, 1), X(−1, 5) and Y(−3, 3).

(i) Work out the midpoint of WY.
(ii) Use this information to work out the coordinates of Z.

⑧ A triangle ABC has vertices at A(3, 2), B(4, 0) and C(8, 2).

(i) Show that the triangle is right-angled.
(ii) Work out the coordinates of the point D such that ABCD is a rectangle.

⑨ The three points P(−2, 3), Q(1, q) and R(7, 0) are collinear (i.e. they lie on the same straight line).

(i) Work out the value of q.
(ii) Work out the ratio of the lengths PQ : QR.

⑩ A quadrilateral has vertices A(−2, 8), B(−5, 5), C(5, 3) and D(3, 7).

(i) Draw the quadrilateral.
(ii) Show by calculation that it is a trapezium.
(iii) Work out the coordinates of E when ABCE is a parallelogram.

4 Equation of a straight line

Prior knowledge

In Chapter 3 you used these three facts.

- The equation of the line with gradient m cutting the y-axis at the point $(0, c)$ is $y = mx + c$.
- The equation of the line with gradient m passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- The equation of the line passing through (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Example 5.4

An isosceles triangle with AB = AC has vertices at A(2, 3), B(8, 5) and C(4, 9).

Work out the equation of the line of symmetry.

Solution

Figure 5.7 shows the triangle ABC with the line of symmetry joining A to the midpoint of BC.

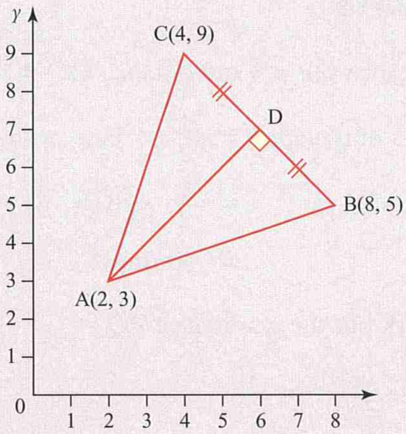


Figure 5.7

The coordinates of D are $\left(\dfrac{8 + 4}{2}, \dfrac{5 + 9}{2}\right) = (6, 7)$

Let (x_1, y_1) be $(2, 3)$ and (x_2, y_2) be $(6, 7)$.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \quad \frac{y - 3}{7 - 3} = \frac{x - 2}{6 - 2}$$

$$\Rightarrow \quad \frac{y - 3}{4} = \frac{x - 2}{4}$$

$$\Rightarrow \quad y = x + 1$$

Chapter 5 Coordinate geometry

5

Example 5.5

The straight line with equation $5x - 4y = 40$ intersects the x-axis at P and the y-axis at Q.

(i) Work out the area of triangle OPQ where O is the origin.

(ii) Work out the equation of the line that passes through Q and is perpendicular to PQ.

Solution

(i) Work out the coordinates of P and Q.

Substitute $y = 0$ in equation of line $\quad 5x - 0 = 40$

$x = 8 \qquad$ P(8, 0)

Substitute $x = 0$ in equation of line $\quad 0 - 4y = 40$

$y = -10 \qquad$ Q(0, −10)

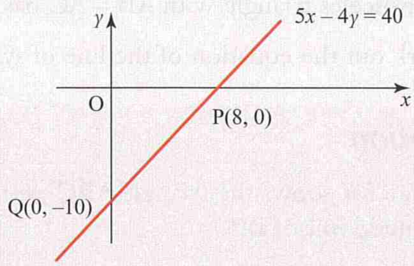


Figure 5.8

A sketch graph will often be useful.

Distance OP = 8 and distance OQ = 10.

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times 8 \times 10\\ &= 40 \text{ units}^2\end{aligned}$$

(ii) Work out the gradient of PQ

$$\frac{0 - (-10)}{8 - 0} = \frac{10}{8} = \frac{5}{4}$$

Gradient of line perpendicular to PQ $= -\frac{4}{5}$

Line passes through (0, −10) $\quad y = -\frac{4}{5}x - 10$

Exercise 5B

① By calculating the gradients of the following pairs of lines, state whether they are parallel, perpendicular or neither.

(i) $x = 2$
$y = -2$

(ii) $y = 2x$
$y = -2x$

(iii) $x + 2y = 1$
$2x - y = 1$

(iv) $y = x - 3$
$x - y + 4 = 0$

(v) $y = 3 - 4x$
$y = 4 - 3x$

(vi) $x + y = 5$
$x - y = 5$

(vii) $x - 2y = 3$
$y = \frac{1}{2}x - 1$

(viii) $x + 3y - 4 = 0$
$y = 3x + 4$

(ix) $2y = x$
$2x + y = 4$

(x) $2x + 3y - 4 = 0$
$2x + 3y - 6 = 0$

(xi) $x + 3y = 1$
$y + 3x = 1$

(xii) $2x = 5y$
$5x + 2y = 0$

② Work out the equations of these lines.

(i) Parallel to $y = 3x$ and passing through $(3, -1)$.

(ii) Parallel to $y = 2x + 3$ and passing through $(0, 7)$.

(iii) Parallel to $y = 3x - 4$ and passing through $(3, -7)$.

(iv) Parallel to $4x - y + 2 = 0$ and passing through $(5, 0)$.

(v) Parallel to $3x + 2y - 1 = 0$ and passing through $(3, -2)$.

(vi) Parallel to $2x + 4y - 5 = 0$ and passing through $(0, 5)$.

③ Work out the equations of these lines.

(i) Perpendicular to $y = 2x$ and passing through $(0, 0)$.

(ii) Perpendicular to $y = 3x - 1$ and passing through $(0, 4)$.

(iii) Perpendicular to $y + x = 2$ and passing through $(3, -1)$.

(iv) Perpendicular to $2x - y + 4 = 0$ and passing through $(1, -1)$.

(v) Perpendicular to $3x + 2y + 4 = 0$ and passing through $(3, 0)$.

(vi) Perpendicular to $2x + y - 1 = 0$ and passing through $(4, 1)$.

④ Points P and Q have coordinates P(3, −1) and Q(5, 7).

(i) Work out the gradient of PQ.

(ii) Work out the coordinates of the midpoint of PQ.

(iii) The perpendicular bisector of a line PQ is the line which is perpendicular to PQ and passes through its midpoint. Work out the equation of the perpendicular bisector of PQ.

⑤ A triangle has vertices P(2, 5), Q(−2, −2) and R(6, 0).

(i) Sketch the triangle.

(ii) Work out the coordinates of L, M and N, which are the midpoints of PQ, QR and RP respectively.

(iii) Work out the equations of the lines LR, MP and NQ (these are the medians of the triangle).

(iv) Show that the point (2, 1) lies on all three of these lines. (This shows that the medians of a triangle are concurrent.)

⑥ The straight line with equation $2x + 3y - 12 = 0$ cuts the x-axis at A and the y-axis at B.

(i) Sketch the line.

(ii) Work out the coordinates of A and B.

(iii) Work out the area of triangle OAB where O is the origin.

(iv) Work out the equation of the line which passes through O and is perpendicular to AB.

(v) Work out the length of AB and, using the result in (iii), calculate the shortest distance from O to AB.

⑦ A quadrilateral has vertices at the points A(−7, 0), B(2, 3), C(5, 0) and D(−1, −6).

(i) Sketch the quadrilateral.

(ii) Work out the gradient of each side.

(iii) Work out the equation of each side.

(iv) Work out the length of each side.

(v) Work out the area of the quadrilateral.

⑧ £10 000 is invested and simple interest of 2% per annum is received on this investment. (Simple interest is when the interest received each year is calculated on the initial investment in the account only.)

(i) Calculate the interest received after each of the first three years.

(ii) Sketch the graph of interest against time and write down its equation.

(iii) Use the equation to work out how long it would take for the investment to reach £11 000

⑨ A spring has an unstretched length (often called the natural length) of 20 cm. When it is hung with a load of 50 g attached, its stretched length is 25 cm.

Assuming that the extension of the spring is proportional to the load at all times

(i) calculate the load corresponding to an extension of 12.5 cm

(ii) calculate the extension corresponding to a load of 75 g

(iii) calculate the extension corresponding to a load of 800 g and comment on your answer.

Prior knowledge

Students learned to solve simultaneous linear equations in Chapter 4.

5 The intersection of two lines

You can work out the point of intersection of any two lines (or curves) by solving their equations simultaneously.

Example 5.6

(i) Sketch the lines $x + 3y - 6 = 0$ and $y = 2x - 5$ on the same axes.

(ii) Work out the coordinates of the point where they intersect.

Solution

(i) The line $x + 3y - 6 = 0$ passes through (0, 2) and (6, 0).

The line $y = 2x - 5$ passes through (0, –5) and has a gradient of 2.

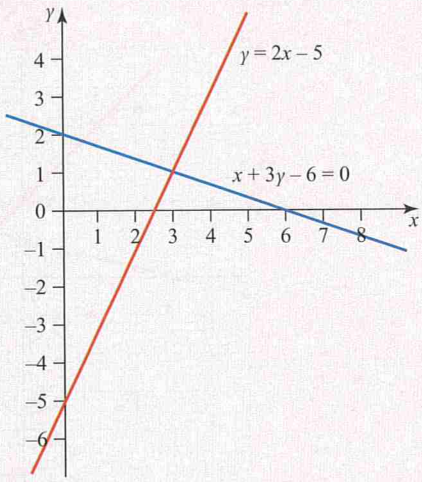


Figure 5.9

(ii) $x + 3y - 6 = 0 \Rightarrow 2x + 6y - 12 = 0$ (multiplying by 2) ①

$y = 2x - 5 \Rightarrow 2x - y - 5 = 0$ ②

① – ② $\Rightarrow 7y - 7 = 0$

$\Rightarrow y = 1$

Substituting $y = 1$ in ① gives $2x + 6 - 12 = 0$

$\Rightarrow x = 3$

The coordinates of the point of intersection are therefore (3, 1).

Discussion point

➜ Graphical methods such as this will have limited accuracy. What factors would affect the accuracy of your solution in this case?

An alternative method for solving these equations simultaneously would be to plot both lines on graph paper and read off the coordinates of the point of intersection.

Example 5.7

(i) Plot the lines $x + y - 2 = 0$ and $4y - x = 4$ on the same set of axes, for $-4 \leqslant x \leqslant 4$, using 1 cm to represent 1 unit on both axes.

(ii) Read off the solution to the simultaneous equations

$x + y - 2 = 0$

$4y - x = 4$

Solution

(i) For each line choose three values of x and calculate the corresponding values of y. Then plot the lines and read off the coordinates of the point of intersection.

$x + y - 2 = 0$

x	–2	0	2
y	4	2	0

$4y - x = 4$

x	–4	0	4
y	0	1	2

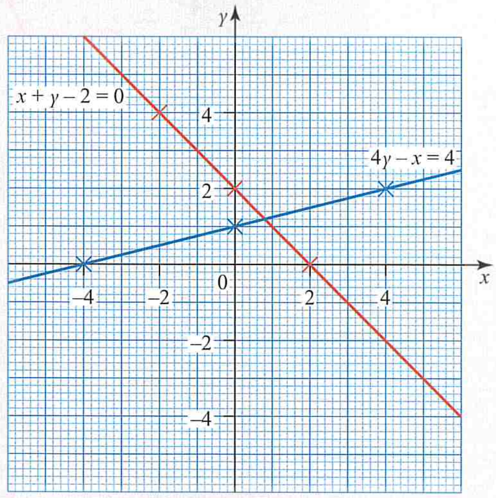


Figure 5.10

(ii) The point of intersection is (0.8, 1.2), so the solution to the simultaneous equations is

$x = 0.8, y = 1.2$

Discussion points

- Why should you plot three points for each line?
- Two lines may not intersect. When is this the case?

Exercise 5C

You will need graph paper for this exercise.

① Solve these pairs of simultaneous equations by plotting their graphs. In each case you are given a suitable range of values of x.

(i) $x = 3y + 1$ $\quad y = x - 1$ $\quad 0 \leqslant x \leqslant 3$

(ii) $3x + 2y = 5$ $\quad x + y = 3$ $\quad -2 \leqslant x \leqslant 2$

② Solve these pairs of simultaneous equations by plotting their graphs. In each case you are given a suitable range of values of x.

(i) $y = 2x - 4$ $\quad 3x + 4y = 17$ $\quad 0 \leqslant x \leqslant 6$

(ii) $6x + y = 1$ $\quad 4x - y = 4$ $\quad 0 \leqslant x \leqslant 2$

③ (i) Plot the lines $x = 4$, $y = x + 4$ and $4x + 3y = 12$ on the same axes for $-1 \leqslant x \leqslant 5$

(ii) State the coordinates of the three points of intersection, and for each point give the pair of simultaneous equations that are satisfied there.

(iii) Work out the area of the triangle enclosed by the three lines.

④ (i) Using the same scale for both axes, plot the lines $2y + x = 4$ and $2y + x = 10$ on the same axes for $0 \leqslant x \leqslant 6$, and say what you notice about them. Why is this the case?

(ii) Add the line $y = 2x$ to your graph. What do you notice now? Can you justify what you see?

(iii) State the coordinates of the two points of intersection, and for each point give the pair of simultaneous equations that are satisfied there.

⑤ A triangle has vertices A(0, 3), B(3, 6) and C(3, 0).

(i) Work out the lengths of the sides of the triangle ABC.

(ii) Work out the equations of the sides of the triangle ABC.

(iii) Describe the triangle ABC.

⑥ A(1, 2), B(2, 5), C(5, 4) and D(4, 1) are the vertices of a quadrilateral ABCD.

(i) Work out the gradients of the sides of the quadrilateral and state two pieces of information that this gives you.

(ii) Work out the lengths of AB and BC.

(iii) What type of quadrilateral is ABCD?

⑦ Alpha and Beta are two rival taxi firms which have the following price structures:

Alpha: A fixed charge of £2 plus 60p per mile.

Beta: A fixed charge of £3 plus 40p per mile.

(i) On the same axes sketch the graph of price (vertical axis) against distance travelled (horizontal axis) for each firm.

(ii) Write down the equation of each line.

(iii) Which firm would you use for a distance of 7 miles?

(iv) For what distance do both firms charge the same?

⑧ When the market price £p of an article varies, so does the number demanded, D and the number supplied, S.

In one case $D = 15 + 0.5p$ and $S = p - 10$

(i) Sketch both lines on the same graph with D and S both on the vertical axis.

The equilibrium position for the market is when the supply and the demand are equal.

(ii) Work out the equilibrium price and the number bought and sold in equilibrium.

6 Dividing a line in a given ratio

You can use similar triangles to work out the coordinates of a point that divides a line in a given ratio if you know the coordinates of the end points of the line.

Look at Figure 5.11. A is (4, 7) and B is (19, 27).

C divides line AB in the ratio 2 : 3, i.e. AC : CB = 2 : 3

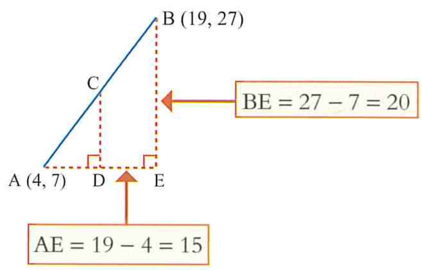


Figure 5.11

$$AC = \frac{2}{5}AB \quad \text{so} \quad AD = \frac{2}{5}AE$$
$$= \frac{2}{5}(15)$$
$$= 6$$

$$\text{Also} \quad CD = \frac{2}{5}BE$$
$$= \frac{2}{5}(20)$$
$$= 8$$

The x-coordinate of C is x-coordinate of A + AD i.e. $4 + 6 = 10$

The y-coordinate of C is y-coordinate of A + CD i.e. $7 + 8 = 15$

Point C is (10, 15).

This result can be generalised.

If C divides line AB in the ratio $p : q$ where A is (x_1, y_1) and B is (x_2, y_2) then

$$\text{C is } \left(\frac{qx_1 + px_2}{p + q}, \frac{qy_1 + py_2}{p + q}\right)$$

ACTIVITY 5.2

Show that the generalisation given on the right is true.
Use Figure 5.12. A is (x_1, y_1) and B is (x_2, y_2). C divides line AB in the ratio $p : q$.

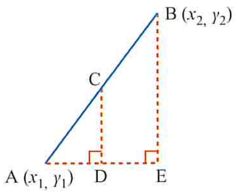


Figure 5.12

Example 5.8

AB is a straight line. A is (−3, 2) and B is (4, −19).

C is a point on AB such that AC : CB is 4 : 3

Work out the coordinates of C.

Solution

Taking (−3, 2) as (x_1, y_1) and (4, −19) as (x_2, y_2) and $p : q$ as 4 : 3

x-coordinate of C is $\dfrac{3(-3) + 4(4)}{3 + 4} = \dfrac{-9 + 16}{7}$

$= 1$

y-coordinate of C is $\dfrac{3(2) + 4(-19)}{3 + 4} = \dfrac{6 - 76}{7}$

$= -10$

C is (1, −10).

Example 5.9

In Figure 5.13, PQR is a straight line and PQ : QR is 2 : 5

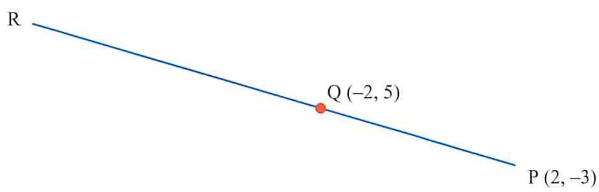


Figure 5.13

Work out the coordinates of R.

Solution

Take $(2, -3)$ as (x_1, y_1) and R as (x_2, y_2). To work out the x-coordinate

$$-2 = \frac{5(2) + 2(x_2)}{2 + 5}$$

$$-2 = \frac{10 + 2x_2}{7}$$

$$-14 = 10 + 2x_2 \quad \text{(multiplying by 7)}$$

$$-24 = 2x_2 \quad \text{(subtracting 10)}$$

$$-12 = x_2$$

To work out the y-coordinate

$$5 = \frac{5(-3) + 2(y_2)}{2 + 5}$$

$$5 = \frac{-15 + 2y_2}{7}$$

$$35 = -15 + 2y_2 \quad \text{(multiplying by 7)}$$

$$50 = 2y_2 \quad \text{(adding 15)}$$

$$25 = y_2$$

R is $(-12, 25)$.

Two shapes are said to be **similar** if corresponding sides are in the same ratio.

For example, the two triangles below are similar:

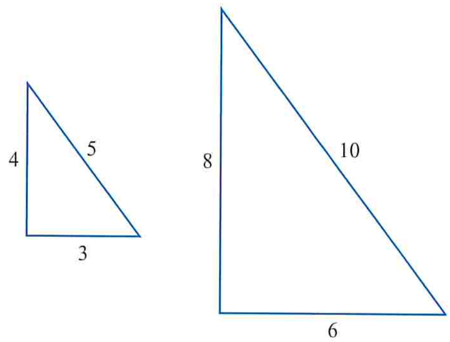


Figure 5.14

Discussion points

- What is the ratio of the lengths of the corresponding sides in Figure 5.14?
- What is the ratio of the areas of the two triangles

! Remember that when increasing or decreasing a length using a scale factor of n, corresponding areas will be increased using a scale factor of n^2.

Exercise 5D

① In each part, AB is a straight line and C is a point on AB. Work out the coordinates of C.

(i)	A is (8, 3)	B is (3, 18)	AC : CB is 3 : 2
(ii)	A is (12, −1)	B is (3, 5)	AC : CB is 1 : 2
(iii)	A is (−2, 4)	B is (14, −4)	AC : CB is 3 : 5
(iv)	A is (11, 9)	B is (−1, 19)	AC : CB is 4 : 1
(v)	A is (0, −6)	B is (−18, −15)	AC : CB is 5 : 4

② In each part, DEF is a straight line.

(i) D is (4, 3) E is (8, 5) DE : EF is 2 : 3 Work out the coordinates of F.

(ii) D is (19, −5) E is (7, 3) DE : EF is 4 : 3 Work out the coordinates of F.

(iii) E is (4, 9) F is (16, 33) DE : EF is 1 : 4 Work out the coordinates of D.

(iv) E is (2, −8) F is (7, −19) DE : EF is 3 : 5 Work out the coordinates of D.

(v) D is (−15, −8) E is (−3, −2) DE : EF is 6 : 5 Work out the coordinates of F.

③ ABC is a straight line. AB is 25% longer than BC.

(i) Work out the ratio AB : BC in its simplest form.

(ii) Work out the coordinates of C.

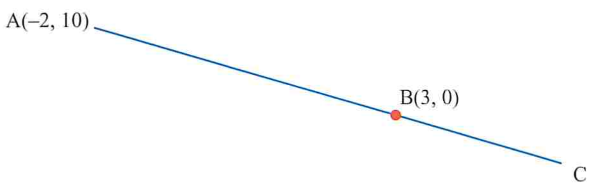


Figure 5.15

④ PRQ is a straight line.

PQ = 4PR

Work out the coordinates of R.

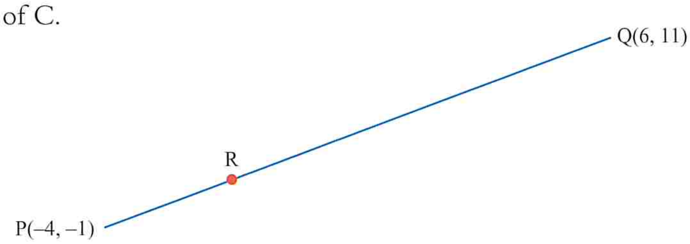


Figure 5.16

⑤ ABC is a straight line.

AC : BC = 8 : 5

Work out the coordinates of B.

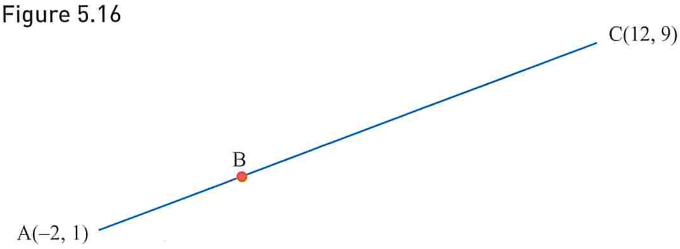


Figure 5.17

⑥ A very important application of ratios is when an architect is drawing a plan for a new building. This will require a scale drawing of the building together with a plan for each room. The plan is for the bedroom plus ensuite bathroom. His drawing uses a scale of 1 : 50.

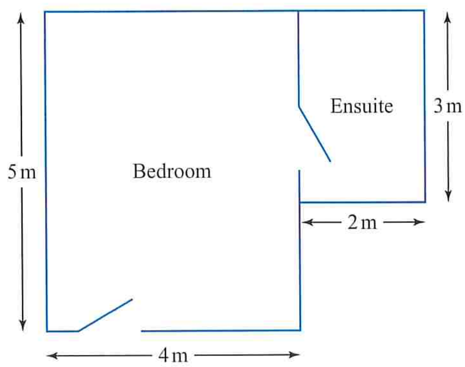


Figure 5.18

(i) What are the dimensions of the bedroom and the ensuite on the scale drawing.

(ii) What is the ratio of the area of the actual rooms to the area on the scale drawing?

7. The size of a computer monitor or TV screen is quoted by giving the length of the diagonal across the screen.
   (i) My computer monitor has a rectangular screen with dimensions 50 cm by 30 cm. To the nearest cm, what size would this be quoted as?
   (ii) My television has a 100 cm screen which is a rectangle with sides in the ratio 33 : 20. What is the height and width of the TV screen to the nearest cm?

8. Triangle ABC has a right angle at B and the lengths of the sides are shown in the diagram.
   A triangle A′B′C′ is inscribed inside ABC so that A′ is the midpoint of BC, B′ is the midpoint of CA and C′ is the midpoint of AB.
   (i) Calculate the length of AC.
   (ii) Show that triangle A′B′C′ is also a right-angled triangle and calculate the lengths of its sides.
   (iii) What do you notice about the ratio of the sides of the smaller triangle to the larger one?
   (iv) What do you notice about the ratio of the area of the smaller triangle to the larger one?

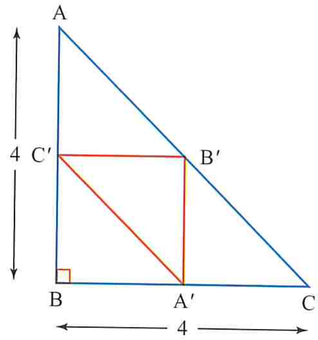


Figure 5.19

7 Equation of a circle

When you draw a circle, you open your compasses to a fixed distance (the radius) and choose a position (the centre) for the point of your compasses. These facts are used to derive the *equation* of the circle.

Circles with centre (0, 0)

Figure 5.20 shows a circle with centre O(0, 0) and radius 4. P(x, y) is a general point on the circle.

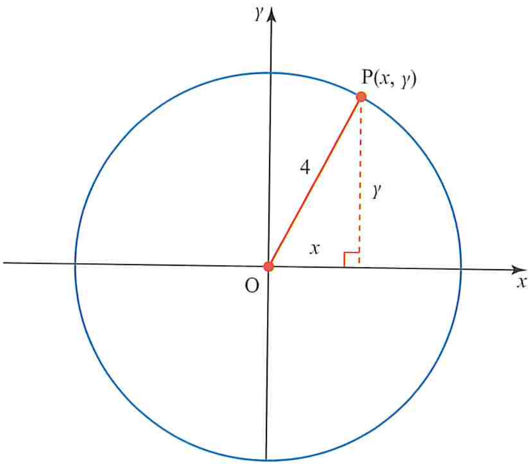


Figure 5.20

Using Pythagoras' theorem, OP $= \sqrt{x^2 + y^2} = 4$

This simplifies to $x^2 + y^2 = 16$, which is the equation of the circle.

This can be generalised. A circle with centre (0, 0), radius r has equation

$$x^2 + y^2 = r^2.$$

Circles with centre (*a*, *b*)

Figure 5.21 shows a circle with centre C(4, 5) and radius 3. P(x, y) is a general point on the circle.

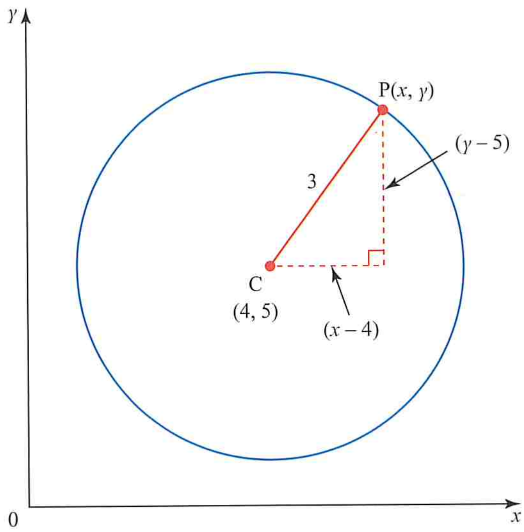


Figure 5.21

$$\text{CP} = \sqrt{(x-4)^2 + (y-5)^2} = 3$$

This simplifies to $(x-4)^2 + (y-5)^2 = 9$, which is the equation of the circle.

This can be generalised. A circle with centre (a, b), radius r has equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

Note

Multiplying out this equation gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2.$$

This rearranges to

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

This form of the equation highlights some of the important characteristics of the equation of a circle. In particular

- the coefficients of x^2 and y^2 are equal
- there is no xy term.

Example 5.10

Write down the centre and radius of the circle

$$x^2 + (y + 3)^2 = 25$$

Solution

Comparing with the general equation for a circle with radius r and centre (a, b),

$$(x - a)^2 + (y - b)^2 = r^2$$

gives $a = 0$, $b = -3$ and $r = 5$

$\Rightarrow$ the centre is $(0, -3)$, the radius is 5

Example 5.11

Figure 5.22 shows a circle with centre $(1, -2)$, which passes through the point $(4, 2)$.

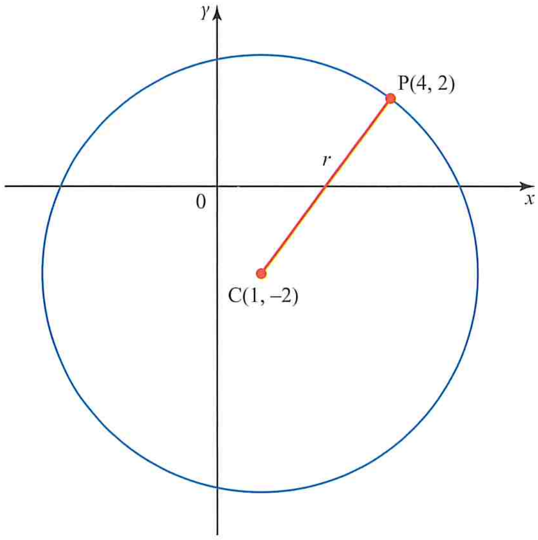


Figure 5.22

(i) Work out the radius of the circle.

(ii) Write down the equation of the circle.

Solution

(i) Use the two points you are given to work out the radius of the circle.

$$r^2 = (4 - 1)^2 + (2 - (-2))^2$$
$$= 25$$

$\Rightarrow$ radius $= 5$

(ii) Now using $(x - a)^2 + (y - b)^2 = r^2$

$\Rightarrow$ $(x - 1)^2 + (y + 2)^2 = 25$

is the equation of the circle.

Equation of a circle

Example 5.12

Show that the equation $x^2 + y^2 + 4x - 6y - 3 = 0$ represents a circle.

Hence give the coordinates of the centre and the radius of the circle.

Solution

Using completing the square

$$x^2 + 4x + y^2 - 6y = 3$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 - 6y + 9 = 3 + 4 + 9$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 3 + 4 + 9$$

$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 16$$

This represents a circle with centre $(-2, 3)$, radius 4

Exercise 5E

① Write down the equations of these circles.

(i) centre $(1, 2)$, radius 3
(ii) centre $(4, -3)$, radius 4
(iii) centre $(1, 0)$, radius 5
(iv) centre $(-2, -2)$, radius 2
(v) centre $(-4, 3)$, radius 1

② For each of the circles given below

(a) state the coordinates of the centre
(b) state the radius
(c) sketch the circle, paying particular attention to its position in relation to the origin and the coordinate axes.

(i) $x^2 + y^2 = 25$
(ii) $(x-3)^2 + y^2 = 9$
(iii) $(x+4)^2 + (y-3)^2 = 25$
(iv) $(x+1)^2 + (y+6)^2 = 36$
(v) $(x-4)^2 + (y-4)^2 = 16$

③ Work out the equation of the circle with centre $(2, -3)$ which passes through $(1, -1)$.

④ A and B are $(4, -4)$ and $(2, 6)$ respectively. Work out

(i) the midpoint C of AB
(ii) the distance AC
(iii) the equation of the circle that has AB as its diameter.

⑤ Show that the equation $x^2 + y^2 - 4x - 8y + 4 = 0$ represents a circle.

Hence give the coordinates of the centre and the radius of the circle, and sketch the circle.

⑥ Why does the equation $x^2 - 4x + y^2 - 4y + 22 = 0$ not represent a circle?

⑦ A circle of radius 5 cm passes through the points $(0, 0)$ and $(0, 6)$.

Sketch two possible positions of the circle and write down the equation in each case.

⑧ $(6, 3)$ is a point on the circle with centre $(11, 8)$.

(i) Work out the radius of the circle.
(ii) Work out the equation of the circle.
(iii) Work out the coordinates of the other point where the diameter through $(6, 3)$ meets the circle.

Prior knowledge

Students met a number of circle facts at GCSE and a reminder of some of these is given below. See section 6.4 on circle theorems.

Circle geometry facts

The angle in a semi-circle is 90°

An alternative way of expressing this is to say that the angle subtended by the diameter at any point on the circumference is 90°.

AB is a diameter.

P is a point on the circumference.

Angle APB = 90°.

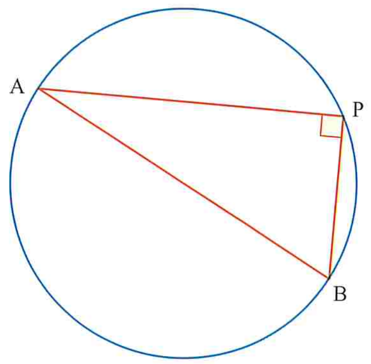


Figure 5.23

Discussion point

➜ Can you prove RM = MS by using congruent triangles?

The perpendicular from the centre to a chord bisects the chord

C is the centre.

RS is a chord.

M is the midpoint of RS.

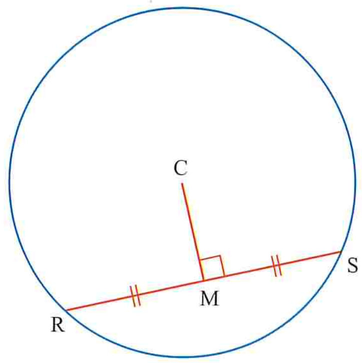


Figure 5.24

The angle between tangent and radius is 90°

C is the centre.

TQ is a tangent, touching the circle at Q.

QC is a radius.

Angle TQC = 90°.

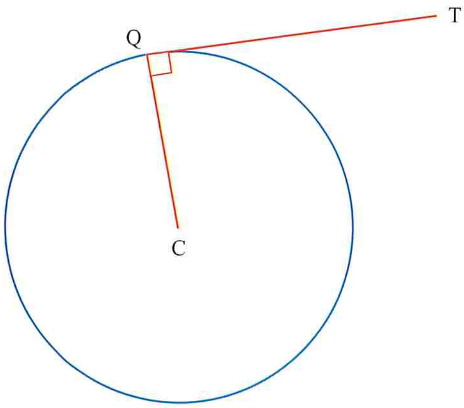


Figure 5.25

Two tangents from a point to a circle are equal in length

From any point outside a circle it is possible to draw two tangents to that circle.

C is the centre.

TA and TB are tangents, touching the circle at A and B.

AC and BC are radii.

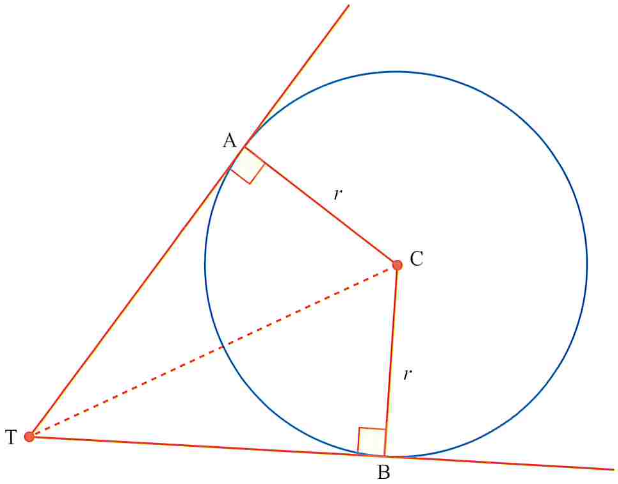


Figure 5.26

Discussion point

→ Can you prove TA = TB by using congruent triangles?

Example 5.13

The circle in Figure 5.27 has centre C.

PT is a tangent that touches the circle at P.

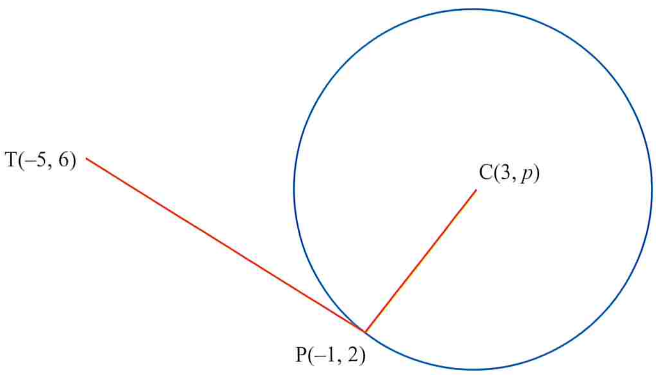


Figure 5.27

Work out the value of p.

Solution

Work out gradient of PT.

Gradient of PT is

$$\frac{2-6}{-1-(-5)} = \frac{-4}{-1+5}$$

$$= \frac{-4}{4}$$

$$= -1$$

Lines PT and PC are perpendicular since the angle between tangent and radius is 90°, so gradient of PC is 1.

$$\frac{p-2}{3-(-1)} = 1$$

$$\frac{p-2}{4} = 1$$

$$p - 2 = 4$$

$$p = 6$$

Example 5.14

A circle has centre C and passes through A(−2, 1) and B(4, 2).

Work out the equation of the line that is perpendicular to AB and passes through C.

Give your answer in the form $ax + by = c$.

Solution

Sketch the circle.

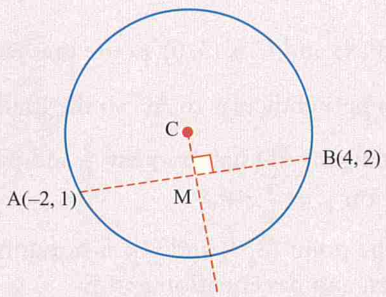


Figure 5.28

The perpendicular from the centre to a chord bisects the chord, so the required line will pass through the midpoint M of the chord AB.

Coordinates of M are $\left(\frac{-2+4}{2}, \frac{1+2}{2}\right) = \left(1, \frac{3}{2}\right)$

Gradient of AB is $\frac{2-1}{4-(-2)} = \frac{1}{6}$

Lines CM and AB are perpendicular.

Using $m_1 m_2 = -1$

Gradient of required line is −6.

Using $y - y_1 = m(x - x_1)$

$$y - \frac{3}{2} = -6(x - 1)$$

The question asks for the answer in the form $ax + by = c$.

$$y - \frac{3}{2} = -6x + 6$$

$$2y - 3 = -12x + 12$$

Multiply both sides by 2.

$$12x + 2y = 15$$

Example 5.15

(i) Using the fact that a tangent is perpendicular to the radius passing through the point of contact, work out the equation of the tangent to the circle $(x-3)^2 + y^2 = 25$ at the point A(0, 4).

(ii) Work out the coordinates of the point D where this tangent intersects the tangent to the circle through the point B(8, 0).

(iii) Show that these two tangents are equal in length.

Solution

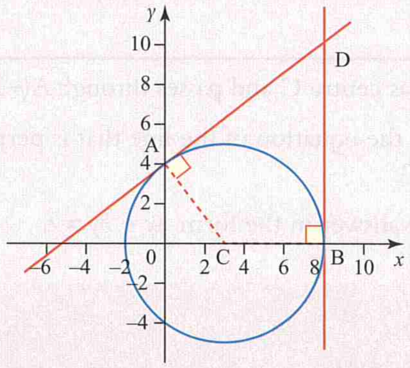


Figure 5.29

(i) A is (0, 4) and C is (3, 0) so the gradient of AC $= -\frac{4}{3}$.

AD is perpendicular to AC so the gradient of AD is $+\frac{3}{4}$.

The tangent AD has gradient $\frac{3}{4}$ and passes through (0, 4) so has equation $y = \frac{3}{4}x + 4$

(ii) B is the point (8, 0) and CB is horizontal, so the tangent at B is vertical and has equation $x = 8$

Solving $y = \frac{3}{4}x + 4$ and $x = 8$ simultaneously gives

$y = \frac{3}{4}(8) + 4 = 10$, so D is the point (8, 10).

(iii) A(0, 4) and D(8, 10) $\Rightarrow$ AD $= \sqrt{(8-0)^2 + (10-4)^2} = 10$

B(8, 0) and D(8, 10) $\Rightarrow$ BD = 10 so the two tangents are equal in length.

Exercise 5F

If a diagram is not given, drawing a sketch may help.

① AB is a diameter of a circle. P is a point on the circumference of the circle.
A is (2, 8) and P is (4, −2).
Work out the gradient of BP.

② A circle has centre C.
RS is a chord of the circle and R is (−1, 6).
Y(2, 3) is a point on RS such that angle CYR = 90°.
Work out the coordinates of S.

③ Look at Figure 5.30.

AB is a diameter of the circle.

A is $(k, 5)$, P is $(3, 8)$ and B is $(7, 2)$.

Work out the value of k.

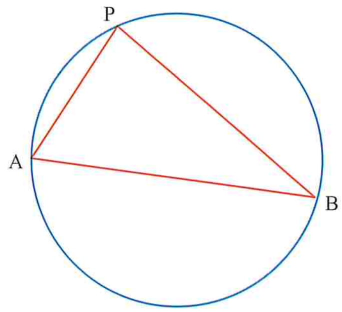


Figure 5.30

④ Figure 5.31 shows a circle, centre C.

AB is a chord of a circle.

D is a point on AB such that angle ADC is 90°.

Work out the equation of the line that passes through A and C.

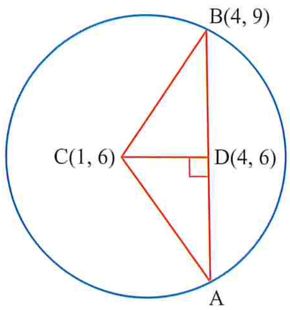


Figure 5.31

⑤ Look at Figure 5.32. T$(3, -4)$ is a point on the circumference of the circle $x^2 + y^2 = 25$

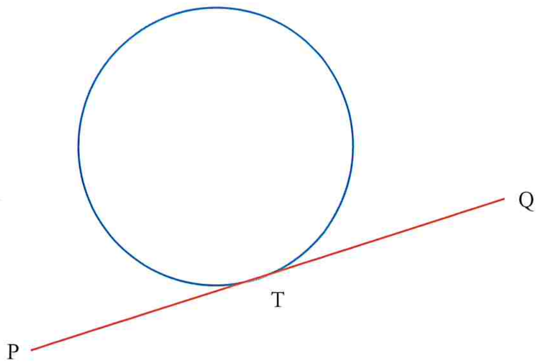


Figure 5.32

Work out the equation of the tangent PTQ.

Give your answer in the form $y = mx + c$.

⑥ Figure 5.33 shows a circle that intersects the x-axis at $(-2, 0)$ and $(6, 0)$.

The centre of the circle is $(a, 3)$.

Work out the equation of the circle.

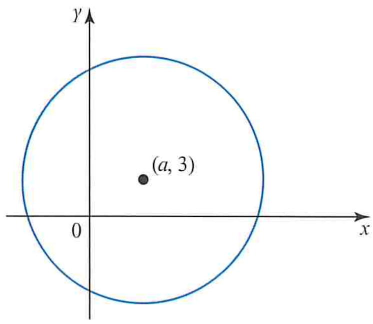


Figure 5.33

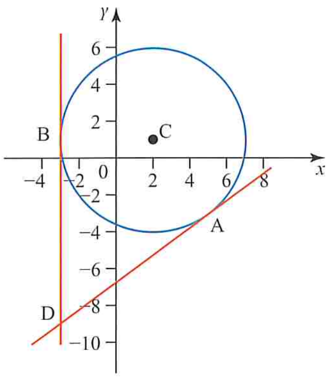


Figure 5.34

⑦ Figure 5.34 shows a circle with centre C(2, 1) and radius 5 units together with the tangent at the point A(5, –3).

(i) Write down the equation of the circle.

(ii) Work out the equation of the tangent and verify that the point D(–3, –9) lies on the tangent.

(iii) Write down the equation of the other tangent through the point D(–3, –9) and state the coordinates of the point B where this touches the circle.

(iv) Work out the lengths of the two tangents from D to the circle.

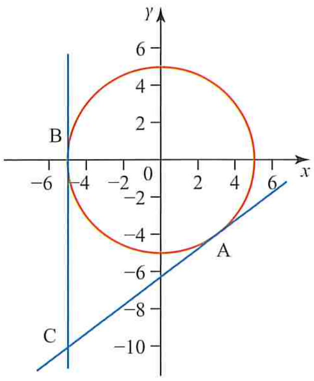


Figure 5.35

⑧ Figure 5.35 shows a circle $x^2 + y^2 = 25$ together with the tangent to the circle at the point A(3, –4).

(i) Show that the tangent at A has equation $3x - 4y - 25 = 0$

(ii) Write down the equation of the tangent to the circle at the point B(–5, 0).

(iii) Work out the coordinates of the point of intersection, C, of these tangents.

(iv) Show, by calculation, that CA = CB.

FUTURE USES

- At A-Level Maths and Further Maths you will do more work on circles and also study hyperbolas, parametric equations and polar coordinates.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- identify parallel and perpendicular lines
- work out the equation of a line given the gradient of a line parallel or perpendicular to it and at least one point that it passes through
- calculate the distance between two points
- work out the coordinates of the midpoint of a line joining two points
- recognise different forms for the equation of a straight line
- find the point of intersection of two lines
- divide a straight line in a given ratio
- recognise the equation of a circle
- write down the equation of a circle given the radius and the coordinates of the centre
- solve a variety of problems involving tangents to circles.

KEY POINTS

1 Two lines are parallel when their gradients are equal.
2 Two lines are perpendicular when the product of their gradients is -1.
3 When the points A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively then

distance AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

midpoint of AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

4 The coordinates of the point of intersection of two lines are found by solving their equations simultaneously.
5 If C divides line AB in the ratio $p:q$ where A is (x_1, y_1) and B is (x_2, y_2) then

C is $\left(\dfrac{qx_1 + px_2}{p + q}, \dfrac{qy_1 + py_2}{p + q}\right)$.

6 The equation of a circle with centre (h, k) and radius r is

$(x - h)^2 + (y - k)^2 = r^2$.

When the centre is at the origin $(0, 0)$ this simplifies to

$x^2 + y^2 = r^2$.

7 Circle facts:
- The angle in a semi-circle is $90°$.
- The perpendicular from the centre to a chord bisects the chord.
- The angle between tangent and radius is $90°$.
- The tangents to a circle from an external point are equal in length.

6 Geometry I

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds.
John Maynard Keynes

Much of the work in this chapter will already have been covered in your GCSE studies.

Knowledge of geometry topics will be needed within many sections of the specification.

This section provides a summary of the main facts that are required.

1 Mensuration

You need to recall these formulae.

Area of triangle $= \frac{1}{2} \times$ base $\times$ height

Area of parallelogram = base $\times$ height

Area of a trapezium $= \frac{1}{2} \times$ sum of parallel sides $\times$ distance between them

Circumference of circle $= \pi d = 2\pi r$ Area of circle $= \pi r^2$

Volume of prism = area of cross section $\times$ length

These formulae will be given in the examination in the relevant question.

Volume of pyramid $\frac{1}{3} \times$ base area $\times$ height

Volume of cone $= \frac{1}{3}\pi r^2 h$ Curved surface area of cone $= \pi r l$

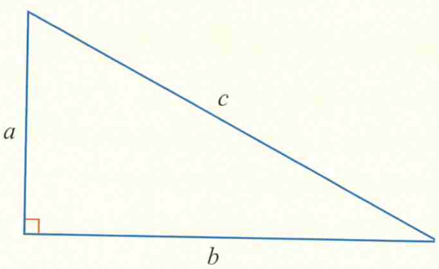
Volume of sphere $= \frac{4}{3}\pi r^3$ Surface area of sphere $= 4\pi r^2$

ACTIVITY 6.1

Write down the square of all the integers from 1 to 25 inclusive.
Check that $5^2 = 3^2 + 4^2$.
Write down as many other examples of $c^2 = a^2 + b^2$ as you can find.
How is each set of a, b and c linked to a right-angled triangle?

Prior knowledge

2 Pythagoras' theorem



$c^2 = a^2 + b^2$

Figure 6.1

Pythagorean triples

The following are all Pythagorean triples as each set of three numbers satisfies $c^2 = a^2 + b^2$.

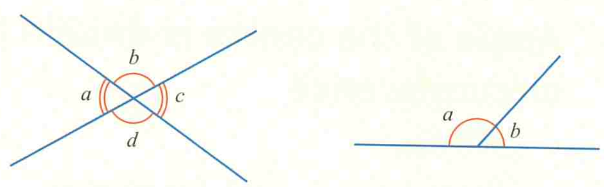
3, 4, 5 5, 12, 13 8, 15, 17 7, 24, 25

Using similar triangles, any multiple or fraction of each set will also be a Pythagorean triple.

For example, 9, 12, 15 2.5, 6, 6.5 16, 30, 34 1.4, 4.8, 5

3 Angle facts

- When two lines intersect, there are two angle facts to remember:



$a = c$
$b = d$

Vertically opposite angles are equal

$a + b = 180°$

Adjacent angles on a straight line add up to 180°

Figure 6.2

- Parallel lines have the same gradient (slope). When a third line intersects a pair of parallel lines, there are three angle properties relating to these that you need to remember:

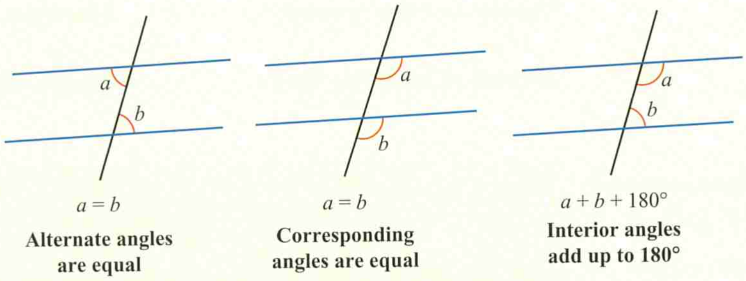


Figure 6.3

Angle properties of polygons

- Angle sum of a triangle is $180°$.
- Angle sum of a quadrilateral (4 sides) is $360°$.
- Angle sum of a pentagon (5 sides) is $540°$
- Angle sum of an n-sided polygon is $(n - 2) \times 180°$.

Special quadrilaterals

- A parallelogram has opposite angles equal.
- A rhombus is a parallelogram with all four sides of equal length.
- A trapezium is a quadrilateral with one pair of parallel sides.
- A kite has one line of symmetry and the diagonals intersect at $90°$.

Special polygons

- A regular polygon is a figure with all sides of equal length and all angles of equal size.

4 Circle theorems

The following angle properties should be known.

Angle at the centre is double the angle at the circumference

C is the centre.

A, B and D are points on the circumference.

Angle ACB = 2 × angle ADB.

Both angles must be subtended from the same arc.

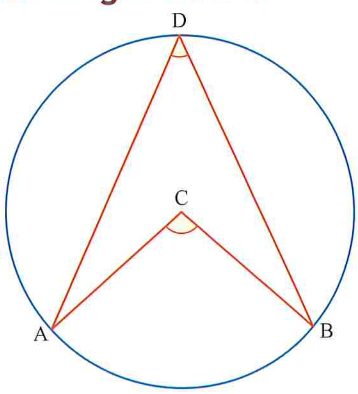


Figure 6.4

S is the centre.

P, Q and R are points on the circumference.

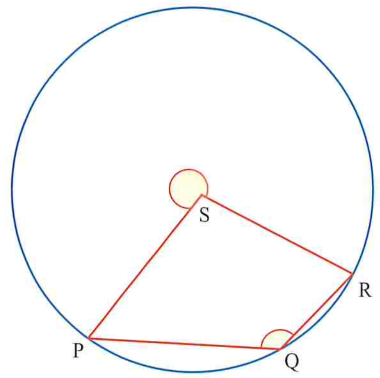


Figure 6.5

Reflex angle PSR = 2 × angle PQR.

Angle in a semi-circle = 90°

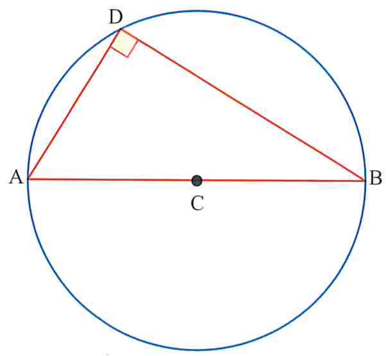


Figure 6.6

Angle ADB is referred to as the angle in a semi-circle and is always 90°.

Angles in the same segment are equal

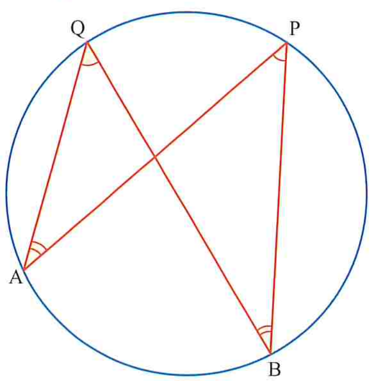


Figure 6.7

Can also be referred to as **Angles subtended by the same arc are equal**.

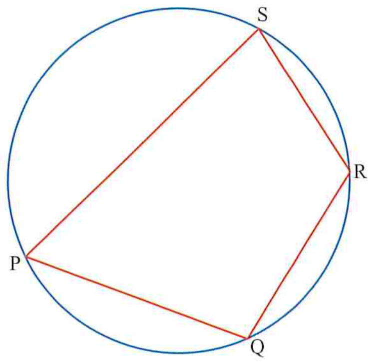
A, B, P and Q are points on the circumference.

Angle APB = angle AQB.

Also, angle QAP = angle QBP.

6 Chapter 6 Geometry I

Opposite angles of a cyclic quadrilateral add up to 180°



A cyclic quadrilateral has all four vertices on the circumference.

Figure 6.8

P, Q, R and S are points on the circumference.

Angle PQR + angle RSP = 180°.

Also, angle SPQ + angle QRS = 180°.

Alternate segment theorem

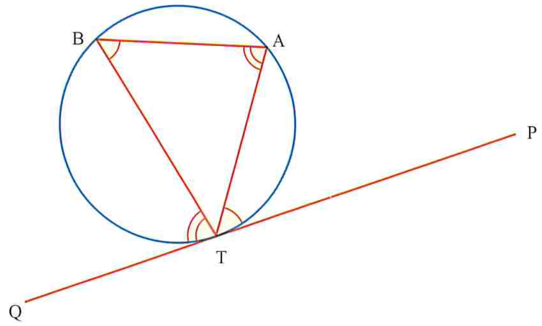


Figure 6.9

PTQ is a tangent, touching the circle at T.

A and B are points on the circumference.

The angle between the tangent PT and the chord TA is equal to the angle in the other segment.

Angle ATP = angle TBA.

Also, Angle QTB = Angle BAT.

The angle between the tangent QT and the chord TB is equal to the angle in the other segment.

Circle theorems and other angle facts will be needed in the Geometric proof section later in this chapter.

Exercise 6A

1. Work out angle x and angle y. C is the centre of the circle.

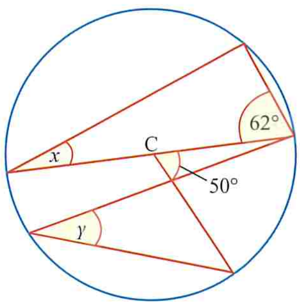


Figure 6.10

② Work out angle x. C is the centre of the circle.

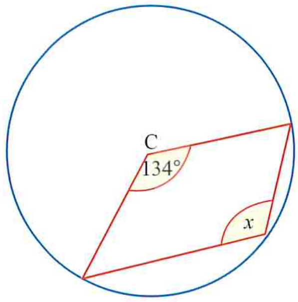


Figure 6.11

③ Work out angle x and angle y. QTP is a tangent.

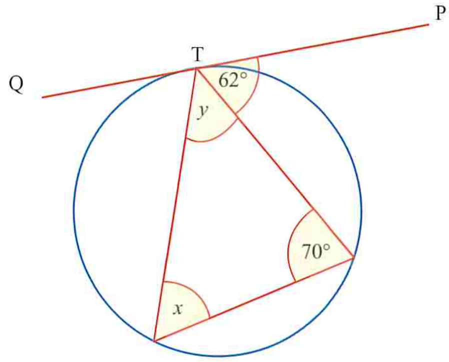


Figure 6.12

④ Work out angle x and angle y. C is the centre of the circle.

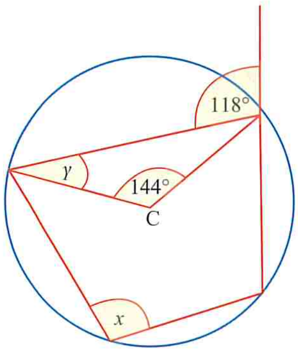


Figure 6.13

⑤ Work out angle x. C is the centre of the circle.

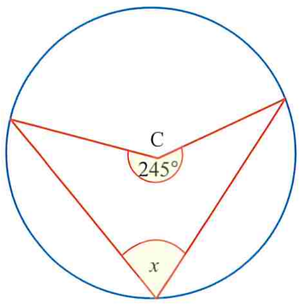


Figure 6.14

6

Chapter 6 Geometry I

⑥ PTQ is a tangent. Work out angle x.

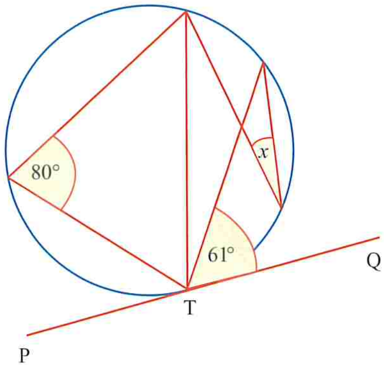


Figure 6.15

⑦ Work out angle x. AB is a diameter.

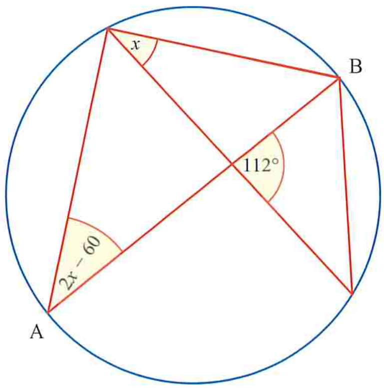


Figure 6.16

⑧ Work out angle c. AB is a tangent.

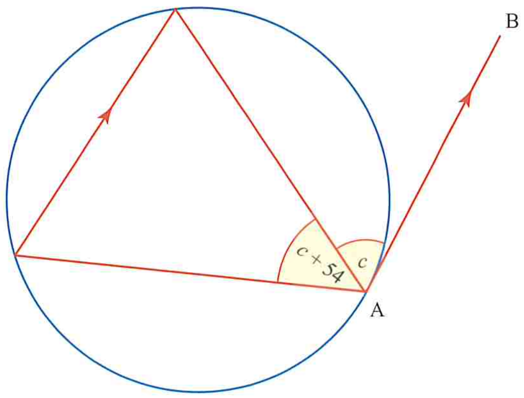


Figure 6.17

⑨ Work out x and y.

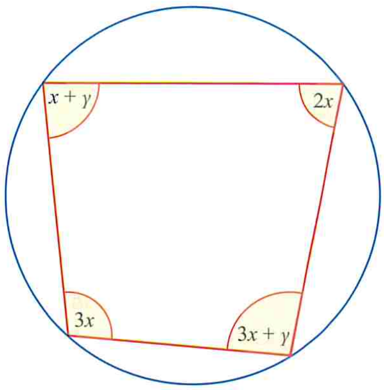


Figure 6.18

5 Geometric proof

In this section you will need to construct formal proofs.

The geometrical properties used must be stated using correct notation and vocabulary.

Example 6.1

In triangle BCD, BC = BD.

ABC is a straight line.

Prove that angle ABD = $2x$.

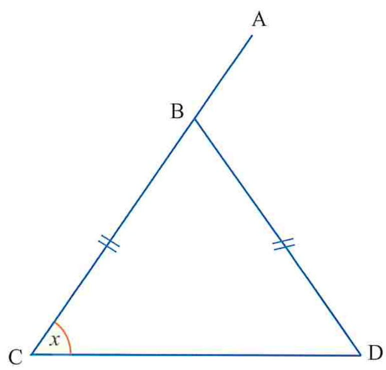


Figure 6.19

Solution

Abbreviations may be used but the reasons must be unambiguous.

Method 1

angle CDB = x (base angles of isosceles triangle)

angle CBD = $180 - 2x$ (angle sum of triangle)

angle ABD = $2x$ (adjacent angles on a straight line)

Method 2

angle CDB = x (base angles of isosceles triangle)

angle ABD = $2x$ (exterior angle of triangle = sum of interior opposite angles)

There will often be more than one method in a geometric proof. You only need to provide one.

Example 6.2

AP is a tangent that touches the circle at P.

AP is parallel to QR.

Prove that triangle PQR is isosceles.

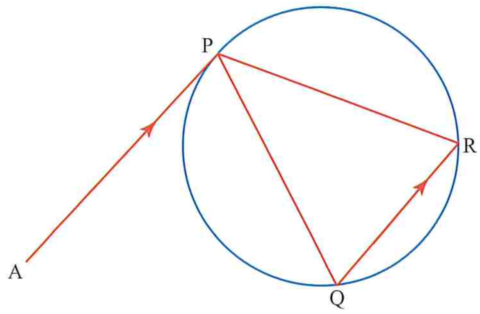


Figure 6.20

Solution

Method 1

angle APQ = angle PQR (alternate angles)

angle APQ = angle PRQ (alternate segment theorem)

Therefore, angle PQR = angle PRQ

Triangle with two equal angles is isosceles.

Method 2

This is very similar to method 1 but starts by introducing a lower case letter.

angle APQ = x

angle PQR = x (alternate angles)

angle PRQ = x (alternate segment theorem)

Therefore, angle PQR = angle PRQ

Triangle with two equal angles is isosceles.

Example 6.3

PQRS is a cyclic quadrilateral.

C is the centre.

Angle QPS = y

Angle QCR = $2x$

Angle SQR = 40°

Prove that $y = x + 40$

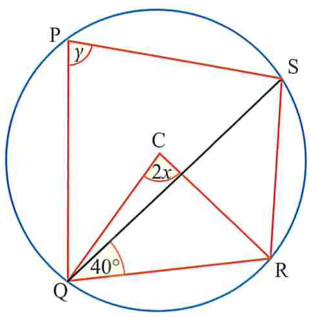


Figure 6.21

Note

The use of congruent triangles will not be required.

The examples show the recommended way to present a proof.

Solution

angle QSR = x (angle at circumference is half angle at centre)

angle SRQ = $180 - y$ (opposite angles of cyclic quadrilateral)

In triangle QRS, $x + 180 - y + 40 = 180$ (angle sum of triangle)

Rearranging $x + 40 = y$

Exercise 6B

① AC is a diameter. B is a point on the circumference.

Prove that $x = 90 - y$.

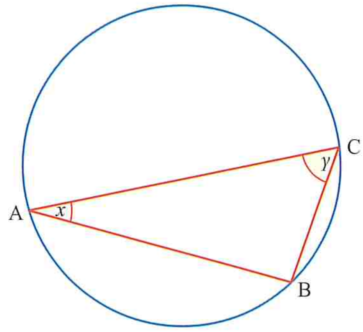


Figure 6.22

② ABCD is parallel to EFG.

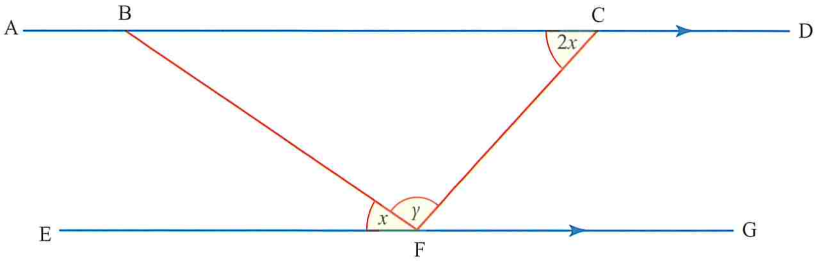


Figure 6.23

Prove that $3x + y = 180$.

③ AB is a diameter. X and Y are points on the circumference.

Prove that $a + b = 90$

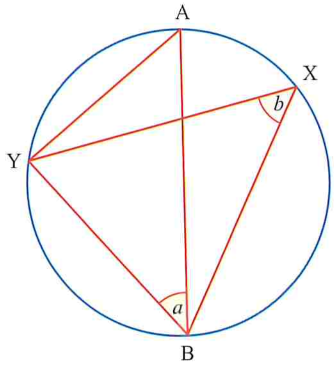


Figure 6.24

④ CBE and DBF are straight lines.

CD is parallel to AB.

BC = BD

Prove that angle ABC = angle ABF.

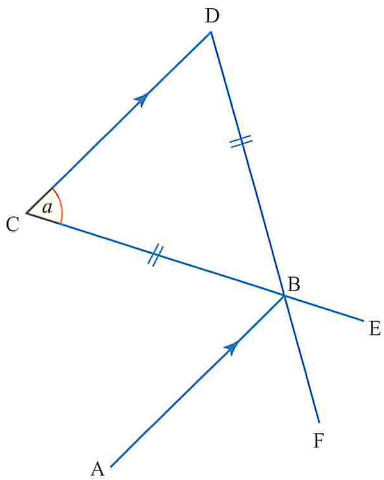


Figure 6.25

⑤ PT is a tangent, touching the circle at T. C is the centre.

M and N are points on the circumference.

Prove that angle TMN = $45 - y$.

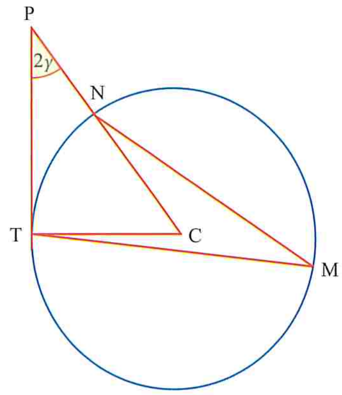


Figure 6.26

⑥ AB is a tangent, touching the circle at B.

ADC is a straight line.

AB = BC

Prove that triangle ABD is isosceles.

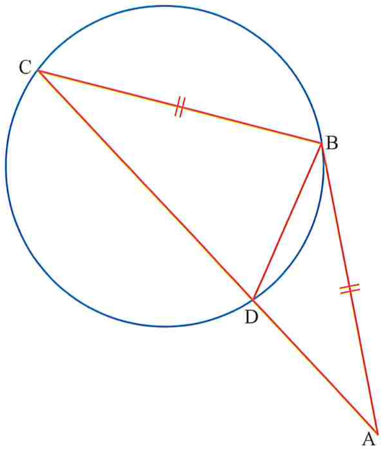


Figure 6.27

⑦ DEFG is a cyclic quadrilateral. C is the centre.

Prove that $x = 2y - 80$

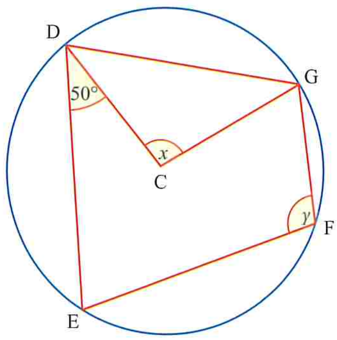


Figure 6.28

⑧ C is the centre.

P, Q and R are points on the circumference with PQ = QR.

Prove that $y = 2x$.

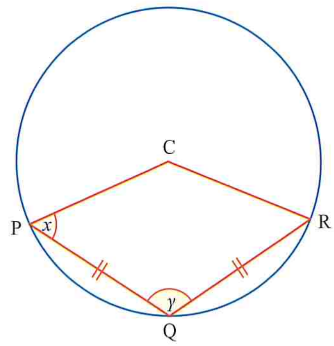


Figure 6.29

6 Trigonometry in two dimensions

You have met definitions of the three trigonometric functions, sin, cos and tan, using the sides of a right-angled triangle.

sin is an abbreviation of sine, cos of cosine and tan of tangent.

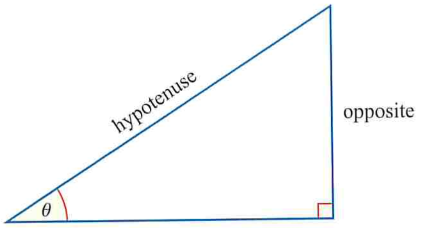


Figure 6.30

Discussion point

➜ Do these definitions work for angles of any size?

In Figure 6.30

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

ACTIVITY 6.2

(i) Using only a pencil, ruler and protractor, estimate $\sin 62°$.

(ii) Use your calculator to check your percentage error.

(iii) Suggest a way of reducing the percentage error when using this method.

Example 6.4

Work out the length of the side marked a in the triangle in Figure 6.31.

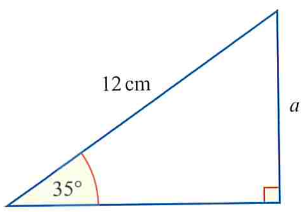


Figure 6.31

Solution

Side a is *opposite* the angle of 35°, and the *hypotenuse* is 12 cm, so we use sin 35°.

$$\sin 35° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{a}{12}$$

$$\Rightarrow \quad a = 12 \sin 35°$$

$$\Rightarrow \quad a = 6.9\,\text{cm (1 d.p.)}$$

Example 6.5

RWC

The diagram represents a ladder leaning against a wall.

Work out the length of the ladder.

Give your answer to 3 significant figures.

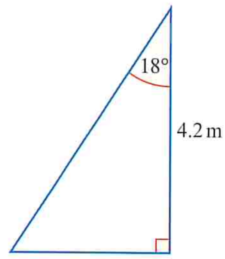


Figure 6.32

Solution

The side of length 4.2 m is *adjacent* to the angle of 18°, and we want the *hypotenuse* so use cos 18°.

$$\begin{aligned}\cos 18° &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ &= \frac{4.2}{\text{hypotenuse}}\\ \text{hypotenuse} &= \frac{4.2}{\cos 18°}\\ &= 4.42\,\text{m (3 s.f.)}\end{aligned}$$

Example 6.6

Work out the size of the angle marked θ in the triangle in Figure 6.33.

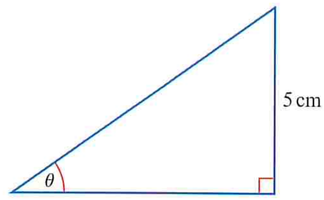


Figure 6.33

Solution

The sides whose lengths are known are those *opposite* and *adjacent* to θ so we use tan θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{7}$$

$$\Rightarrow \qquad \theta = 35.5°\ (1\text{ d.p.})$$

Discussion point

➜ The full calculator value for $\frac{5}{7}$ has been used to work out the value of θ.

What is the least number of decimal places that you could use to give the same value for the angle (to 1d.p.) in this example?

Example 6.7

RWC

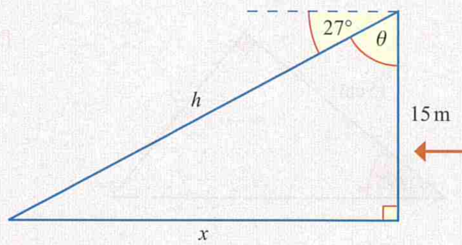
A bird flies straight from the top of a 15 m tall tree, at an angle of depression of 27°, to catch a worm on the ground.

(i) How far does the bird fly?

(ii) How far was the worm from the bottom of the tree?

Solution

First draw a sketch, labelling the information given and using letters to mark what you want to find.



Remember, *angles of depression* are measured down from the horizontal and angles of elevation are measured up from the horizontal.

Figure 6.34

(i) $\theta + 27° = 90°$

$\Rightarrow \quad \theta = 63°$

$$\cos 63° = \frac{15}{h}$$

$$\Rightarrow \quad h = \frac{15}{\cos 63°} = 33.040\,338\,97$$

The bird flies 33 m.

! Make sure that you record the full calculator value of h for future use.

(ii) Using Pythagoras' theorem

$$h^2 = x^2 + 15^2$$

$$\Rightarrow \quad x^2 = 33.040\,338\,97^2 - 15^2 = 866.663\,999$$

$$\Rightarrow \quad x = 29.439\,157\,58$$

The worm is 29.4 m from the bottom of the tree.

Note

Questions involving right-angled triangles will often entail applying trigonometry in a context. Examples and exercises include some questions without a context to provide practice of the skills needed in applications questions.

Discussion point

→ If you used trigonometry for part (ii) of this question, which would be the best function to use? Why?

Historical note

The word for trigonometry is derived from three Greek words.

Tria: *three* gonia: *angle* metron: *measure*

(τρια) (γονια) (μετρον)

This shows how trigonometry developed from studying angles, often in connection with astronomy, although the subject was probably discovered independently by a number of people. Hipparchus (150 BC) is believed to have produced the first trigonometric tables which gave lengths of chords of a circle of unit radius. His work was further developed by Ptolemy in AD100.

Exercise 6C

① Work out the length marked x in each of these triangles. Give your answers correct to 1 decimal place.

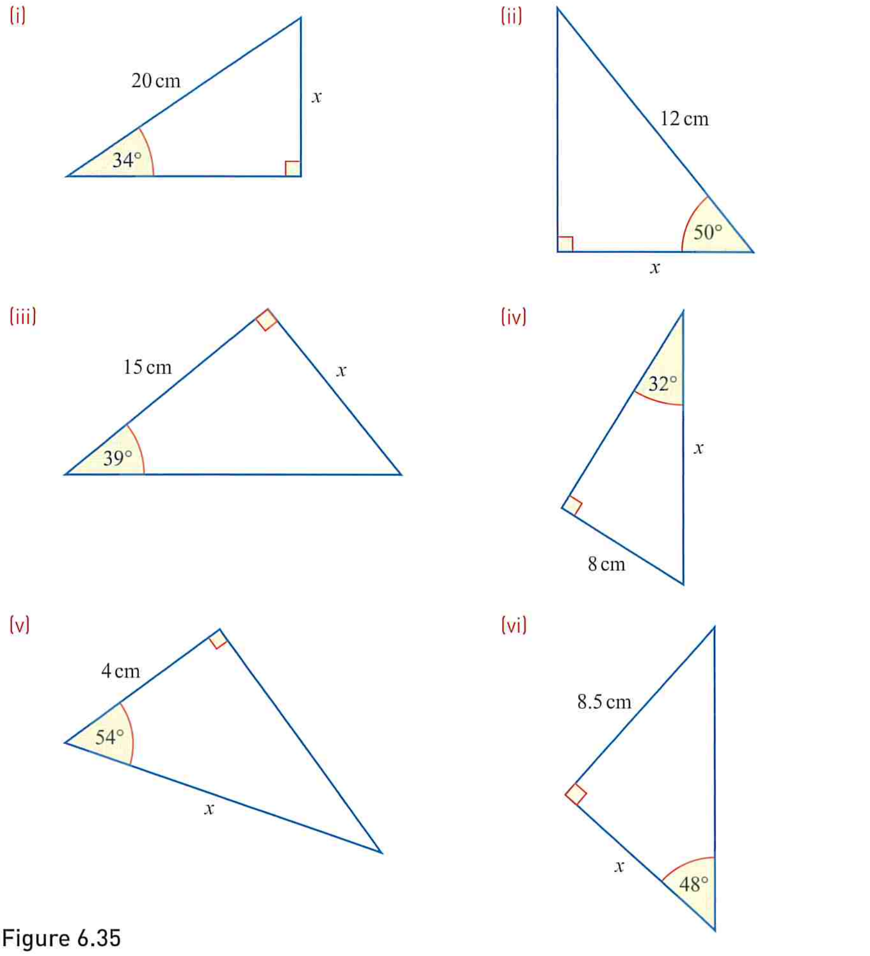
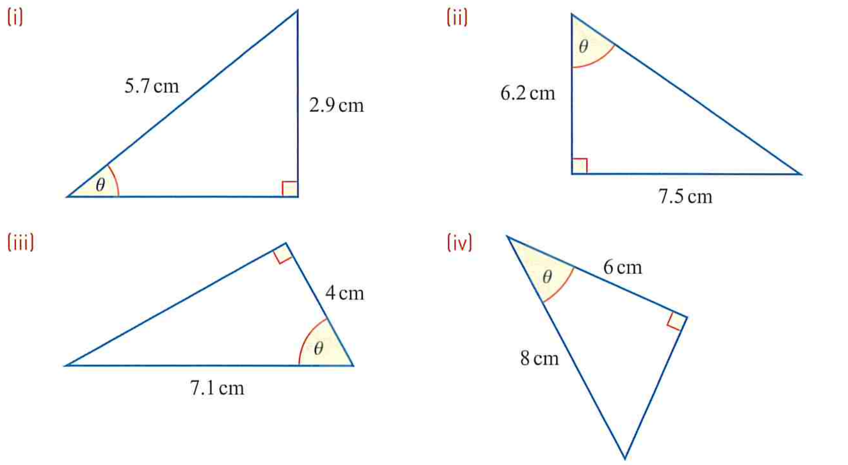
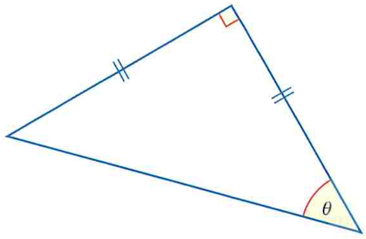


Figure 6.35

② Work out the size of the angle marked θ in each of these triangles. Give your answers correct to 1 decimal place.



(v)



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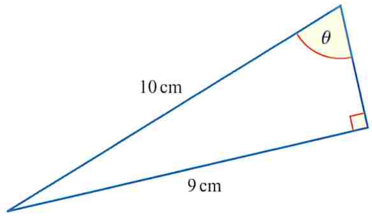
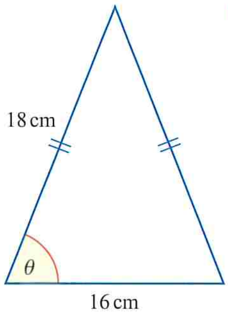


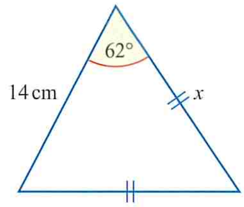
Figure 6.36

③ In an isosceles triangle, the line of symmetry bisects the base of the triangle. Use this fact to work out the angle θ and the lengths x and y in these diagrams.

(i)



(ii)



(iii)

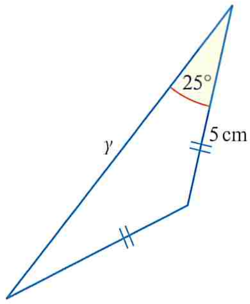


Figure 6.37

RWC ④ A ladder 5 m long rests against a wall. The foot of the ladder makes an angle of 65° with the ground.

How far up the wall does the ladder reach?

RWC ⑤ From the top of a vertical cliff 30 m high, the angle of depression of a boat at sea is 21°.

How far is the boat from the bottom of the cliff?

RWC ⑥ From a point 120 m from the base of an office block, the angle of elevation of the top of the block is 67°.

How tall is the block?

⑦ A rectangle has sides of length 12 cm and 8 cm.

What angle does the diagonal make with the longest side?

RWC ⑧ The diagram shows the positions of three airports:

E (East Midlands), M (Manchester) and L (Leeds).

The distance from M to L is 65 km on a bearing of 060°.

Angle LME = 90° and ME = 100 km.

(i) Calculate, correct to 3 significant figures, the distance LE.

(ii) Calculate, correct to the nearest degree, the size of angle MEL.

(iii) An aircraft leaves M at 10.45 am and flies direct to E, arriving at 11.03 am. Calculate, correct to 3 significant figures, the average speed of the aircraft in kilometres per hour.

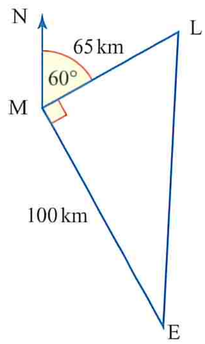


Figure 6.38

Angles of 45°, 30° and 60°

The sine, cosine and tangent of these angles have exact values.

When working without a calculator, the exact values should be known or derived.

Consider an isosceles right-angled triangle with AB = BC = 1 unit.

Using Pythagoras' theorem

$$AC^2 = 1^2 + 1^2$$

$$AC = \sqrt{2}$$

$$\sin 45° = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = 1$$

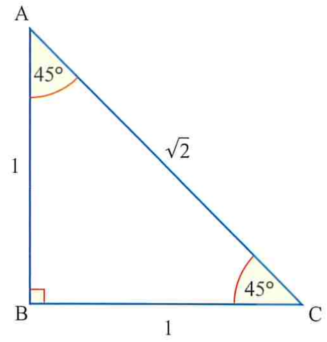


Figure 6.39

Discussion point

➜ What would the results be if you used AB = BC = 2 units?

Consider an equilateral triangle of side length 2 (Figure 6.40(a)).

By adding an angle bisector we get two congruent triangles (Figure 6.40(b)).

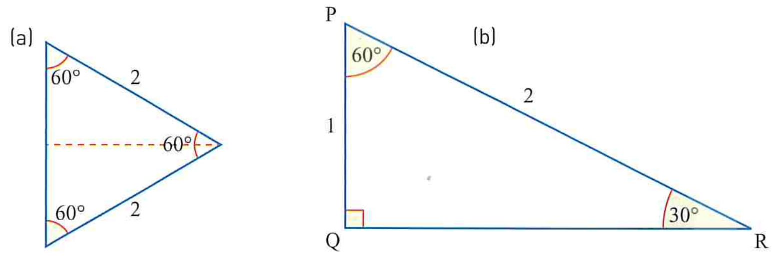


Figure 6.40

Using Pythagoras' theorem

$$QR^2 = 2^2 - 1^2$$

$$QR = \sqrt{3}$$

Using the trig ratios this gives us

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

Example 6.8

Do not use a calculator for this question.

Work out the exact value of y.

Give your answer in the form $p + q\sqrt{3}$ where p and q are integers.

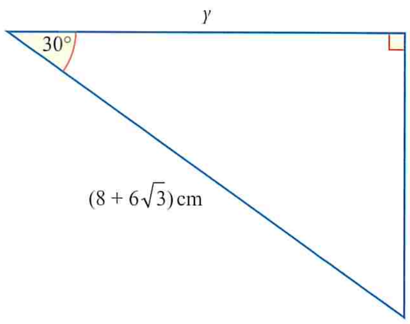


Figure 6.41

Solution

$$\cos 30° = \frac{y}{8+6\sqrt{3}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{y}{8+6\sqrt{3}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} \times (8+6\sqrt{3}) = y$$

$$\Rightarrow \quad 4\sqrt{3}+9 = y$$

$$y = 9+4\sqrt{3}$$

Exercise 6D

Use of a calculator is not allowed.

① Work out the exact value of x in each of the following.
Give answers in their simplest form.

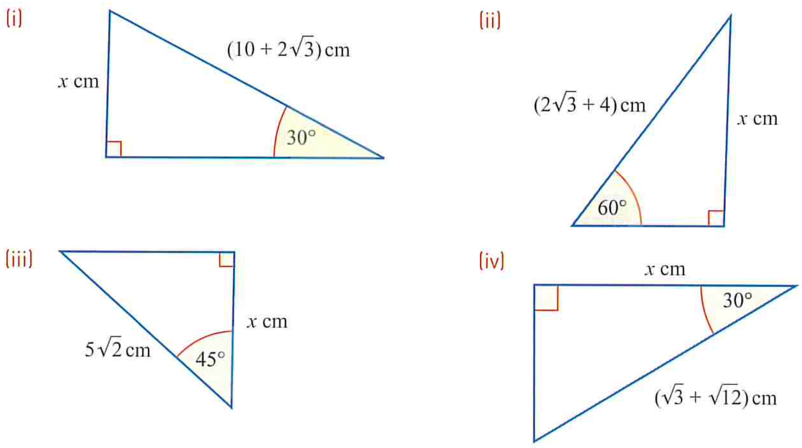


Figure 6.42

② Look at Figure 6.43. Show that y is an integer.

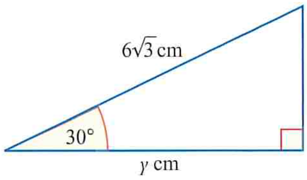


Figure 6.43

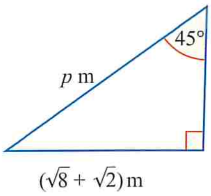


Figure 6.44

③ Look at Figure 6.44. Show that p is an integer.

④ Look at Figure 6.45.

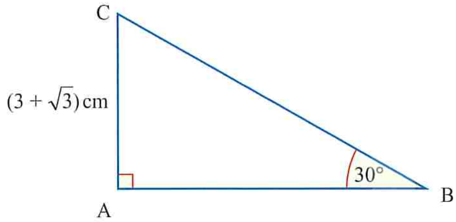


Figure 6.45

Work out the area of triangle ABC.

Give your answer in the form $p + q\sqrt{3}$ where p and q are integers.

⑤ Look at Figure 6.46. Work out the exact value of CD.

Give your answer in the form $k\sqrt{6}$ where k is an integer.

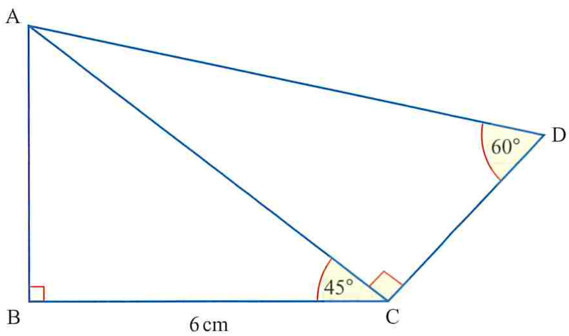


Figure 6.46

RWC ⑥ A ski-lift is spanning a valley in the Alps, rising from a height of 2039 m to a height of 2364 m over a horizontal distance of 325 m. What is the angle of elevation of the ski lift?

RWC ⑦ The centrepiece of a show garden has been designed as a square of side 3 metres surrounded by four equilateral triangles as shown in Figure 6.47.

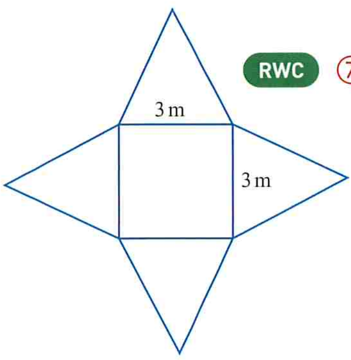


Figure 6.47

(i) The centre square is to be planted with small shrubs which each require a square area of side 30 cm. How many shrubs are required?

(ii) The triangular areas are to be planted with bedding plants, each requiring an area of approximately 100 cm^2. Approximately how many bedding plants will be required?

(iii) The bedding plants are sold in boxes of 12 and the head gardener decides to order 5% extra plants to allow for ones which might not be up to standard. How many boxes does he need to order?

RWC ⑧ Figure 6.48 shows a vertical building standing on horizontal ground. The points A, B and C are in a straight line on horizontal ground and AC = 30 m. The point T is at the top of the building and CT is vertical. The angles of elevation of T from A and B are 30° and 60° respectively.

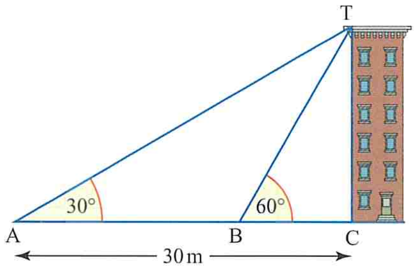


Figure 6.48

(i) Calculate the exact value of the height CT of the building.

(ii) Work out the distances BC and AB.

7 Trigonometric functions for angles of any size

By convention, angles are measured anticlockwise from the positive x-axis (Figure 6.49). Anticlockwise is taken to be positive and clockwise to be negative.

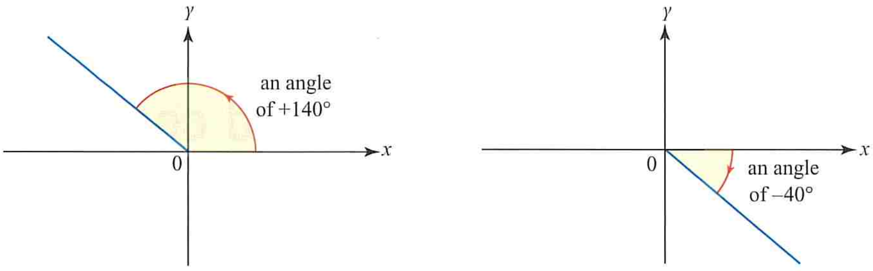


Figure 6.49

The only exception is for compass bearings, which are measured clockwise from the north.

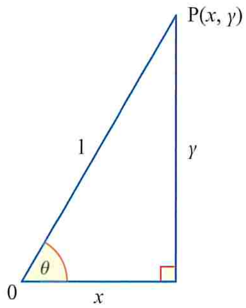


Figure 6.50

Definitions of the trigonometric functions, sin, cos and tan

First look at the right-angled triangle in Figure 6.50 which has a hypotenuse of unit length.

Sin, cos and tan are defined as the following ratios.

$$\sin\theta = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{1} = x \qquad \tan\theta = \frac{y}{x}$$

We can extend these definitions to angles beyond 90°.

Imagine the angle θ situated at the origin, as in Figure 6.51, and allow θ to take any value. The vertex marked P has coordinates $(\cos\theta, \sin\theta)$ and can now be anywhere on the unit circle.

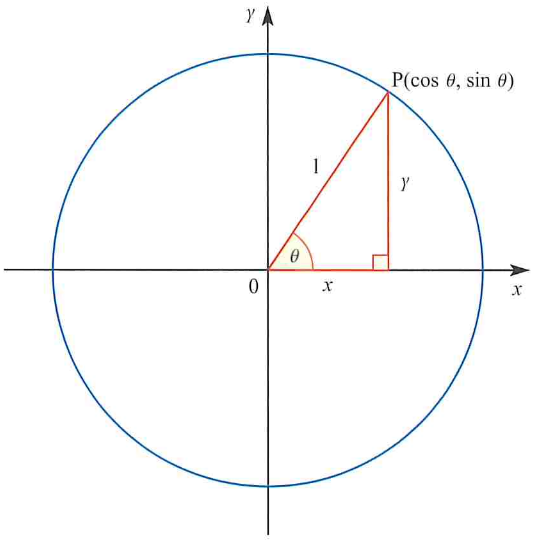


Figure 6.51

You can now see that these definitions can be applied to *any* angle θ, whether it is positive or negative, and whether it is less than or greater than 90°.

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x}$$

For some angles, x or y (or both) will take a negative value, so the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ will vary accordingly.

8 The sine and cosine graphs

Look at Figure 6.52. There is a unit circle and angles have been drawn at intervals of 30°. The resulting y-coordinates are plotted relative to the axes on the right. They have been joined with a continuous curve to give the graph of $\sin\theta$ for $0 \leqslant \theta \leqslant 360°$.

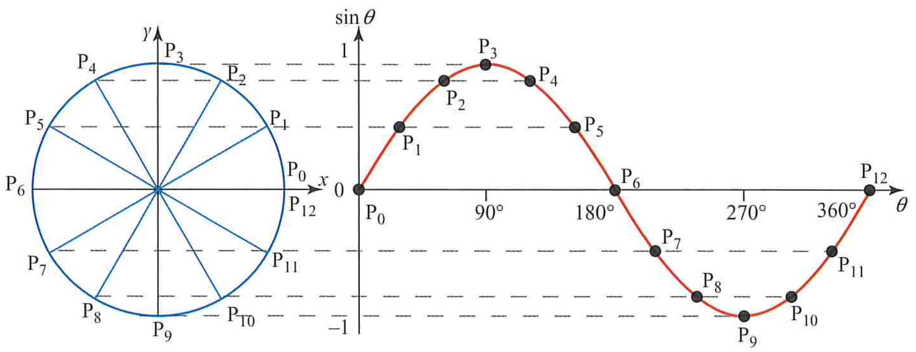


Figure 6.52

Continuing this process for angles 390°, 420°, ... and angles −30°, −60°, ... you get the graph of $y = \sin\theta$.

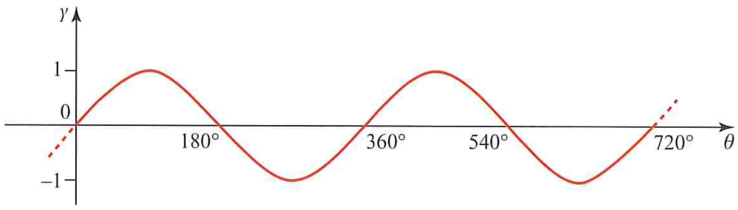


Figure 6.53

Since the curve repeats itself every 360°, as shown in Figure 6.53, the sine function is described as *periodic* with *period* 360°.

In a similar way you can transfer the x-coordinates onto a set of axes to obtain the cosine graph. This is most easily illustrated if you first rotate the circle through 90° anticlockwise.

Figure 6.54 shows this new orientation, together with the resulting graph.

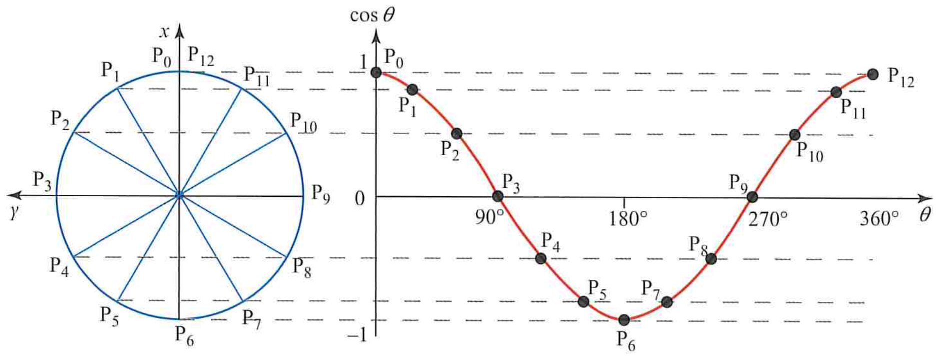


Figure 6.54

For angles beyond this interval the cosine graph repeats itself periodically, with a period of 360°.

Notice that the graphs of $\sin\theta$ and $\cos\theta$ have exactly the same shape. The cosine graph can be obtained by translating the sine graph 90° to the left, as shown in Figure 6.55.

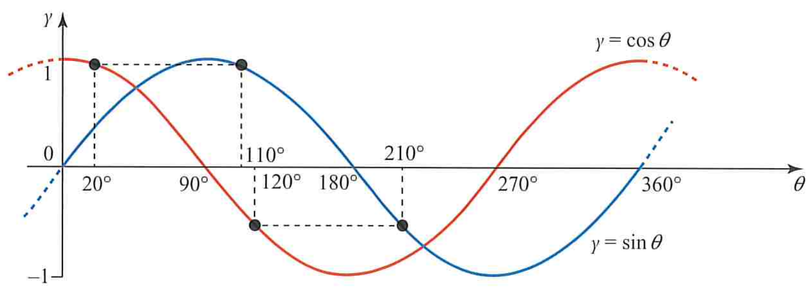


Figure 6.55

9 The tangent graph

The value of $\tan\theta$ can be worked out from the definition $\tan\theta = \frac{y}{x}$ or by using $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Discussion points

- The function $\tan\theta$ is undefined for $\theta = 90°$. What does *undefined* mean?
- How can you tell that $\tan 90°$ is undefined?
- For which **other** values of θ is $\tan\theta$ undefined?

The graph of $\tan\theta$ is shown in Figure 6.56. The dotted lines $\theta = \pm 90°$ and $\theta = 270°$ are *asymptotes*; they are not actually part of the curve.

Discussion point

- How would you describe an asymptote to a friend?

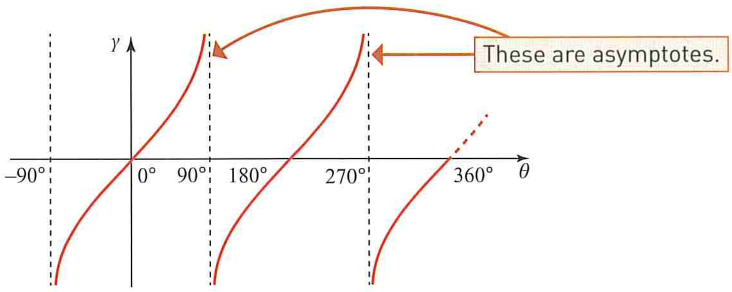


Figure 6.56

Note

It is important to learn the graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$.

Discussion points

- The graph of $\tan\theta$ is periodic, like those for $\sin\theta$ and $\cos\theta$. What is the period of this graph?
- Show how the part of the curve for $0° \leqslant \theta \leqslant 90°$ can be used to generate the rest of the curve using rotations and translations.

10 Solution of trigonometric equations

Suppose that you want to solve the equation

$$\sin\theta = 0.5$$

You start by pressing the calculator keys

shift | sin | 0 | . | 5 | =

and the answer comes up as 30

Note

The $\sin^{-1}$ key may also be labelled invsin or arcsin.

If your calculator does not give the answer 30 then it might be in the wrong angle setting. Check for a D (or DEG) at the top of the screen. If not, then select DEG whilst in SETUP mode.

However, look at the graph of $y = \sin\theta$ (Figure 6.57). You can see that there are other roots as well.

Discussion point

→ How many roots does the equation have?

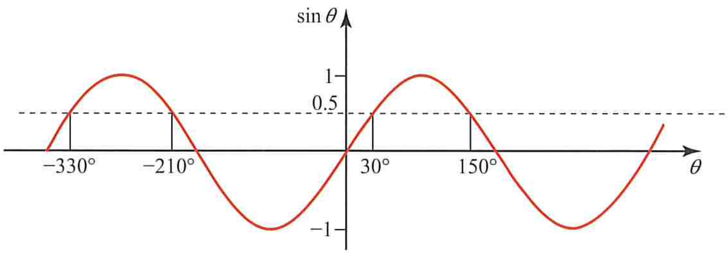


Figure 6.57

The root 30° is called the *principal value*.

Other roots can be found by looking at the graph. The roots for $\sin\theta = 0.5$ are seen to be:

$$\theta = \ldots, -330°, -210°, 30°, 150°, \ldots$$

As the graph is periodic, then the roots repeat every 360°.

Note

A calculator always gives the principal value of the solution. These values are in the range

$0° \leqslant \theta \leqslant 180°$ (cos) $\quad -90° \leqslant \theta \leqslant 90°$ (sin) $\quad -90° < \theta < 90°$ (tan)

Example 6.9

Work out the values of θ in the interval $0° \leqslant \theta \leqslant 360°$ for which $\cos\theta = 0.4$

Solution

$\cos\theta = 0.4 \quad \Rightarrow \quad \theta = 66.4°$ (principal value)

Figure 6.58 shows the graph of $y = \cos\theta$.

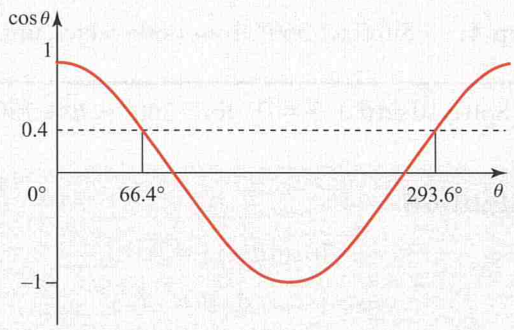


Figure 6.58

The values of θ for which $\cos\theta = 0.4$ are 66.4°, 293.6°

Discussion points

→ How do we get 293.6° from 66.4°?

→ Is there a general rule for finding a second angle between $0° \leqslant \theta \leqslant 360°$?

Example 6.10

Work out the values of x in the interval $-360° \leqslant x \leqslant 360°$ for which $6 + 2\tan x = 0$

Solution

$$6 + 2\tan x = 0$$

$$\Rightarrow \quad 2\tan x = -6$$

$$\Rightarrow \quad \tan x = -3$$

$$\Rightarrow \quad x = -71.6° \quad \text{(principal value)}$$

Figure 6.59 shows the graph of $y = \tan x$.

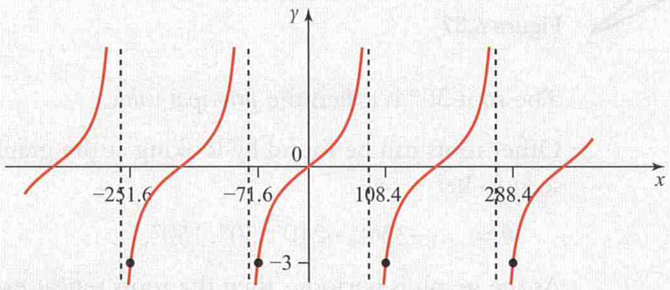


Figure 6.59

The values of x for which $\tan x = -3$ are

$-251.6°, -71.6°, 108.4°, 288.4°$

Short method for solving trigonometric equations for any angle:

Step 1: Use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ to find the principal value, θ.

Step 2: Work out a second angle using one of the following

$\sin^{-1}$:	$180° - \theta$
$\cos^{-1}$:	$360° - \theta$
$\tan^{-1}$:	$\theta + 180°$

Step 3: Add 360° to both values, until the upper limit is reached.

Step 4: Subtract 360° from both values, until the lower limit is reached.

Example 6.11

Solve $10\sin\theta + 3 = 0$ for $-360° \leqslant \theta \leqslant 720°$.

Solution

$$10\sin\theta + 3 = 0$$

$$\Rightarrow \quad \sin\theta = -0.3$$

Step 1: Principal value $= \sin^{-1}(-0.3)$

$= -17.5°$

Step 2: The second angle is $180° + 17.5° = 197.5°$

Step 3: $-17.5° + 360° = 342.5°$ and $342.5° + 360° = 702.5°$

$197.5° + 360° = 557.5°$

Step 4: $-17.5° - 360° = -377.5°$ (too low)

$197.5° - 360° = -162.5°$

$\therefore \ \theta = -162.5°, -17.5°, 197.5°, 342.5°, 557.5°$ or $702.5°$

Make sure the graph plotter is set to 'degrees'.

ACTIVITY 6.3

- Use a graph plotter to plot the graph of $y = \sin(x + 10)$.
- How does the graph of $y = \sin(x + 10)$ compare with the graph of $y = \sin x$?
- Then do the same with the graph of $y = \sin(x + 20)$.
- Is it possible to find a graph of the form $y = \sin(x + c)$ which is the same as the graph of $y = \cos x$?
- Write down an identity in the form $\cos x \equiv \sin(x + c)$ where c is a number to be found.
- Go through the same process with the cosine graph to find a similar identity of the form $\sin x \equiv \cos(x + d)$.

Note

Knowledge of these graphs and the corresponding identities will not be examined in this specification. However, students who go on to study Mathematics at A-Level will meet them again.

Exercise 6E

Give answers to 1 decimal place where necessary.

① Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $\cos\theta = 0.5$ (ii) $\tan\theta = 1$ (iii) $\sin\theta = \frac{\sqrt{3}}{2}$

(iv) $\sin\theta = -0.5$ (v) $\cos\theta = 0$ (vi) $\tan\theta = -5$

(vii) $\tan\theta = 0$ (viii) $\cos\theta = -0.54$ (ix) $\sin\theta = 1$

② Solve the following equations for $-180° \leqslant \theta \leqslant 180°$.

(i) $3\cos\theta = 2$ (ii) $7\sin\theta = 5$ (iii) $3\tan\theta = 8$

(iv) $6\sin\theta + 5 = 0$ (v) $5\cos\theta + 2 = 0$ (vi) $5 - 9\tan\theta = 10$

PS ③ Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $\sin^2\theta = 0.75$ (ii) $\cos^2\theta = 0.5$ (iii) $\tan^2\theta = 1$

PS ④ (i) Factorise $2x^2 + x - 1$

(ii) Hence solve $2x^2 + x - 1 = 0$

(iii) Use your results to solve these equations for $-360° \leqslant \theta \leqslant 360°$.

(a) $2\sin^2\theta + \sin\theta - 1 = 0$

(b) $2\cos^2\theta + \cos\theta - 1 = 0$

(c) $2\tan^2\theta + \tan\theta - 1 = 0$

$\sin^2\theta$ is alternative notation for $(\sin\theta)^2$

PS ⑤ Solve the following equations for $-180° \leqslant x \leqslant 180°$.

(i) $\tan^2 x - 3\tan x = 0$ (ii) $1 - 2\sin^2 x = 0$

(iii) $3\cos^2 x + 2\cos x - 1 = 0$ (iv) $2\sin^2 x = \sin x + 1$

PS ⑥ *Do not use a calculator in this question.*

Solve the following equations for $-360° < x < 360°$.

(i) $\tan x = \sqrt{3}$ (ii) $2\sin x = 1$

(iii) $\sqrt{2}\cos x - 1 = 0$ (iv) $2\sin x = \sqrt{3}$

(v) $\tan^2 x - \tan x = 0$ (vi) $4\cos x = \sqrt{12}$

PS ⑦ Solve $(\cos\theta - 1)(\cos\theta + 2)(2\cos\theta - 1) = 0$ for $0° \leqslant \theta \leqslant 360°$.

PS ⑧ (i) Given that $f(x) = 2x^3 - x^2 - 3x - 1$, calculate $f\left(-\frac{1}{2}\right)$.

(ii) Hence solve $2\sin^3\theta - \sin^2\theta - 3\sin\theta - 1 = 0$ for $-180° \leqslant \theta \leqslant 180°$.

11 Trigonometric identities

Remember the earlier definitions for trigonometric functions of angles of any magnitude

$$\sin\theta = y \quad \cos\theta = x \quad \tan\theta = \frac{y}{x}$$

where the angle θ was defined by a point P(x, y) on a circle of unit radius (Figure 6.60).

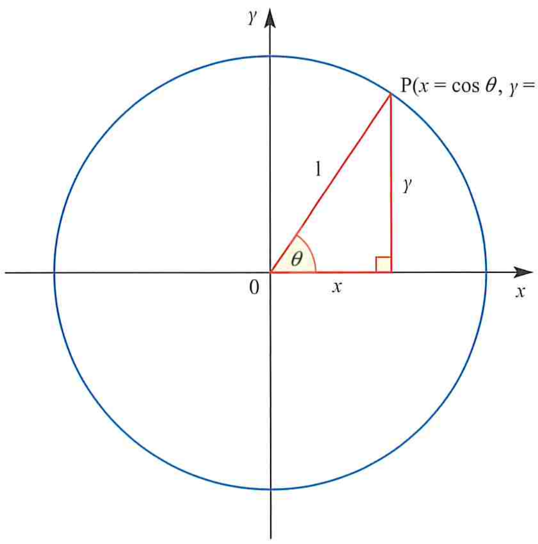


Figure 6.60

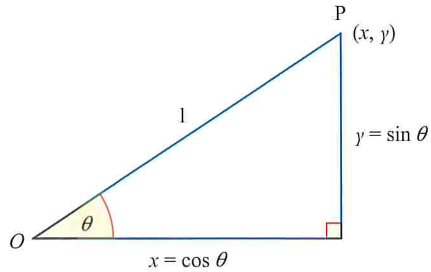


Figure 6.61

Applying Pythagoras' theorem to the triangle it can be seen that

$$\sin^2\theta + \cos^2\theta = 1$$

If θ is not acute, then x and/or y would be negative, but squaring produces the same result.

So the rule is correct for any value of θ.

As is the rule $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ which was seen earlier in this chapter.

Example 6.12

Solve the equation $2\cos^2\theta - \sin\theta - 1 = 0$ for values of θ in the range $0°$ to $360°$.

Solution

$\sin^2\theta + \cos^2\theta = 1$ can be rearranged into $\cos^2\theta = 1 - \sin^2\theta$.
So, here the $\cos^2\theta$ has been replaced with $1 - \sin^2\theta$, leaving an equation in $\sin\theta$.

$$
\begin{aligned}
& 2\cos^2\theta - \sin\theta - 1 = 0 \\
\Rightarrow\quad & 2\left(1 - \sin^2\theta\right) - \sin\theta - 1 = 0 \\
\Rightarrow\quad & 2 - 2\sin^2\theta - \sin\theta - 1 = 0 \\
\Rightarrow\quad & 0 = 2\sin^2\theta + \sin\theta - 1 \\
\Rightarrow\quad & 0 = \left(2\sin\theta - 1\right)\left(\sin\theta + 1\right) \\
\Rightarrow\quad & \sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1 \\
\Rightarrow\quad & \theta = 30°,\ 150° \quad \text{or} \quad \theta = -90°,\ 270° \\
\therefore\quad & \theta = 30°,\ 150° \quad \text{or} \quad 270°
\end{aligned}
$$

Example 6.13

Show that $\dfrac{\tan x \cos x}{\sqrt{1 - \cos^2 x}}$ simplifies to 1

Solution

$$
\begin{aligned}
\frac{\tan x \cos x}{\sqrt{1 - \cos^2 x}} &= \frac{\frac{\sin x}{\cos x}\cos x}{\sqrt{\sin^2 x}} \\
&= \frac{\sin x}{\sin x} \\
&= 1
\end{aligned}
$$

Example 6.14

(i) Prove that $\cos^2 x - \sin^2 x \equiv 2\cos^2 x - 1$

(ii) Hence, solve $\cos^2 x - \sin^2 x = 0.5$ for $0° \leqslant x \leqslant 360°$.

Solution

(i) Start with one side of the identity and, step-by-step, change it into the other side.

$$
\begin{aligned}
\cos^2 x - \sin^2 x &\equiv \cos^2 x - (1 - \cos^2 x) \\
&\equiv \cos^2 x - 1 + \cos^2 x \\
&\equiv 2\cos^2 x - 1
\end{aligned}
$$

The equivalence (or identity) sign is used to indicate that the equation is true for all values of x.

(ii) Using part (i), the equation can be rewritten as

$$\begin{aligned} 2\cos^2 x - 1 &= 0.5 \\ \Rightarrow \quad 2\cos^2 x &= 1.5 \\ \Rightarrow \quad \cos^2 x &= \frac{3}{4} \\ \Rightarrow \quad \cos x &= \pm\sqrt{\frac{3}{4}} \\ \Rightarrow \quad \cos x &= \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos x = -\frac{\sqrt{3}}{2} \\ \Rightarrow \quad x &= 30°, 330° \quad \text{or} \quad x = 150°, 210° \end{aligned}$$

Summary

$\sin^2\theta + \cos^2\theta \equiv 1$ and $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin\theta \equiv \cos\theta\tan\theta$ and $\cos\theta \equiv \dfrac{\sin\theta}{\tan\theta}$

Exercise 6F

① For each of the equations (i)–(v):

(a) use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function

(b) factorise, and hence solve, the resulting equation for $0° \leqslant \theta \leqslant 360°$.

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

(ii) $\sin^2\theta + \cos\theta + 1 = 0$

(iii) $2\sin^2\theta - \cos\theta - 1 = 0$

(iv) $\cos^2\theta + \sin\theta = 1$

(v) $1 + \sin\theta - 2\cos^2\theta = 0$

② For each of the equations (i)–(iii):

(a) use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to rewrite the equation in a form involving only one trigonometric function

(b) use the quadratic formula to solve the resulting equation for $0° \leqslant \theta \leqslant 180°$.

(i) $\sin^2\theta - 2\cos\theta + 1 = 0$

(ii) $\cos^2\theta - \sin\theta = 0$

(iii) $\sin^2\theta - 3\cos\theta = 0$

③ (i) Use the identity

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

to rewrite the equation $\sin\theta = 2\cos\theta$ in terms of $\tan\theta$.

(ii) Hence solve the equation $\sin\theta = 2\cos\theta$ for $0° \leqslant \theta \leqslant 180°$.

④ Use the identity

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

to solve the following equations for $0° \leqslant \theta \leqslant 360°$.

(i) $2\sin\theta + \cos\theta = 0$

(ii) $\sqrt{3}\tan\theta = 2\sin\theta$

(ii) $4\cos\theta\tan\theta = 1$

PS ⑤ Write the following in terms of $\sin x$.

(i) $\cos^2 x \tan^2 x$ (ii) $\tan x \cos^3 x$ (iii) $\cos x\,(2\cos x - 3\tan x)$

PS ⑥ Show that $(3\sin x)(\sin x + 2) - 3(2\sin x - \cos^2 x)$ simplifies to an integer.

PS ⑦ Prove the following identities.

(i) $\tan x\sqrt{1 - \sin^2 x} \equiv \sin x$

(ii) $\dfrac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \tan^2 x$

(iii) $(1 + \sin x)(1 - \sin x) \equiv \cos^2 x$

(iv) $\dfrac{2\sin x\cos x}{\tan x} \equiv 2 - 2\sin^2 x$

PS ⑧ Solve $5\sin x\left(\sin x + \cos x\right) = 3$ for $0° < x < 360°$.

Hint: Replace 3 with 3×1 and then replace the 1 with $\sin^2 x + \cos^2 x$.

FUTURE USES

Trigonometric functions are explored to a greater depth in A-Level Mathematics, including the use of various trig identities and the study of inverse trig functions. Trigonometry is used in many areas, including mechanics when resolving vectors such as forces and velocities.

It is also used extensively in A-Level Further Mathematics, to describe complex numbers (a combination of real and imaginary numbers), to describe transformations, and many other applications.

In A-Level Further Mathematics, you will also study the hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$.

REAL-WORLD CONTEXT

Trigonometry has many real-world applications, including every aspect of engineering. It is also essential to architects and surveyors. Space exploration and the motion/positioning of satellites would not be possible without trigonometry. Mobile telephones, video games, and computers in general, make much use of this vital area of mathematics. In fact, ancient civilisations were aware of its usefulness, and made use of it to achieve amazing feats of construction, many of which are still standing to this day.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- use trigonometry in a right-angled triangle
  - to find an angle when you know any two sides
  - to find the other sides or angle when you know the length of one side and an angle
- use Pythagoras' theorem in two dimensions
  - in the form $a^2 + b^2 = c^2$
- apply the following angle facts
  - vertically opposite angles are equal
  - adjacent angles on a straight line add up to 180°
  - alternate angles are equal
  - corresponding angles are equal
  - interior angles add up to 180°
- apply the following circle theorems
  - the angle at the centre is double the angle at the circumference
  - angle in a semi-circle = 90°
  - angles in the same segment are equal
  - opposite angles of a cyclic quadrilateral add up to 180°
  - alternate segment theorem
- construct a formal geometric proof for problems involving triangles and circles
- solve practical problems in two dimensions (e.g. a ladder against a wall)
- use a calculator to find
  - the sin, cos or tan of any angle
  - an angle, given the sin, cos or tan ratio
- sketch and recognise the graphs of sin, cos or tan for any angle
- solve trigonometric equations
- recognise and use the trigonometric identities linking sin, cos and tan.

KEY POINTS

1 In a right-angled triangle Pythagoras' theorem gives

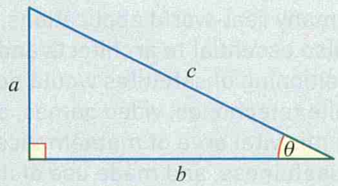


Figure 6.62

$c^2 = a^2 + b^2$

2 Using the triangle above gives the definitions:

$\sin\theta = \dfrac{a}{c} \qquad \cos\theta = \dfrac{b}{c} \qquad \tan\theta = \dfrac{a}{b}$

4 $\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

5 In a geometrical proof, show all your working and give unambiguous reasons for each stage.

6 Trig graphs:

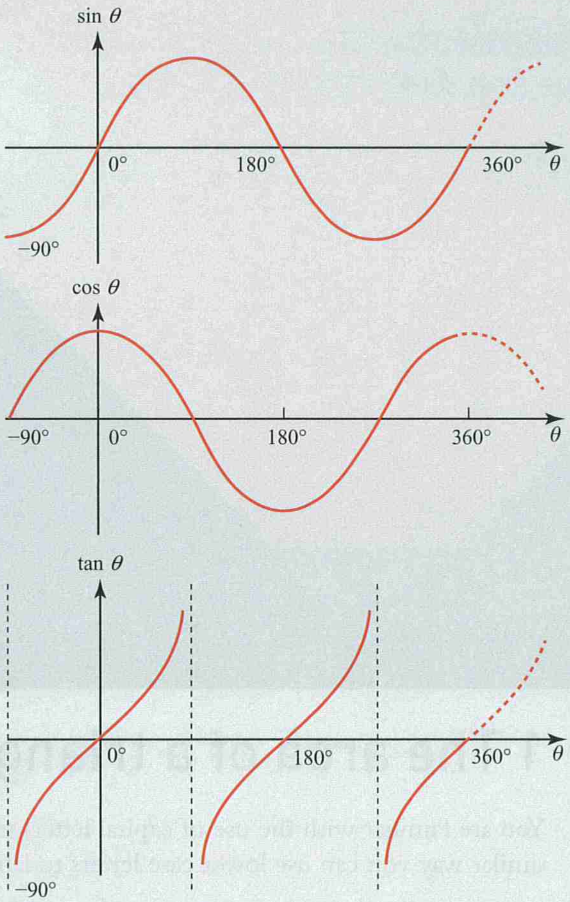


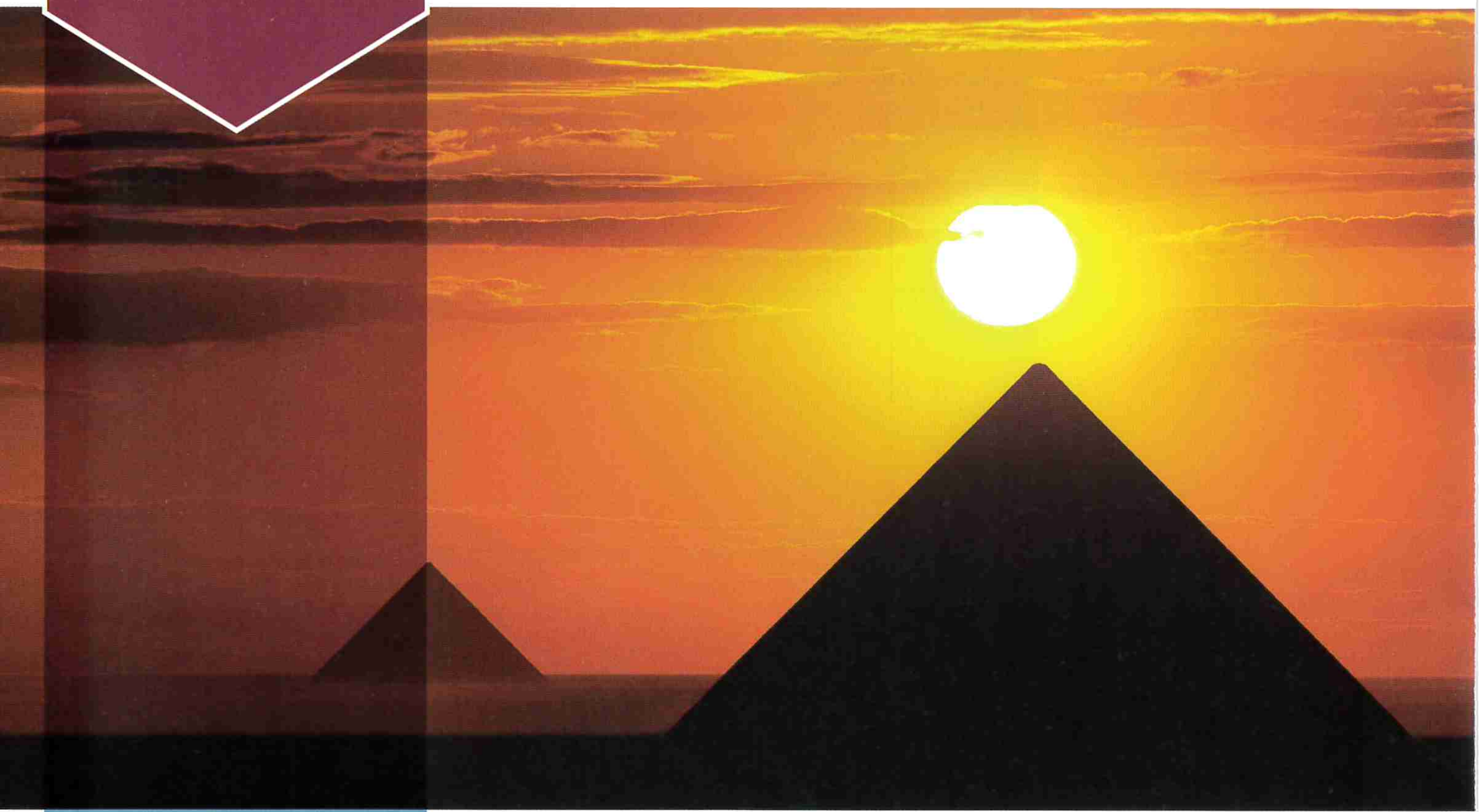
Figure 6.63

7 Trig identities:

$\sin^2\theta + \cos^2\theta \equiv 1$ and $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$

$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sin\theta \equiv \cos\theta\tan\theta$ and $\cos\theta \equiv \frac{\sin\theta}{\tan\theta}$

7 Geometry II



What we know is a drop; what we don't know is an ocean.

Isaac Newton

1 The area of a triangle

You are familiar with the use of capital letters to label the vertices of a triangle. In a similar way you can use lower case letters to label the sides.

a denotes the length of the side opposite angle A, b is the length of the side opposite angle B, and c is the length of the side opposite angle C.

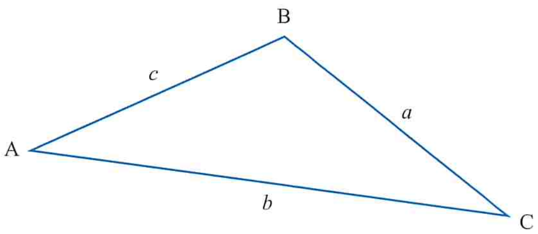


Figure 7.1

Using this notation, for any triangle ABC the area is given by the formula

$$\text{area} = \frac{1}{2}bc\sin A.$$

Proof

Figure 7.2 shows a triangle ABC. The perpendicular CD is the height h corresponding to AB as base.

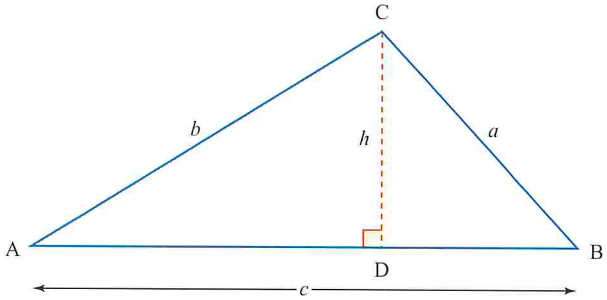


Figure 7.2

Using area of a triangle equals half its base times its height,

$$\text{area} = \frac{1}{2}ch \qquad ①$$

In triangle ACD

$$\sin A = \frac{h}{b}$$

$$\Rightarrow \qquad h = b \sin A$$

Substituting in ① gives

$$\text{area} = \frac{1}{2}bc \sin A$$

Note

Taking the other two points in turn as the top of the triangle gives equivalent results:

$$\text{area} = \frac{1}{2}ca \sin B$$

and

$$\text{area} = \frac{1}{2}ab \sin C.$$

The formula may be easier to remember as half the product of two sides times the sine of the angle between them.

Example 7.1

Figure 7.3 shows a regular pentagon, PQRST, inscribed in a circle, centre C, radius 8 cm. Calculate the area of

(i) triangle CPQ

(ii) the pentagon.

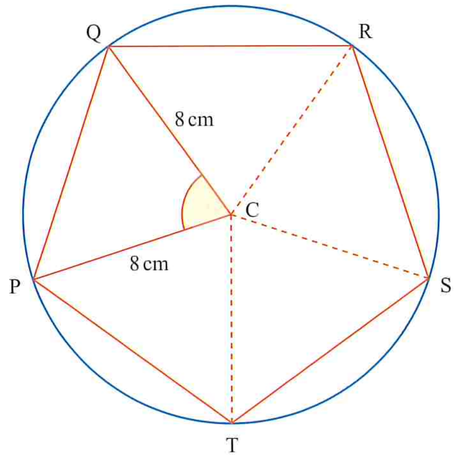


Figure 7.3

Solution

(i) angle PCQ $= 360° \div 5$

$= 72°$

area PCQ $= \frac{1}{2} \times 8 \times 8 \times \sin 72°$

$= 30.4338...$

$= 30.4\,\text{cm}^2$ (1 d.p.)

(ii) area PQRST $= 5 \times 30.4338...$

$= 152.169...$

$= 152.2\,\text{cm}^2$ (1 d.p.)

Example 7.2

Figure 7.4 shows an isosceles triangle with an area of $24\,\text{cm}^2$ and one angle of 40°. Calculate the lengths of the two equal sides.

Solution

Let the equal sides be of length x cm.

Using area $= \frac{1}{2}ab\sin C$

$\therefore \quad 24 = \frac{1}{2} \times x \times x \times \sin 40°$

$\Rightarrow \quad x^2 = \frac{48}{\sin 40°}$

$\Rightarrow \quad x = 8.64\,\text{cm}$ (3 s.f.)

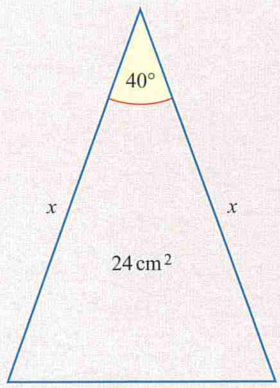


Figure 7.4

Exercise 7A

Where necessary leave answers approximated to 3 significant figures.

① Work out the area of each of the following triangles.

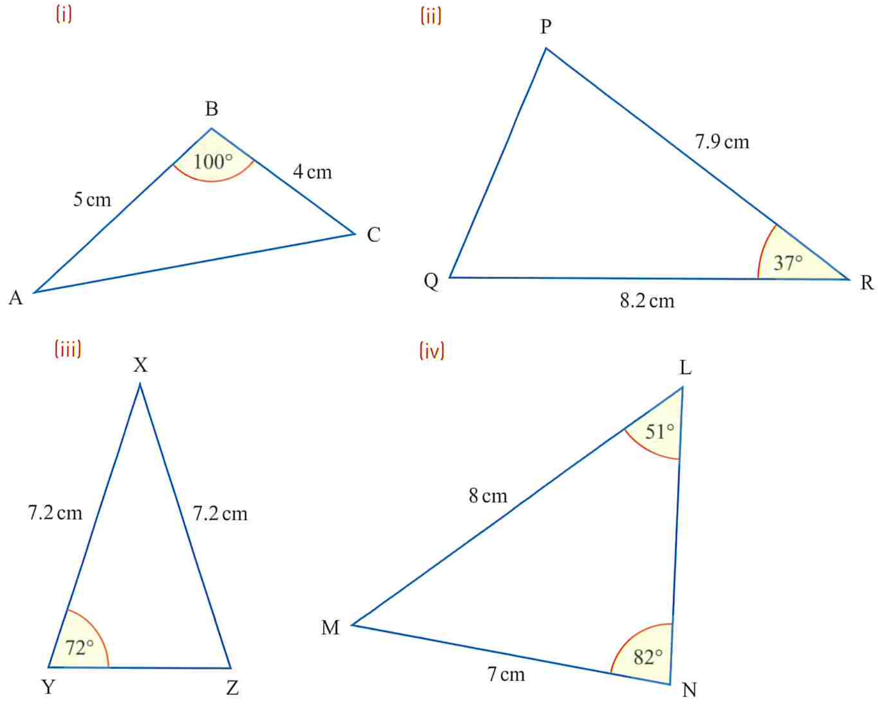


Figure 7.5

PS ② A regular hexagon is made up of six equilateral triangles. Work out the area of a regular hexagon of side 7 cm.

PS ③ A pyramid on a square base has four identical triangular faces which are isosceles triangles with equal sides 9 cm and equal angles 72°.

(i) Work out the area of a triangular face.

(ii) Work out the length of a side of the base.

(iii) Hence work out the total surface area of the pyramid.

PS ④ A tiler wishes to estimate the number of triangular tiles needed to tile an area of $10\,\text{m}^2$. The dimensions of each tile are shown in the diagram.

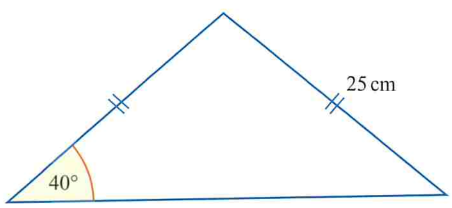


Figure 7.6

(i) Work out the area of a tile.

The tiler then divides $10\,\text{m}^2$ by this area and rounds to the next whole number.

(ii) What result would this give?

(iii) Explain what is wrong with this estimate.

PS ⑤ A regular tetrahedron has four faces, each of which is an equilateral triangle of side 10 cm. Work out the total surface area of the tetrahedron.

PS ⑥ The area of a rhombus is $\sqrt{48}\,\text{cm}^2$. Given also that one of its interior angles is 120°, work out the length of its shortest diagonal.

PS ⑦ A square with sides of length 2 cm has the same area as an equilateral triangle. Work out the side-length of the triangle.

PS ⑧ A circle is drawn inside a square, so that they touch at four points as shown.

A rectangle of dimensions 1 cm × 2 cm is drawn in the corner of the square and touches the circle once. The sides of the rectangle are parallel to the sides of the square. A radius of the circle is drawn to the point where the rectangle meets the circle.

Work out the size of the angle marked θ.

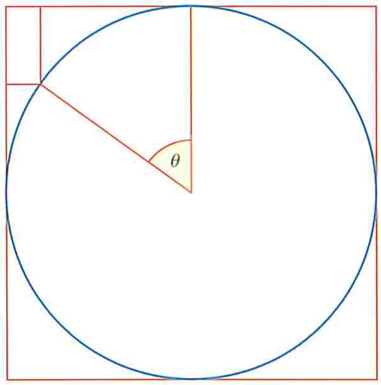


Figure 7.7

The following two trigonometric rules can be used in any triangle, which makes them particularly useful when dealing with scalene triangles.

2 The sine rule

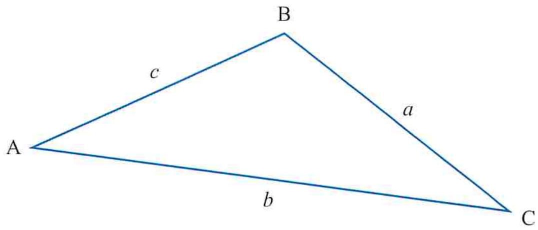


Figure 7.8

You have already seen that for any triangle ABC

$$\text{area} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Rightarrow \quad \frac{bc\sin A}{abc} = \frac{ca\sin B}{abc} = \frac{ab\sin C}{abc}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This is one form of the *sine rule* and is the version that is easier to use if you want to work out the size of an angle.

Inverting this gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

which is better when you need to work out the length of a side.

Discussion point

➜ Why is the inverted form of the sine rule better when you want to work out the length of a side?

Note

If a triangle is right-angled then it is much simpler to use the basic trig ratios and/or Pythagoras' theorem. However, the sine and cosine rules are still applicable.

Example 7.3

Work out the length of the side BC in the triangle shown in Figure 7.9.

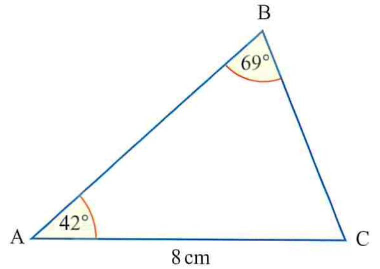


Figure 7.9

> **!** When using the sine rule to work out the size of an angle, you need to be careful because sometimes there are two possible answers, as in Example 7.4. The reason this problem occurs is that for any positive sine ratio there are two possible angles in the range 0° to 180°, except sin 90° = 1.

Solution

Using the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin 42°} = \frac{8}{\sin 69°}$$

$$\Rightarrow \quad a = \frac{8 \sin 42°}{\sin 69°}$$

$$= 5.733\,887\ldots$$

$$\therefore \quad \text{side BC} = 5.7\,\text{cm}\,(1\,\text{d. p.})$$

> It is advisable to do the calculation entirely on your calculator, and round only the final answer.

Example 7.4

Work out the size of the angle P in the triangle PQR, given that R = 32°, $r = 4$ cm and $p = 7$ cm where r and p are the lengths of the sides opposite angles R and P respectively.

> **!** Always check that the second option is a valid solution. Sometimes one of the options gives an impossible triangle. For example, if a solution has an angle sum which is greater than 180° then we can reject it. Also, the longest side must be opposite the largest angle, and the shortest side must be opposite the smallest angle.

Solution

The sine rule for ΔPQR is

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

$$\therefore \quad \frac{\sin P}{7} = \frac{\sin 32°}{4}$$

$$\Rightarrow \quad \sin P = 0.927\,358\,712$$

$$\Rightarrow \quad P = 68.0°\ (1\text{ d.p.}) \text{ or } P = 180° - 68.0° = 112.0°\ (1\text{ d.p.})$$

Both solutions are possible as indicated in Figure 7.10(b).

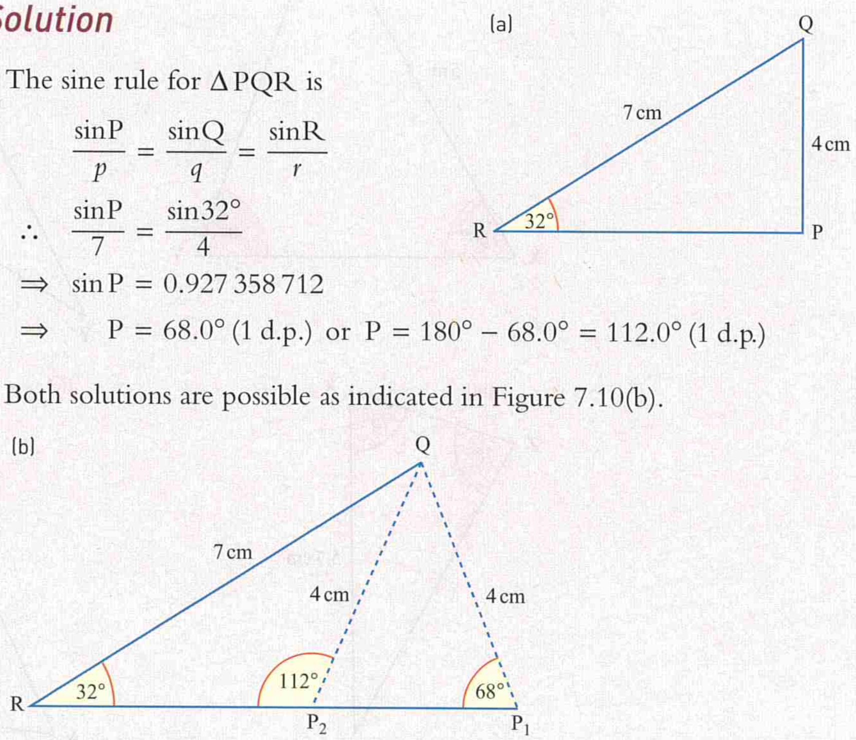


Figure 7.10

Note

Students may have met this situation when studying congruent triangles in GCSE maths.

If two triangles have SSS, SAS, ASA (or AAS) or RHS in common then they are congruent. However, if they have ASS in common (as in the above example), then they are not necessarily congruent, as there are two possible triangles.

ACTIVITY 7.1

Figure 7.11 shows triangle XYZ with $\mathrm{XY} = 6\,\mathrm{cm}$, $\mathrm{XZ} = 8\,\mathrm{cm}$ and $\angle\mathrm{XYZ} = 78°$. What happens when you use the sine rule to calculate the remaining angles?

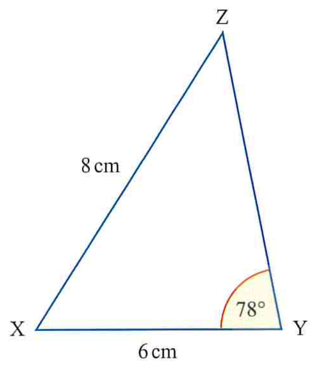


Figure 7.11

Prior knowledge

Many trigonometric problems involve the use of bearings, which is covered in the GCSE specification.

Exercise 7B

Where necessary leave answers approximated to 3 significant figures.

① Work out the length x in each of these triangles.

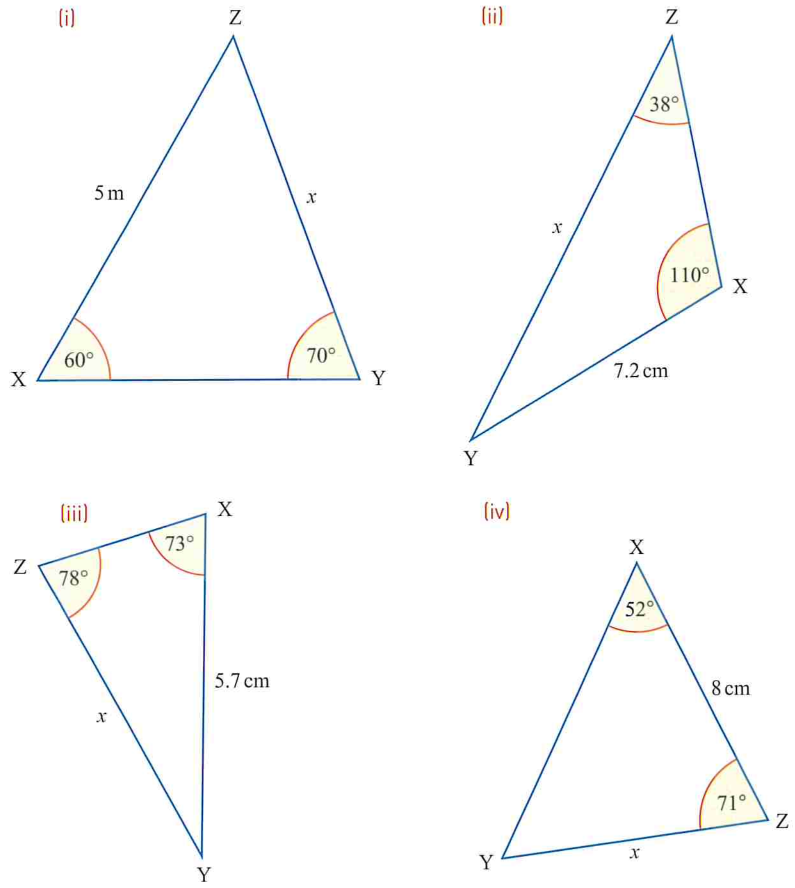


Figure 7.12

② Work out the size of the angle θ in each of these triangles.

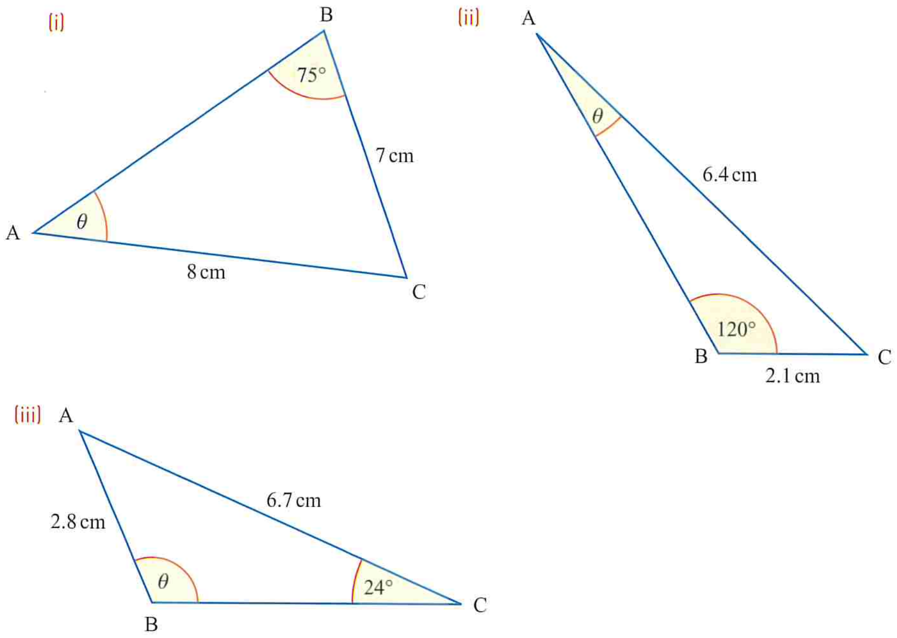


Figure 7.13

③ Work out the size of the angle marked x in the quadrilateral shown.

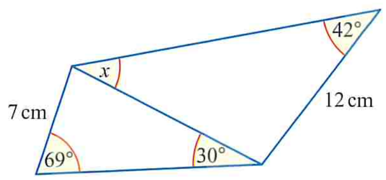


Figure 7.14

④ Tracey walks on a bearing of 132° for 4 km.

She then changes direction and walks on a bearing of 017° until she is due east of her starting position.

How far is she from her starting position?

⑤ The angles of a kite are 122°, 102°, 102° and 34°.

The diagonal which lies along the kite's line of symmetry is 12 cm in length.

Work out the lengths of each of the kite's four sides.

⑥ Anna and Julia are at point P.

Point Q is due North of point P.

They disagree about the shortest route from P to Q.

Anna walks on a bearing of 330°, and then changes to a bearing of 040°, which takes her straight to Q.

Julia walks 3 km on a bearing of 020°, after which she walks on a bearing of 300°, which then takes her straight to Q.

Who took the shorter route?

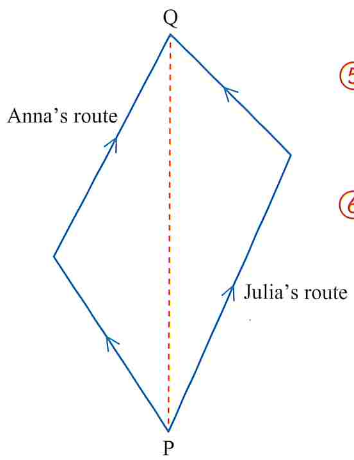


Figure 7.15

3 The cosine rule

Sometimes it is not helpful to use the sine rule with the information you have about a triangle, for example, if you know all three side lengths but none of the angles.

Like the sine rule, the cosine rule can be applied to any triangle, and again there are equivalent versions.

Use this version to work out a side length.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Use this version to work out the size of an angle.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Proof

For the triangle ABC, line CD is perpendicular to side AB as shown in Figure 7.16.

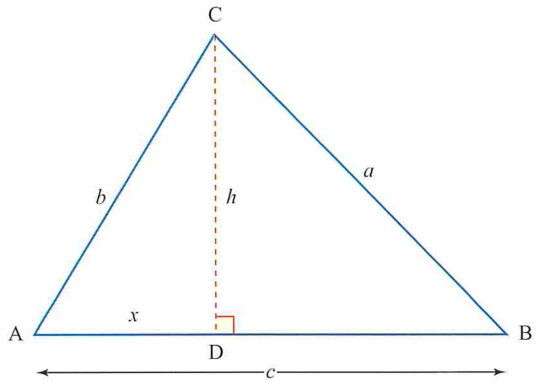


Figure 7.16

This is shorthand notation for 'triangle'.

Pythagoras' theorem.

In Δ ACD

$$b^2 = x^2 + h^2 \qquad ①$$

and $\quad \cos A = \frac{x}{b}$ so $x = b\cos A \qquad ②$

Pythagoras' theorem.

In Δ BCD

$$a^2 = (c - x)^2 + h^2$$

$$\Rightarrow \quad a^2 = c^2 - 2cx + x^2 + h^2$$

$$\Rightarrow \quad a^2 = c^2 - 2cx + b^2 \qquad \text{using ①}$$

$$\Rightarrow \quad a^2 = c^2 - 2cb\cos A + b^2 \qquad \text{using ②}$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{(as required)}$$

Rearranging this gives

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

which is the second form of the cosine rule.

Note

Starting with a perpendicular from a different vertex would give the following similar results.

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{and} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 7.5

Work out the length of the side AB in the triangle shown in Figure 7.17.

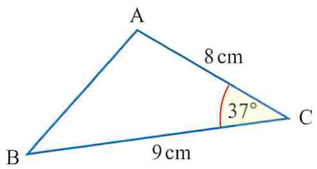


Figure 7.17

Solution

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 9^2 + 8^2 - 2 \times 9 \times 8 \times \cos 37$$

$$= 29.996$$

$$AB = 5.48\,\text{cm} \quad (3\text{ s.f.})$$

There are two common errors when using this formula.

- In a non-calculator paper, evaluate the three terms a^2, b^2 and $2ab\cos C$ separately. A common error is to calculate $a^2 + b^2 - 2ab$ and then multiply by $\cos C$. However, these questions are usually in calculator papers, in which case the whole calculation can be typed into a scientific calculator – this will deal with the priority of operations correctly.
- Another common error is to forget to square root after calculating $a^2 + b^2 - 2ab\cos C$.

Example 7.6

Work out the size of the angle P in the triangle shown in Figure 7.18.

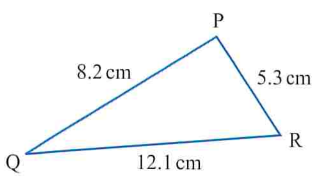


Figure 7.18

Solution

The cosine rule for this triangle can be written as

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos P = \frac{5.3^2 + 8.2^2 - 12.1^2}{2 \times 5.3 \times 8.2}$$

$$\cos P = -0.588$$

$$P = 126.0° \quad (1\text{ d.p.})$$

The cosine rule

Exercise 7C

Where necessary leave answers approximated to 3 significant figures.

① Work out the length x in each of these triangles.

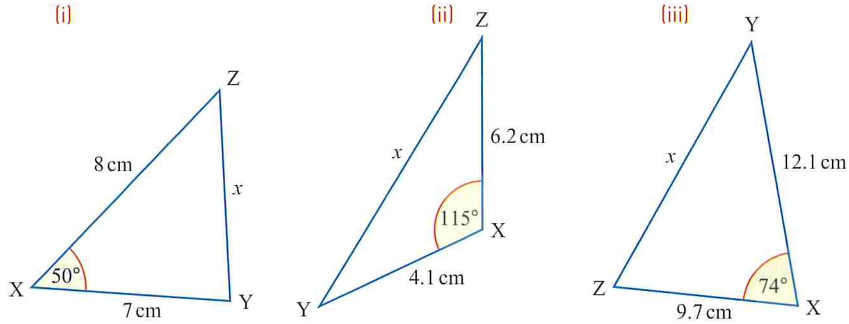


Figure 7.19

② Work out the size of the angle θ in each of the following triangles.

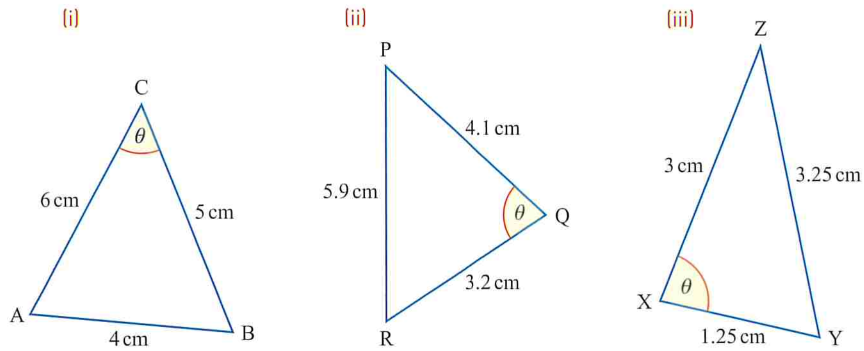


Figure 7.20

③ The diagonals of a parallelogram have lengths of 12 cm and 18 cm and the angle between them is 72°. Work out the lengths of the sides of the parallelogram.

④ Figure 7.21 shows a quadrilateral ABCD with AB = 8 cm, BC = 6 cm, CD = 7 cm, DA = 5 cm and ∠ABC = 90°.
Calculate

(i) AC

(ii) ∠ADC.

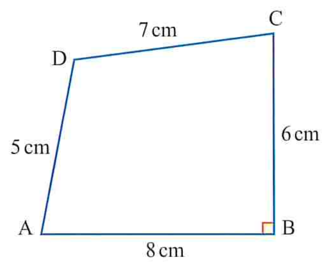


Figure 7.21

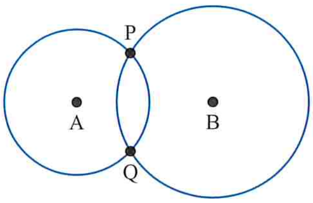


Figure 7.22

⑤ Figure 7.22 shows two circles. One has centre A and a radius of 8 cm. The other has centre B and a radius of 10 cm. AB = 12 cm and the circles intersect at P and Q. Calculate ∠PAB.

⑥ A parallelogram has sides of length 5 cm and 10 cm, and an angle of 130°.
Work out the length of the longest diagonal.

⑦ A triangle has sides of length 6 cm, 7 cm and 11 cm.
Work out its area.

⑧ Alan walks 7 km on a bearing of 054°.
He then walks 5 km on a bearing of 122°.
How far is he from his starting point?

⑨ In triangle ABC, AB = 8 cm, BC = 5 cm and ∠BAC = 35°.
Use the cosine rule to work out the possible lengths of AC.

4 Using the sine and cosine rules together

Note

Three angles are not independent measurements, as a third angle can be calculated from the other two.

When solving any triangle, three independent measurements are required.

Given **3 side-lengths**, use the cosine rule to work out the size of an angle.

Given **2 side-lengths and an included angle**, use the cosine rule to work out the length of the third side.

Given **2 side-lengths and an angle (not included)**, use the sine rule to work out the size of another angle from which you can calculate the included angle, and you can then use the cosine rule to work out the missing length. This situation can sometimes produce two possible solutions.

Given **2 angles and one side-length**, use the sine rule to work out another side-length. If the given side-length is between the two angles, then first calculate the size of the third angle using the angle sum of a triangle.

Once a fourth independent measurement of a triangle has been calculated, then the other two can be calculated using either the cosine rule or the sine rule.

Example 7.7

Figure 7.23 shows the positions of three towns, Aldbury, Bentham and Chorton.

Bentham is 8 km from Aldbury on a bearing of 037° and Chorton is 9 km from Bentham on a bearing of 150°. Work out

(i) the size of the angle ABC

(ii) the distance of Chorton from Aldbury (to the nearest 0.1 km)

(iii) the bearing of Chorton from Aldbury (to the nearest 1°).

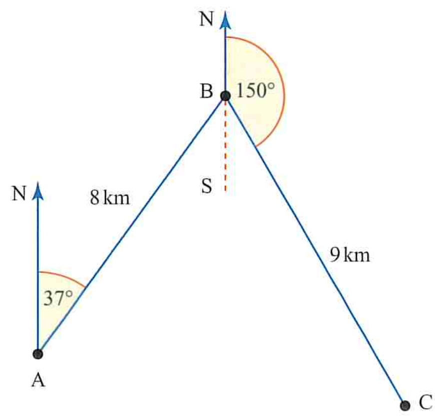


Figure 7.23

Solution

(i) $\angle ABS = 37°$ (alternate angles)
and $\angle SBC = 30°$ (adjacent angles on a straight line)
so $\angle ABC = 67°$

(ii) Using the cosine rule

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$= 9^2 + 8^2 - 2 \times 9 \times 8\cos 67°$$
$$= 88.7347\ldots$$
$$b = 9.4199\ldots$$

Don't clear this from your calculator as you will need it later.

Chorton is 9.4 km (1 d.p.) from Aldbury.

(iii) Using the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin A}{9} = \frac{\sin 67°}{9.4199\ldots}$$
$$\sin A = 0.87947\ldots$$
$$A = 61.57\ldots°$$

The bearing of Chorton from Aldbury is 099°.

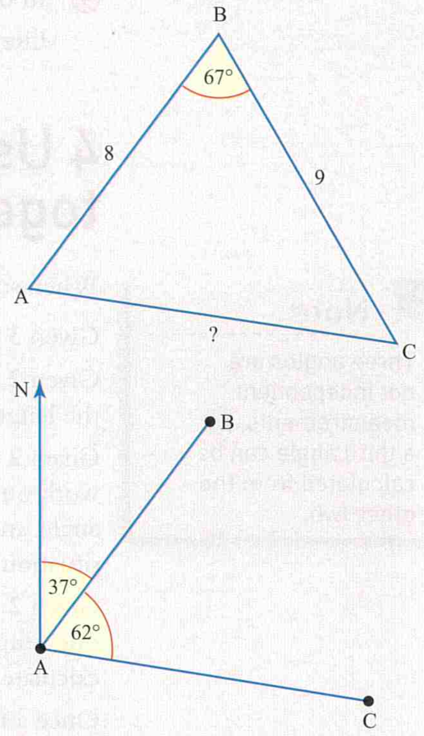


Figure 7.24

Discussion point

➜ The other value of A that gives $\sin A = 0.87947\ldots$ is $118.42\ldots°$
Why does this not give an alternative solution to this problem?

Example 7.8

A triangular plot of land has sides of length 70 m, 80 m and 95 m.

Work out its area in hectares. (1 hectare is 10 000 m^2.)

Solution

First draw a sketch and label the sides.

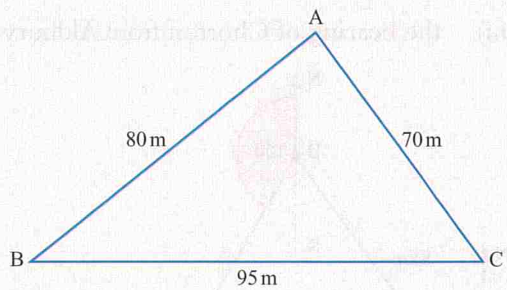


Figure 7.25

You can now see that the first step is to work out the size of one of the angles, and this will need the cosine rule.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{70^2 + 80^2 - 95^2}{2 \times 70 \times 80}$$

$$= \frac{13}{64}$$

$$\Rightarrow \quad A = 78.28°$$

$$\text{Area} = \frac{1}{2}bc \sin A$$

$$\text{Area} = \frac{1}{2} \times 70 \times 80 \times \sin 78.28°$$

$$= 2741 \text{m}^2$$

$$= 0.27 \text{ hectares (2 d.p.)}$$

Exercise 7D

Where necessary leave answers approximated to 3 significant figures.

① The hands of a clock have lengths 6 cm and 8 cm.

Work out the distance between the tips of the hands at 8 p.m.

② From a lighthouse L, a ship A is 4 km away on a bearing of 340° and a ship B is 5 km away on a bearing of 065°.

Work out the distance AB.

③ When I am at a point X, the angle of elevation of the top (T) of a vertical tree is 27°, but if I walk 20 m towards the tree along horizontal ground, to point Y, the angle of elevation is then 47°.

(i) Work out the distance TY.
(ii) Work out the height of the tree.

④ Two adjacent sides of a parallelogram have lengths 9.3 cm and 7.2 cm, and the shorter diagonal is of length 8.1 cm.

(i) Work out the sizes of the angles of the parallelogram.
(ii) Work out the length of the other diagonal of the parallelogram.

⑤ A yacht sets off from A and sails for 5 km on a bearing of 067° to a point B so that it can clear the headland before it turns onto a bearing of 146°. It then stays on that course for 8 km until it reaches a point C.

(i) Work out the distance AC.
(ii) Work out the bearing of C from A.

⑥ Two ships leave the docks, D, at the same time. *Princess Pearl*, P, sails on a bearing of 160° at a speed of 18 km h^{-1}, and *Regal Rose*, R, sails on a bearing of 105°. After 2 hours the angle DRP is 80°.

(i) Work out the distance between the ships at this time.
(ii) Work out the speed of the *Regal Rose*.

⑦ The diagram in Figure 7.26 represents a simplified drawing of the timber cross-section of a roof.

(i) Work out the lengths of the struts BD and EG.

(ii) Work out the length DE.

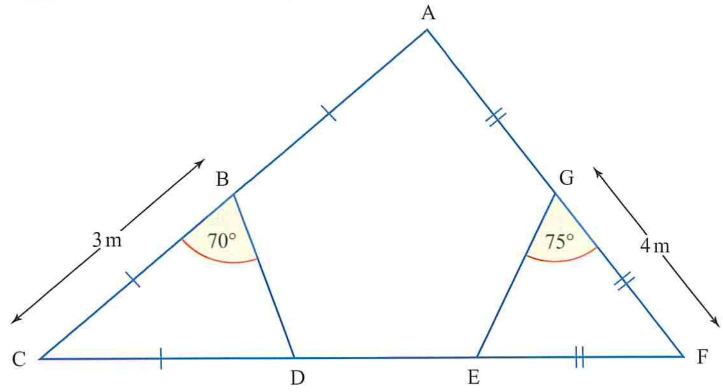


Figure 7.26

⑧ Sam and Aziz cycle home from school.

Sam cycles due east for 4 km, and Aziz cycles due south for 3 km and then for 2 km on a bearing of 125°.

How far apart are their homes?

5 Problems in three dimensions

Discussion point

➜ An aircraft flying between two places at the same latitude doesn't usually follow a route along the line of latitude. Why?

Figure 7.27

When you are solving three-dimensional problems it is important to draw good diagrams (although you will not be assessed on this in the exam, a clear diagram does benefit understanding). There are two types:

- representations of three-dimensional objects
- true shape diagrams of two-dimensional sections within a three-dimensional object.

Representations of three-dimensional objects

Figures 7.28 and 7.29 illustrate ways in which you can draw a clear diagram.

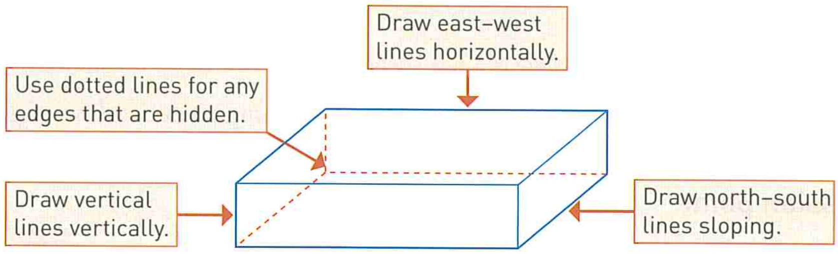


Figure 7.28

Open up the diagram as much as possible by choosing a suitable direction for your *north–south* axis; (a) is clearer than (b).

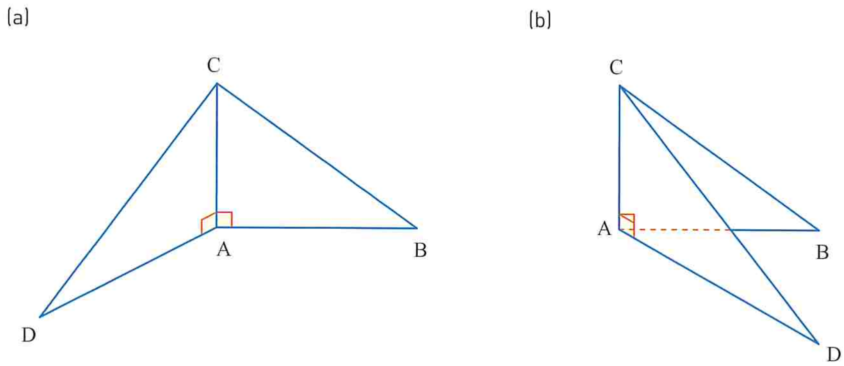


Figure 7.29

True shape diagrams

In a two-dimensional representation of a three-dimensional object, right angles do not always appear to be 90°, so draw as many true shape diagrams as necessary.

For example, if you need to do calculations on the triangular cross-section BCD in Figure 7.30(a), you should draw the triangle so that the right angle really does look 90° as in Figure 7.30(b).

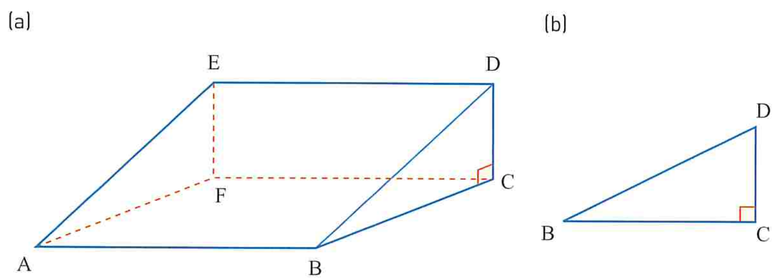


Figure 7.30

6 Lines and planes in three dimensions

A *plane* is a flat surface (not necessarily horizontal).

A *line of greatest slope* of a sloping plane is a line of greatest gradient, i.e. the line that a ball would follow if allowed to roll down it. This is shown in Figure 7.31.

Discussion point

→ Give an example of a sloping plane from everyday life.

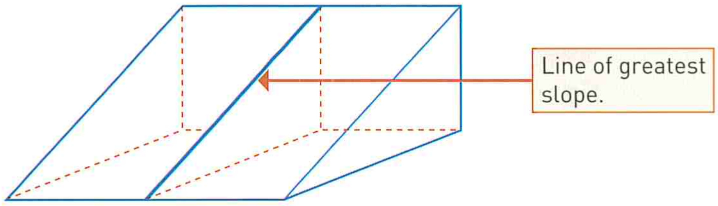


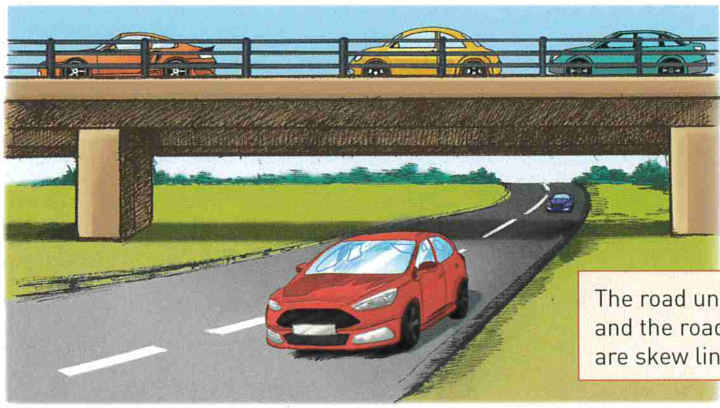
Figure 7.31

In three-dimensional problems you need to be aware of the relationships between lines and planes.

Two lines

In two dimensions, two lines either meet (when extended if necessary), or they are parallel.

In three dimensions, there is a third option: they are *skew*, as in Figure 7.32.



The road under the bridge and the road over the bridge are skew lines.

Figure 7.32

A line and a plane

In three dimensions there are three options, as shown in Figure 7.33.

(a) The line and the plane are *parallel*. A curtain rail is *parallel* to the floor.

(b) The line meets the plane at a *single point*. When you are writing, your pen meets the paper at a *single point*.

(c) The line *lies in* the plane. When you put your pen down, your pen *lies in* the plane of the paper.

(a) (b) (c)

Figure 7.33

Angle between a line and a plane

Draw a perpendicular from the line to the plane.

Line PQ meets the plane ABCD at Q.

PR is perpendicular to the plane.

QR is in the plane.

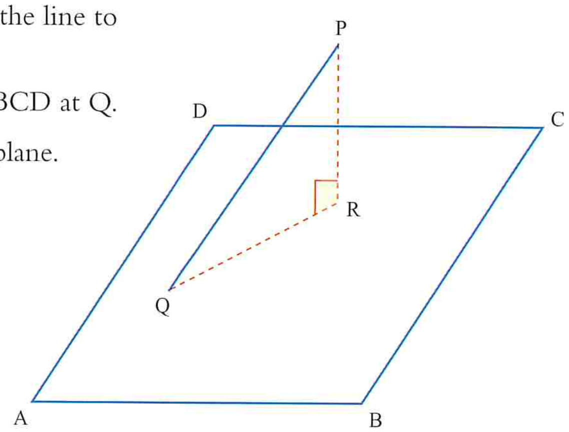


Figure 7.34

Angle between line and plane is angle PQR.

> **Discussion point**
>
> ➜ Give other examples of these cases.

Two planes

In three dimensions there are two options.

(a) The two planes are parallel. Opposite walls of a room are usually parallel.

(b) The two planes meet *in a line*. The ceiling meets each wall of a room *in a line*. An open gate and a wall meet *in a line*.

(a) (b)

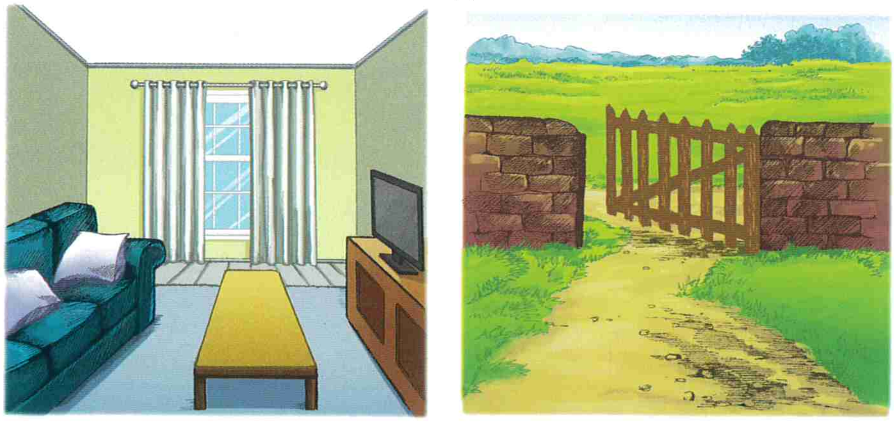


Figure 7.35

Angle between two planes

Identify the line where the planes meet.

Draw a line in each plane that is perpendicular to the line where the planes meet.

The angle between these two lines is the angle between the planes.

Planes ABCD and APQD meet along AD.

The dashed lines are each perpendicular to AD.

x is the angle between the planes.

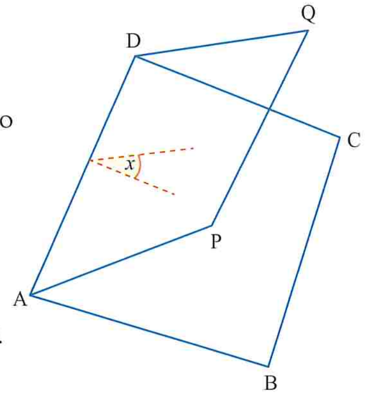


Figure 7.36

Example 7.9

Figure 7.37 shows a wedge ABCDEF with AB = 8 cm, BC = 6 cm and CD = 2 cm. The angle BCD is 90°.

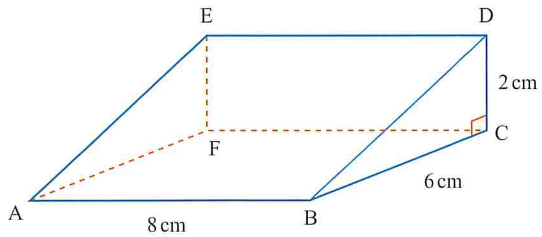


Figure 7.37

Work out

(i) the length AC (ii) the length AD

(iii) the size of the angle between DA and ABCF

(iv) the size of the angle between ABDE and ABCF.

Solution

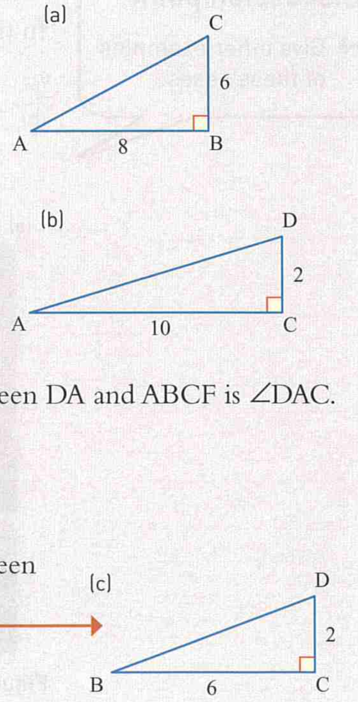


Figure 7.38

(i) From Figure 7.38(a)

$AC^2 = 8^2 + 6^2$ (Pythagoras)

$\Rightarrow \quad AC = 10\,\text{cm}$

(ii) From Figure 7.38(b)

$AD^2 = AC^2 + 2^2$ (Pythagoras)

$\Rightarrow \quad AD = 10.2\,\text{cm}$ (1 d.p.)

DA and ABCF meet at A. DC is perpendicular to ABCF.

(iii) From Figure 7.38(b), the angle between DA and ABCF is ∠DAC.

$\tan \angle DAC = \dfrac{2}{10}$

$\Rightarrow \angle DAC = 11.3°$ (1 d.p.)

ABDE and ABCF meet along AB. BD is perpendicular to AB. BC is perpendicular to AB.

(iv) From Figure 7.38(c), the angle between ABDE and ABCF is ∠DBC

$\tan \angle DBC = \dfrac{2}{6}$

$\Rightarrow \quad \angle DBC = 18.4°$ (1 d.p.)

Example 7.10

RWC

Figure 7.39 shows a straight level road AB, 400 m long. A vertical radio mast XY stands some distance from the road, and the bottom of the mast, X, is on the same level as the road. The angle of elevation of Y from A is 30°, $\angle XAB = 25°$ and $\angle AXB = 90°$. Calculate

(i) the distance AX

(ii) the height of the mast

(iii) the distance of X from the road.

Give your answers to 3 significant figures.

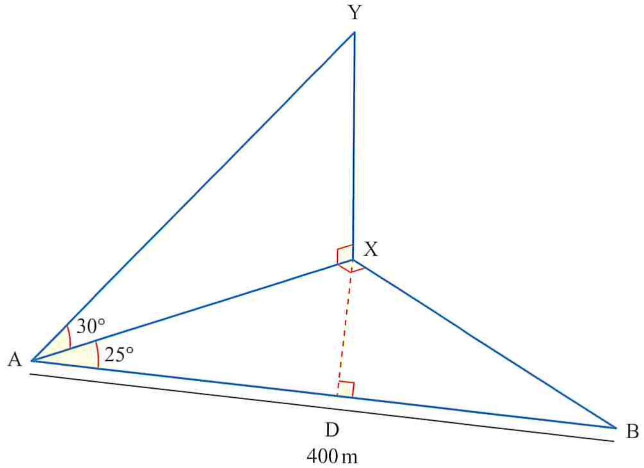


Figure 7.39

Solution

(i) From Figure 7.40(a)

$$\frac{AX}{400} = \cos 25°$$

$\Rightarrow$ $AX = 362.523\ldots$

$\Rightarrow$ The distance AX = 363 m.

(ii) From Figure 7.40(b)

$$\frac{XY}{362.523\ldots} = \tan 30°$$

$\Rightarrow$ $XY = 209.302\ldots$

$\Rightarrow$ The height of the mast XY = 209 m.

(iii) From Figure 7.40(c)

$$\frac{DX}{362.523\ldots} = \sin 25°$$

$\Rightarrow$ $DX = 153.208\ldots$

$\Rightarrow$ The distance of X from the road = 153 m.

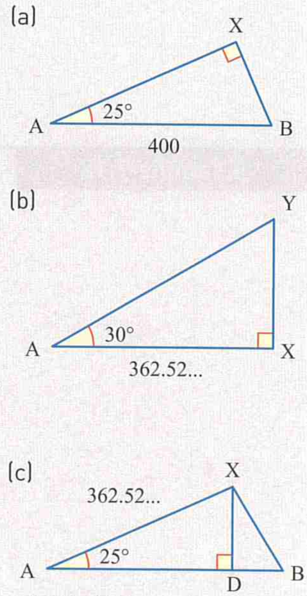


Figure 7.40

Example 7.11

The pyramid VABCD has square horizontal base ABCD.

The vertex, V, is directly above the centre, X, of the base.

M is the midpoint of BC.

AB = 8 metres and VX = 15 metres.

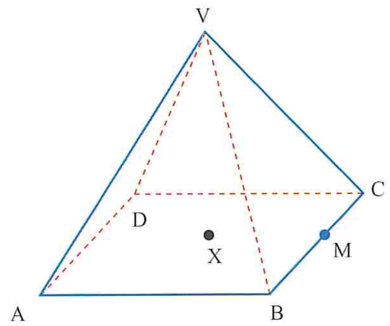


Figure 7.41

Work out the angle between the planes ABCD and VBC.

Solution

The planes meet along BC.

MX and VM are both perpendicular to BC.

Angle VXM is 90°.

$$XM = 8 \div 2$$
$$= 4\,\text{m}$$
$$\tan VMX = \frac{15}{4}$$
$$\text{angle } VMX = 75.1° \quad \text{(1 d.p.)}$$

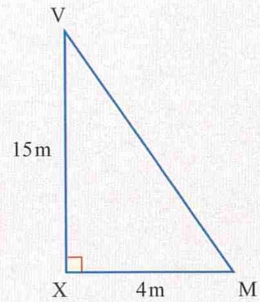


Figure 7.42

Example 7.12

The cuboid has a square base ABCD of side 8 cm and a height of 4 cm.

M is the midpoint of AC.

(i) Calculate the exact length of DM.

(ii) Work out the angle between the planes ABCD and ACH.

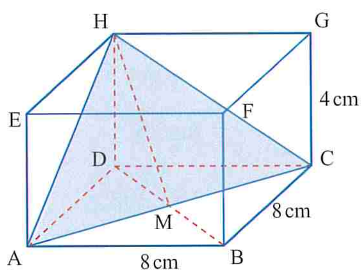


Figure 7.43

Solution

(i) DM = half the length of the diagonal of the square base

$$= \frac{1}{2}\sqrt{8^2 + 8^2}$$

$$4\sqrt{2}\text{ cm}$$

(ii) The angle required is ∠HMD.

$$\tan \text{HMD} = \frac{\text{HD}}{\text{DM}}$$

$$= \frac{4}{4\sqrt{2}}$$

The required angle is 35.3°.

Pythagoras' theorem in three dimensions

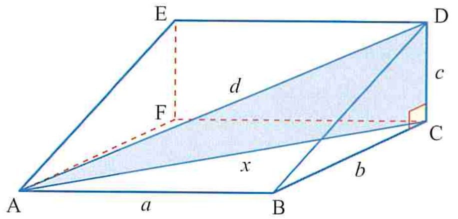


Figure 7.44

ACTIVITY 7.2

Look for integer values of a, b, c and d such that $a^2 + b^2 + c^2 = d^2$
You could start with $3^2 + 4^2 = 5^2$ and then use $5^2 + 12^2 = 13^2$
Can you find at least two examples of values of a, b, c and d?

In Figure 7.44, the base is rectangular, so using Pythagoras' theorem in 2 dimensions

$$a^2 + b^2 = x^2$$

The triangle ACD has a right angle at C, giving

$$x^2 + c^2 = d^2$$

Substituting for x^2 from the first equation gives

$$a^2 + b^2 + c^2 = d^2$$

This is the 3-D version of Pythagoras' theorem.

Example 7.13

ABCDEFGH is a cuboid with side-lengths as shown in the diagram.

Calculate the length of the diagonal AF.

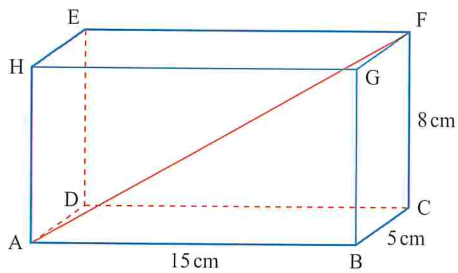


Figure 7.45

Solution

$$\text{AF} = \sqrt{15^2 + 5^2 + 8^2}$$

$$= \sqrt{314}$$

$$= 17.7\text{ cm} \quad (3\text{ s.f.})$$

Use of the sine and cosine rules in 3-D problems

Example 7.14

ABCDEFGH is a cuboid with side-lengths as shown in the diagram.

(i) Calculate the size of angle HDF.

(ii) Hence, calculate the size of angle DHF.

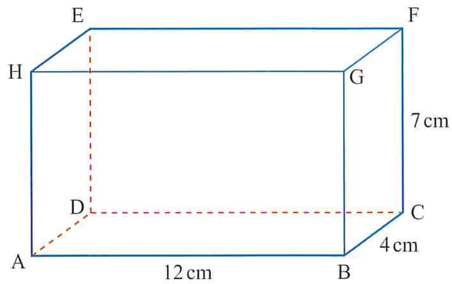


Figure 7.46

Solution

Consider triangle HDF.

(i) $HD = \sqrt{4^2 + 7^2} = \sqrt{65}$

$FD = \sqrt{12^2 + 7^2} = \sqrt{193}$

$HF = \sqrt{4^2 + 12^2} = \sqrt{160}$

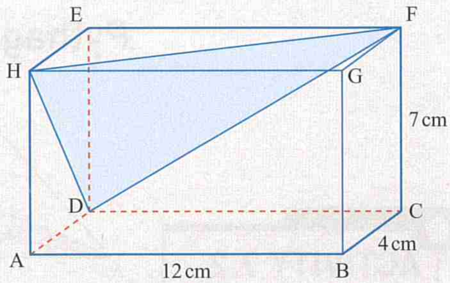


Figure 7.47

Using the cosine rule: $\cos HDF = \dfrac{HD^2 + FD^2 - HF^2}{2 \times HD \times FD}$

$$\cos HDF = \frac{65 + 193 - 160}{2 \times \sqrt{65} \times \sqrt{193}}$$

$$\cos HDF = 0.437$$

$$HDF = 64.1° \ \text{(1 d.p.)}$$

(ii) Using the sine rule: $\dfrac{\sin H}{FD} = \dfrac{\sin D}{HF}$

$$\frac{\sin H}{\sqrt{193}} = \frac{\sin 64.1}{\sqrt{160}}$$

$$\sin H = 0.988$$

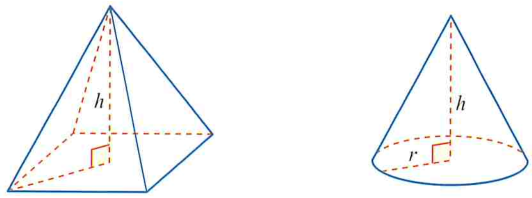
$$DHF = 81.0° \ \text{(1 d.p.)}$$

Pyramids and cones

Pyramids and cones are three-dimensional shapes. Although a cone is not a pyramid, it has the same properties. Pyramids must have a polygonal base.

In both cases their volumes are given by the formula

$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$



Volume $= \frac{1}{3} \times$ base area $\times$ height   Volume $= \frac{1}{3} \times$ base area $\times$ height $= \frac{1}{3}\pi r^2 h$

Figure 7.48

Example 7.15

A cone with a base radius of 6 cm and a height of 3π cm has the same volume as a pyramid with a square base of side 2π cm. What is the height of the pyramid?

Solution

Let h = height of pyramid

Volume of pyramid = Volume of cone

$$\therefore \quad \frac{1}{3}(2\pi)^2 h = \frac{1}{3}\pi 6^2 \times 3\pi$$

$$\Rightarrow \quad \frac{4\pi^2 h}{3} = \frac{108\pi^2}{3}$$

$$\Rightarrow \quad 4h = 108$$

$$\Rightarrow \quad h = 27\text{ cm}$$

Exercise 7E

① The cube ABCDEFGH shown in the diagram has sides of length 10 cm.

Calculate

(i) the length AC
(ii) the length AG
(iii) the angle GAC.

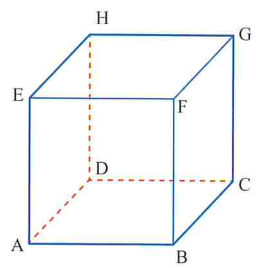


Figure 7.49

② Figure 7.50 represents a pyramid ABCD with a horizontal base ABC.

AB = AC = 5 cm and BD = CD = 13 cm.

D is vertically above A and $\angle$BAD = $\angle$CAD = 90°.

M is the midpoint of BC.

Calculate

(i) the length AM
(ii) the angle BCD
(iii) the angle between the planes BCA and BCD.

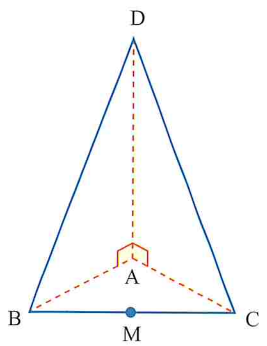


Figure 7.50

RWC ③ Figure 7.51 shows a wedge ABCDEF which has been made to hold a door open.

AB = 5 cm, BC = 12 cm and FC = 4 cm.

Calculate

(i) the angle FBC

(ii) the length AC

(iii) the angle between the line FA and the plane ABCD.

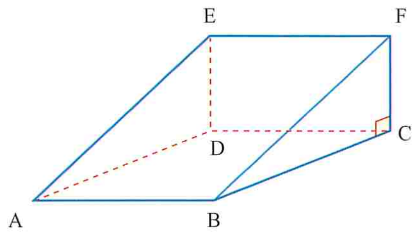


Figure 7.51

There is a gap of 2 cm between the door and the floor.

(iv) How far along BF will the base of the door meet the wedge?

④ A, B and C are points on a horizontal plane.

A is 75 m from C on a bearing of 210° and the bearing of B from C is 120°. The bearing of B from A is 075°.

From A, the angle of elevation of the top T of a vertical tower at C is 42°.

Calculate

(i) the distance BC

(ii) the height of the tower

(iii) the angle of elevation of T from B.

RWC ⑤ C is the foot of a vertical tower CT 28 m high.

A and B are points in the same horizontal plane as C and CA = CB.

P is the point on AB that is nearest to C.

The angle of elevation of the top of the tower from P is 40° and ∠ACB = 120°.

Calculate

(i) the length CP

(ii) the length CB

(iii) the length AB

(iv) the angle of elevation of the top of the tower from B.

RWC ⑥ The waste-paper basket shown in Figure 7.52 has a top ABCD that is a square of side 30 cm and a base PQRS that is a square of side 20 cm.

The line joining the centres of the top and base is perpendicular to both and is 40 cm long.

Calculate

(i) the length PR

(ii) the length AC

(iii) the length AP.

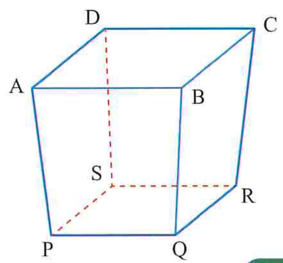


Figure 7.52

RWC ⑦ In Egypt, pyramids were used as burial chambers for the Pharaohs.

The largest of these, shown in the diagram and built about 2500 BC for Cheops, is 146 m high and has a square base of side 231 m.

X is the centre of the base and VX = 146 m.

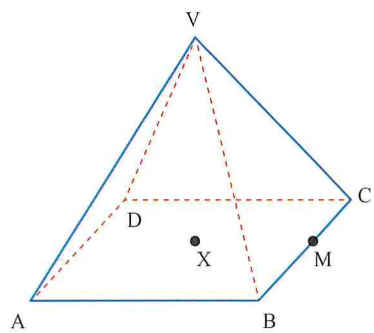


Figure 7.53

Calculate

(i) the angle between VA and ABCD
(ii) the length VA
(iii) the length VM where M is the midpoint of AB
(iv) the angle between VAB and ABCD.

RWC ⑧ The tent shown in Figure 7.54 has a base that is 2.2 m wide and 3.6 m long. The ends are isosceles triangles, inclined at an angle of 80° to the base. $\angle AEB = \angle DFC = 70°$ and M is the midpoint of AB.

Calculate

(i) the length of EM
(ii) the height of EF above the base
(iii) the length of EF.

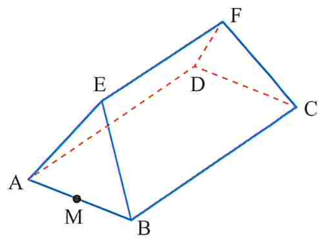


Figure 7.54

⑨ The right pyramid VABCD has rectangular base ABCD. The vertex, V, is directly above the centre, X, of the base. M is the midpoint of BC.

AB = 12 metres, BC = 9 metres and VA = 18 metres.

Work out

(i) the length AC
(ii) the length VX
(iii) the angle between VA and ABCD
(iv) the angle between VBC and ABCD.

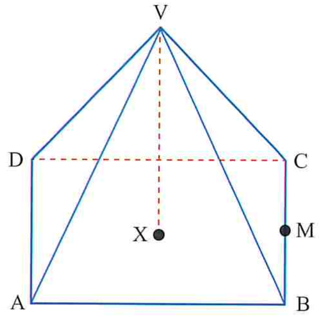


Figure 7.55

RWC ⑩ A new perfume is to be packaged in a box that is in the shape of a regular tetrahedron VABC of side 6 cm standing on a triangular prism ABCDEF as shown in the diagram.

The height of the prism is 12 cm.

M is the midpoint of BC.

Calculate

(i) the length AM
(ii) the length VM
(iii) the angle VAM
(iv) the total height of the box.

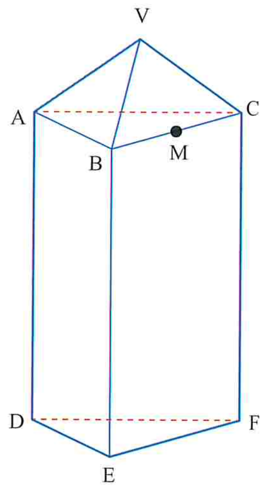


Figure 7.56

⑪ The cuboid has a square base ABCD of side 6 cm and a height of 3 cm. M is the midpoint of EG.

(i) Calculate the length of BM.
(ii) Work out the area of triangle BEG.
(iii) Work out the angle between triangle BEG and the plane ABCD.

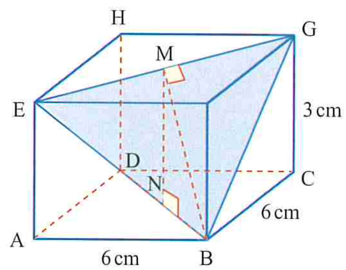


Figure 7.57

⑫ The cube has sides of 12 cm and M is the midpoint of AC.

(i) Calculate the length of DM.
(ii) Work out the angle between the planes ABCD and ACH.
(iii) Calculate the area of the largest triangle that would fit inside this cube.
(iv) What is the area of the largest triangle that would fit inside a cube of side 20 cm? Give your answer in exact form.

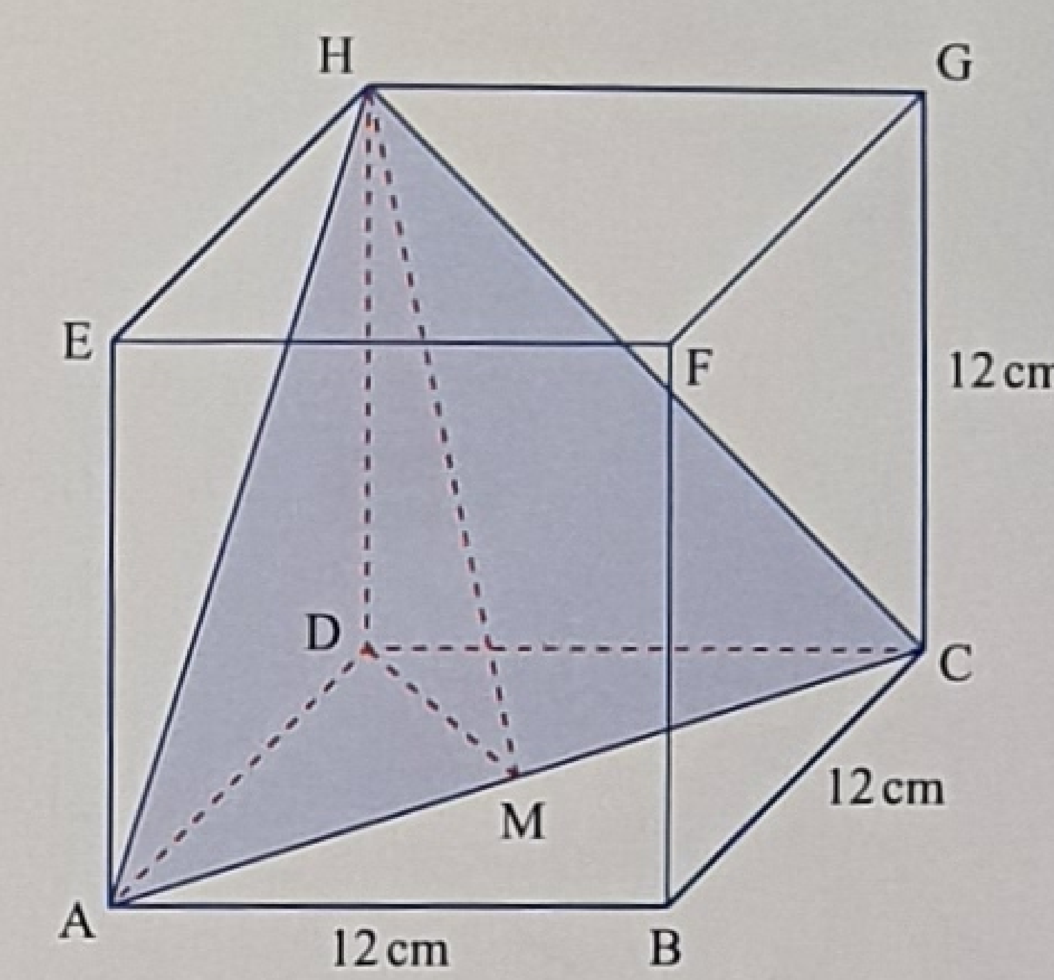


Figure 7.58

⑬ A cuboid ABCDEFGH has edges of length 8 cm, 3 cm and 5 cm as shown.

Calculate the size of the smallest angle in triangle AEG.

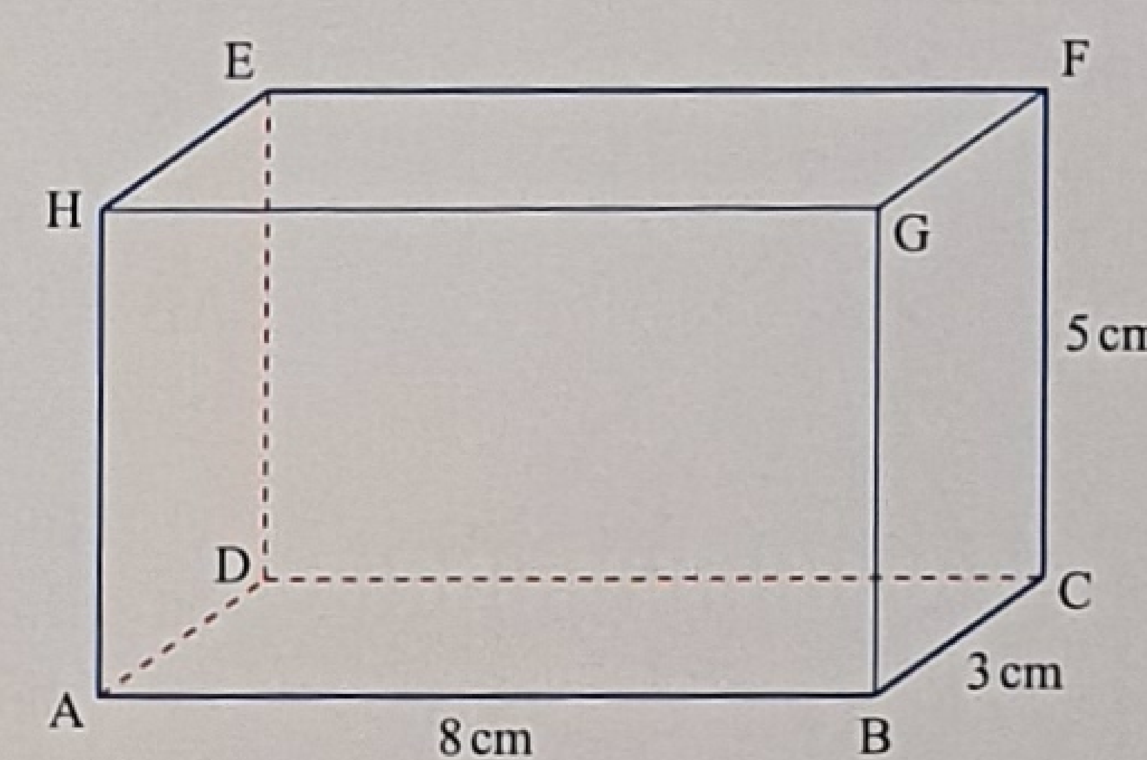


Figure 7.59

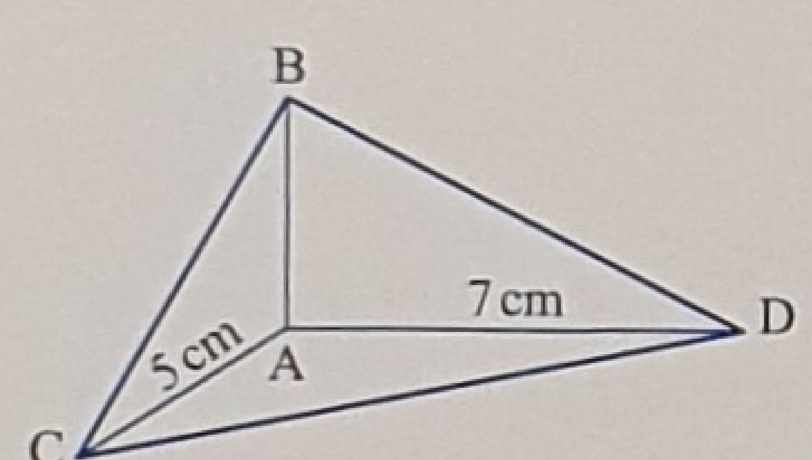


Figure 7.60

⑭ A tetrahedron ABCD has lengths AC = 5 cm and AD = 7 cm. Given that angle CAD = 130°, angle BCD = 65° and angle BDC = 55°, calculate the length of edge BC.

⑮ Tetrahedron PQRS has lengths PQ = 8 cm, PS = 10 cm and QR = 5 cm, and angles QPS = 64° and QRS = 73°. Calculate the size of angle QSR.

FUTURE USES

You will use the principles introduced here in the study of lines and planes in vector form at A-Level.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- calculate the area of a triangle given two sides and an included angle
- use the sine rule to calculate the size of an angle or a side-length
- use the cosine rule to calculate the size of an angle or side-length
- draw a 2-D representation of a 3-D object
- calculate the angle between a line and a plane or the angle between two planes
- use Pythagoras' theorem to calculate lengths in three dimensions
- solve practical problems in three dimensions using the knowledge above.

REAL-WORLD CONTEXT

Angles between two lines, a line and a plane or two planes are of prime importance to architects and engineers in the design of buildings and machinery.

There are applications in navigation, for both ships and aircraft.

There are also applications in software engineering.

KEY POINTS

1 Area of a triangle $= \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

2 Sine rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ and $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

3 Cosine rule : $a^2 = b^2 + c^2 - 2bc\cos A$ and $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

4 When solving three-dimensional problems always draw a clear diagram where:

- vertical lines are drawn vertically
- east–west lines are drawn horizontally
- north–south lines are drawn sloping
- edges that are hidden are drawn as dotted lines.

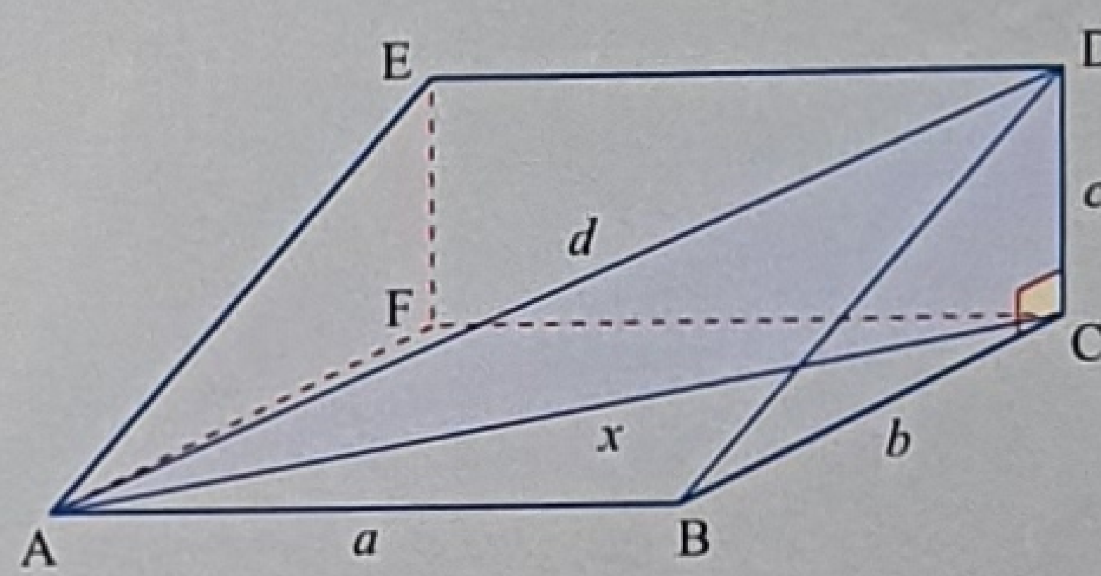


Figure 7.61

5 In three dimensions, Pythagoras' theorem extends to

$a^2 + b^2 + c^2 = d^2$

6 Volume of a pyramid (or cone) $= \frac{1}{3} \times$ base area $\times$ perpendicular height