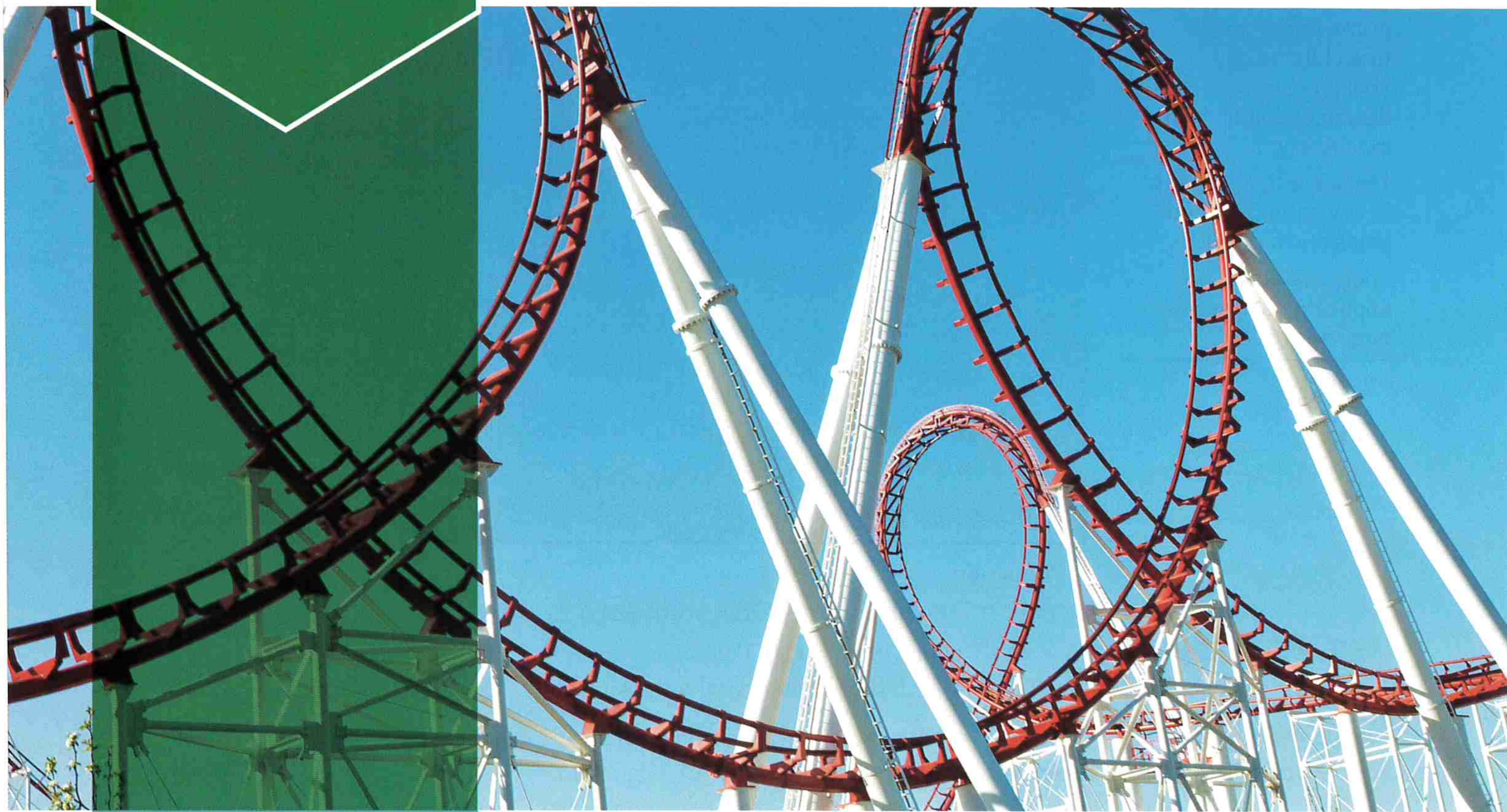
8 Calculus



I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton

Prior knowledge

The formula 'gradient $= \frac{y_2 - y_1}{x_2 - x_1}$' was introduced in Chapter 3 for the gradient of a straight line joining the two points (x_1, y_1) and (x_2, y_2).

$m = \frac{y_2 - y_1}{x_2 - x_1}$ leads to a general equation of a straight line $(y - y_1) = m(x - x_1)$.

1 The gradient of a curve

In Figure 8.1 the curve has a zero gradient at A, a positive gradient at B and a negative gradient at C.

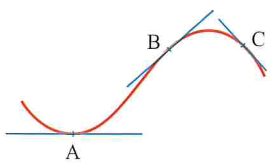


Figure 8.1

One way of finding these gradients is to draw the tangents and use two points on each one to calculate its gradient. This is time-consuming and the results depend on the accuracy of your drawing and measuring. If you know the equation of the curve, then *differentiation* provides another method of calculating the gradient.

2 Differentiation

Instead of trying to draw an accurate tangent, this method starts by calculating the gradients of chords PQ_1, PQ_2, ... As the different positions of Q get closer to P, the values of the gradient of PQ get closer to the gradient of the tangent at P. The first few positions of Q are shown in Figure 8.2.

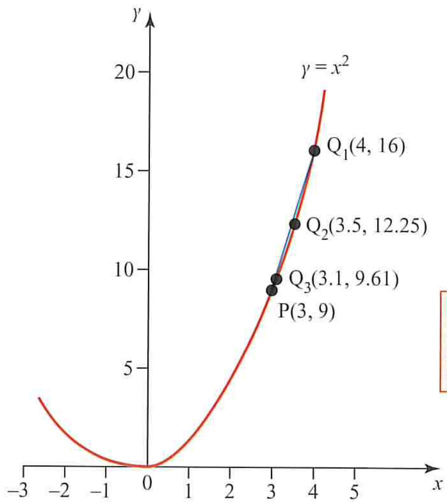


Figure 8.2

The line PQ_1 joining two points on the curve is called a chord.

For P at (3, 9)

chord	coordinates of Q	gradient of PQ
PQ_1	(4, 16)	$\frac{16-9}{4-3} = 7$
PQ_2	(3.5, 12.25)	$\frac{12.25-9}{3.5-3} = 6.5$
PQ_3	(3.1, 9.61)	$\frac{9.61-9}{3.1-3} = 6.1$
PQ_4	(3.01, 9.0601)	$\frac{9.0601-9}{3.01-3} = 6.01$
PQ_5	(3.001, 9.006 001)	$\frac{9.006\,001-9}{3.001-3} = 6.001$

ACTIVITY 8.1

Take points R_1 to R_5 on the curve $y = x^2$ with x-coordinates 2, 2.5, 2.9, 2.99, and 2.999 respectively and work out the gradients of the chords joining each of these points to P(3, 9).

In this process the gradient of the chord PQ gets closer and closer to that of the tangent, and hence the gradient of the curve at (3, 9).

Look at the sequence formed by the gradients of the chords.

7, 6.5, 6.1, 6.01, 6.001, ...

It looks as though this sequence is converging to 6

The table and the activity show that the gradient of the curve $y = x^2$ at (3, 9) seems to be 6 or about 6 but do not provide conclusive proof of its value. To do that you need to apply the method in more general terms.

Take the point P(3, 9) and another point Q close to (3, 9) on the curve $y = x^2$. Let the x-coordinate of Q be $(3 + h)$ where h is small. Since $y = x^2$ at all points on the curve, the y-coordinate of Q will be $(3 + h)^2$.

Figure 8.3 shows Q in a position where h is positive. Negative values of h would put Q to the left of P.

From Figure 8.3, the gradient of PQ is $\dfrac{(3+h)^2 - 9}{h}$

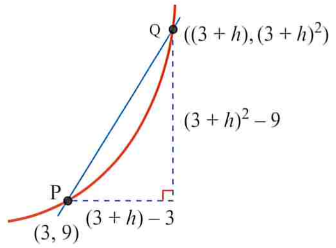


Figure 8.3

$$\begin{aligned} &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{6h + h^2}{h} \\ &= \frac{h(6+h)}{h} \\ &= 6 + h. \end{aligned}$$

For example, when $h = 0.001$, the gradient of PQ is 6.001 and when $h = -0.001$, the gradient of PQ is 5.999. The gradient of the tangent at P is between these two values. Similarly the gradient of the tangent at P would be between $6 - h$ and $6 + h$ for all small non-zero values of h.

For this to be true, the gradient of the tangent at (3, 9) must be *exactly* 6.

In this case, 6 was the *limit* of the gradient values, whether you approached P from the right or the left.

ACTIVITY 8.2

Using a similar method, work out the gradient of the tangent to the curve at

(i) (2, 4)

(ii) (−1, 1)

(iii) (−3, 9).

What do you notice?

The gradient function

The work so far has involved calculating the gradient of the curve $y = x^2$ at just one particular point. It would be very tedious if you had to do this every time and so instead you can consider a general point (x, y) and then substitute the value(s) of x and/or y corresponding to the point(s) of interest. The gradient function is a measure of how the function is changing – often referred to as 'the rate of change of the function'.

Example 8.1

Calculate the gradient of the curve $y = x^3$ at the general point (x, y).

Solution

Let P have the general value x as its x-coordinate, so P is the point (x, x^3) (since it is on the curve $y = x^3$).

Let the x-coordinate of Q be $(x + h)$ so Q is the point $((x + h), (x + h)^3)$.

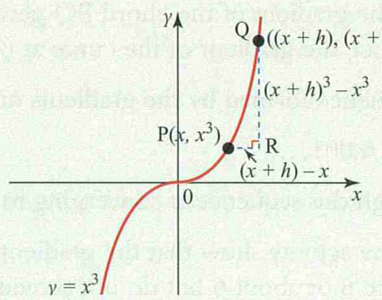


Figure 8.4

The gradient of the chord PQ is given by

$$\frac{QR}{PR} = \frac{(x+h)^3 - x^3}{(x+h) - x}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= 3x^2 + 3xh + h^2$$

As Q gets closer to P, h takes smaller and smaller values and the gradient approaches the value of $3x^2$, which is the gradient of the tangent at P.

The gradient of the curve $y = x^3$ at the point (x, y) is equal to $3x^2$.

ACTIVITY 8.3

Use the method in Example 8.1 to prove that the gradient of the curve $y = x^4$ at the point (x, y) is equal to $4x^3$.

An alternative notation

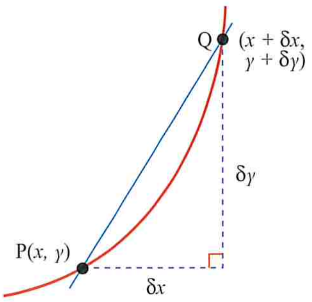


Figure 8.5

So far, h has been used to denote the difference between the x-coordinates of our points P and Q, where Q is close to P.

h is sometimes replaced by δx. The Greek letter δ (delta) is shorthand for 'a small change in' and so δx represents a small change in x, δy a small change in y and so on.

In Figure 8.5 the gradient of the chord PQ is $\frac{\delta y}{\delta x}$.

In the limit as δx tends towards 0, δx and δy both become infinitesimally small and the value obtained for $\frac{\delta y}{\delta x}$ approaches the gradient of the tangent at P.

Read this as 'the limit as δx tends towards 0'.

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is written as $\frac{dy}{dx}$.

Using this notation, you have a rule for differentiation.

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

The gradient function, $\frac{dy}{dx}$, is sometimes called the *derivative* of y with respect to x and when you find it you have *differentiated* y with respect to x.

Because of the connection with gradient, $\frac{dy}{dx}$ is also referred to as the rate of change of y with respect to x.

Notation

An alternative way of expressing a function such as $y = 2x^2 + x - 3$ is to replace y by $f(x)$ and write $f(x) = 2x^2 + x - 3$. When discussing a function written in this form you just say 'f of x equals two x squared plus x minus three'. $f'(x)$ is the notation used for the differential of $f(x)$, so when $y = f(x)$ you would write $\frac{dy}{dx} = f'(x)$. The derivative of $f(x)$ is written as $f'(x)$, pronounced as 'f dashed x'.

ACTIVITY 8.4

From earlier work you know that all lines of the form $y = x + c$, (where c can be positive, negative or zero) are parallel.

Using any software at your disposal, sketch graphs of $y = x^2$, $y = x^2 + 5$ and $y = x^2 - 3$ on the same axes setting your axes to $-4 < x < 4$ and $-5 < y < 25$

What do you notice?

Repeat this for the graphs of $y = x^2 + 2x$, $y = x^2 + 2x + 5$ and $y = x^2 + 2x - 3$

3 Differentiation using standard results

Finding the gradient from first principles establishes a formal basis for differentiation but in practice you would use the differentiation rule. This also includes the results obtained by differentiating (i.e. finding the gradient of) equations which represent straight lines.

The gradient of the line $y = x$ is 1

The gradient of the line $y = c$ is 0, where c is a constant, since this line is parallel to the x-axis.

The rule can be extended further to include functions of the type $y = kx^n$ for any constant k, to give

$$y = kx^n \Rightarrow \frac{dy}{dx} = nkx^{n-1}.$$

You may find it helpful to remember the rule as

'multiply by the power of x and reduce the power by 1'.

Reflecting on Activity 8.4 and using this rule:

$$y = x^2 + 2x \Rightarrow \frac{dy}{dx} = 2x + 2$$

$$y = x^2 + 2x + 5 \Rightarrow \frac{dy}{dx} = 2x + 2$$

$$y = x^2 + 2x - 3 \Rightarrow \frac{dy}{dx} = 2x + 2$$

The three graphs have the same gradient function so are parallel.

Example 8.2

Write down the gradient function for each of the following functions.

(i) $y = x^7$ (ii) $y = 4x^3$ (iii) $y = 5x^2$

Solution

(i) $\dfrac{dy}{dx} = 7x^6$ (ii) $\dfrac{dy}{dx} = 12x^2$ (iiii) $\dfrac{dy}{dx} = 10x$

Exactly the same rule, 'multiply by the power of x and reduce the power by 1' applies when the power is zero (i.e. y = a constant) or is negative. Remember that, for example, when you subtract 1 from 0, the answer is −1 and when you subtract 1 from −3 the answer is −4

For $y = x^0$, the rule gives $\dfrac{dy}{dx} = 0 \times x^{-1}$ which $= 0$

Example 8.3

Work out the gradient function for each of the following functions.

(i) $y = x^{-3}$ (ii) $y = 2x^{-4}$ (iii) $y = \dfrac{3}{x^2}$ (iv) $y = \dfrac{3}{4x^2}$

Solution

(i) $\dfrac{dy}{dx} = -3x^{-4}$

(ii) $\dfrac{dy}{dx} = -8x^{-5}$

(iii) First write $y = \dfrac{3}{x^2}$ as $y = 3x^{-2} \rightarrow \dfrac{dy}{dx} = -6x^{-3} = -\dfrac{6}{x^3}$

(iv) First write $y = \dfrac{3}{4x^2}$ as $y = \dfrac{3}{4}x^{-2} \rightarrow \dfrac{dy}{dx} = \dfrac{3}{4}(-2x^{-3}) = -\dfrac{3}{2x^3}$

You must leave the 4 in the denominator until you can simplify at the end.

Sums and differences of functions

Many of the functions you will meet are sums or differences of simpler ones. For example, the function $(4x^3 + 3x)$ is the sum of the functions $4x^3$ and $3x$. To differentiate a function such as this you differentiate each part separately and then add the results together.

Example 8.4

Differentiate $y = 4x^3 + 3x$.

Solution

$\dfrac{dy}{dx} = 12x^2 + 3$

Example 8.5

Differentiate $y = \dfrac{x^2}{2} - \dfrac{2}{3x^2}$.

Solution

Start by writing the expression in the form $y = \frac{1}{2}x^2 - \frac{2}{3}x^{-2}$

Differentiating, $\dfrac{dy}{dx} = \frac{1}{2}(2x) - \frac{2}{3}\left(-2x^{-3}\right)$

$$= x + \frac{4}{3}x^{-3}$$

$$= x + \frac{4}{3x^3}$$

Example 8.6

Given that $y = 2x^3 - 3x + 4$, work out

(i) $\dfrac{dy}{dx}$

(ii) the gradient of the curve at the point (2, 14)

(iii) the rate of change of y with respect to x when $x = -3$

Solution

(i) $\dfrac{dy}{dx} = 6x^2 - 3$

(ii) At (2, 14), $x = 2$

Substituting $x = 2$ in the expression for $\dfrac{dy}{dx}$ gives

$$\frac{dy}{dx} = 6 \times (2)^2 - 3 = 21$$

(iii) $\dfrac{dy}{dx}$ is the rate of change of y with respect to x.

Substituting $x = -3$ in the expression for $\dfrac{dy}{dx}$ gives

$$\frac{dy}{dx} = 6 \times (-3)^2 - 3$$

$$= 51$$

Example 8.7

Given that $y = 5x^2 - \dfrac{5}{x^2} + 2$, work out

(i) $\dfrac{dy}{dx}$

(ii) the gradient of the curve at the point (1, 2)

(iii) the rate of change of y with respect to x when $x = -1$

Solution

(i) $y = 5x^2 - \dfrac{5}{x^2} + 2 \Rightarrow y = 5x^2 - 5x^{-2} + 2$

$$\Rightarrow \frac{dy}{dx} = 10x - 5(-2)x^{-3}$$

$$\Rightarrow 10x + \frac{10}{x^3}$$

(ii) At the point $(1, 2)$, $\dfrac{dy}{dx} = 10(1) + \dfrac{10}{1} = 20$

(iii) $\dfrac{dy}{dx}$ is the rate of change of y with respect to x.

Substituting $x = -1$ in the expression for $\dfrac{dy}{dx}$ gives

$$\frac{dy}{dx} = 10(-1) + \frac{10}{(-1)^3} = -10 + (-10) = -20$$

Exercise 8A

① Differentiate the following functions.

(i) $y = x^4$ (ii) $y = 2x^3$ (iii) $y = 5x^2$

(iv) $y = 7x^9$ (v) $y = -3x^6$ (vi) $y = 5$

(vii) $y = 10x$ (viii) $y = \frac{1}{4}x^3$ (ix) $y = 2\pi x$

(x) $y = \pi x^2$

② Differentiate the following functions.

(i) $y = 2x^5 + 4x^2$ (ii) $y = 3x^4 + 8x$ (iii) $y = x^3 + 4$

(iv) $y = x - 5x^3$ (v) $y = 4x^3 + 2x$ (vi) $y = 2x + 6$

(vii) $y = 3x^5 + 2$

③ Differentiate the following functions.

(i) $y = 3x^5 + 4x^4 - 3x^2 + 2$

(ii) $y = x^5 + 12x^3 + 3x$

(iii) $y = x^3 + 42x^2 - 5x + 24$

④ Write down the rate of change of the following functions with respect to y.

(i) $y = x^{-4}$ (ii) $y = 3x^{-2}$ (iii) $y = 3x^2 + 4x^{-1}$

(iv) $y = 2x^{-3} - 4$ (v) $y = x^2 + x^{-2}$ (vi) $y = 3x^{-2} + 2x^{-3}$

⑤ Differentiate the following functions.

(i) $y = 3x^2 + \dfrac{2}{x^3}$ (ii) $y = x^2 + \dfrac{1}{x^2}$ (iii) $y = 3x^3 + \dfrac{3}{x^3}$

(iv) $y = \dfrac{2}{x} - \dfrac{3}{x^2}$ (v) $y = \dfrac{1}{2x} - \dfrac{1}{3x^2}$ (vi) $y = \dfrac{2}{3x} - \dfrac{3}{4x^2}$

⑥ A rectangle has length $6x$ and width $3x$.

The area of the rectangle is y.

(i) Write down y in terms of x.

(ii) Work out $\dfrac{dy}{dx}$.

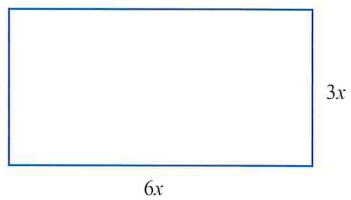


Figure 8.6

7. When a stone is thrown into a lake circular ripples appear centred on the point at which the stone entered the water and spreading outwards. After a time, t seconds, the radius of the circle is r cm where $r = 10t^2$.
   (i) Work out the rate at which the radius is increasing (include the units).

   With time, the definition of the ripples becomes negligible so that after 8 seconds they cannot be seen by the human eye.
   (ii) What is the area of the largest ripple that you can see? Give your answer to the nearest 10 square metres.
8. An expanding sphere has radius $2x$.
   (i) Show that the volume, y, of the sphere is given by the formula $y = \frac{32}{3}\pi x^3$.
   (ii) Work out the rate of change of y with respect to x when $x = 2$

Expressions that first need expanding or dividing

In this case you will need to manipulate the expression into a sum or difference before differentiating.

Example 8.8

Work out $\frac{dy}{dx}$.

(i) $y = x^3(x^2 - 4)$

(ii) $y = \frac{x^5 + x^2}{x}$

Solution

(i) Expand to give $y = x^5 - 4x^3$

$$\frac{dy}{dx} = 5x^4 - 12x^2$$

(ii) Make into two fractions $y = \frac{x^5}{x} + \frac{x^2}{x}$

Cancel to give $y = x^4 + x$

$$\frac{dy}{dx} = 4x^3 + 1$$

Exercise 8B

1. Work out the gradient function for each of the following functions.
   (i) $y = x(x^2 + 2)$
   (ii) $y = 2x^2(3x - 4)$
   (iii) $y = (x + 3)(x + 2)$
   (iv) $y = (x + 5)(x + 2)$
   (v) $y = x^3(4 + x - x^2)$
   (vi) $y = (x + 2)(x - 5)$
2. Work out an expression for the rate of change of y with respect to x for each of the following.
   (i) $y = \frac{x^5 + x^3}{4}$
   (ii) $y = \frac{x^7 + x^3}{x^2}$
   (iii) $y = \frac{4x^6 - 2x^2}{x^2}$
   (iv) $y = (3x + 1)(x - 2)$
   (v) $y = x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$
   (vi) $y = x^{\frac{1}{2}}\left(x^{\frac{7}{2}} + x^{-\frac{1}{2}}\right)$

3 (i) Simplify $\dfrac{3x^3 - 2x^2}{x}$.

(ii) Use your answer to (i) to differentiate $y = \dfrac{3x^3 - 2x^2}{x}$.

4 Work out the gradient of the curve $y = x^3(x - 2)$ at the point $(3, 27)$.

5 Work out the rate of change of y with respect to x for $\dfrac{6x^4 + 2x^5}{2x^3}$ when $x = -1$

6 Work out the rate of change of y with respect to x for $y = x^{\frac{1}{3}}\left(x^{\frac{5}{3}} - x^{\frac{2}{3}}\right)$ when $x = -3$

7 Work out the gradient of the curve $y = \dfrac{3x^4 + x^2 - 5x}{x}$ at the point $(1, -1)$.

8 Work out the gradient of the curve $y = 3\sqrt{x} - \dfrac{3}{\sqrt{x}}$ at the point $(4, 4.5)$.

Note

The rule

$y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$

is valid for all values of n but will only be examined when n is an integer.

4 Tangents and normals

Now that you know how to calculate the gradient of a curve at any point you can use this to work out the equation of the tangent at any particular point on the curve.

Example 8.9

(i) Work out the equation of the tangent to the curve $y = 3x^2 - 5x - 2$ at the point $(1, -4)$.

(ii) Sketch the curve and show the tangent on your sketch.

Solution

(i) First work out the gradient function $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = 6x - 5$$

Substitute $x = 1$ into this gradient function to calculate the gradient, m, of the tangent at $(1, -4)$

$$m = 6 \times 1 - 5$$
$$= 1$$

The equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = 1(x - 1)$$

$x_1 = 1$, $y_1 = -4$ and $m = 1$

$$\Rightarrow \quad y = x - 5$$

(ii) $y = 3x^2 - 5x - 2$ is a ∪-shaped quadratic curve.

It crosses the x-axis when $3x^2 - 5x - 2 = 0$.

$$\Rightarrow \quad (3x + 1)(x - 2) = 0$$

$$\Rightarrow \quad x = -\frac{1}{3} \quad \text{or} \quad x = 2$$

It crosses the y-axis when $y = -2$

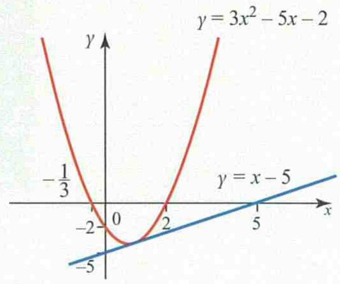


Figure 8.7

Tangents and normals

The *normal* to a curve at a particular point is the straight line that is at right angles to the tangent at that point (see Figure 8.8). Remember that for perpendicular lines $m_1 m_2 = -1$

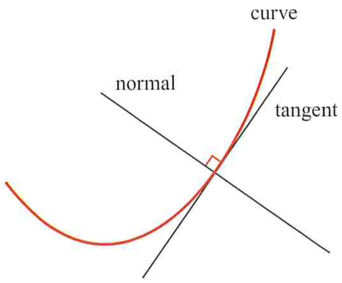


Figure 8.8

Example 8.10

Figure 8.9 is a sketch of the curve $y = x^3 - 3x^2 + 2x$ and the point P(3, 6). Work out the equation of the normal to the curve $y = x^3 - 3x^2 + 2x$ at P.

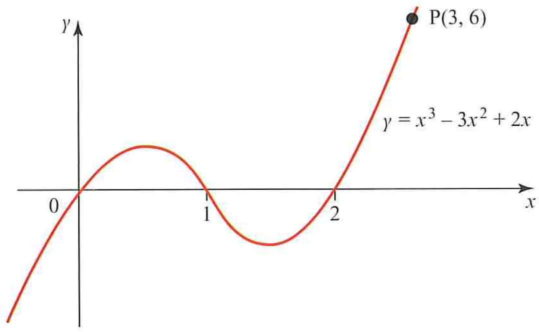


Figure 8.9

Solution

$$y = x^3 - 3x^2 + 2x \Rightarrow \frac{dy}{dx} = 3x^2 - 6x + 2$$

Substitute $x = 3$ to work out the gradient, m_1, of the tangent at the point (3, 6)

$$m_1 = 3 \times (3)^2 - 6 \times 3 + 2 = 11$$

The gradient, m_2, of the normal to the curve at this point is given by

$$m_2 = -\frac{1}{m_1} = -\frac{1}{11}.$$

$m_1 m_2 = -1$

The equation of the normal is given by

$$y - y_1 = m_2(x - x_1)$$

(x_1, y_1) is (3, 6).

$$\Rightarrow \quad y - 6 = -\frac{1}{11}(x - 3)$$

Multiply by 11 to eliminate the fraction.

$$\Rightarrow \quad 11y - 66 = -x + 3$$

$$\Rightarrow \quad x + 11y - 69 = 0$$

Example 8.11

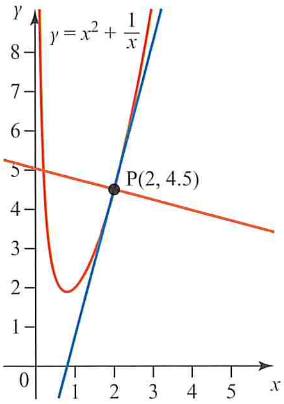


Figure 8.10

Figure 8.10 is a sketch of the curve $y = x^2 + \dfrac{1}{x}$ for $0 \leqslant x \leqslant 4$ where P is the point (2, 4.5).

(i) Work out the equation of the tangent to the curve at P

(ii) Work out the equation of the normal to the curve at P.

Solution

(i) $y = x^2 + \dfrac{1}{x} = x^2 + x^{-1} \Rightarrow \dfrac{dy}{dx} = 2x - x^{-2}$

At (2, 4.5) $\dfrac{dy}{dx} = 4 - \dfrac{1}{4} = 3.75$ which is the gradient of the tangent.

Using $(y - y_1) = m(x - x_1)$ the equation of the tangent is

$$y - 4.5 = 3.75(x - 2)$$
$$\Rightarrow \quad y - 4.5 = 3.75x - 7.5$$
$$\Rightarrow \quad y = 3.75x - 3$$

(ii) The gradient of the tangent $= 3.75 = \dfrac{15}{4}$ so the gradient of the normal is $-\dfrac{4}{15}$.

Using $(y - y_1) = m(x - x_1)$ the equation of the normal is

$$y - 4.5 = -\frac{4}{15}(x - 2)$$
$$\Rightarrow \quad 15(y - 4.5) = -4(x - 2)$$
$$\Rightarrow \quad 15y - 67.5 = -4x + 8$$
$$\Rightarrow \quad 4x + 15y - 75.5 = 0$$

Exercise 8C

① The sketch shows the graph of $y = 5x - x^2$.

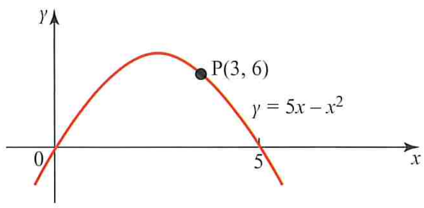


Figure 8.11

The marked point, P, has coordinates (3, 6). Work out

(i) the gradient function $\dfrac{dy}{dx}$

(ii) the gradient of the curve at P

(iii) the equation of the tangent at P

(iv) the equation of the normal at P.

② The sketch shows the graph of $y = 3x^2 - x^3$.

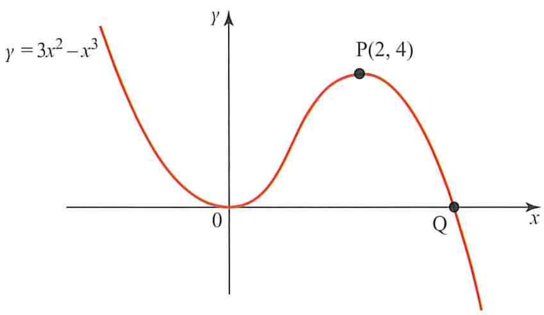


Figure 8.12

(i) The marked point, P, has coordinates (2, 4). Work out

(a) the equation of the tangent at P

(b) the equation of the normal at P.

(ii) The graph touches the x-axis at the origin O and crosses it at the point Q. Work out the equation of the tangent at Q.

(iii) Without further calculation, state the equation of the tangent to the curve at O.

③ The sketch shows the graph of $y = x^5 - x^3$.

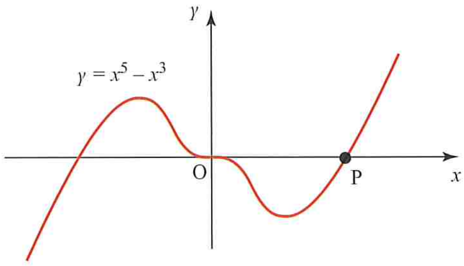


Figure 8.13

(i) Work out the coordinates of the point P where the curve crosses the positive x-axis.

(ii) Work out the equation of the tangent at P.

(iii) Work out the equation of the normal at P.

The tangent at P meets the y-axis at Q and the normal meets the y-axis at R.

(iv) Work out the coordinates of Q and R and hence calculate the area of triangle PQR.

④ (i) Given that $y = x^3 - 3x^2 + 4x + 1$, work out the gradient function $\frac{dy}{dx}$.

(ii) The point P is on the curve $y = x^3 - 3x^2 + 4x + 1$ and its x-coordinate is 2.

(a) Work out the equation of the tangent at P.

(b) Work out the equation of the normal at P.

(iii) Work out the values of x for which the curve has a gradient of 13

⑤ The sketch shows the graph of $y = x^3 - 9x^2 + 23x - 15$

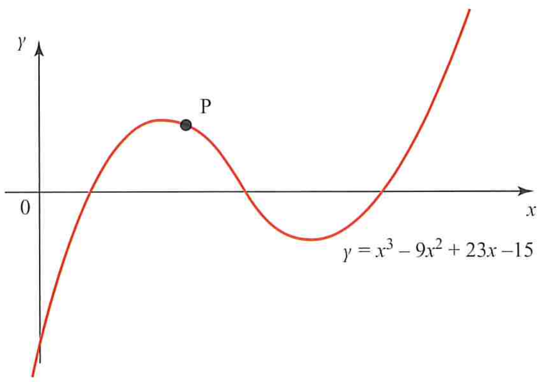


Figure 8.14

The point P marked on the curve has its x-coordinate equal to 2

(i) Work out the equation of the tangent at P.

Q is a point on the curve where the tangent is parallel to the tangent at P.

(ii) Work out the equation of the tangent at Q.

6 The point $(2, -8)$ is on the curve $y = x^3 - px + q$.

(i) Identify a relationship between p and q.

The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis.

(ii) Work out the value of p.

(iii) Work out the coordinates of the other point where the tangent is parallel to the x-axis.

(iv) Work out the equation of the normal to the curve at the point where it crosses the y-axis.

7 The sketch shows the graph of $y = x^2 - x - 1$

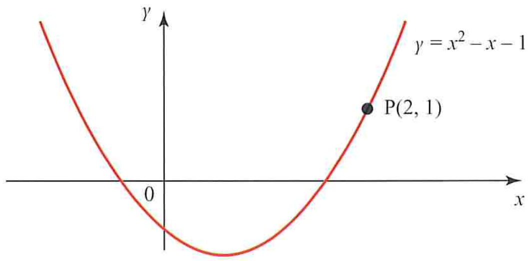


Figure 8.15

(i) Work out the equation of the tangent at the point P.

The normal at a point Q on the curve is parallel to the tangent at P.

(ii) Work out the coordinates of the point Q.

8 A curve has the equation $y = (x - 3)(7 - x)$.

(i) Work out the equation of the tangent at the point $(6, 3)$.

(ii) Work out the equation of the normal at the point $(6, 3)$.

(iii) Which one of these lines passes through the origin?

9 A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.

(i) Show that the curve passes through the points $(0, 0)$ and $(1, 0)$.

(ii) Work out the equations of the tangents and normals at each of these points.

(iii) What shape is formed by the four lines in part (ii)?

10 Figure 8.16 shows the curve with the equation $y = x^2 + \frac{2}{x}$ for $x > 0$

(i) Work out the gradient function $\frac{dy}{dx}$ and calculate the coordinates of the minimum point.

(ii) State the equations of the tangent and the normal at that minimum point.

(iii) Work out the equations of the tangent and normal at the point where $x = 2$

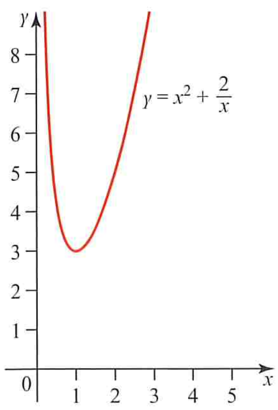


Figure 8.16

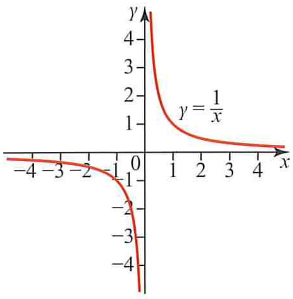


Figure 8.17

Discussion point

➜ The curve on the left has equation $y = \frac{1}{x}$ for $x \neq 0$

(i) Work out the equations of the tangents at the points $(-1, -1)$ and $(1, 1)$.

(ii) What do you notice about these lines?

(iii) Work out the equations of the normals at the points $(-1, -1)$ and $(1, 1)$.

(iv) What do you notice about these?

5 Increasing and decreasing functions

A function $y = f(x)$ is

- increasing if $\frac{dy}{dx} > 0$
- decreasing if $\frac{dy}{dx} < 0$

Some functions are increasing or decreasing over their whole domain.

For example, $y = 3 - 2x$ is a decreasing function for all real values of x because $\frac{dy}{dx} = -2$ which is < 0

Other functions are increasing over parts of their domain and decreasing over others.

Example 8.12

Work out the values of x for which the function $y = x^2 - 4x + 1$ is an increasing function.

Solution

First work out $\frac{dy}{dx}$ $\qquad \frac{dy}{dx} = 2x - 4$

To be an increasing function $\frac{dy}{dx} > 0$ $\qquad 2x - 4 > 0$

$$2x > 4$$

$$x > 2$$

Exercise 8D

① Work out the values of x for which the following functions are increasing.

(i) $y = x^2 + 4$ (ii) $y = 2x - 3$

(iii) $y = x^2 + 2x - 5$ (iv) $y = x^2 - 3x$

(v) $y = 3x^2 + 4x + 7$ (vi) $y = (x + 6)(x - 2)$

(vii) $y = x^3 - 2x^2$ (viii) $y = x^3 + 6x^2 - 15x$

(ix) $y = x^3 - 3x^2 - 9x + 1$

② Work out the values of x for which the following functions are decreasing.

(i) $y = 4x^2$ (ii) $y = x^2 - 6x + 2$

(iii) $y = x(x + 2)$ (iv) $y = 3 + 4x - x^2$

(v) $y = 12 - x$ (vi) $y = (2x + 1)^2$

(vii) $y = \frac{1}{3}x^3 + x^2$ (viii) $y = 2x^3 - 3x^2 - 72x$

(ix) $y = 27x - x^3$

③ Prove that $y = \frac{1}{3}x^3 + 2x^2 + 7x + 1$ is an increasing function for all values of x.

④ Prove that $y = x^3 - 6x^2 + 27x - 4$ is an increasing function for all values of x.

⑤ Work out the values of x for which $y = x^2 + \frac{2}{x}$ is an increasing function.

⑥ Prove that $y = 12 - 2x - x^3$ is a decreasing function for all values of x.

⑦ Prove that $y = \frac{1}{x}$ is a decreasing function for all $x \neq 0$

⑧ Work out the values of x for which the following functions are

(a) increasing (b) decreasing.

(i) $y = x + \frac{1}{x}$ (ii) $y = x - \frac{1}{x}$

(iii) $y = x^2 + \frac{1}{x^2}$ (iv) $y = x^2 - \frac{1}{x^2}$

⑨ Air is being pumped into a spherical balloon at the rate of $1000\,\text{cm}^3\,\text{s}^{-1}$. Initially the balloon contains no air. (The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$).

(i) Calculate the volume V of the balloon after 10 seconds.

(ii) Calculate the volume of the balloon after t seconds.

(iii) State the value of $\frac{dV}{dt}$.

(iv) Calculate the radius of the balloon after t seconds.

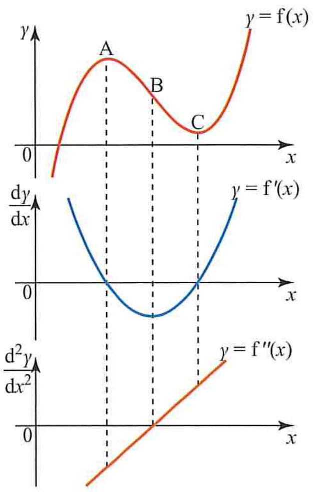


Figure 8.18

6 The second derivative

Figure 8.18 shows a sketch of a function $y = f(x)$ with a sketch of the corresponding gradient function, $\frac{dy}{dx} = f'(x)$ below it.

The third graph shows the gradient of the function $y = f'(x)$, denoted by $y = f''(x)$.

The gradient of any point on the curve of $\frac{dy}{dx}$ is found by differentiating $\frac{dy}{dx}$ and is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. This is written as $\frac{d^2y}{dx^2}$ or $y = f''(x)$ and is called the second derivative.

! $\frac{d^2y}{dx^2}$ is not the same as $\left(\frac{dy}{dx}\right)^2$.

Example 8.13

Given that $y = 2x^3 - 4x^2 + 3x - 1$, work out $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dy}{dx} = 6x^2 - 8x + 3$$

$$\frac{d^2y}{dx^2} = 12x - 8$$

Example 8.14

A ball is thrown upwards with a speed of $20\,\text{ms}^{-1}$. Its height h m above the ground after a time of t seconds is given by $h = 1 + 20t - 5t^2$.

(i) Work out $\frac{dh}{dt}$ and say what this represents.

(ii) Calculate the maximum height reached by the ball and the time at which this height is reached.

(iii) Work out the rate of change of $\frac{dh}{dt}$, written as $\frac{d^2h}{dt^2}$, and say what this represents.

(iv) Sketch the graph of h against t.

Solution

(i) $\frac{dh}{dt} = 20 - 10t$

This represents the velocity of the stone.

(ii) The maximum height is reached when the ball is instantaneously at rest. This means that $\frac{dh}{dt} = 0$ giving $20 - 10t = 0$, so $t = 2$

When $t = 2$, $h = 1 + 20(2) - 5(4) = 21$

The maximum height is 21 m above the ground after a time of 2 seconds.

When the velocity is positive the stone is moving upwards and when it is negative it is moving downwards. When it is zero it is stationary at the highest point.

(iii) $\frac{dh}{dt} = 20 - 10t \quad \Rightarrow \quad \frac{d^2h}{dt^2} = -10$

The rate of change of velocity is acceleration, and the positive direction is measured upwards, so this means that the acceleration of the ball is $-10\,\text{ms}^{-2}$ upwards, which is the same as saying that the ball is decelerating, i.e. slowing down, at a rate of $10\,\text{ms}^{-2}$ as it travels upwards. On its descent it will accelerate at a rate of $10\,\text{ms}^{-2}$ downwards.

(iv) $h = 1 + 20t - 5t^2$ is represented by a quadratic graph passing through (0, 1) and having a maximum point at (2, 21).

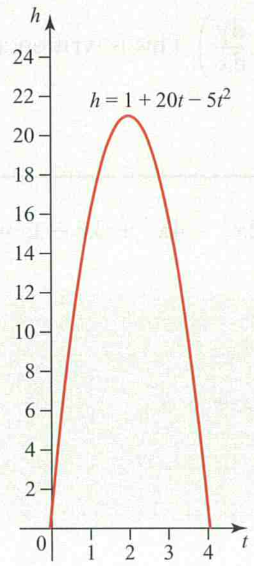


Figure 8.19

Exercise 8E

① Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3x^3 + 3x$ (ii) $y = x^5 - 25$

(iii) $y = 3x - 5x^4$

② Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = x^4 - 2x^2 + 5x - 4$ (ii) $y = 2x^3 + 3x - 4$

(iii) $y = x^3 - 2x^2 + 1$

③ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions. Remember that when an expression involves brackets you need to multiply out before differentiating.

(i) $y = (2x - 1)(x + 2)$ (ii) $y = (2x - 1)^2$

(iii) $y = (1 - 3x)(2x - 3)$

④ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.

(i) $y = 3(x - 2)\left(x^2 - 2x + 3\right)$ (ii) $y = 2x^2(x - 1)^2$

(iii) $y = x^3(3x + 1)^2$

⑤ The sum of two numbers x and y is 13 and their product P is 40.

(i) Write down an expression for y in terms of x.

(ii) Write down an expression for P in terms of x.

(iii) Write down expressions for $\frac{dy}{dx}$ and $\frac{dP}{dx}$.

(iv) Write down the rate of change of $\frac{dP}{dx}$.

⑥ For the curve $y = 3x^3 - 2x^2 - 6x - 4$

(i) write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) work out the gradient of the curve at the points $(-1, -7)$, $(1, -9)$ and $(2, 0)$.

(iii) work out the rate of change of the gradient at each of these points.

⑦ A formula which you will meet in Mechanics or Physics is $s = ut + \frac{1}{2}at^2$, where the letters in this case are t = time, u is the initial velocity (which will be a constant, or zero if starting from rest), a is the acceleration (which must also be constant, for this formula) and s is the distance travelled. The only variables in the formula are s and t. Using this formula $\frac{ds}{dt}$ will give the velocity after a time t has elapsed.

(i) Work out $\frac{ds}{dt}$ and hence the velocity after 12 seconds when the distance is measured in metres and time in seconds.

(ii) Work out $\frac{d^2s}{dt^2}$.

7 Stationary points

ACTIVITY 8.5

(i) Plot the graph of $y = x^4 - 3x^3 - x^2 + 3x$, taking values of x from -1.5 to $+3.5$ in steps of 0.5

You will need your y-axis to go from -10 to $+20$

Alternatively, if you have access to a graphics calculator or graphing software you could use that.

(ii) Describe the curve as x goes from -1.5 to 3.5

A *stationary point* on a curve is one where the gradient is zero. This means that the tangents to the curve at these points are horizontal. Figure 8.20 shows a curve with two stationary points A and B.

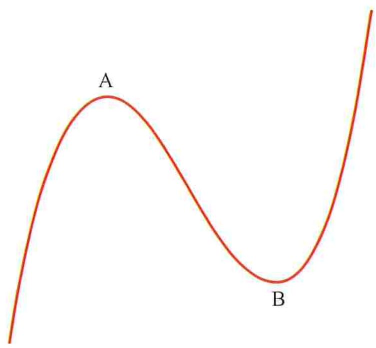


Figure 8.20

Note

Maximum and minimum points are turning points. Questions on this specification will not use the term 'turning points'.

As the curve passes through the points A and B it changes direction completely. At A the gradient changes from positive to negative and at B from negative to positive. A is called a *maximum* point and B is a *minimum* point.

ACTIVITY 8.6

Figure 8.21 shows the graph of $y = \cos x$.
Describe the gradient of the curve, using the words 'positive', 'negative', 'zero', 'increasing' and 'decreasing', as x increases from 0° to 360°.

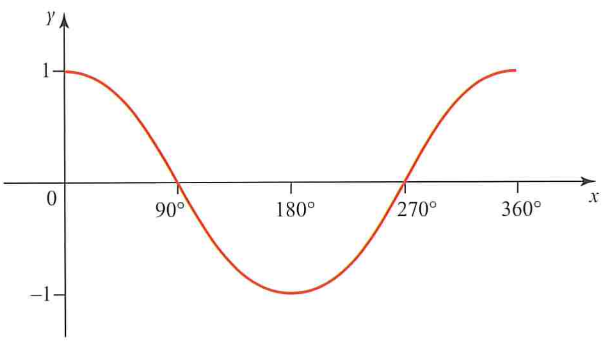


Figure 8.21

Maximum and minimum points

Figure 8.22 shows the graph of $y = 4x - x^2$. It has a maximum point at (2, 4).

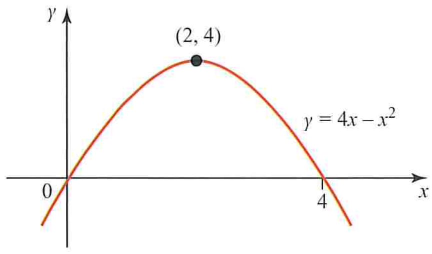


Figure 8.22

You can see that

- at the maximum point the gradient $\frac{dy}{dx}$ is zero
- the gradient is positive to the left of the maximum and negative to the right of it.

This is true for any maximum point (see Figure 8.23).

In the same way, for any minimum point (see Figure 8.24)

- the gradient is zero at the minimum
- the gradient goes from negative to zero to positive.

You can see that the gradient function is decreasing $(+, 0, -)$ through a maximum point and increasing $(-, 0, +)$ through a minimum point.

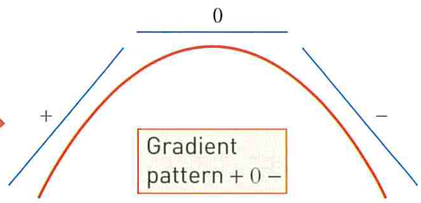


Figure 8.23

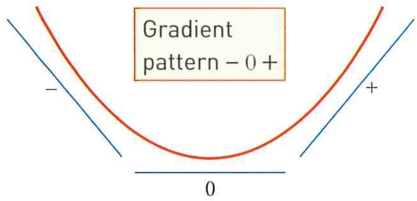


Figure 8.24

Once you have found the position and type of any stationary points, you can use this information to sketch the curve.

Example 8.15

For the curve $y = x^3 - 12x + 3$

(i) work out $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 0$

(ii) classify the points on the curve with these x-values

(iii) work out the corresponding y-values

(iv) sketch the curve.

Solution

(i) $\frac{dy}{dx} = 3x^2 - 12$

When $\frac{dy}{dx} = 0$

$$3x^2 - 12 = 0$$

$$\Rightarrow \quad 3(x^2 - 4) = 0$$

$$\Rightarrow \quad 3(x + 2)(x - 2) = 0$$

$$\Rightarrow \quad x = -2 \quad \text{or} \quad x = 2$$

(ii) For $x = -2$

$$x = -3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(-3)^2 - 12 = +15$$

$$x = -1 \quad \Rightarrow \quad \frac{dy}{dx} = 3(-1)^2 - 12 = -9$$

Gradient pattern + 0 –

$\Rightarrow$ maximum point when $x = -2$

For $x = +2$

$$x = 1 \quad \Rightarrow \quad \frac{dy}{dx} = 3(1)^2 - 12 = -9$$

$$x = 3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(3)^2 - 12 = +15$$

Gradient pattern – 0 +

$\Rightarrow$ minimum point when $x = +2$

(iii) When $x = -2$, $y = (-2)^3 - 12(-2) + 3 = 19$

When $x = +2$, $y = (2)^3 - 12(2) + 3 = -13$

(iv) There is a maximum at $(-2, 19)$ and a minimum at $(2, -13)$.

The only other information you need to sketch the curve is the value of y when $x = 0$. This tells you where the curve crosses the y-axis.

When $x = 0$, $y = (0)^3 - 12(0) + 3 = 3$

The graph of $y = x^3 - 12x + 3$ is shown in Figure 8.25.

Discussion point

➜ Why can you be confident about continuing the sketch of the curve beyond the x-values of the stationary points?

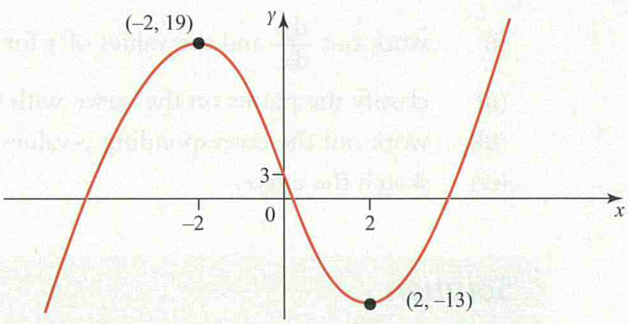


Figure 8.25

Discussion point

➜ In Example 8.15 you did not work out the coordinates of the points where the curve crosses the x-axis.

(i) Why was this?

(ii) Under what circumstances would you work out these points?

Example 8.16

Identify all the stationary points on the curve of $y = x^4 - 2x^3 + x^2 - 2$ and sketch the curve.

Solution

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 2x\left(2x^2 - 3x + 1\right) = 0$$

$$\Rightarrow \quad 2x(2x - 1)(x - 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = 0.5 \text{ or } x = 1$$

You may find it helpful to summarise your working in a table. You can find the various signs, + or –, by taking a test point in each interval, for example, $x = 0.25$ in the interval $0 < x < 0.5$

	$x < 0$	0	$0 < x < 0.5$	0.5	$0.5 < x < 1$	1	$x > 1$
sign of $\frac{dy}{dx}$	–	0	+	0	–	0	+
stationary point		min		max		min	

When $x = 0$: $\quad y = (0)^4 - 2(0)^3 + (0)^2 - 2 = -2$

When $x = 0.5$: $y = (0.5)^4 - 2(0.5)^3 + (0.5)^2 - 2 = -1.9375$

When $x = 1$: $\quad y = (1)^4 - 2(1)^3 + (1)^2 - 2 = -2$

Therefore (0.5, –1.9375) is a maximum stationary point and (0, –2) and (1, –2) are both minimum stationary points.

The graph of $y = x^4 - 2x^3 + x^2 - 2$ is shown in Figure 8.26.

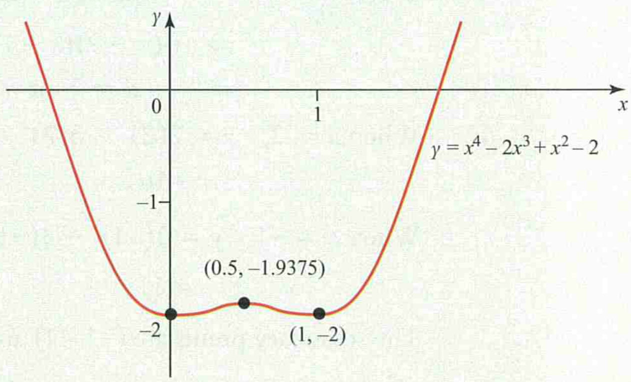


Figure 8.26

The method above, setting out a table of values, is rather tedious but there is an alternative way of identifying whether a stationary point is a maximum or a minimum using the second derivative. Recall that $\frac{d^2y}{dx^2}$ represents the rate of change of $\frac{dy}{dx}$, i.e. the rate of change of the gradient.

Figures 8.23 and 8.24 illustrated that at a minimum/maximum the gradient function is increasing/decreasing. At a point where $\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} > 0$ gives a minimum

$\frac{d^2y}{dx^2} < 0$ gives a maximum

If $\frac{d^2y}{dx^2}$ is positive at a stationary point (i.e. where $\frac{dy}{dx} = 0$), then the gradient must go from negative to positive, in which case the turning point will be a minimum.

Conversely, if $\frac{d^2y}{dx^2}$ is negative at a stationary point, then the gradient must go from positive to negative which will indicate a maximum turning point.

Note

If $\frac{d^2y}{dx^2} = 0$ at the stationary point, it is not possible to use this method and you will have to go back to the method of checking the gradient on each side of the stationary point.

Example 8.17

Given that $y = 2x^3 - 3x^2 - 12x + 4$

(i) work out $\frac{dy}{dx}$, and the values where $\frac{dy}{dx} = 0$

(ii) work out the coordinates of each of the stationary points

(iii) work out the value of $\frac{d^2y}{dx^2}$ at each of the stationary points and hence determine the nature of each one

(iv) sketch the graph of $y = 2x^3 - 3x^2 - 12x + 4$

Solution

(i) $\frac{dy}{dx} = 6x^2 - 6x - 12$

When $\frac{dy}{dx} = 0$, $\quad 6(x^2 - x - 2) = 0$

$\Rightarrow \quad 6(x - 2)(x + 1) = 0$

$\Rightarrow \quad x = 2 \text{ or } x = -1$

(ii) When $x = 2$, $y = 2(2)^3 - 3(2)^2 - 12(2) + 4$

$= -16$

When $x = -1$, $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 4$

$= 11$

The stationary points are $(-1, 11)$ and $(2, -16)$

(iii) $\frac{d^2y}{dx^2} = 12x - 6$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$ so $(-1, 11)$ is a maximum point.

When $x = 2$, $\frac{d^2y}{dx^2} = +18 > 0$ so $(2, -16)$ is a minimum point.

(iv) The curve crosses the y-axis when $x = 0$, i.e. the point $(0, 4)$. This information, together with the positions of the stationary points, is sufficient to enable you to sketch the curve.

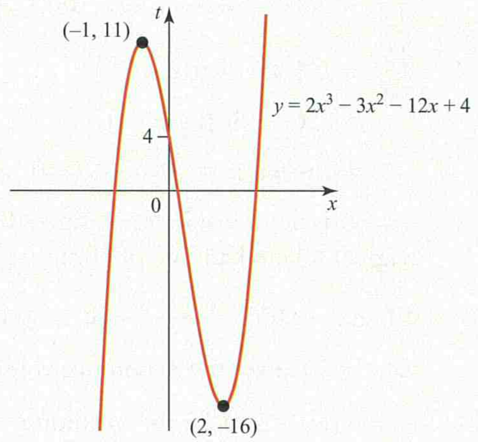


Figure 8.27

Applications of maxima and minima

Example 8.18

An open box is made from a square sheet of cardboard, with sides 60 cm long, by cutting out a square from each corner, folding up the sides and joining the cut edges.

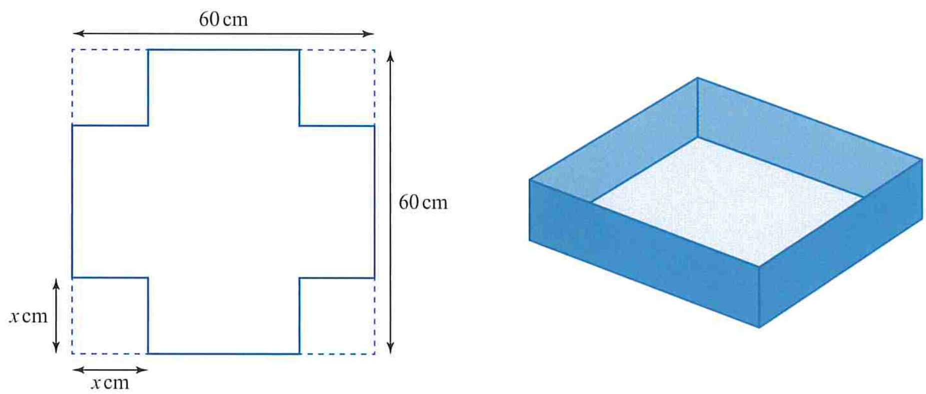


Figure 8.28

(i) Write down an expression for the volume V of the box in terms of x.

(ii) Work out $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$ and use these to calculate the value of x giving the maximum volume.

(iii) What is the maximum volume?

Solution

(i) Volume = area of base × depth

$$= (60 - 2x)^2 \times x$$

$$= x(60 - 2x)^2 \text{ cm}^3$$

(ii) Expanding this, $V = x(3600 - 240x + 4x^2)$

$$= 3600x - 240x^2 + 4x^3$$

$$\frac{dV}{dx} = 3600 - 480x + 12x^2 \text{ and } \frac{d^2V}{dx^2} = -480 + 24x$$

$$\frac{dV}{dx} = 12(300 - 40x + x^2)$$
$$= 12(x - 30)(x - 10)$$
$$= 0 \text{ when } x = 10 \text{ or } x = 30$$

$x = 30$ is not a viable result, since there would be no box (the original square had side of 60 cm) so this value must be rejected.

When $x = 100$, $\frac{d^2V}{dx^2} = -480 + 24(10) = -240$ (i.e. negative)

$\Rightarrow x = 10$ gives the maximum volume.

(iii) $V = x(60 - 2x)^2$ so the maximum volume is 10×40^2 cm^3

Maximum volume is 16 000 cm^3.

Example 8.19

The perimeter of a rectangle has a fixed length. Show that the area of the rectangle is greatest when the rectangle is a square.

Solution

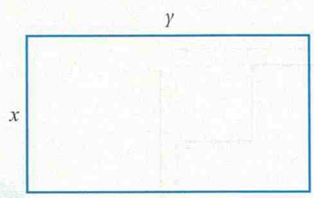


Figure 8.29

The perimeter $2x + 2y = c$, where c is a constant.

$$\Rightarrow 2y = c - 2x$$

$$\Rightarrow y = \frac{1}{2}c - x$$

Area $A = xy = x\left(\frac{1}{2}c - x\right)$

$$= \frac{1}{2}cx - x^2$$

Differentiating: $\frac{dA}{dx} = \frac{1}{2}c - 2x$

$$= 0 \text{ when } x = \frac{c}{4}$$

$\frac{d^2A}{dx^2} = -2 \Rightarrow$ stationary point is a maximum.

Since $y = \frac{1}{2}c - x$, when $x = \frac{c}{4}$, then $y = \frac{c}{4}$ also, so the shape is a square.

Consequently the area is greatest when the rectangle is a square.

Example 8.20

A closed cardboard box with a square base is to be constructed to have a capacity of 216 000 cm³. What dimensions use a minimum amount of cardboard?

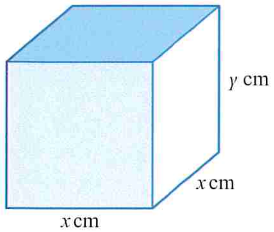


Figure 8.30

Solution

Area of base = area of top = x^2.

Area of sides = $4 \times xy$.

The area of cardboard used $A = 2x^2 + 4xy$.

Volume of the box $V = x^2y \Rightarrow 216\,000 = x^2y$

$$\Rightarrow y = \frac{216\,000}{x^2} = 216\,000x^{-2}.$$

Substituting for y in the expression for A gives

$$A = 2x^2 + 4x \times 216\,000x^{-2}$$
$$= 2x^2 + 864\,000x^{-1}.$$

Differentiating: $\dfrac{dA}{dx} = 4x - 864\,000x^{-2}$

$= 0$ when $4x = 864\,000x^{-2}$

$\Rightarrow x^3 = 216\,000$

$\Rightarrow x = 60$

Differentiating again: $\dfrac{d^2A}{dx^2} = 4 + 1728\,000x^{-3}$

> 0 when $x = 60$ so a minimum.

When $x = 60$, $y = \dfrac{216\,000}{60^2} = 60$

The dimensions of the box using the minimum amount of cardboard are 60 cm × 60 cm × 60 cm.

Example 8.21

$P = 32x^2 + \dfrac{8}{x}$.

Show that P has a minimum value when $x = \dfrac{1}{2}$ and work out the minimum value of P.

Solution

$$P = 32x^2 + \frac{8}{x}$$
$$= 32x^2 + 8x^{-1}$$

$\dfrac{dP}{dx} = 64x - 8x^{-2}$

$= 0$ when $64x = 8x^{-2}$

$\Rightarrow x^3 = \dfrac{8}{64} = \dfrac{1}{8}$

$\Rightarrow x = \dfrac{1}{2}$

$\dfrac{d^2P}{dx^2} = 64 + 16x^{-3}$

$x = \dfrac{1}{2} \Rightarrow \dfrac{d^2P}{dx^2} = 192$ (i.e. positive)

P is a minimum when $x = \dfrac{1}{2}$

The minimum value of P is $32 \times 0.5^2 + \dfrac{8}{0.5} = 24$

Exercise 8F

If you have access to a graphic calculator you will find it helpful to use it to check your answers.

① For each of the curves given below

(a) work out $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$

(b) work out the value(s) of $\frac{d^2y}{dx^2}$ at those points

(c) classify the point(s) on the curve with these x-values

(d) work out the corresponding y-value(s)

(e) sketch the curve.

(i) $y = 1 + x - 2x^2$

(ii) $y = 12x + 3x^2 - 2x^3$

(iii) $y = x^3 - 4x^2 + 9$

(iv) $y = x(x - 1)^2$

(v) $y = x^2(x - 1)^2$

(vi) $y = x^3 - 48x$

(vii) $y = x^3 + 6x^2 - 36x + 25$

(viii) $y = 2x^3 - 15x^2 + 24x + 8$

② The graph of $y = px + qx^2$ passes through the point $(3, -15)$ and its gradient at that point is -14

(i) Work out the values of p and q.

(ii) Calculate the maximum value of y and state the value of x at which it occurs.

③ (i) Identify the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.

(ii) Sketch the curve $y = f(x)$.

④ The curve $y = ax^2 + bx + c$ crosses the y-axis at the point $(0, 2)$ and has a minimum point at $(3, 1)$.

(i) Work out the equation of the curve.

(ii) Check that the stationary point is a minimum.

⑤ The sum of two positive numbers p and q is 12

(i) Write q in terms of p.

(ii) S is the sum of the squares of the two numbers. Write down an expression for S in terms of p.

(iii) Work out the least value of S, checking that it is a minimum.

⑥ The sum of two positive numbers a and b is 40

(i) Write $2ab$ in terms of a.

(ii) Work out values of a and b when $2ab$ is a maximum, checking that it is a maximum.

(iii) Work out the maximum value of $2ab$.

⑦ x and y are two positive numbers whose sum is 10

(i) Express $P = xy^2$ in terms of x.

(ii) Work out the values of x for which $\frac{dP}{dx} = 0$

(iii) Use the second derivative test to identify which one gives the maximum value of P.

(iv) Comment on the implication of the other value of x that you calculated.

8. Netty and Mackenzie are going to climb a mountain and the equation of their path is given by $10y = x + 4x^2 - x^3$ for $x \geqslant 0$. Distances horizontally and vertically are measured in units of 1000 metres. Give all answers to 3 significant figures.

   (i) How far away, horizontally, is the summit?

   (ii) How much higher is the summit than where they are now?

9. The base of a cuboid is x cm by x cm and its height is y cm. Its volume is $216\,\text{cm}^3$.

   (i) Write down an expression for the surface area in terms of x.

   (ii) Work out the dimensions that give the minimum surface area, proving that this is a minimum.

FUTURE USES

- This work will be extended if you study Mathematics at a higher level.
- At A-Level you will learn additional formulae to deal with more complex algebraic products and quotients.
- You will also use integration to calculate the area under a curve or between two curves and to calculate the volume when a curve is rotated about the x- or y-axes.
- There are also applications in other subjects, for example, Kinematics, Physics and Economics.

REAL-WORLD CONTEXT

Differentiation is used in the study of motion.
It is also the basis of differential equations which can be used to solve problems involving growth and decay.
Navier-Stokes equations, which are a particular form of differential equation, are vital to video-gaming and also help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution and many other things.

LEARNING OUTCOMES

Now you have finished the chapter, you should be able to

- differentiate positive and negative powers of a variable such as x
- differentiate sums and differences of functions of x
- differentiate functions of x that first need expanding or dividing
- use differentiation to work out the gradient of a curve
- use this information to identify stationary points on a polynomial curve
- derive the equation of a tangent or a normal to a curve
- identify when a function is increasing and when it is decreasing
- calculate the position of any stationary points on the curve
- use the second derivative to determine the nature of any stationary points.

KEY POINTS

1 $y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$

$y = c \quad \Rightarrow \quad \frac{dy}{dx} = 0$

where n is a positive integer and k and c are constants.

2 $y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$

3 For the tangent and normal at (x_1, y_1)

- the gradient of the tangent, $m_1 =$ the value of $\frac{dy}{dx}$
- the gradient of the normal, $m_2 = -\frac{1}{m_1}$
- the equation of the tangent is $y - y_1 = m_1(x - x_1)$
- the equation of the normal is $y - y_1 = m_2(x - x_1)$.

4 A function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$

A function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$

5 The second derivative is obtained by differentiating $\frac{dy}{dx}$ and is denoted by $\frac{d^2y}{dx^2}$.

6 At a stationary point, $\frac{dy}{dx} = 0$

The nature of the stationary point can be determined by looking at the sign of the gradient just either side of it.

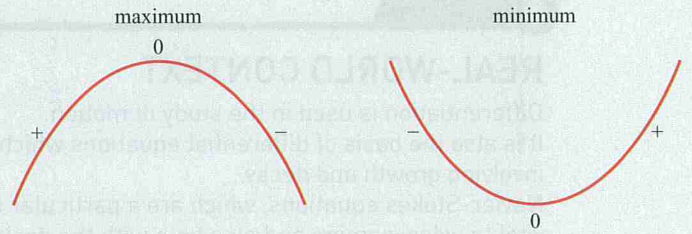


Figure 8.31

7 The sign of $\frac{d^2y}{dx^2}$ is an alternative way of determining the nature of the stationary point:

- If $\frac{d^2y}{dx^2} < 0$ at the stationary point, the point is a maximum.
- If $\frac{d^2y}{dx^2} > 0$ at the stationary point, the point is a minimum.
- If $\frac{d^2y}{dx^2} = 0$ at the stationary point, the result is inconclusive and you need to check the values of $\frac{dy}{dx}$ as in point 6.

9 Matrices

Nothing tends so much to the advancement of knowledge as the application of a new instrument.

Sir Humphry Davy

Prior knowledge

- Students should be familiar with enlargements, reflections and rotations which were met when studying transformations at GCSE.
- It is helpful to be familiar with vectors which are used to describe translations in GCSE.
- Some problem-style questions require students to be comfortable solving equations, including quadratic and simultaneous equations – see Chapters 1, 2 and 4.

An arrangement of information presented in columns and rows is called a matrix.

As an example, the number of male students and female students in two tutor groups are shown in the following matrix.

$$\begin{array}{l c} & \begin{array}{cc}\text{Male} & \text{Female}\end{array} \\ \begin{array}{l}\text{Tutor group A} \\ \text{Tutor group B}\end{array} & \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} \end{array}$$

Each of the four numbers in the matrix is called an **element**.

The matrix on the previous page has 2 rows and 2 columns and so is referred to as a 2×2 (read as 'two by two') matrix.

$\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has 2 rows and 1 column. It is an example of a 2×1 matrix.

1 Multiplying matrices

Multiplication of a matrix by a scalar

In this context, a scalar is a number.

Each element of the matrix is multiplied by the scalar.

Example 9.1

Given that $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$, what is $4\mathbf{A}$?

Solution

$$4\begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ -4 & 20 \end{bmatrix}$$

Multiplication of a 2×2 matrix by a 2×1 matrix

Each row of the 2×2 matrix is multiplied by the column of the 2×1 matrix.

Example 9.2

Given that $\mathbf{P} = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, calculate the product $\mathbf{PQ}$.

$\boxed{\mathbf{PQ} = \mathbf{P} \times \mathbf{Q}}$

Solution

$$\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} (-3) \times 6 + (-1) \times 5 \\ 4 \times 6 + 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} -23 \\ 34 \end{bmatrix}$$

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix. So the product

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ cannot be calculated.

Multiplication of a 2×2 matrix by a 2×2 matrix

Each row of the first 2×2 matrix is multiplied by each column of the second 2×2 matrix.

Example 9.3

Given that $\mathbf{C} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$, calculate the product **CD**.

Solution

$$\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0\times 2+(-1)\times(-3) & 0\times 4+(-1)\times 1 \\ (-2)\times 2+3\times(-3) & (-2)\times 4+3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -13 & -5 \end{bmatrix}$$

Equating elements

If two matrices are equal, then their corresponding elements can be equated.

This principle is used in the following example.

Example 9.4

Given that $\begin{bmatrix} 3 & a \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -3 & b \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ -7 & 0 \end{bmatrix}$

work out the values of a and b.

Solution

Multiplying out the left-hand side gives $\begin{bmatrix} 6-3a & -6+ab \\ -7 & 4+b \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ -7 & 0 \end{bmatrix}$

Equating elements in row 1 column 1 gives

$$\begin{aligned} 6-3a &= 12 \\ 6-12 &= 3a \\ -6 &= 3a \\ a &= -2 \end{aligned}$$

Equating elements in row 2 column 2 gives

$$\begin{aligned} 4+b &= 0 \\ b &= -4 \end{aligned}$$

Check by comparing the elements in row 1 column 2:
$-6+ab = -6+(-2)\times(-4) = 2$

Discussion point

➜ Can you identify matrices $\mathbf{P}$ and $\mathbf{Q}$ for which $\mathbf{PQ} = \mathbf{QP}$?

ACTIVITY 9.1

$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ $\quad$ $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

Calculate the products **AB** and **BA**. What do you notice?

Multiplying matrices

Exercise 9A

① $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 6 & -2 \\ -3 & -1 \end{bmatrix}$ $\quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$

$\mathbf{E} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \mathbf{F} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\mathbf{G} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\quad \mathbf{H} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Work out

(i) 4**A** (ii) 2**D** (iii) **AF**
(iv) **CE** (v) **DH** (vi) **BH**
(vii) **AB** (viii) **BA** (ix) **BC**
(x) **CB** (xi) **DA** (xii) **BD**
(xiii) **AC** (xiv) **DC**

② Work out the value of p in each of the following.

(i) $\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -1 \\ p & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

(iii) $\begin{bmatrix} p & 1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ p \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}$

(iv) $\begin{bmatrix} p & 4p \\ p & -2p \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ p & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 16 & 9 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2p \\ -1 & p \end{bmatrix} = \begin{bmatrix} 9 & -14 \\ 0 & 0 \end{bmatrix}$

③ Work out the values of x and y in each of the following.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & y \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & x \\ 2y & 3y \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

(iii) $\begin{bmatrix} -3 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ x & y \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -4 & 10 \end{bmatrix}$

(iv) $\begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ x & y \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ -2 & -4 \end{bmatrix}$

④ Given that $\begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

(i) write down two equations in x and y

(ii) work out x and y by solving the pair of simultaneous equations.

⑤ Work out the values of a and b in each of the following.

(i) $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 21 \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = 6\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & b \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & 2a \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ 2 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 5 & 1 \\ 3a & b+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 3 \end{bmatrix}$

PS ⑥ Given that $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the values of a, b, c and d.

PS ⑦ $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 0 & k \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$. Work out the value of k such that $\mathbf{AB} = \mathbf{BA}$.

PS ⑧ Given that $\begin{bmatrix} p & 4 \\ 1 & q \end{bmatrix}\begin{bmatrix} r & p-4 \\ q+8 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 3p+4 \\ -17 & -3p \end{bmatrix}$, work out the values of p, q and r.

Discussion point

Are there any values of a, b, c and d, for which the matrix $\mathbf{M}$ in question 9 does not exist?

PS ⑨ (i) Calculate the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(ii) Hence write down the matrix $\mathbf{M}$, in terms of a, b, c and d, such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2 Transformations

If a point P(x, y) is transformed to point P′(x', y'), then P is said to have been mapped to the **image** P′.

For example,

(i) when the point (4, 2) is transformed by reflection in the y-axis, the image point is (−4, 2). So (4, 2) has been mapped to (−4, 2).

(ii) when the point (3, −1) is transformed by rotation through 270°, centre O, the image point is (−1, −3). So (3, −1) has been mapped to (−1, −3).

Note

Assume that a rotation is anticlockwise unless told otherwise.

Matrix transformations

A transformation can be defined by a matrix.

For a transformation that maps $(1, 0)$ to (a, c), and $(0, 1)$ to (b, d)

the transformation matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

When such a matrix acts on a point, we use the point's position vector.

The position vector of any point is measured from the origin, and is the point's coordinates in vertical form with the x-coordinate on top.

So the position vector of $(-1, 5)$ is $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

When a transformation acts on a point, multiply its matrix by the point's position vector. The product is the position vector of the image.

For a transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acting on the point (x, y), the image (x', y') is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

Example 9.5

Work out the image of point $(-2, 5)$ for the transformation defined by matrix $\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$$

The image point is $(11, 9)$.

ACTIVITY 9.2

(i) For the transformation defined by matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, work out the image of each of the following points.

$(3, 2)$ $(-1, 5)$ $(6, 0)$ $(-3, -4)$ and (x, y)

How would you describe the transformation?

(ii) For the transformation defined by matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, work out the image of each of the following points.

$(2, 1)$ $(-4, 3)$ $(0, 4)$ $(-5, -1)$ and (x, y)

How would you describe the transformation?

3 The identity matrix

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix **I**.

When multiplied by another matrix **A**: $\quad \mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

When **I** is used as a transformation matrix, no movement occurs.

Exercise 9B

① Work out the image of point (4, 2) for the transformation defined by matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

② Work out the image of point (1, −3) for the transformation represented by matrix $\begin{bmatrix} 0 & -3 \\ -1 & 5 \end{bmatrix}$.

③ Work out the image of point (−2, −3) for the transformation defined by matrix $\begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$.

④ The image of point (4, 3) under the transformation matrix $\begin{bmatrix} 2 & 1 \\ c & 3 \end{bmatrix}$ is (11, 1). Work out the value of c.

⑤ The image of point $(a, 1)$ under the transformation matrix $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ is (7, 17). Work out the value of a.

⑥ The image of point (3, −2) under the transformation matrix $\begin{bmatrix} a & 2a \\ b & 3 \end{bmatrix}$ is (b, b). Work out the values of a and b.

⑦ The transformation matrix $\begin{bmatrix} 2c & d \\ c & -d \end{bmatrix}$ maps the point (2, 5) to the point (−6, 12). Work out the values of c and d.

⑧ Given that $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, show that $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

> $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$, i.e. **AA**

⑨ Given that $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & p \end{bmatrix} = \mathbf{I}$, work out the value of p.

PS ⑩ Under the transformation matrix $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, point D is mapped to (11, 10). Work out the coordinates of D.

PS ⑪ The matrices $\mathbf{M} = \begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} a+1 & b+2 \\ -a & 2 \end{bmatrix}$ satisfy the equation $\mathbf{MN} = c\mathbf{I}$. Work out the values of a, b and c.

PS ⑫ (i) Under the transformation matrix $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, which of the following points is **not** invariant?

(1, 3) (2, 4) (3, 9) (5, 15)

(ii) If the point (x, y) is invariant under $\begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}$, identify a condition for all of the invariant points of this transformation.

Note

Question 12 is for interest only as this specification does not assess a candidate's knowledge of invariant points, i.e. a point which does not move (see GCSE Maths).

FUTURE USES

At A-Level, invariant points (and invariant lines) are investigated to a much greater depth.

4 Transformations of the unit square

The unit square has vertices O(0, 0), A(1, 0), B(1, 1) and C(0, 1).

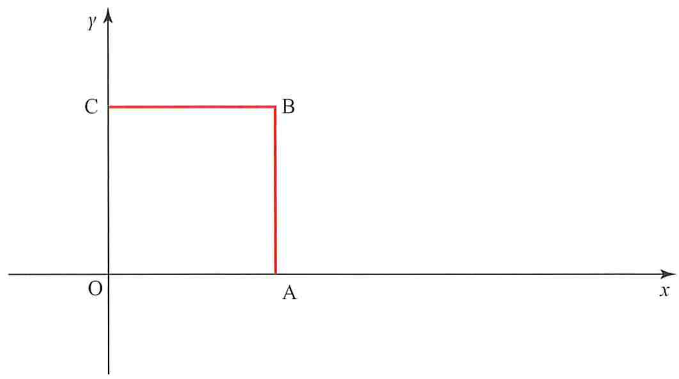


Figure 9.1

The following transformations of the unit square are the only ones that you will be expected to carry out.

Reflections

in the x-axis in the y-axis in the line $y = x$ in the line $y = -x$

Rotations about the origin

through 90° through 180° through 270°

A rotation of 270° is the same as a rotation of 90° clockwise.

Enlargements, centre the origin

with positive scale factors with negative scale factors

You will need to be able to work out the matrix that represents any of the above transformations.

Note

Rotations are anticlockwise unless the question states otherwise.

Example 9.6

Work out the 2×2 matrix that represents each of the following transformations.

(i) Rotation through 270° about the origin.

(ii) Enlargement of scale factor 3, centre the origin.

Solution

In both parts, consider the images of the position vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(i)

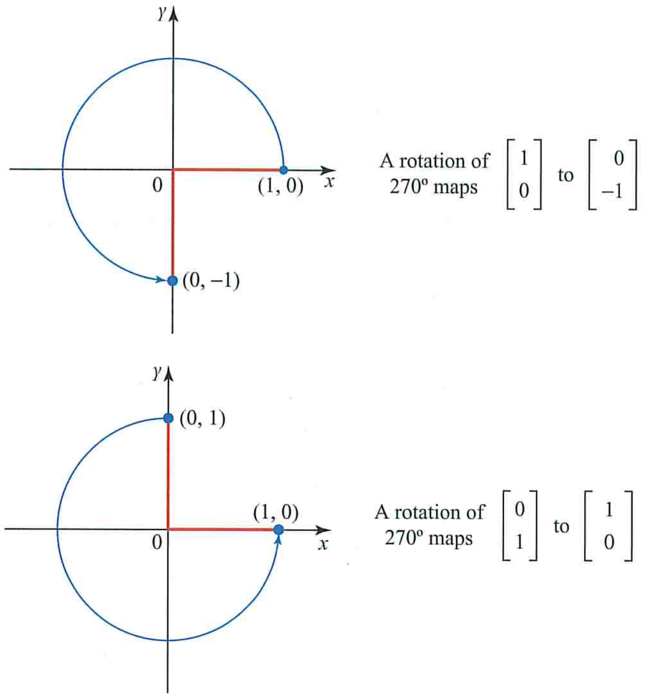


Figure 9.2

Write these two position vectors side by side: $\begin{bmatrix} 0 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

to give the transformation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(ii)

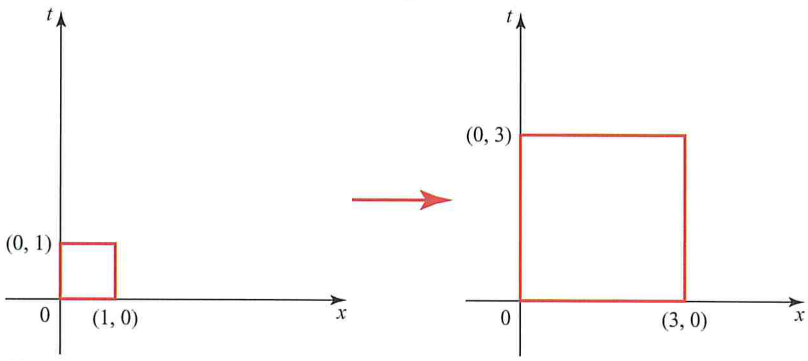


Figure 9.3

Point (1, 0) maps to (3, 0), which has position vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Point (0, 1) maps to (0, 3), which has position vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Write these two position vectors side by side: $\begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

to give the transformation matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Example 9.7

The unit square OABC is transformed by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ to OA′B′C′.

Show the image on a diagram, labelling each vertex.

Solution

Method 1

Use the matrix to work out the image of each vertex.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Image of O(0, 0) is O(0, 0).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ Image of A(1, 0) is A′(−1, 0).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ Image of B(1, 1) is B′(−1, −1).

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ Image of C(0, 1) is C′(0, −1).

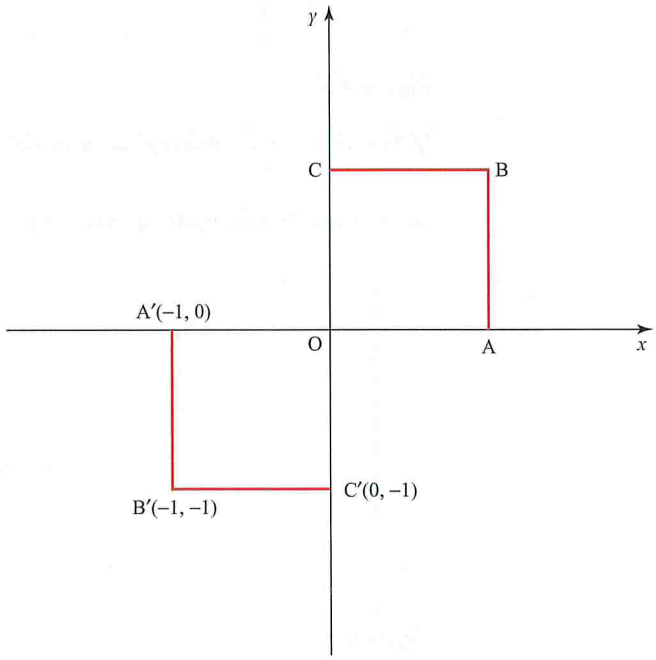


Figure 9.4

Method 2 (alternative method)

If the matrix is recognised as representing a rotation through 180°, the diagram can be drawn without first calculating the points.

Exercise 9C

① Work out the 2×2 matrix that represents each of the following transformations.

(i) Reflection in the x-axis.

(ii) Rotation of 90° about O.

(iii) Enlargement, scale factor 2, centre the origin.

(iv) Reflection in the y-axis.

(v) Reflection in the line $y = x$.

(vi) Rotation by 180°, centre the origin.

(vii) Reflection in the line $y = -x$.

(viii) Enlargement, scale factor -3, centre O.

(ix) Enlargement, centre O, scale factor $\frac{1}{2}$.

② The unit square OABC is transformed by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to OA′B′C′. Show the image on a diagram, labelling each vertex.

③ The unit square OABC is transformed by the matrix $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ to OA′B′C′.

Show the image on a diagram, labelling each vertex.

④ Describe fully the transformations given by the following matrices.

(i) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

⑤ The unit square OABC is transformed to OA′B′C′.
OA′B′C′ is shown on the diagram.

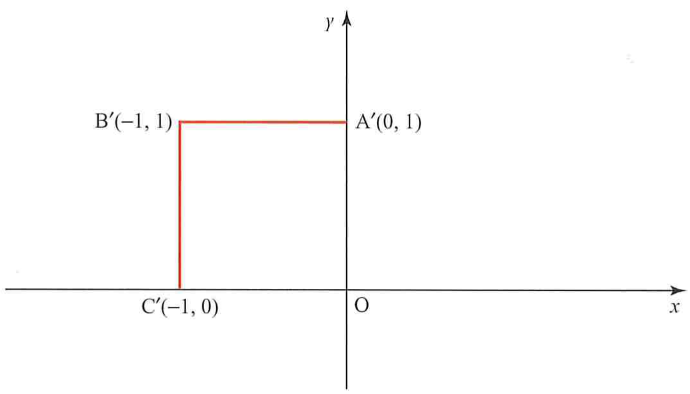


Figure 9.5

Work out the matrix for the transformation.

⑥ The unit square OABC is transformed by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ to OA′B′C′.

Work out the area of OA′B′C′.

⑦ The unit square OABC is transformed by the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ to OA′B′C′.

The area of OA′B′C′ is 64 square units.

Work out the two possible values of k.

⑧ (i) Draw a diagram to show the unit square OABC rotated 45° about the origin.

(ii) Work out the coordinates of A′ and C′ (the images of A and C).

(Hint: $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$.)

(iii) Hence write down the transformation matrix for a rotation of 45° about the origin.

Note

Question 8 is for interest only as this specification only includes rotations of 90°, 180° and 270°.

5 Combining transformations

Two transformations may be applied successively.

The two transformations can be combined into a single transformation that maps point A to point A″.

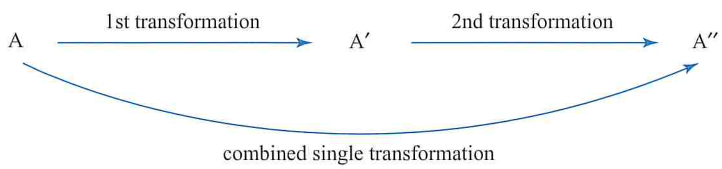


Figure 9.6

When a transformation represented by matrix **P** is followed by a transformation represented by matrix **Q**, the matrix for the combined transformation is **QP**.

Example 9.8

Point L is transformed by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ to the point M.

Point M is then transformed by the matrix $\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$ to the point N.

Work out the matrix that transforms point L to point N.

Solution

Note that the matrix for the second transformation is written first when working out the combined transformation matrix.

Multiply the matrices in the correct order.

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 1 \end{bmatrix}$$

Example 9.9

The unit square is transformed by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ followed by a further transformation by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Work out the matrix for the combined transformation, and describe it geometrically.

Solution

Multiply the matrices in the correct order.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

This is the transformation matrix for which

(1, 0) maps to (0, −1), and (0, 1) maps to (−1, 0).

Discussion point

Can you identify a pair of transformations (described geometrically) for which the result is the same regardless of the order in which they are applied?

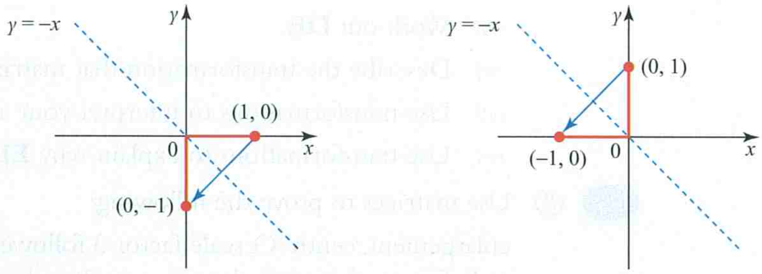


Figure 9.7

The combined matrix represents a reflection in the line $y = -x$.

Exercise 9D

① Point P(3, −2) is transformed by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the x-coordinate of the image point of P.

② Point W(−1, 4) is transformed by $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$, followed by a further transformation $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$.

(i) Work out the matrix for the combined transformation.

(ii) Work out the coordinates of the image point of W.

③ The unit square is reflected in the x-axis followed by a rotation through 180°, centre the origin.

Work out the matrix for the combined transformation.

4. The unit square is enlarged, centre the origin, scale factor 2, followed by a reflection in the line $y = x$.

   Work out the matrix for the combined transformation.

5. The unit square is rotated by 90°, centre the origin, followed by a reflection in the y-axis.

   Work out the matrix for the combined transformation.

6. Matrix $\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (i) Describe the transformation that this matrix represents.

   (ii) Work out $\mathbf{P}^2$.

   (iii) Use transformations to interpret your answer to part (ii).

7. Matrix $\mathbf{D} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Matrix $\mathbf{E} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

   (i) Describe the transformation that matrix $\mathbf{D}$ represents.

   (ii) Describe the transformation that matrix $\mathbf{E}$ represents.

   (iii) Work out $\mathbf{DE}$.

   (iv) Describe the transformation that matrix $\mathbf{DE}$ represents.

   (v) Use transformations to interpret your answer to part (iv).

   (vi) Use transformations to explain why $\mathbf{ED} = \mathbf{DE}$.

PS 8. Use matrices to prove the following:

   enlargement, centre O, scale factor 3 followed by enlargement, centre O, scale factor −2 is equivalent to a single enlargement, centre O, scale factor k, where k is an integer to be found.

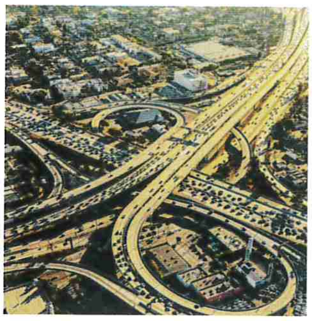
PS 9. (i) Calculate the combined transformation matrix of $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, followed by $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and finally $\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$.

   (ii) Under this combined transformation, point A is mapped to the point $(-14, 7)$. Work out the coordinates of point A.

PS 10. Identify three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ such that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

FUTURE USES

At A-Level (Further Maths only) 3 × 3 matrices are used to describe 3-dimensional transformations. They can also be used to solve simultaneous equations in three unknowns.



REAL-WORLD CONTEXT

Matrices are used in discrete mathematics to represent networks for a variety of situations such as water or traffic flow, or a rail network, e.g. the London Underground. They can also be used to solve problems which require costs to be minimised, or profits to be maximised. You will meet these applications if you study mathematics at A-Level.
Matrices have applications in most scientific fields, including quantum mechanics and electromagnetism. Computer programmers also make use of matrices, e.g. when coding graphics.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- multiply a 2×2 matrix by a scalar
- multiply a 2×2 matrix by a 2×1 matrix
- multiply a 2×2 matrix by a 2×2 matrix
- recognise the identity matrix
- recognise and use the following transformations when written as matrices:
  - reflection in the x-axis
  - reflection in the y-axis
  - reflection in the line $y = x$
  - reflection in the line $y = -x$
  - rotation of $90°$ about the origin
  - rotation of $180°$ about the origin
  - rotation of $270°$ about the origin
  - enlargement of scale factor k, centred on the origin
- apply a combination of transformations
- work out the matrix which represents a combination of transformations.

KEY POINTS

1 To multiply a 2×2 matrix by a 2×1 matrix, each row of the 2×2 matrix is multiplied by the column of the 2×1 matrix.
To multiply a 2×2 matrix by a 2×2 matrix, each row of the first 2×2 matrix is multiplied by each column of the second matrix.

2 If two matrices are equal, the corresponding elements can be equated.

3 A point P(x, y) can be transformed to an image point P$'(x', y')$.
If the transformation matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

4 $\mathbf{I}$ is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5 A transformation of matrix **A** followed by a transformation of **B** is a combined transformation. The matrix for the combined transformation is **BA**.

Key words

Calculate	Work out the numerical value (often used after making a substitution)
Draw (a graph)	Draw axes on graph paper, plot points accurately and join with a straight line or smooth curve
Evaluate	Give a numerical value for your answer
Expand	Remove brackets
Expand and simplify	Remove brackets and collect like terms
Explain	Give reasons, either in words, or using mathematical symbols, or both
Expression	One or more terms, for example one side of a formula
Factorise	Write as a product
Give your answer in its simplest form	Cancel answers given as ratios or fractions or collect like terms
Hence	Use earlier work to deduce the result
Hence or otherwise	Using previous work to deduce the result is an option
Plot	Mark points (usually on graph paper) and join with a straight line or curve
Prove	Show all relevant steps (include explanations of facts used in geometrical proofs)
Show that	Show all relevant steps to reach a given result
Sketch (a graph)	Do not use graph paper. Draw axes and show the correct shape in each quadrant. Label appropriate points (e.g. intersection with axes, stationary points)
Verify	'Check out' a statement or result that you have been given
Work out the exact value of	Give the answer as an integer, fraction, recurring decimal, in terms of π, etc. or a surd
Write down	The answer should be obvious (no working is necessary)

Practice questions 1

Calculator NOT allowed $1\frac{3}{4}$ **hours**

① $\begin{bmatrix} 4 & p \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

Work out the value of p. [3 marks]

② Simplify $\dfrac{2x - 6}{x^2 + 5x - 24}$. [3 marks]

③ A triangle has sides 3 cm, 4 cm and 5 cm. The vertices of the triangle lie on the circle. Work out the exact area inside the circle not covered by the triangle. [4 marks]

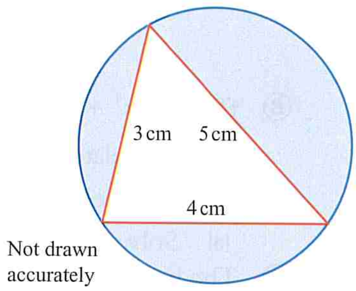


Figure 1

④ A palindromic number reads the same backwards as forwards. For example, 20 502 and 351 153 are palindromic. How many seven-digit palindromic numbers are odd? [2 marks]

⑤ The graph of $y = ab^x$ passes through the points $(1, 6)$ and $(2, 12)$.

(a) Work out the values of a and b. [2 marks]

(b) Sketch the graph of $y = ab^x$. [2 marks]

⑥ A, B and C lie on a circle. BD is a tangent. AC is parallel to BD.

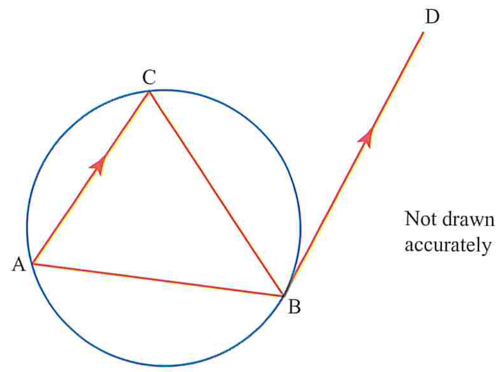


Figure 2

Prove that triangle ABC is isosceles. [3 marks]

⑦ A circle has centre C and passes through A(−7, 3) and B(5, 3), as shown in the diagram.

AC = BC = 10

Work out the equation of the circle. [4 marks]

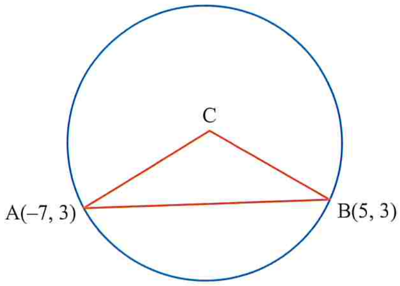


Figure 3

⑧ $f(x) = 2x^4 + x^3 - 7x^2 + 5x - 1$

(a) Calculate f(1). [1 mark]

(b) Use the factor theorem to show that $(2x - 1)$ is a factor of $f(x)$. [2 marks]

(c) Solve $f(x) = 0$. [4 marks]

⑨ The line $3y + x = 10$ and the circle $x^2 + y^2 = 20$ intersect at P and Q.

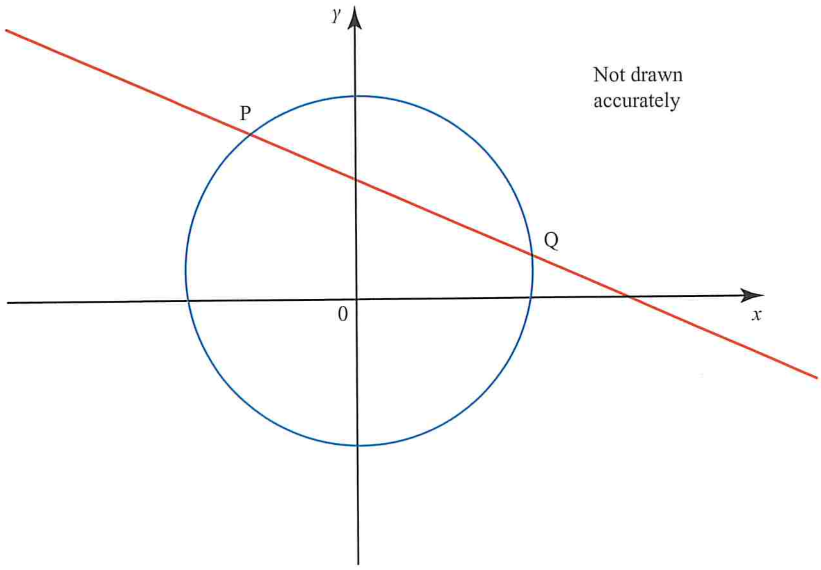


Figure 4

Work out the coordinates of P and Q. [6 marks]

⑩ Rationalise the denominator of $\dfrac{4 - \sqrt{3}}{\sqrt{3} + 1}$. [3 marks]

⑪ (a) Write $\sqrt{\dfrac{x^{\frac{4}{3}} \times x^{\frac{8}{3}}}{x}}$ as a single power of x. [2 marks]

(b) Solve $x^{-\frac{1}{3}} = \dfrac{2}{5}$. [2 marks]

⑫ $(x + 4)^2 + p \equiv x^2 + qx + 28$

Work out the values of p and q. [3 marks]

⑬ (a) Sketch the graph of $y = \cos x$ for $-180° \leqslant x \leqslant 540°$. [2 marks]

(b) Solve $\cos x\ x = \dfrac{\sqrt{3}}{2}$ for $-180° \leqslant x \leqslant 540°$. [3 marks]

⑭ A sequence has nth term $= \dfrac{3n + 2}{1 - 6n}$.

Prove that the limiting value of the nth term as $n \rightarrow \infty$ is $-\dfrac{1}{2}$. [3 marks]

⑮ Work out the equation of the tangent to the curve $y = (x + 5)(x - 3)$ at the point where $x = -3$. Give your answer in the form $y = mx + c$. [5 marks]

⑯ The nth term of a sequence is $an^2 + bn + c$. The 1st, 5th and 7th terms are -2, 2 and 16 respectively.

(a) Substituting $n = 1, 5$ and 7, write three equations in a, b and c. [2 marks]

(b) Solve your equations to work out the nth term of the sequence. [5 marks]

⑰ (a) Factorise $y^2 - 3y + 2$ [2 marks]

(b) Hence solve $y^2 - 3y + 2 = 0$ [1 mark]

(c) Hence solve $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$ [2 marks]

⑱ $f(x) = x^2 + 6, \ x \geqslant 0$

$g(x) = 2x + 1$, for all x

(a) Work out the inverse function f^{-1} and its domain. [2 marks]

(b) Solve $fg(x) = gf(x)$. [4 marks]

⑲ $14xy^6$ is one of the terms in the expansion of $(ax + y)^n$.

(a) Write down the value of n. [1 mark]

(b) Calculate the value of a. [2 marks]

Practice questions 2

Calculator allowed **$1\frac{3}{4}$ hours**

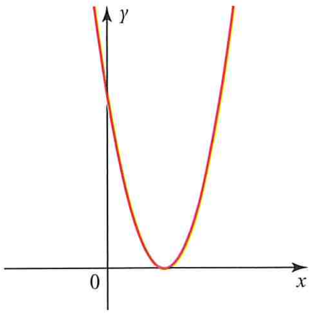


Figure 1

① Which of the following equations matches the graph shown in Figure 1? [1 mark]

$y = x^2 - 3$ $\quad$ $y = x^2 + 3$ $\quad$ $y = (x - 3)^2$ $\quad$ $y = (x + 3)^2$

② Simplify $\dfrac{6xy^2}{8x^2} \div \dfrac{10x^3y}{12y^4}$. [2 marks]

③ Expand and simplify $(x^2 - 3x + 4)(2x^2 + 5x - 7)$. [2 marks]

④ Work out the equation of the line that is perpendicular to $2x + 3y = 5$ and crosses the x-axis at $(-5, 0)$. Write your answer in the form $ax + by = c$. [3 marks]

⑤ $y = 4x^3 + x^2 - 7x$

(a) Work out $\dfrac{dy}{dx}$. [2 marks]

(b) Work out the rate of change of y with respect to x when $x = 1.5$ [2 marks]

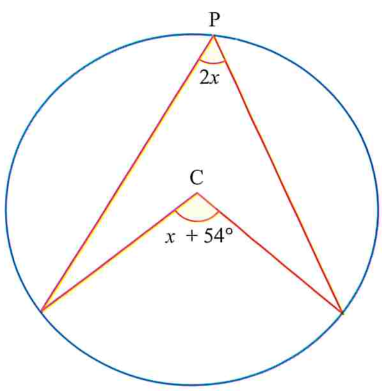


Figure 2

⑥ C is the centre of the circle, shown in Figure 2. P is a point on the circumference.

Work out the value of x. [3 marks]

⑦ h increased by 50% is equal to 60% of m.

Express the ratio $h : m$ in its simplest terms. [2 marks]

⑧ A ship travels for 25 km on a bearing of 065° from A to B.

It then travels for 18 km from B to C on a bearing of 135°.

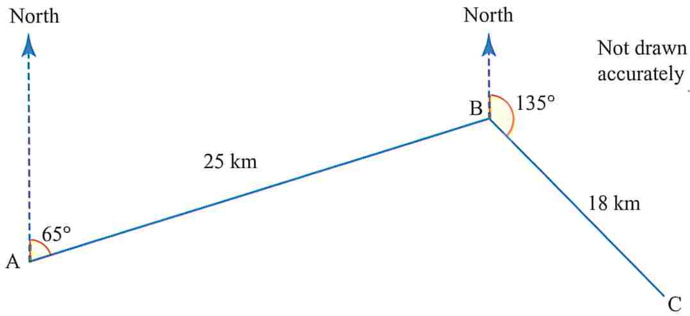


Figure 3

(a) Work out the distance AC. [3 marks]

(b) Work out the bearing of C from A. [4 marks]

⑨ Figure 4 shows the straight line PQR.

PR = 5PQ

Work out the coordinates of Q. [4 marks]

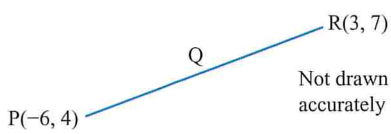


Figure 4

⑩ Draw the graph of $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \begin{cases} 3x & 0 \leqslant x \leqslant 2 \\ 6 & 2 < x \leqslant 5 \\ 16 - 2x & 5 < x \leqslant 8 \end{cases}$$

[3 marks]

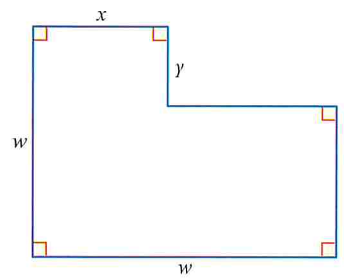


Figure 6

⑪ Figure 6 shows an L-shape. (All lengths are in centimetres.)

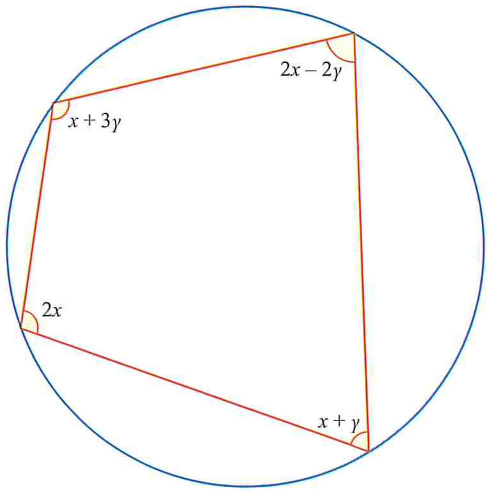
The shape has area A cm^2.

(a) Show that $A = w^2 + xy - wy$. [2 marks]

(b) Rearrange the formula to make y the subject. [3 marks]

⑫ Figure 7 shows a cyclic quadrilateral. Each angle of the quadrilateral is given in terms of x and y.

Work out x and y. [5 marks]



Not drawn accurately

Figure 7

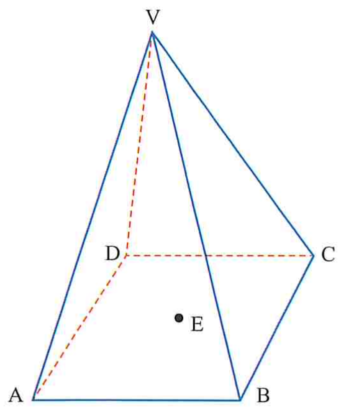


Figure 8

⑬ A paperweight is in the shape of a square-based pyramid (see Figure 8). The base ABCD has edge 5 cm. The vertex V is directly above the centre of the base, E.

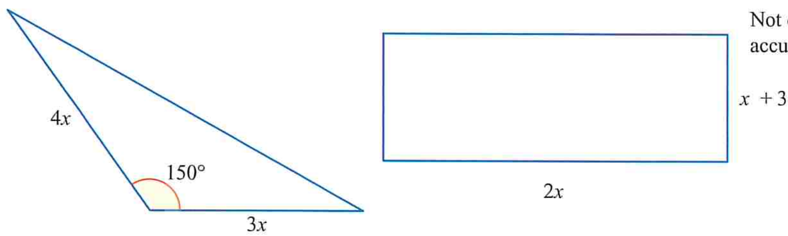
VE = 7 cm.

(a) Work out VA. [3 marks]

(b) Work out the angle between VA and ABCD. [3 marks]

(c) Work out the angle between VAB and ABCD. [3 marks]

⑭ A triangle and a rectangle are shown in Figure 9. All dimensions are in centimetres.



Not drawn accurately

Figure 9

(a) Show that the area of the triangle is $3x^2$ cm^2. [2 marks]

(b) Work out the range of values of x for which area of triangle $<$ area of rectangle. [5 marks]

⑮ Point P is transformed by $\begin{bmatrix} 3 & 1 \\ 6 & 3 \end{bmatrix}$ followed by a further transformation $\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$.

(a) Work out the matrix for the combined transformation. [2 marks]

(b) The image of point P after the combined transformation is (2, 3). Work out the coordinates of P. [3 marks]

⑯ Prove that $\sin x \tan x \equiv \frac{1}{\cos x} - \cos x$. [3 marks]

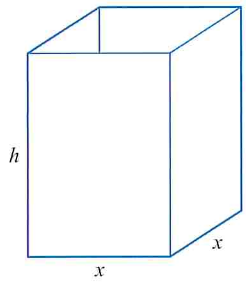


Figure 10

⑰ The square-based open container in Figure 10 is a cuboid with a volume of 1000 cm^3.

The open end is opposite the square base as shown.

Each side of the square base is x cm, and the height of the container is h cm.

(a) Write h in terms of x. [1 mark]

(b) Hence show that the container's external surface area, A, is given by

$A = x^2 + \frac{4000}{x}$. [2 marks]

(c) Use calculus to work out the container's minimum surface area. [3 marks]

(d) Use the second derivative to justify that your answer to part (c) is a minimum. [3 marks]

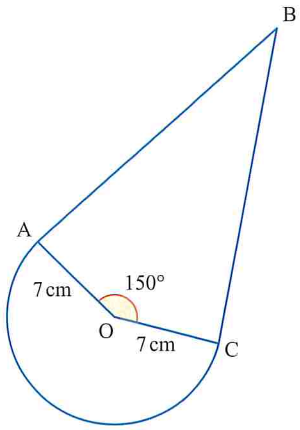


Figure 11

⑱ A kite and a sector of a circle are shown in Figure 11.

The two shapes share the sides OA and OC, where O is the centre of the circle, and A and C are points on the circle.

The radius of the circle is 7 cm and the angle AOC is 150° as shown.

The other two sides, AB and CB, of the kite are tangents to the circle.

Calculate the total area of the shape. [7 marks]

Answers

Chapter 1

Exercise 1A (page 3)

1 (i) 5:2
(ii) 2:3
(iii) 11:2
(iv) 1:2
(v) 3:1

2 (i) £69
(ii) 260
(iii) 11.2 cm

3 (i) 7163
(ii) 6.72
(iii) £6.30

4 (i) 84
(ii) £420
(iii) £12
(iv) 12

5 (i) 741
(ii) 3136.25
(iii) £1314
(iv) £48.85

6 (i) $\frac{52}{45}$ or $1\frac{7}{45}$
(ii) $\frac{1}{32}$
(iii) $\frac{53}{20}$ or $2\frac{13}{20}$

7 (i) 10.7
(ii) 4.9
(iii) 0.04
(iv) 0.60

8 35

9 80

Discussion point (page 5)

Factorise means the expression must be written as a product of factors.

Factorise *fully* means that each factor cannot be factorised any further.

Exercise 1B (page 5)

1 (i) $10a - b - 2c$
(ii) $6x - 3y - 4z$
(iii) $19x + 5y$
(iv) $p + 14q$
(v) $5x$
(vi) $2a^2 + 12a - 12$
(vii) $3q^2 - 3p^2$
(viii) $10fg + 10fh - 5gh$

2 (i) $2(4 + 5x^2)$
(ii) $2b(3a + 4c)$
(iii) $2a(a + 2b)$
(iv) $pq(q^2 + p^2)$
(v) $3xy(x + 2y^3)$
(vi) $2pq(3p^2 - 2pq + q^2)$
(vii) $3lm^2(5 - 3l^2m + 4lm^2)$
(viii) $12a^4b^4(7a - 8b)$

3 (i) $4(5x - 4y)$
(ii) $6(x + 1)$
(iii) $z(x - y)$
(iv) $2q(p - r)$
(v) $k(l + n)$
(vi) $-4(a + 2)$
(vii) $3(x^2 + 2y^2)$
(viii) $2(a + 4)$

4 (i) $10a^3b^4$
(ii) $12p^3q^4r$
(iii) lm^2n^2p
(iv) $36r^4s^3$
(v) $64ab^2c^2d^2e$
(vi) $60x^3y^3z^3$
(vii) $84a^5b^9$
(viii) $42p^3q^8r^5$

5 (i) $2a$
(ii) pq
(iii) $\frac{4b}{a}$
(iv) $\frac{bd}{ac}$
(v) $\frac{2xy^2z}{3}$
(vi) $\frac{5}{2a^2b^2}$
(vii) $\frac{7p^2r^3}{6q^4s^2}$

6 (i) $\frac{11a}{12}$
(ii) $\frac{13x}{20}$
(iii) $\frac{7p}{12}$
(iv) $\frac{s}{3}$
(v) $\frac{5b}{12}$
(vi) $\frac{7a}{3b}$
(vii) $\frac{(5q - 3p)}{2pq}$
(viii) $-\frac{5x}{6y}$

7 (i) $16x - 10$
(ii) $7x^2 + x - 3$

8 (i) $16x + 2$
(ii) $11x^2 + x - 2$
(iii) $x(3x + 2)(x - 1)$ or $3x^3 - x^2 - 2x$

Discussion points (page 7)

An equation is used to show that two expressions or numbers are equal, e.g. $5x + 2 = 3x + 8$. An equation must contain an equals sign.

Solving an equation finds the values of any unknown(s) which satisfy the equation.

Discussion point (page 7)

The letter and number do not have to change sides, but convention puts the letter on the left. This also makes it easier to read, as we read from left to right.

Exercise 1C (page 8)

1 (i) $x = 7$
(ii) $a = -2$
(iii) $x = 2$
(iv) $y = 2$
(v) $c = 5$
(vi) $p = 10$
(vii) $x = -5$
(viii) $x = -6$
(ix) $y = 7$
(x) $k = 42$
(xi) $t = 60$
(xii) $p = -55$
(xiii) $p = 0$

2 (i) $2l + 2(l - 80) = 600$
(ii) $l = 190$; Area $= 20\,900\,\text{m}^2$

3 (i) $2(s + 4) + s = 17$
(ii) Ben = 7, Chris = 7, Stephen = 3

4 (i) $5c - a$
(ii) $5c - 15 = 40$; $c = 11$

5 (i) $3j + 12$ or $2(j + 12)$
(ii) $3j + 12 = 2(j + 12)$; $j = 12$

6 (i) $8a$
(ii) $6a + 6$
(iii) $a = 3$

7 (i) $m - 2, m - 1, m, m + 1, m + 2$
(ii) $m - 2 + m - 1 + m + m + 1 + m + 2 = 105$; $m = 21$
(iii) 19, 20, 21, 22, 23

8 (i) $2(x + 2) = 5(x - 3)$; $x = 6\frac{1}{3}$
(ii) $16\frac{2}{3}\,\text{cm}^2$

Discussion point (page 9)

Let the price of a large ice cream be l and the price of a small be s. Form two simultaneous equations in l and s using the two pieces of information given, and solve.

A small ice cream costs 80p.
A large ice cream costs £1.20.

Discussion point (page 9)

The second sentence duplicates the information given in the first sentence.

So there is an infinite number of solutions.

Exercise 1D (page 11)

1 (i) $0.3b$ *or* $\frac{3b}{10}$
(ii) $\frac{9y}{2}$
(iii) $\frac{cd}{100}$

2 $66\frac{2}{3}\%$

3 (i) $1.2a$
(ii) $1.05b$
(iii) $0.65k$
(iv) $0.98m$

4 $1.8a = 1.5b$

$\frac{1.8}{1.5} = \frac{b}{a}$

$1.2 = \frac{b}{a}$

5 60%

6 8 : 12 : 27

7 (i) $a = \frac{5b}{2}$
(ii) 6 : 1
(iii) 5 : 4

8 15 : 8

9 24 : 25

10 $100 + m : 100 - m$

11 40 boys and 50 girls

Discussion point (page 12)

When you multiply out the double bracket, there are terms in x^2 and x and a number, but no other terms (e.g. no x^3, no $\sqrt{x}$, no $\frac{1}{x}$).

Exercise 1E (page 13)

1 (i) $x^2 + 9x + 20$
(ii) $x^2 + 4x + 3$
(iii) $2a^2 + 9a - 5$
(iv) $6p^2 + 5p - 6$
(v) $x^2 + 6x + 9$
(vi) $4x^2 - 9$
(vii) $14m - 3m^2 - 8$
(viii) $12 + 4t - 5t^2$
(ix) $16 - 24x + 9x^2$
(x) $m^2 - 6mn + 9n^2$

2 (i) $x^5 - x^4 + 2x^3 - 3x^2 + x - 2$
(ii) $x^6 + 2x^5 - 3x^4 - 4x^3 + 5x^2 + 6x - 3$
(iii) $2x^5 - 4x^4 - x^3 + 11x^2 - 13x + 5$
(iv) $x^6 - 1$
(v) $x^3 + 4x^2 + x - 6$
(vi) $2x^3 + 5x^2 - 14x - 8$
(vii) $x^3 + 3x^2 + 3x + 1$
(viii) $p^3 - 15p^2 + 75p - 125$
(ix) $8a^3 + 36a^2 + 54a + 27$
(x) $2x^3 - 17x - 18$
(xi) $2x^2 - 2x$

3 (i) $2x^3 - 5x^2 - 3x$
(ii) $10x^2 - 14x - 6$

4 (i) $3x^3 - \frac{1}{2}x^2 - \frac{21}{2}x + 5$
(ii) $11x^2 - 15x - 4 + 4xy - 2y$

5 (i) $a^2 + 2ab + b^2$
(ii) $a^3 + 3a^2b + 3ab^2 + b^3$
(iii) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
(iv) $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

6 (i) $x^3 + 18x^2 + 108x + 216$
(ii) $p^3 - 6p^2 + 12p - 8$
(iii) $16y^4 + 32y^3 + 24y^2 + 8y + 1$
(iv) $x^4 - 12x^3 + 54x^2 - 108x + 81$
(v) $243w^5 - 1620w^4 + 4320w^3 - 5760w^2 + 3840w - 1024$

Activity 1.1 (page 14)

(i) 7
(ii) 1
(iii) 6

(iv) $a^6 + 6a^5b + 15a^4b^2$
$+ 20a^3b^3 + 15a^2b^4$
$+ 6ab^5 + b^6$

Exercise 1F (page 17)

1 (i) $1 + 3x + 3x^2 + x^3$
(ii) $y^4 + 4y^3 + 6y^2$
$+ 4y + 1$
(iii) $x^5 + 5x^4y + 10x^3y^2$
$+ 10x^2y^3 + 5xy^4 + y^5$
(iv) $125 + 75w + 15w^2 + w^3$
(v) $p^4 + 16p^3 + 96p^2$
$+ 256p + 256$
(vi) $32 + 80m + 80m^2$
$+ 40m^3 + 10m^4 + m^5$

2 (i) $x^3 - 3x^2y + 3xy^2 - y^3$
(ii) $1 - 8x + 24x^2$
$- 32x^3 + 16x^4$
(iii) $32 - 80y + 80y^2$
$-40y^3 + 10y^4 - y^5$
(iv) $125 - 150p + 60p^2$
$-8p^3$
(v) $81x^4 - 432x^3 + 864x^2$
$-768x + 256$
(vi) $1024x^5 - 1280x^4$
$+ 640x^3 - 160x^2$
$+ 20x - 1$

3 $1 + 6x + 15x^2$

4 $x^7 + 14x^6 + 84x^5$

5 -4320

6 10

7 1

8 36

9 $81x^8 + 108x^5 + 54x^2$
$+ \dfrac{12}{x} + \dfrac{1}{x^4}$

10 (i) $1 + 10x + 40x^2 + 80x^3$
$+ 80x^4 + 32x^5$
(ii) $1 - 10x + 40x^2 - 80x^3$
$+ 80x^4 - 32x^5$
(iii) $20x + 160x^3 + 64x^5$

11 (i) $27 + 27w + 9w^2 + w^3$
(ii) $27 + 27x + 54y + 9x^2$
$+ 36xy + 36y^2 + x^3$
$+ 6x^2y + 12xy^2 + 8y^3$

12 (i) 6
(ii) ± 4
(iii) 3840

13 160

14 3360

Activity 1.2 (page 18)

(i) They are all powers of 11
(ii) 2^n
(iii) n
(iv) $\frac{1}{2}n(n-1)$
(v) Palindromic

Future uses (page 18)

${}^nC_r$ is the $(r + 1)$th number in the nth row of Pascal's triangle

Note: Candidates will not be expected to have knowledge of the ${}^nC_r$ function.

Discussion point (page 19)

A rational number is a number which can be expressed in the form $\frac{a}{b}$ where a and b are integers. So a rational number can be a fraction or an integer (when $b = 1$); and can be positive or negative.

Exercise 1G (page 20)

1 (i) $4\sqrt{2}$
(ii) $5\sqrt{5}$
(iii) $5\sqrt{3}$
(iv) $\sqrt{2}$
(v) $3\sqrt{3}$
(vi) $7\sqrt{2} - 3$
(vii) $10\sqrt{2}$
(viii) $36 + 3\sqrt{3}$
(ix) $16\sqrt{5}$
(x) $\sqrt{3}$

2 (i) $3 - 2\sqrt{2}$
(ii) $3 + 2\sqrt{5}$
(iii) $3\sqrt{7} - 9$
(iv) 2
(v) $11 - \sqrt{2}$
(vi) $5 - 3\sqrt{7}$
(vii) $24 - 13\sqrt{3}$
(viii) $8 - 2\sqrt{15}$
(ix) $13\sqrt{2} - 17$
(x) $17 + 12\sqrt{2}$

3 (i) $\dfrac{\sqrt{3}}{3}$
(ii) $\sqrt{5}$
(iii) $\dfrac{4\sqrt{6}}{3}$
(iv) $\dfrac{\sqrt{6}}{3}$
(v) 1
(vi) $\dfrac{\sqrt{21}}{7}$
(vii) $3\sqrt{7}$
(viii) $\dfrac{\sqrt{5}}{3}$
(ix) $\dfrac{\sqrt{15}}{5}$
(x) $\dfrac{\sqrt{2}}{2}$

4 (i) $\dfrac{12}{7}$
(ii) $4 + 8\sqrt{3}$
(iii) $20 - 4\sqrt{6}$
(iv) $\dfrac{12 - 5\sqrt{3}}{13}$ or
$\dfrac{12}{13} - \dfrac{5}{13}\sqrt{3}$

5 (i) $45 + 29\sqrt{2}$
(ii) $38 + 17\sqrt{5}$
(iii) $97 - 56\sqrt{3}$
(iv) $73 + 28\sqrt{6}$
(v) $176 + 80\sqrt{5}$
(vi) $682 - 305\sqrt{5}$

6 (i) $v = 2$
(ii) $w = \frac{5}{3}$
(iii) $y = \frac{3}{2}$
(iv) $x = \frac{3}{5}$

7 $1351 + 780\sqrt{3}$

8 (i) $m = \frac{5}{2}$
(ii) $n = \frac{16}{5}$
(iii) $x = 6$
(iv) $x = -\frac{144}{7}$

9 $x = \frac{4\sqrt{2}}{3}$

10 $8 + \sqrt{48}$

Exercise 1H (page 22)

1 (i) $\frac{10\sqrt{3} - 2\sqrt{6}}{23}$
(ii) $\frac{4\sqrt{7} + \sqrt{14}}{14}$
(iii) $\frac{9 - 3\sqrt{3}}{2}$
(iv) $\frac{8 + 5\sqrt{2}}{7}$
(v) $\frac{\sqrt{7} - 2}{3}$
(vi) $10\sqrt{3} + 3 - 10\sqrt{2} - \sqrt{6}$

2 $12 + 9\sqrt{2}$

3 $18\sqrt{5} - 40$

4 $1 - \frac{1}{3}\sqrt{3}$

5 $3 - \sqrt{5}$

6 $3\sqrt{3} + 2\sqrt{2}$

7 $10 - 4\sqrt{2}$

8 $1 + \sqrt{3}$

Activity 1.3 (page 23)

The factorial function is only defined for positive integers and zero.

A calculator will produce an error message if a negative number, or a non-integer, is input.

A calculator will interpret $-5!$ as $-(5!)$, giving the answer -120.

Discussion point (page 24)

The answer to part (i) can be calculated as $\frac{12!}{61} = 665280$.

So the answer to part (ii) can be written as $\frac{12!}{6! \times 6!} = 924$.

Exercise 1I (page 25)

1 120
2 5040
3 (i) 5040
(ii) 117 649
4 12
5 180
6 3 628 800
7 (i) 360
(ii) 1296
8 (i) 720
(ii) 48
9 (i) 90
(ii) 90
(iii) 1999
(iv) 100
10 120
11 40 320
12 (i) 360
(ii) 240
13 $n(n-1)(n-2) \times \ldots \ldots \times 2 \times 1$ (which could be written more simply as $n!$ but the factorial notation is not assessed in this specification)
14 (i) 120
(ii) 1287

Chapter 2

Discussion point (page 29)

Yes, except that the rows and columns could be interchanged.

Discussion point (page 29)

Yes, but the brackets will be in the reverse order. (Work it through to check for yourself.)

Discussion point (page 31)

Factorising gives $(2x - 5)(x - 3)$. The factors are the other way around.

Exercise 2A (page 32)

1 (i) $(a + d)(b - c)$
(ii) $(2x + w)(y + 1)$
(iii) $(2p - 3r)(q - 4)$
(iv) $(5 - 2n)(1 - m)$
2 (i) $(x + 2)(x + 3)$
(ii) $(y - 1)(y - 4)$
(iii) $(m - 4)^2$
(iv) $(m - 3)(m - 5)$
(v) $(x + 5)(x - 2)$
(vi) $(a + 12)(a + 8)$
(vii) $(x - 3)(x + 2)$
(viii) $(y - 12)(y - 4)$
(ix) $(k + 6)(k + 4)$
(x) $(k - 12)(k + 2)$
3 (i) $(x + 2)(x - 2)$
(ii) $(a + 5)(a - 5)$
(iii) $(3 + p)(3 - p)$
(iv) $(x + y)(x - y)$
(v) $(t + 8)(t - 8)$
(vi) $(2x + 1)(2x - 1)$
(vii) $(2x + 3)(2x - 3)$
(viii) $(2x + y)(2x - y)$
(ix) $(4x + 5)(4x - 5)$
(x) $(3a + 2b)(3a - 2b)$
4 (i) $(2x + 1)(x + 2)$
(ii) $(2a - 3)(a + 7)$
(iii) $(5p - 1)(3p + 1)$
(iv) $(3x - 1)(x + 3)$
(v) $(5a + 1)(a - 2)$
(vi) $(2p - 1)(p + 3)$
(vii) $(4x - 1)(2x + 3)$
(viii) $(2a - 9)(a + 3)$
(ix) $(3x - 5)^2$
(x) $(2x + 5)(2x - 3)$

5 (i) $(x + y)(x + 2y)$
(ii) $(x + 5y)(x - y)$
(iii) $(a - 4b)(a + 3b)$
(iv) $(c - 3d)(c - 8d)$
(v) $(x + 4y)(x + 5y)$
(vi) $(p + 5r)(p - 3r)$
(vii) $(a + 3r)(a - 5r)$
(viii) $(s - 2t)^2$
(ix) $(m - 6n)(m + n)$
(x) $(r + 4s)(r - 2s)$

6 (i) $(3a + 1)(a + 1)$
(ii) $(4x + 5)(2x - 3)$
(iii) $(3p - 2)(p - 4)$
(iv) $(2 + 5y)(6 - 5y)$
(v) $(a + 1)(3a + 1)$
(vi) $(4x + 5)(2x - 3)$
(vii) $(3p - 2)(p - 4)$
(viii) $12y - 4y^2$

7 (i) $(2x + y)(x + 2y)$
(ii) $(3x - y)(x + 2y)$
(iii) $(5a - 3b)(a - b)$
(iv) $(3c + 4d)(2c - d)$
(v) $(6p - q)(p - 6q)$
(vi) $(7g - 2h)(g + h)$
(vii) $(3h - 4k)(2h + k)$
(viii) $(4w - x)(2w - x)$

8 (i) $x(x + 2)(x - 2)$
(ii) $a^2(a + 4)(a - 4)$
(iii) $y^3(3 + y)(3 - y)$
(iv) $2x(x + 1)(x - 1)$
(v) $p^2(2p + 3)(2p - 3)$
(vi) $x(10 + x)(10 - x)$
(vii) $2c(3c + 1)(3c - 1)$
(viii) $2x(2x + 5y)(2x - 5y)$

Activity 2.1 (page 32)

(i) 81 and a^4
(ii) $(a^2)^2 - 9^2$
(iii) $(a^2 + 9)(a^2 - 9)$
$= (a^2 + 9)(a + 3)(a - 3)$

Activity 2.2 (page 33)

(i) $(5x + 3)(2x + 1)$
(ii) $(5p + 5q + 3)(2p + 2q + 1)$

Discussion point (page 33)

Omit it since length must be positive.

Exercise 2B (page 34)

1 (i) $u = v - at$
(ii) $t = \dfrac{v-u}{a}$; an equation of motion

2 $b = \dfrac{2A}{h}$; area of a triangle

3 $l = \dfrac{P-2b}{2}$; perimeter of a rectangle

4 $r = \sqrt{\dfrac{A}{\pi}}$; area of a circle

5 $c = \dfrac{2A-bh}{h}$; area of a trapezium

6 $h = \dfrac{A-\pi r^2}{2\pi r}$; surface area of a cylinder with a base but no top

7 $l = \dfrac{\lambda e}{T}$; tension of a spring or string

8 (i) $u = \dfrac{2s-at^2}{2t}$
(ii) $a = \dfrac{2(s-ut)}{t^2}$; an equation of motion

9 $x = \dfrac{\sqrt{\omega^2 a^2 - v^2}}{\omega}$; speed of a particle on an oscillating spring

Exercise 2C (page 35)

1 $m = \dfrac{2x}{3-x}$

2 $y = \dfrac{2x}{5-x}$

3 $b = -\dfrac{a}{7}$

4 $h = \dfrac{S-2\pi r^2}{2\pi r}$

5 $x = \dfrac{1-2y}{y-1}$

6 $c = \dfrac{1-2d}{d+3}$

7 (i) $t = \dfrac{3x}{x-1}$
(ii) 4.5

8 (i) $p = \dfrac{2-3r}{2r-3}$
(ii) -1

Activity 2.3 (page 35)

(i) (a) $(x + 3)(x + 3)$
$= x^2 + 3x + 3x + 9$
$= x^2 + 6x + 9$
(b) $y = x^2 + 6x + 9$
$y = (x + 3)^2$
$(x + 3) = \pm\sqrt{y}$
$x = \pm\sqrt{y} - 3$

(ii) (a) $(x - 5)(x - 5) + 4$
$= x^2 - 5x - 5x + 25 + 4$
$= x^2 - 10x + 29$
(b) $p = x^2 - 10x + 29$
$p = (x - 5)^2 + 4$
$p - 4 = (x - 5)^2$
$x - 5 = \pm\sqrt{p-4}$
$x = 5 \pm \sqrt{p-4}$

Discussion points (page 35)

A fraction in arithmetic is one number divided by another number.

The definition of a fraction in algebra is the same but with 'number' replaced by 'expression'.

Discussion points (page 35)

You can cancel a fraction in arithmetic when the numerator and denominator have a common factor.

It is the same for fractions in algebra.

A factor in arithmetic is a number that divides exactly into the given number, i.e. there is no remainder.

The definition of a factor in algebra is the same but with 'number' replaced by 'expression'.

Discussion points (page 36)

x is not a factor of the numerator $(2x + 2)$ or the denominator $(3x + 3)$.

The correct answer involves factorising both the numerator and the denominator:

$\dfrac{2(x+1)}{3(x+1)}$

Cancelling $(x + 1)$ gives $\dfrac{2}{3}$.

Neither a nor a^2 is a factor of the numerator and denominator.

The correct answer involves factorising to get

$$\frac{(a-3)(a+2)}{(a-3)(a-5)} = \frac{a+2}{a-5}$$

Discussion point (page 36)

Individual terms have been cancelled rather than factors.

The correct answer is

$$\frac{(2n+3)(2n-3)}{(n+1)} \times \frac{(n+1)(n-1)}{(2n+3)}$$

$$= (2n-3)(n-1)$$

Exercise 2D (page 37)

1 (i) $\frac{1}{2}$

(ii) $\frac{4}{x+8}$

(iii) $\frac{3}{x-y}$

(iv) $\frac{2x}{3y}$

(v) $\frac{1}{3-p}$

(vi) $\frac{2b^2}{5a^2}$

2 (i) $\frac{x-1}{2}$

(ii) $\frac{x}{x-y}$

(iii) $\frac{1}{a-3}$

(iv) $\frac{3}{2}$

(v) $\frac{3x-1}{3}$

(vi) $\frac{x}{2y}$

3 (i) $\frac{b}{2}$

(ii) x

(iii) $\frac{x(x+y)}{y}$

(iv) $\frac{x}{8(x-1)}$

(v) $2(a+1)$

(vi) $\frac{2(2p+q)}{3(2p-q)}$

4 (i) $\frac{(x-2)}{x(x+2)}$

(ii) $\frac{(2x-1)(x+2)}{(2x+1)(x-1)}$

(iii) $4(p+3)$

(iv) $\frac{3(x-1)(x^2-3)}{(x-3)^2}$

(v) $\frac{3(a+2)}{(a-1)(a-2)}$

(vi) $\frac{t^2-1}{2t}$

5 (i) $\frac{7a}{20}$

(ii) $-\frac{7}{3a}$

(iii) $\frac{(m-3n)}{(m+n)(m-n)}$

(iv) $\frac{5(p+2)}{(p-2)(2p+1)}$

(v) $\frac{5x}{2(x-1)(x+4)}$

(vi) $\frac{7a+8}{6(a-1)(a+4)}$

6 (i) $\frac{(5a+1)}{a(a+1)(a-1)}$

(ii) 2

(iii) $\frac{1}{(p+1)(p-1)}$

(iv) $\frac{2(a^2+b^2)}{(a+b)(a-b)}$

(v) $\frac{10-3x}{x^2-4}$

(vi) $\frac{48-3x}{5(x-2)(x+4)}$

7 (i) $\frac{2(x^2+3x+3)}{(x+1)(x+2)(x+3)}$

(ii) $\frac{5x^2-9x-32}{(x+1)(x-2)(x+3)}$

(iii) $-\frac{1}{x(x+1)^2}$

8 (i) $\frac{7t+3}{(t+1)^2}$

(ii) $\frac{1+3y-3x}{(y+x)(y-x)}$

(iii) $\frac{n^3+6n^2+8n+2}{n(n+1)(n+2)}$

Discussion point (page 38)

If you multiplied both the numerator and the denominator, the multiplier would cancel and the fraction would be unchanged.

Discussion point (page 39)

Not all of the left-hand side has been multiplied by 30.

Exercise 2E (page 39)

1 $x = \frac{5}{6}$

2 $a = \frac{5}{8}$

3 $x = -8$

4 $x = -\frac{2}{3}$

5 $p = \frac{18}{13}$

6 $x = 3$

7 $x = -6$

8 $t = 12$

Exercise 2F (page 41)

1 $a = 4$ $b = -6$

2 $c = 2$ $d = 6$

3 $p = 6$ $q = -40$

4 $a = \frac{5}{2}$ $b = -\frac{33}{4}$

5 $p = 9$ $q = 2$

6 $c = \frac{9}{4}$ $d = \frac{1}{2}$

7 $a = 2$ $b = 8$ $c = -3$

8 $a = 5$ $b = 3$ $c = -35$

9 $p = 3$ $q = -2$ $r = 2$

10 $a = 3$ $b = 24$ $c = -47$

11 $a = 2$ $b = 4$ $c = 8$

12 $p = 23$ $q = 2$ $r = 3$

13 (i) $a = 4$ $b = 4$

(ii) $x = \sqrt{y-4} + 4$

14 (i) $p = 3 \quad q = 1 \quad r = -2$

(ii) $x = \sqrt{\dfrac{y+2}{3}} - 1$

Chapter 3

Discussion point (page 43)

(i) Yes (ii) No

Exercise 3A (page 43)

1 (i) -9 (ii) 0.2
(iii) 15 (iv) -1
(v) -1 (vi) 0

2 (i) 12 (ii) 75
(iii) 3 (iv) -4
(v) 12 (vi) -9

3 (i) 1 (ii) -3
(iii) 0 (iv) 2.25
(v) 0 (vi) 4.2

4 (i) $\frac{8}{3}$ (ii) 2
(iii) $-\frac{4}{7}$

5 (i) $6x - 2$ (ii) $3x + 1$
(iii) $3x^2 - 2$

6 (i) $(x^2 - 1)^2$
(ii) $(x - 1)^4$
(iii) x^4

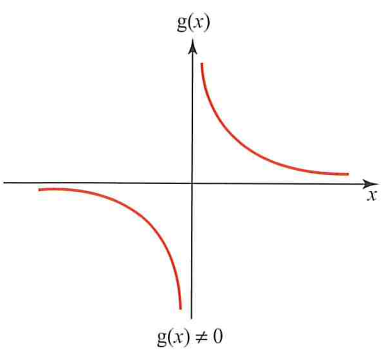
7 (i) $9x^2 + 15x - 1$
(ii) $x^2 + x - 7$

8 (i) 1.5
(ii) 1.2
(iii) 3

Discussion point (page 44)

$\frac{1}{0}$ is not a real number

Activity 3.1 (page 44)



Exercise 3B (page 45)

1 (i) $f(x) < 6$
(ii) $f(x) \geqslant 5$
(iii) $f(x) \geqslant 2$
(iv) $f(x) \geqslant 6$

2 (i) $2 \leqslant f(x) \leqslant 10$
(ii) $-3 < f(x) < 7$
(iii) $f(x) \leqslant 11$
(iv) $-9 < f(x) \leqslant 11$

3 (i) $\frac{5}{2} \leqslant f(x) \leqslant 5$
(ii) $-1.75 \leqslant f(x) \leqslant 0.25$
(iii) $-2.33 \leqslant f(x) \leqslant 3$
(iv) $-7 \leqslant f(x) \leqslant 5$

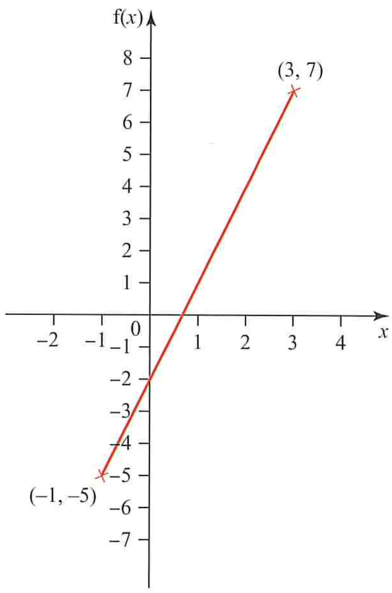
4 (i) $0 \leqslant f(x) \leqslant 4$
(ii) $0 < f(x) < 16$
(iii) $f(x) \geqslant 0$
(iv) $-1 \leqslant f(x) \leqslant 27$

5 (i) $-3 \leqslant f(x) \leqslant 29$
(ii) $-2 \leqslant f(x) \leqslant 46$
(iii) $-5 \leqslant f(x) \leqslant 3$
(iv) $-10 \leqslant f(x) \leqslant 2$

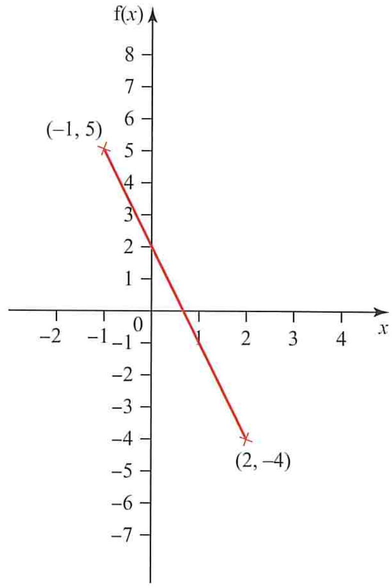
6 (i) $0 \leqslant f(x) \leqslant 15$
(ii) $-1 \leqslant f(x) \leqslant 15$
(iii) $-1 \leqslant f(x) \leqslant 3$
(iv) $0 \leqslant f(x) \leqslant 3$

7 (i) Domain $1 \leqslant x \leqslant 5$
Range $3 \leqslant f(x) \leqslant 8$
(ii) Domain $-4 \leqslant x \leqslant 4$
Range $0 \leqslant f(x) \leqslant 2$
(iii) Domain $-2 \leqslant x \leqslant 3$
Range $0 \leqslant f(x) \leqslant 2$

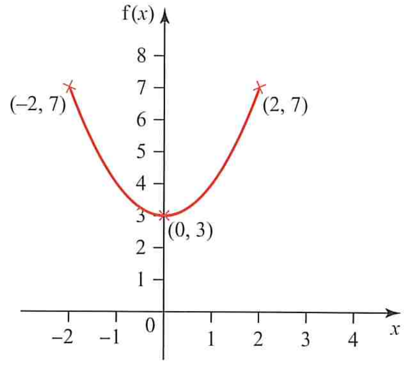
8 (i) $-5 \leqslant f(x) \leqslant 7$



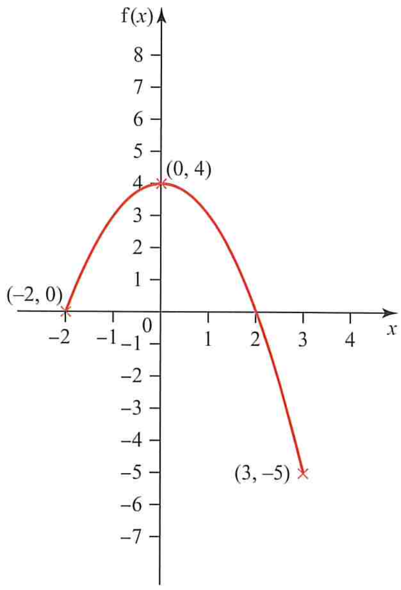
(ii) $-4 \leqslant f(x) \leqslant 5$



(iii) $3 \leqslant f(x) \leqslant 7$



(iv) $-5 \leqslant f(x) \leqslant 4$



Discussion point (page 47)

$x = -0.5$

Activity 3.2 (page 48)

$f(4) = f(-4) = 16; g(4) = 2;$
$fg(4) = gf(4) = gf(-4) = 4$ but $g(-4)$ and $fg(-4)$ gives Math ERROR

There is no real value for the square root of a negative number.

Exercise 3C (page 48)

1 $f(x) = x^2$, $g(x) = (3 + x)$, $fg(x) = (3 + x)^2$

2 (i) (a) $fg(x) = 2x^3 - 1$
(b) $gf(x) = (2x - 1)^3$
(ii) (a) $fg(2) = 15$
(b) $gf(2) = 27$
(c) $fg(-3) = -55$
(d) $gf(-3) = -343$

3 (i) (a) $fg(x) = \frac{1}{x^2}$
(b) $fh(x) = (1 - x)^2$
(c) $gf(x) = \frac{1}{x^2}$
(d) $hf(x) = 1 - x^2$
(ii) In this case, $fg(x) = gf(x)$ but $fh(x) \neq hf(x)$

4 $g(x) = x^3$ and $h(x) = 1 - x$

5 (i) $h(x) = x - 2;\ g(x) = \frac{3}{x}$
(ii) $h(x) = x - 2;\ g(x) = \frac{x}{3}$
(iii) $h(x) = 3x - 1;\ g(x) = x^2$
(iv) $h(x) = 3x - 1;\ g(x) = 2^x$

6 (i) $v(x) = 2x$: $u(x) = \sin x$
(ii) $v(x) = \frac{x}{2}$; $u(x) = \cos x$
(iii) $v(x) = x - 30°$; $u(x) = \tan x$
(iv) $v(x) = \sin x;\ u(x) = x^2$

7 (i) $r(x) = x - 2;\ q(x) = x^4;$ $p(x) = 3x$
(ii) $r(x) = 2x;\ q(x) = x + 3;$ $p(x) = \frac{x}{4}$

8 gdfabec

Discussion point (page 51)

Two points on the line *or* one point and the gradient of the line.

Activity 3.3 (page 51)

Line A: 3

Line B: 0

Line C: $-\frac{2}{5}$

Line D: ∞

Discussion point (page 51)

No, since

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}$$

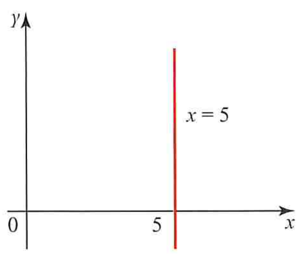
Discussion point (page 54)

(i) $\frac{x}{4} + \frac{y}{3} = 1$
(ii) $a = 4, b = 3$
(iii) a is the intercept on the x-axis and b is the intercept on the y-axis.

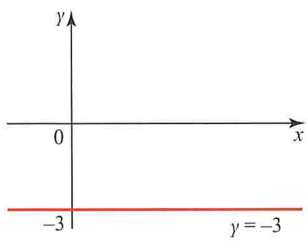
Exercise 3D (page 54)

1 (i) 2 (ii) -3
(iii) $-\frac{11}{5}$ (iv) 3
(v) $7\frac{1}{2}$ (vi) $2\frac{3}{5}$
(vii) $-\frac{1}{5}$ (viii) $-3\frac{2}{3}$

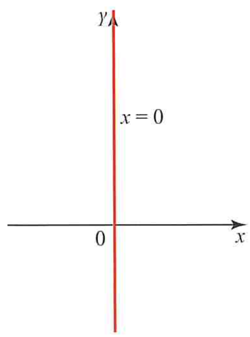
2 (i)



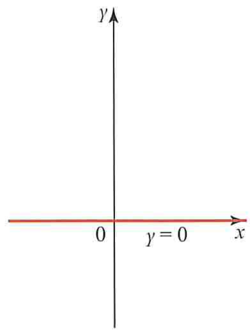
(ii)



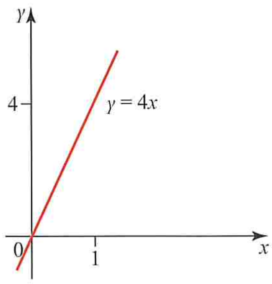
(iii)



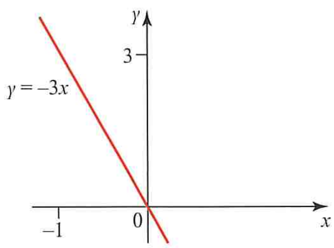
(iv)



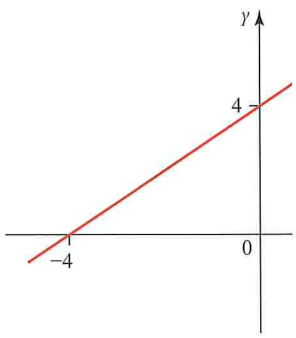
3 (i)



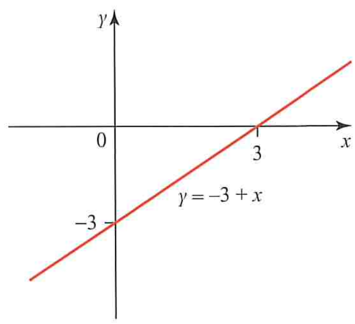
(ii)



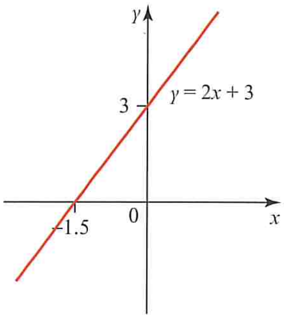
(iii)



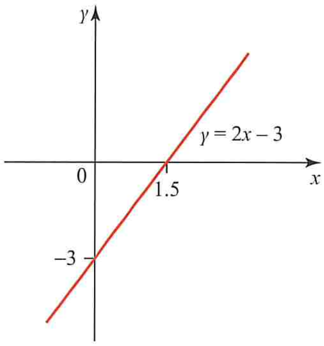
(iv)



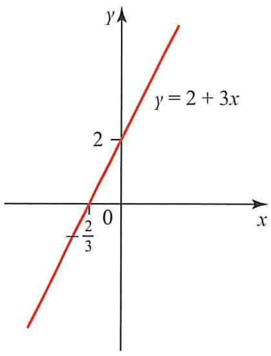
4 (i)



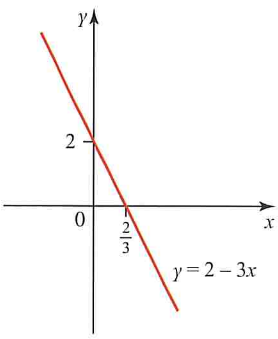
(ii)



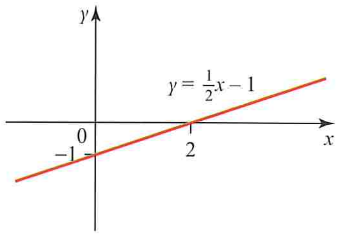
(iii)



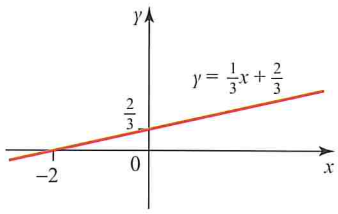
(iv)



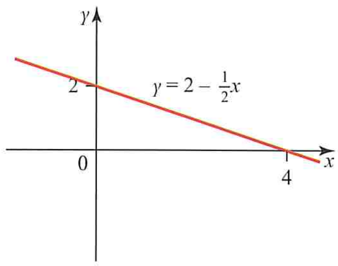
5 (i)



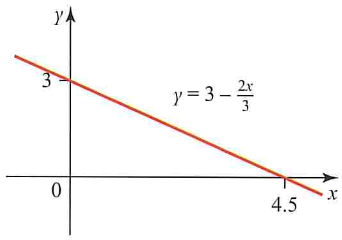
(ii)



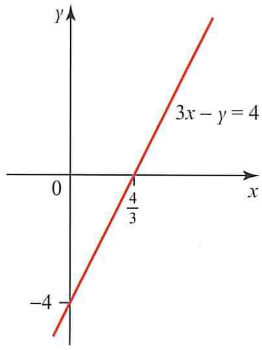
(iii)



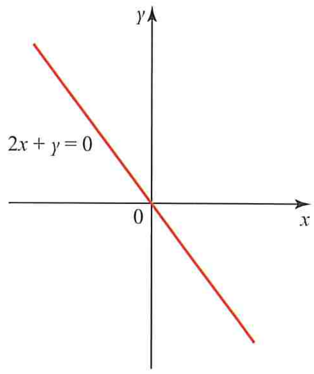
(iv)



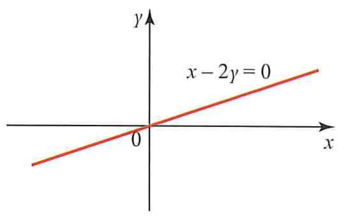
(ii)



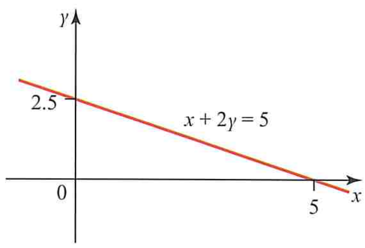
(iii)



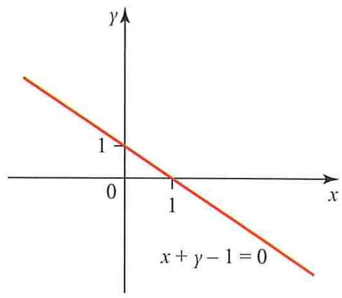
(iv)



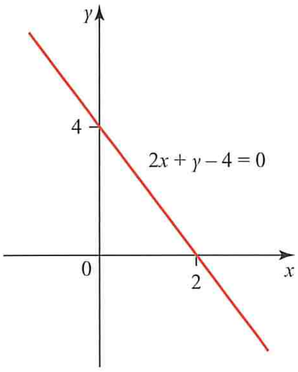
6 (i)



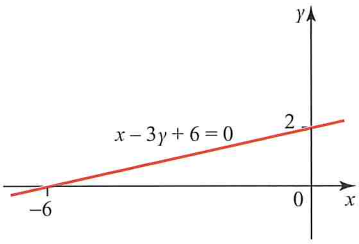
7 (i)



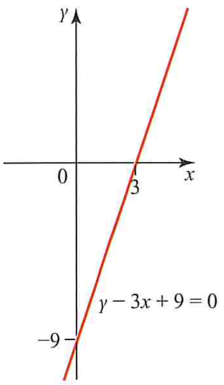
(ii)



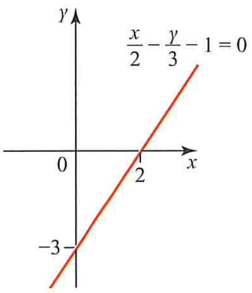
(iii)



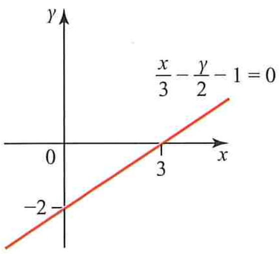
(iv)



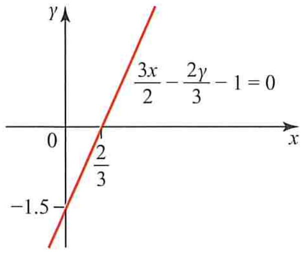
8 (i)



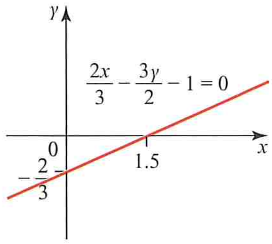
(ii)



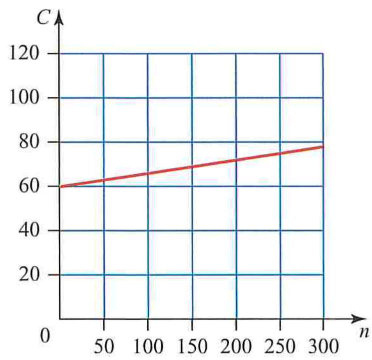
(iii)



(iv)



9 (i) (a) £90 (18p per card)
(b) £360 (7.2p per card)
(ii) £60 set up cost and 6p per card printed.
(iii)



Exercise 3E (page 59)

1 (i) $x = -3$
(ii) $y = 5$
(iii) $y = 2x$
(iv) $2x + y = 4$
(v) $2x + 3y = 12$

2 (i) $x = 5$
(ii) $y = -3$
(iii) $x + 2y = 0$
(iv) $y = x + 4$
(v) $y = 2x - 6$

3 (i) $y = 3x - 7$
(ii) $y = 2x$
(iii) $y = 3x - 13$
(iv) $4x - y - 16 = 0$

4 (i) $y = \frac{1}{3}x$
(ii) $2x - 5y - 42 = 0$
(iii) $3x + 2y + 1 = 0$
(iv) $x + 2y - 12 = 0$

5 (i) $y = x - 2$
(ii) $5x + 3y - 12 = 0$
(iii) $y = x - 5$

6 (i) $3x + 5y - 12 = 0$
(ii) $x + 7y + 32 = 0$
(iii) $y = 2x$

7 (i) $C = 2 + 0.8m$
(ii) £5.20
(iii) 10 miles

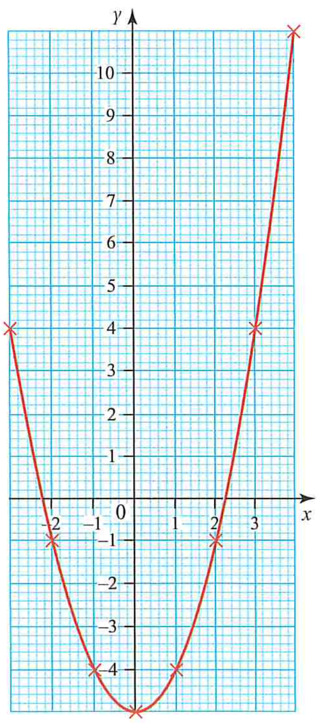
8 (i) $N = 8s + 100$
(ii) £3 030
(ii) Order an extra 80 books instead of 100.

Discussion point (page 60)

A function is of the form $y = f(x)$, enabling you to draw a graph, and the standard form of an equation is $f(x) = 0$, enabling you to find values of x which give the solution of the equation.

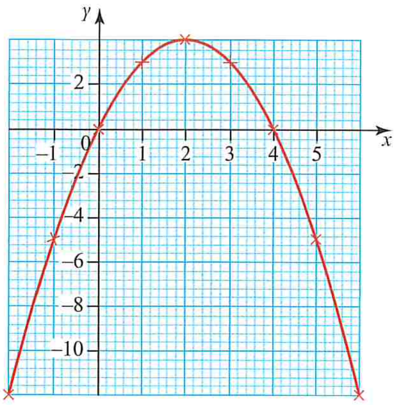
Activity 3.4 (page 60)

x	−3	−2	−1	0	1	2	3	4
y	4	−1	−4	−5	−4	−1	4	11



Activity 3.5 (page 61)

x	-2	-1	0	1	2	3	4	5	6
y	-12	-5	0	3	4	3	0	-5	-12



Exercise 3F (page 63)

1 (i) $y = x^2 - 2x - 3$

(ii) $y = 5 - x^2$

2 (i) $y = 4 - 7x - 2x^2$

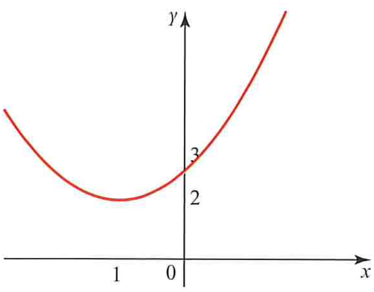
(ii) $y = 4x - x^2$

3 (i) (a) $(-1, 2)$

(b) $x = -1$

(c) $(0, 3)$

(ii)

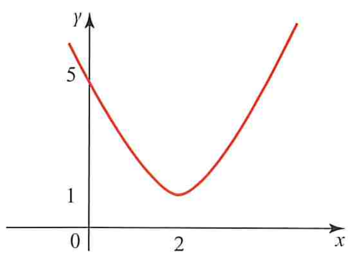


4 (i) (a) $(2, 1)$

(b) $x = 2$

(c) $(0, 5)$

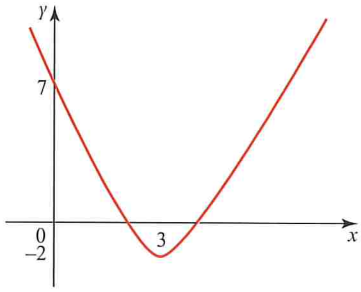
(ii)



5 (i) (a) $(3, -2)$

(b) $x = 3$

(c) $(0, 7)$



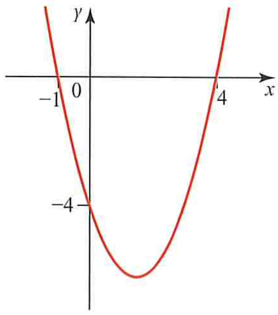
6 (i) (a) $\left(\frac{3}{2}, -\frac{25}{4}\right)$

(b) $x = \frac{3}{2}$

(c) $(0, -4)$

(ii) $(4, 0)$ and $(-1, 0)$

(iii)



7 (i) $x^2 - 6x + 10$

$= (x^2 - 6x + 9) + 1$

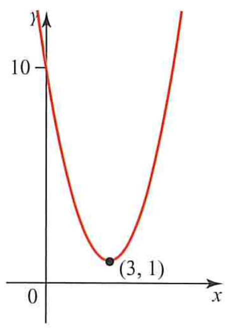
$= (x - 3)^2 + 1$

(ii) $(3, 1)$

(iii) $x = 3$

(iv) $(0, 10)$; it does not intersect the x-axis.

(v)



8 (i) $-(x^2 + 4x - 6)$

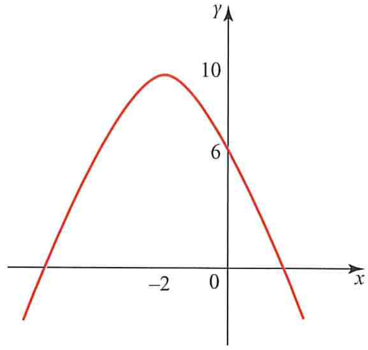
$= -((x + 2)^2 - 10)$

$= 10 - (x + 2)^2$

(ii) $(-2, 10)$

(iii) $x = -2$

(iv) $(0, 6)$, $(-2 - \sqrt{10}, 0)$ and $(-2 + \sqrt{10}, 0)$



9 $y = x^2 - 6x + 2$

10 (i) £19800

(ii) (a) £95 000

(b) £320 000

(c) £500 000

(iii) (a) £19 000

(b) £16 000

(c) £10 000

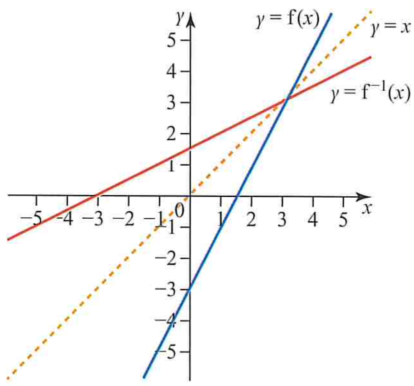
(iv) Would probably make a loss.

Discussion point (page 67)

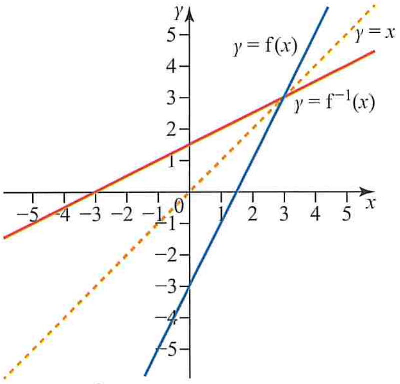
Exercise 3G extends these basic definitions.

Exercise 3G (page 68)

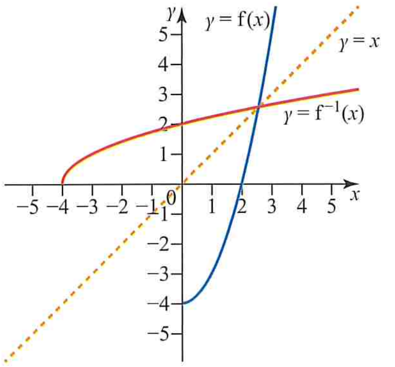
1 $f^{-1}(x) = \dfrac{x + 3}{2}$



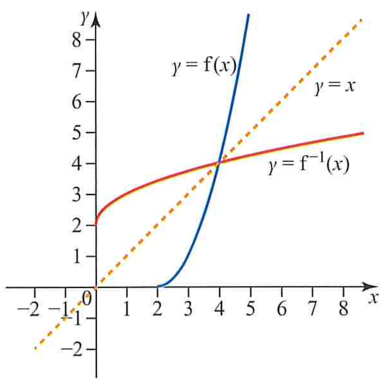
2 $f^{-1}(x) = \dfrac{x+2}{3}$



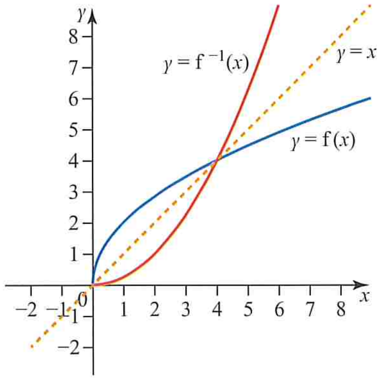
3 $f^{-1}(x) = \sqrt{x+4}$ for $x \geqslant -4$



4 $f^{-1}(x) = \sqrt{x} + 2$ for $x \geqslant 0$

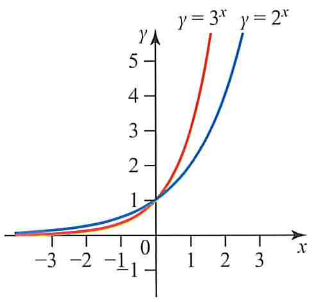


5 $f^{-1}(x) = \dfrac{x^2}{4}$ for $x \geqslant 0$



6 $f^{-1}(x) = \dfrac{1}{x}$ for $x > 0$

NB: Function and inverse are identical.



7 (i) $f^{-1}(x) = \dfrac{x+1}{5}$;

$f^{-1}(4) = 1$

(ii) $f^{-1}(x) = \dfrac{1}{x-2}$ for $x > 2$,

$f^{-1}(3) = 1$

(iii) $f^{-1}(x) = \sqrt{x}$ for $x \geqslant 0$

$f^{-1}(9) = 3$

8 (i)

$f^{-1}(x) = \sqrt{x+3}$ for $x \geqslant -3$

$f^{-1}(-2) = 1$

(ii)

$f^{-1}(x) = \dfrac{\sqrt{x+1}}{2}$ for $x \geqslant -1$

$f^{-1}(3) = 1$

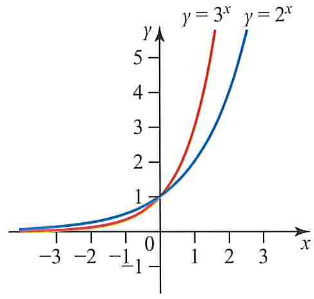
(iii)

$f^{-1}(x) = \dfrac{\sqrt{x-9}}{2}$ for $x \geqslant 9$

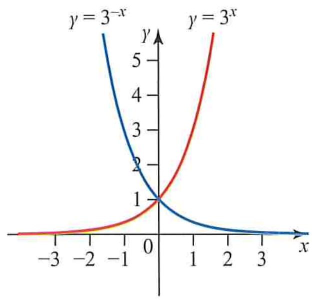
$f^{-1}(13) = 1$

Exercise 3H (page 70)

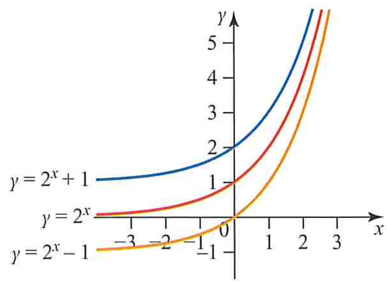
1



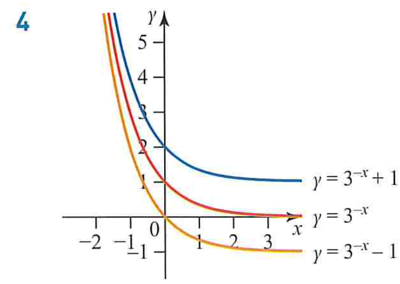
2



3

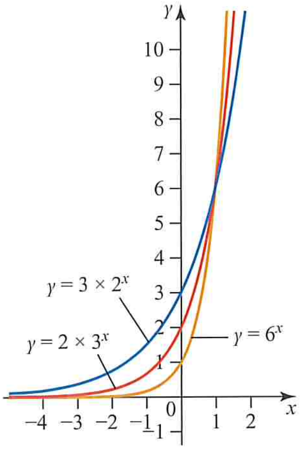


4



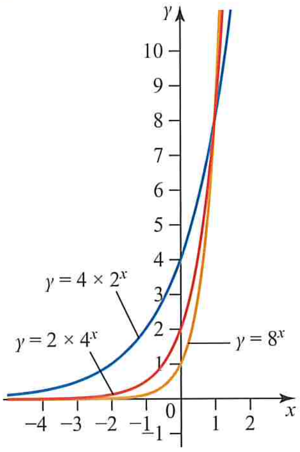
5 (i) 94 (ii) 80 (iii) 5

6



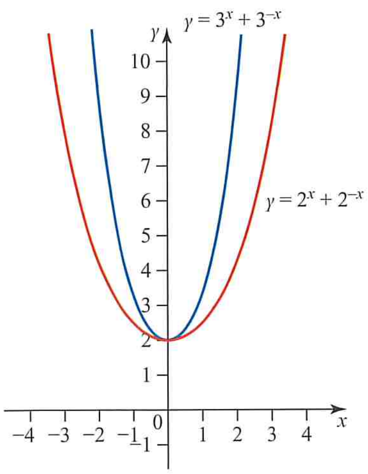
All three graphs intersect at the point (1, 6).

7 (i)



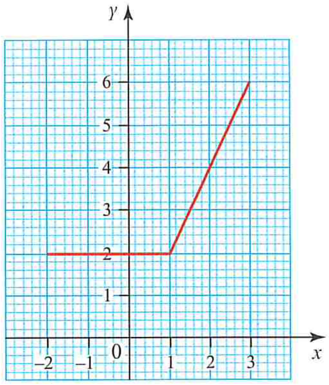
(ii) The shape is similar but the point of intersection is different. It was (1, 6) in Q6 and is (1, 8) in this question.

8 The graphs touch at the point (0, 2).

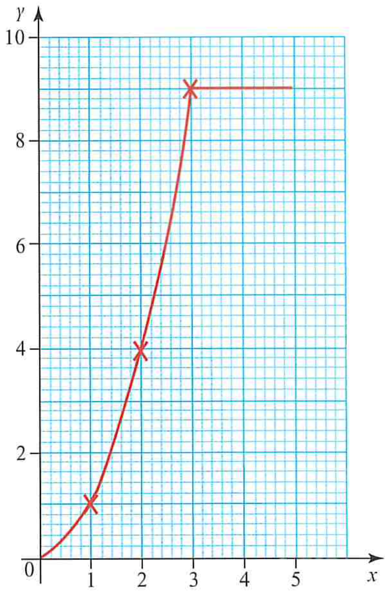


Exercise 3I (page 73)

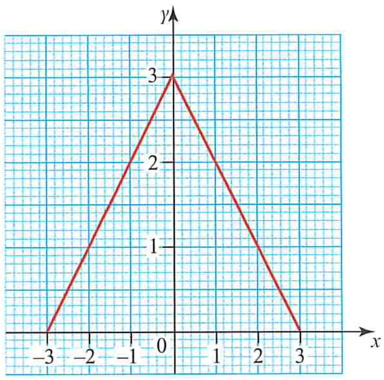
1



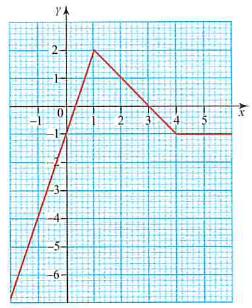
2



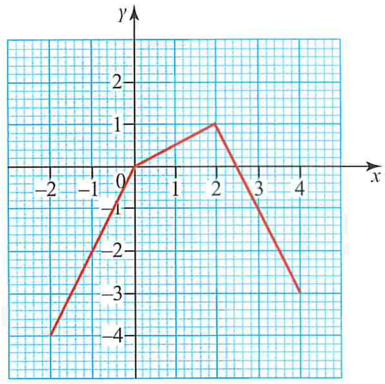
3



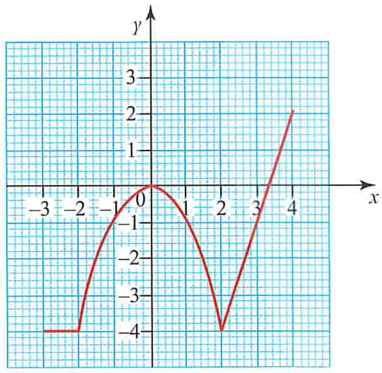
4



5



6



7 (i) $f(x) = x \quad 0 \leqslant x < 3$
$= 3 \quad 3 \leqslant x < 5$
$= 2x - 7 \quad 5 \leqslant x \leqslant 7$

(ii) $0 \leqslant f(x) \leqslant 7$

(iii) $x = 6$

8 (i) $f(x) = 5 \quad -3 \leqslant x < 1$
$= 7 - 2x \quad 1 \leqslant x < 3$
$= x - 2 \quad 3 \leqslant x \leqslant 5$

(ii) $1 \leqslant f(x) \leqslant 5$

(iii) $x = 2, x = 5$

9 (i) $f(x) = x + 3 \quad -3 \leqslant x < 0$
$= 3 \quad 0 \leqslant x < 3$
$= \dfrac{15 - 3x}{2} \quad 3 \leqslant x \leqslant 5$

(ii) $\dfrac{33}{2}$

10 (i) $g(x) = 5x + 15 - 3 \leqslant x < -2$
$= 5 \quad -2 \leqslant x < 2$
$= -\frac{5}{2}x + 10 \quad 2 \leqslant x \leqslant 4$

(ii) $\frac{55}{2}$

11 (i) Walking to the bus stop at a steady speed.
(ii) Waiting for the bus.
(iii) Bus is travelling at a constant speed.
Not very realistic – over a distance of 4 km you would expect the bus to have to slow down for traffic lights/make several stops.

Chapter 4

Discussion points (page 79)

If $b^2 - 4ac = 0$ then both answers are the same.

If $b^2 - 4ac < 0$ then no real answers are possible.

However, students who study Further Maths at A-Level will be introduced to a notation which allows us to square root negative numbers. Such numbers are referred to as imaginary. When combined with real numbers, they are referred to as complex numbers, and have many uses in the real world. For example, when modelling the behaviour of electrical circuits, or the flow of air around aeroplane wings.

Exercise 4A (page 82)

1 (i) $x = 2$ or $x = 6$
(ii) $m = 2$ (repeated)
(iii) $p = 5$ or $p = -3$
(iv) $a = -2$ or $a = -9$
(v) $x = -2$ or $x = -\frac{1}{2}$
(vi) $x = 1$ or $x = -1\frac{3}{4}$
(vii) $t = \frac{1}{5}$ or $t = -\frac{1}{3}$
(viii) $r = -\frac{1}{8}$ or $r = -\frac{2}{3}$
(ix) $x = \frac{1}{3}$ or $x = -3$
(x) $p = \frac{2}{3}$ or $p = 4$

2 (i) $x = 4.32$ or $x = -2.32$
(ii) $x = 1.37$ or $x = -4.37$
(iii) $x = 2.37$ or $x = -3.37$
(iv) $x = 1.77$ or $x = -2.27$
(v) $x = 1.68$ or $x = -2.68$
(vi) $x = 2.70$ or $x = -3.70$
(vii) $x = 4.24$ or $x = -0.24$
(viii) $x = 3.22$ or $x = 0.78$

3 (i) $x = -0.23$ or $x = -1.43$
(ii) $x = -0.41$ or $x = -1.84$
(iii) $x = 0.34$ or $x = -5.84$
(iv) $x = 1.64$ or $x = 0.61$
(v) $x = 1.89$ or $x = 0.11$
(vi) $x = -1.23$ or $x = -2.43$

4 3 cm, 4 cm, 5 cm

5 $x = 1.5$

6 9 and 11

7 (i) $t = 1$ s and $t = 2$ s
(ii) 3 seconds

8 (i) $\frac{1}{2}x(2x + 1) = 68$
$x^2 + 0.5x = 68$
$2x^2 + x = 136$
$2x^2 + x - 136 = 0$
(ii) 17 cm

9 (i) (a) $(x + 6)$ cm
(b) $(x - 10)$ cm
(c) $(x - 16)$ cm
(ii) Vol $= 8(x - 16)(x - 10)$
$= 8(x^2 - 10x - 16x + 160)$
$= 8(x^2 - 26x + 160)$
$= 8x^2 - 208x + 1280$
(iii) Length = 34 cm, width = 28 cm

10 (i) $x = \pm\frac{1}{\sqrt{3}}$
(ii) $x = 2,\ x = -\frac{19}{3}$
(iii) $a = -2$
(iv) $p = \frac{13 \pm 3\sqrt{17}}{2}$

11 (i) $p = 3,\ p = \frac{1}{3}$
(ii) $x = 0,\ x = 3$
(iii) $r = 2,\ r = -\frac{7}{3}$

12 (i) $a = 5,\ a = -\frac{19}{2}$
(ii) $x = 7,\ x = 1$
(iii) $x = -2,\ x = -5$

13 30 cm

14 $3x^2 - 4x - 9 = 0$ (or any multiple of this)

Discussion point (page 84)

Infinitely many possibilities.
x can take any value and, in this example, the corresponding value of y is $4 - x$.

Discussion point (page 86)

In this example the correct solution would be found, but in some cases, e.g. if the curve had equation $y^2 = 4x$, additional values that are not part of the solution can be obtained. Always substitute into the equation of the line. For example

$$y = x - 2$$
$$y^2 = 4x - 8$$

has $x = 2, y = 0$ and $x = 6, y = 4$ as its solution.

Substituting into the equation of the curve would also give the incorrect pair of values $x = 6$, $y = -4$, which is not a solution of both equations.

Discussion point (page 87)

Subtract if the coefficients of the variable to be eliminated have the same sign. Add if they have opposite signs.

Exercise 4B (page 88)

1 (i) $x = 5, y = 2$
(ii) $x = 4, y = -1$
(iii) $x = 2\frac{1}{4},\ y = 6\frac{1}{2}$
(iv) $x = -2, y = -3$
(v) $x = 1\frac{1}{2}, y = 4$

(vi) $x = -\frac{1}{2}, y = -6\frac{1}{2}$

2 (i) $x = 2, y = 3$
(ii) $x = 4, y = 3$
(iii) $x = 6, y = 2$
(iv) $x = -\frac{3}{7}, y = 3\frac{2}{7}$
(v) $x = 2, y = 5$
(vi) $x = -1, y = -2$

3 (i) $x = 1, y = 4$ or $x = 4, y = 1$
(ii) $x = 2, y = 3$ or $x = -\frac{2}{3}, y = \frac{1}{3}$
(iii) $x = 4, y = -2$ or $x = -1, y = -7$
(iv) $x = 1, y = 5$ or $x = 11, y = 25$
(v) $x = 4, y = 2$ or $x = -4, y = -2$
(vi) $x = 1, y = -2$ $x = -2\frac{3}{7}, y = -\frac{2}{7}$

4 (i) $3c + 4l = 72, 5c + 2l = 64$; a chew costs 8p and a lollipop costs 12p.
(ii) $x + 5m = 500$, $x + 7m = 660$; $m = 80, x = 100$; £2.60
(iii) $3c + 2n = 145$, $2c + 5n = 225$; $n = 35$, $c = 25$; £1.65
(iv) $2a + c = 3750$, $a + 3c = 3750$; $c = 750$, $a = 1500$; £67.50

5 A(3, 4), B(4, 3)

6 16 and −6

7 (i) (−2, 2)
(ii) Graph (b) because it has one intersection point, whereas graph (a) has no intersection points and graph (c) has two.

Discussion points (page 91)

The answer is zero in both cases.

Discussion points (page 92)

$f(1) = -4$
No, since you would only try factors of the constant term −1.

Exercise 4C (page 95)

1 (i) Factor
(ii) No
(iii) Factor
(iv) Factor
(v) No
(vi) Factor

2 (i) $(x-1)(x+1)(x-3)$
(ii) $(x+1)(x+2)(x-3)$
(iii) $x(x+1)(x-2)$
(iv) $(x+1)(x+2)(x-5)$
(v) $(x-2)(x+4)(x-3)$
(vi) $(x+1)(x-1)(x-5)(x+2)$
(vii) $(x-1)^4$
(viii) $(x-2)(x+2)(x+3)(x-3)$
(ix) $x(x+1)(x-2)(x+3)(x-6)$
(x) $(x-1)(x+2)(x-3)(x+4)(x-5)$

3 (i) 1, 3, −2
(ii) 2, −1, −4
(iii) −1, −3, 6
(iv) 1, −1, −4
(v) −2.35, 1, 0.85
(vi) 1, −2, 3, −4, 5

4 −5

5 (i) $1 + p + q + 6 = 0$
(ii) $-27 + 9p - 3q + 6 = 0$
(iii) $p = 0 \qquad q = -7$

6 (i) $k = -7$
(ii) $x = 1, x = -3$

7 (i) $\frac{8}{x^2}$
(ii) Surface area $= (x \times x) + 4\left(x \times \frac{8}{x^2}\right)$
$= x^2 + 4\left(\frac{8}{x}\right)$
$= x^2 + \frac{32}{x}$
(iii) $x^2 + \frac{32}{x} = 24$
$x^3 + 32 = 24x$
$x^3 - 24x + 32 = 0$
(iv) $x = 4, x = 1.46$

8 $x = 1, x = -\frac{2}{5}, x = \frac{-1 \pm \sqrt{5}}{2}$

Discussion point (page 96)

$8 \times 10^7 \leqslant x \leqslant 3.8 \times 10^8$

Discussion point (page 96)

An equation contains an = sign and any solution will consist of one or more particular values of any variables involved.

An inequality contains any of the signs $<, \leqslant, >, \geqslant$ and any solution will consist of a range of values of its variable(s).

Discussion points (page 96)

An inequality may be rearranged using addition, subtraction, multiplication by a positive number and division by a positive number in the same way as an equation.

Multiplication or division by a negative number reverses the inequality.

$2 < 3$ and $-2 > -3$
$5 > -1$ and $-5 < 1$.

Exercise 4D (page 97)

1 (i) $x < 5$
(ii) $x \geqslant 2$
(iii) $y \leqslant 4$
(iv) $y < 4$
(v) $x \geqslant -3$
(vi) $b \geqslant -3$
(vii) $x > -3$
(viii) $x < 12$
(ix) $x \geqslant -5$
(x) $x \leqslant -4$
(xi) $2 \leqslant x \leqslant 4$
(xii) $2 \leqslant x \leqslant 5$
(xiii) $-3 < x < 1$
(xiv) $1 < x < 2$

2 $-5 \leqslant p - q \leqslant 1$

3 $-1 < x + y < 7$

4 (i) $-2 \leqslant a + b \leqslant 9$
(ii) $-2 \leqslant a - b \leqslant 9$

5 (i) $-4 \leqslant a + b \leqslant 10$
(ii) $-13 \leqslant a - b \leqslant 1$
(iii) $-9 \leqslant 2a + 3b \leqslant 30$

6 (i) always
(ii) never
(iii) sometimes
(iv) sometimes

(v) never
(vi) always

7 (i) always
(ii) always
(iii) never
(iv) sometimes
(v) never
(vi) sometimes

8 $x > 4$

Exercise 4E (page 100)

1 (i) $x < 1$ or $x > 5$
(ii) $-4 \leqslant a \leqslant 1$
(iii) $-1\frac{1}{2} < y < 1$
(iv) $-2 \leqslant y \leqslant 2$
(v) $x < 2$ or $x > 2$
(vi) $1 \leqslant p \leqslant 2$
(vii) $a < -3$ or $a > 2$
(viii) $-4 \leqslant a \leqslant 2$
(ix) $y < -1$ or $y > \frac{1}{3}$
(x) $y \leqslant -1$ or $y \geqslant 5$

2 (i) $-2 < x < 3$
(ii) $-4 < x < 2$
(iii) $-4 < x < -2$
(iv) $2 < x < 3$
(v) $-\frac{1}{2} < x < 1$
(vi) $x < -3, x > 2$
(vii) $x < -3, x > 7$
(viii) $x < -\frac{1}{3}, x > 2$

3 $x > 5$

4 $0 < p < 1$

5 $3\,\text{m} \leqslant$ *length* $\leqslant 7\,\text{m}$

6 $2.5 \leqslant x < 6$

7 $1.5 < t < 2.5$

8 $\frac{1 - \sqrt{15}}{2} < x < \frac{1 + \sqrt{15}}{2}$

Activity 4.1 (page 101)

(i) $a^3 \times a^0 = a^{3+0} = a^3$; so $a^0 = 1$

(ii) $a^2 \times a^{-2} = a^{2-2} = a^0 = 1$;
so $a^{-2} = \frac{1}{a^2}$

(iii) $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$;
so $a^{\frac{1}{2}}$ is the square root of a.

(iv) $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$
$= a^1 = a$; so $a^{\frac{1}{3}}$ is the cube root of a.

Discussion points (page 104)

Even powers and roots can never be negative. In A-Level Further Maths you will learn how to deal with even powers which are negative.

Odd powers and roots can be negative.

All exponential functions of the form a^x or a^{-x} are always positive.

Exercise 4F (page 105)

1 (i) x^4
(ii) x^{-3}
(iii) $x^{\frac{5}{2}}$
(iv) x^6
(v) x^4
(vi) x^2
(vii) x^{-1}
(viii) x^{10}
(ix) x^2

2 (i) 9
(ii) -8
(iii) $\frac{1}{9}$
(iv) $\frac{1}{4}$
(v) $\frac{1}{64}$
(vi) $\frac{1}{2^{\frac{1}{3}}}$
(vii) 8 or -8
(viii) 9
(ix) $\frac{5}{3}$ or $-\frac{5}{3}$
(x) 9
(xi) $\frac{27}{8}$
(xii) $\frac{1}{10}$ or $-\frac{1}{10}$

3 (i) $x^2 + x$
(ii) $x - 1$
(iii) $x - x^3$
(iv) $x^{-5} + x^{-4}$
(v) $x + x^3$
(vi) $x^3 - 1$

4 (i) $x = 4, x = 1$
(ii) $x = \frac{9}{4}, x = 1$
(iii) $x = 9$
(iv) $x = 1, x = 25$
(v) $x = \frac{1}{4}$
(vi) $x = 1$

5 (i) $x = \pm 2, x = \pm 3$
(ii) $x = \pm 4$
(iii) $x = \pm 2$
(iv) $x = \pm\frac{3}{2}$
(v) $x = \pm 3$

6 (i) $x = -1, x = 2$
(ii) $x = 8, x = -1$
(iii) $x = 1, x = 2$
(iv) $x = 3, x = 2$
(v) $x = 1$
(vi) $x = 3$

7 $x = 2, y = 3$

8 (i) $x = 4$
(ii) $x = -3$
(iii) $x = \frac{5}{2}$
(iv) $x = \frac{3}{2}$
(v) $x = 4$

9 (i) $x = 0, x = -2$
(ii) $x = -2, x = 5$
(iii) $x = \pm 2, x = 4$
(iv) $x = \pm 4, x = 1, x = 2$
(v) $x = \pm 2, x = 4$

Activity 4.2 (page 106)

$(3x^2 - 11x + 7)^{9x^2+27x+18} = 1$ is an example of such an equation, with $x = \pm 1, x = -2, x = 3, x = \frac{2}{3}$, $x = \frac{8}{3}$

Exercise 4G (page 108)

1 $2(m + 7) - 2(5 + m)$
$= 2m + 14 - 10 - 2m$
$= 4$
$=$ a positive integer

2 $5(c - 3) + 3(c + 7)$
$= 5c - 15 + 3c + 21$
$= 8c + 6$
$= 2(4c + 3)$
$= 2 \times$ an integer
$=$ an even number

3 $(y+6)(y+3) - y^2$

$= y^2 + 9y + 18 - y^2$

$= 9y + 18$

$= 9(y + 2)$

$= 9 \times$ an integer

$=$ a multiple of 9

4 (i) $f(n+1) = (n+1)^2$

$= n^2 + 2n + 1$

(ii) $f(n+1) + f(n-1)$

$= (n+1)^2 + (n-1)^2$

$= n^2 + 2n + 1 + n^2 - 2n +$

$= 2n^2 + 2$

$= 2(n^2 + 1)$

$= 2 \times$ an integer

$=$ an even number

(iii) $f(n+1) - f(n-1)$

$= (n+1)^2 - (n-1)^2$

$= n^2 + 2n + 1 - (n^2 - 2n + 1)$

$= 4n$

$= 4 \times$ an integer

$=$ a multiple of 4

5 (i) $x^2 + 2x + 5$

$= (x+1)^2 - 1^2 + 5$

$= (x+1)^2 + 4$

(ii) $(x+1)^2 \geqslant 0$

$\Rightarrow (x+1)^2 + 4 \geqslant 4$

$\Rightarrow x^2 + 2x + 5 \geqslant 4$

$\therefore x^2 + 2x + 5 > 0$

6 $(y-5)^2 \geqslant 0$

$\Rightarrow (y-5)^2 + 1 \geqslant 1$

$\Rightarrow y^2 - 10y + 26 \geqslant 1$

$\therefore y^2 - 10y + 26 > 0$

7 $9m^2(3m-1) + (3m)^2$

$= 27m^3 - 9m^2 + 9m^2$

$= 27m^3$

$= (3m)^3$

$= (\text{integer})^3$

$=$ a cube number

8 $\dfrac{6p-18}{2p-6} = \dfrac{6(p-3)}{2(p-3)}$

$= \dfrac{6}{2}$

$= 3$

$=$ a positive integer

9 $\dfrac{a^2+ab}{ab+b^2} = \dfrac{a(a+b)}{b(a+b)}$

$= \dfrac{a}{b}$

$= \dfrac{\text{positive}}{\text{negative}}$

$=$ negative

10 $f(4x) = (4x)^2 + 2 \times 4x$

$= 16x^2 + 8x$

$= 8x(2x+1)$

Exercise 4H (page 110)

1 (i) $4n + 6$
(ii) $7n - 5$
(iii) $2n - 7$
(iv) $25n - 25$
(v) $8n - 19$
(vi) $0.5n + 2.5$
(vii) $50 - 10n$
(viii) $10 - 3n$
(ix) $1\frac{1}{2} - \frac{1}{2}n$
(x) $-2.5 - 1.5n$

2 (i) 589
(ii) -308
(iii) -1792

3 250.5

4 $9 - 2n + 6 - 3n = 15 - 5n$

5 (i) $p = -52$ $q = 18$
(ii) $36n - 88$

6 If 44 is a term in the sequence, then $7n - 3 = 44$ for a positive integer value of n

$\Rightarrow 7n = 47$

$\Rightarrow n = \dfrac{47}{7}$

$\Rightarrow n = 6\frac{5}{7}$ which is not an integer

$\therefore$ 44 is not a term in the sequence

7 The difference between consecutive terms is $(n+1)^3 - n^3$

$= n^3 + 3n^2 + 3n + 1 - n^3$

$= 3n^2 + 3n + 1$

$= 3(n^2 + n) + 1$

$= 3 \times$ an integer $+1$

$=$ one more than a multiple of 3

$\therefore$ the difference is never a multiple of 3

8 nth term

$= n^2 - 40n + 405$

$= (n-20)^2 - 400 + 405$

$= (n-20)^2 + 5 \geqslant 5$

$\therefore$ all terms are positive

Exercise 4I (page 112)

1 (i) $n^2 + 2n + 1$
(ii) $n^2 + 3n - 4$
(iii) $n^2 + 6n - 3$
(iv) $3n^2 + 4n + 1$
(v) $2n^2 + 3n - 1$
(vi) $2n^2 - 6n$
(vii) $-n^2 + 2n + 10$
(viii) $-2n^2 + 100$

2 (i) $4n - 3$
(ii) $16n^2 - 24n + 9$

3 (i) $n^2 + 2n - 1$
(ii) $n^2 + 2n + 2$

4 (i) $n^2 - 4n - 2$
(ii) $3n^2 - 12n - 6$
(iii) $3n^2 - 12n + 9$

5 (i) 77

(ii) 1

Only the first term will be even. All other terms are the result of adding 3 to an even number which will always produce an odd term.

6 $n^2 - 2n + 12$

7 $n^2 - 2n - 4$

Activity 4.3 (page 113)

Most people would guess that the 6th term is 32.

However, it's actually 31.

The nth term is **not** 2^{n-1} as you might expect.

In fact, the nth term is

$$\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

Activity 4.4 (page 113)

(i)

1.5	2	2.25	2.4
2.5	2.571	2.625	
2.667	2.7	2.727	2.75
2.769	2.786	2.8	2.813

(ii)

2.857	2.903	2.927
2.941	2.970	2.985
2.994		

(iii) The terms are increasing in size and appear to be getting closer to 3.

Exercise 4J (page 114)

1 (i) $\frac{2}{3}$ $\frac{3}{5}$ $\frac{4}{7}$

(ii) 12th

2 (i) 15th

(ii) $\frac{4n-1}{2n-5} = 1$

$4n - 1 = 2n - 5$

$2n = -4$

$n = -2$

n has to be a positive integer

3 (i) 2

(ii) 1

(iii) $\frac{1}{3}$

(iv) $\frac{1}{2}$

(v) $\frac{3}{4}$

(vi) -1

(vii) $-\frac{1}{2}$

(viii) 3

4 nth term $= \frac{5n+1}{2n+1}$

$$= \frac{5 + \frac{1}{n}}{2 + \frac{1}{n}} \to \frac{5+0}{2+0}$$

as $n \to \infty$

$\therefore$ the limit is $\frac{5}{2}$

5 nth term $= \frac{10-6n}{8n-3}$

$$= \frac{\frac{10}{n} - 6}{8 - \frac{3}{n}} \to \frac{0-6}{8-0}$$

$= -0.75$ as $n \to \infty$

$\therefore$ the limit is $\frac{-6}{8} = -0.75$

6 2

7 $\frac{19}{3}$

8 $\frac{2}{7}$

Exercise 4K (page 117)

1 (i) $x = 2, y = 1, z = 5$

(ii) $x = 3, y = -1, z = 2$

(iii) $x = 4, y = 1, z = -3$

2 (i) $x = 2, y = -2, z = 5$

(ii) $x = 4, y = -2, z = 3$

(iii) $x = 7, y = -1, z = -3$

$x = 2, y = -1, z = \frac{1}{2}$

3 (i) $a = -5, b = 4, c = -3$

(ii) $p = -3, q = 4, r = -1$

(iii) $\alpha = 5, \beta = -5, \gamma = 3$

4 (i) $x = 4, y = -5, z = 6$

(ii) $x = 7, y = -1, z = 2$

(iii) $x = 2, y = 1, z = -3$

5 $a = 7, b = 2, c = -3$

6 (i) $a + b + c = 7$,

$4a + 2b + c = 9$,

$9a + 3b + c = 13$

(ii) nth term $= n^2 - n + 7$

7 $2n^2 - 3n + 7$

8 $(1, 3, 4)$

Activity 4.5 (page 119)

(i) Two planes would never meet if they were parallel.

(ii) Two planes meet at a line, so there are an infinite number of common points.

(iii) e.g. $3x + 2y - z = d$ where d is any constant other than 5.

(iv) No – there could be three different, but parallel, lines on each pair of planes.

(v) Yes – they could all share the same common line, giving an infinite number of common points.

Chapter 5

Discussion point (page 122)

When the increase in x is the same for both lines, then the increase in y is also the same for both lines.

Activity 5.1 (page 122)

(i) As Figure 5.2

(ii) $\angle ABE = \angle BCD$ and

$\angle BCD + \angle CBD = 90°$

$\Rightarrow \angle ABE + \angle CBD = 90°$

i.e. $\angle ABC = 90°$

(iii) $m_1 = \frac{q}{p}$ and $m_2 = -\frac{p}{q}$

(iv) Follows from (iii).

Discussion point (page 124)

$\sqrt{4a^2 + 16b^2}$

$= \sqrt{4(a^2 + 4b^2)}$

$= 2\sqrt{a^2 + 4b^2}$

Exercise 5A (page 125)

1 (i) (a) $-\frac{1}{2}$

(b) $\sqrt{80} = 4\sqrt{5}$

(c) $(6, 7)$

(ii) (a) $\frac{1}{3}$

(b) $\sqrt{90} = 3\sqrt{10}$

(c) $\left(1\frac{1}{2}, 8\frac{1}{2}\right)$

(iii) (a) $\frac{5}{11}$

(b) $\sqrt{146}$

(c) $\left(7\frac{1}{2}, -2\frac{1}{2}\right)$

(iv) (a) $-\frac{1}{3}$

(b) $\sqrt{490} = 7\sqrt{10}$

(c) $\left(-2\frac{1}{2}, -3\frac{1}{2}\right)$

(v) (a) $-\frac{2}{15}$

(b) $\sqrt{229}$

(c) $\left(7, 7\frac{1}{2}\right)$

(vi) (a) $-\frac{5}{13}$

(b) $\sqrt{194}$

(c) $\left(\frac{1}{2}, 2\frac{1}{2}\right)$

(vii) (a) 5

(b) $\sqrt{26}$

(c) $\left(-\frac{1}{2}, -6\frac{1}{2}\right)$

(viii) (a) $\frac{3}{11}$

(b) $\sqrt{130}$

(c) $\left(5\frac{1}{2}, 1\frac{1}{2}\right)$

2 (i) Gradient AB $= -1$; gradient AC $= 1$; product $= -1$

(ii) AB $= \sqrt{32}$; AC $= \sqrt{8}$; BC $= \sqrt{40}$; $BC^2 = AB^2 + AC^2$

3 Gradient AB $= -\frac{1}{2}$; gradient AC $= 2$; product $= -1$; AB $=$ AC $= \sqrt{20}$

4 (i) 19.73 units

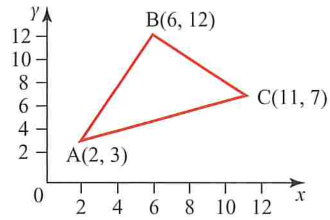
(ii) 9 units^2

5 (i) PQ $\sqrt{173}$; QR $\sqrt{173}$; RS $\sqrt{173}$; PS $\sqrt{173}$

(ii) $\left(3\frac{1}{2}, \frac{1}{2}\right)$

(iii) Gradient PQ $= -\frac{2}{13}$; gradient QR $= -\frac{13}{2}$, so PQ is not perpendicular to QR; rhombus

6 (i)



(ii) AB $=$ AC $= \sqrt{97}$; BC $= \sqrt{50}$

(iii) $\left(8\frac{1}{2}, 9\frac{1}{2}\right)$

(iv) 32.5 units^2

7 (i) $\left(-\frac{1}{2}, 2\right)$

(ii) $(0, -1)$

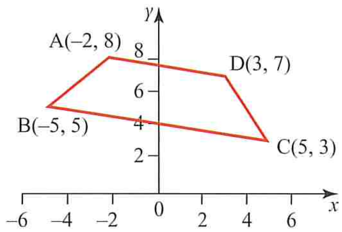
8 (i) Gradient AB $= -2$; gradient BC $= \frac{1}{2}$

(ii) $(7, 4)$

9 (i) $q = 2$

(ii) $1 : 2$

10 (i)



(ii) Gradient AD $=$ gradient BC $= -\frac{1}{5}$; Gradient AB $\neq$ gradient DC.

(iii) $(8, 6)$

Exercise 5B (page 129)

1 (i) $0, \infty$; perpendicular

(ii) $2, -2$; neither

(iii) $-\frac{1}{2}, 2$; perpendicular

(iv) $1, 1$; parallel

(v) $-4, -3$; neither

(vi) $-1, 1$; perpendicular

(vii) $\frac{1}{2}, \frac{1}{2}$; parallel

(viii) $-\frac{1}{3}, 3$; perpendicular

(ix) $\frac{1}{2}, -2$ perpendicular

(x) $-\frac{2}{3}, -\frac{2}{3}$; parallel

(xi) $-\frac{1}{3}, -3$; neither

(xii) $\frac{2}{5}, -\frac{5}{2}$; perpendicular

2 (i) $y = 3x - 10$

(ii) $y = 2x + 7$

(iii) $y = 3x - 16$

(iv) $y = 4x - 20$

(v) $3x + 2y - 5 = 0$

(vi) $x + 2y - 10 = 0$

3 (i) $x + 2y = 0$

(ii) $x + 3y - 12 = 0$

(iii) $y = x - 4$

(iv) $x + 2y + 1 = 0$

(v) $2x - 3y - 6 = 0$

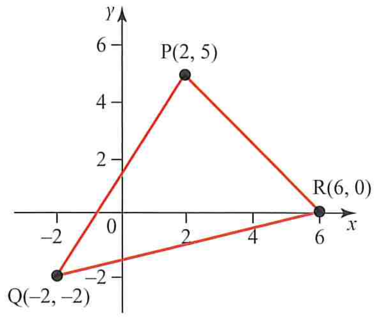
(vi) $x - 2y - 2 = 0$

4 (i) 4

(ii) $(4, 3)$

(iii) $x + 4y - 16 = 0$

5 (i)



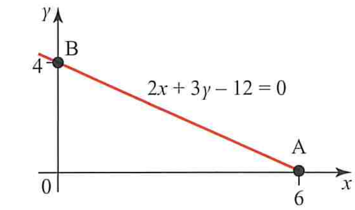
(ii) $L\left(0, 1\frac{1}{2}\right)$, $M(2, -1)$, $N\left(4, 2\frac{1}{2}\right)$

(iii) LR: $x + 4y - 6 = 0$
MP: $x = 2$
NQ: $3x - 4y - 2 = 0$

(iv) Substitute $x = 2$ *and* $y = 1$ into the three equations
LR: $x + 4y - 6$
$= 2 + 4 - 6 = 0$
MP: $x = 2$

NQ: $3x - 4y - 2$
$= 6 - 4 - 2$
$= 0$

6 (i)



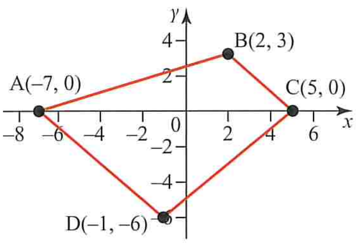
(ii) A(6, 0), B(0, 4)

(iii) 12 units^2

(iv) $3x - 2y = 0$

(v) AB $= \sqrt{52}$ units; shortest distance $= 3.33$ units (2 d.p.)

7 (i)



(ii) AB: $\frac{1}{3}$; BC: -1; CD: 1; DA: -1

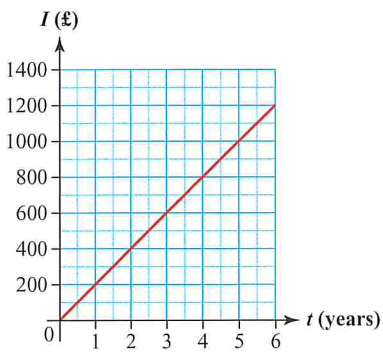
(iii) AB: $x - 3y + 7 = 0$
BC: $x + y - 5 = 0$
CD: $x - y - 5 = 0$
DA: $x + y + 7 = 0$

(iv) AB: $3\sqrt{10}$ units
BC: $3\sqrt{2}$ units
CD: $6\sqrt{2}$ units
DA: $6\sqrt{2}$ units

(v) 54 units^2

8 (i) £200 after each year

(ii) $I = 200t$



(iii) 5 years

9 (i) 125 g

(ii) 7.5 cm

(iii) 80 cm – it is likely that the spring would have reached its elastic limit by then (i.e. it would have been stretched too much and does not function as a spring any more)

Discussion point (page 131)

You need to choose a scale that makes it easy to plot the points and read off the coordinates of the point of intersection. It is particularly difficult to get an accurate solution when it is not represented by a point on the grid.

Discussion points (page 132)

You can always join two points with a straight line. Using three points alerts you if one of your calculated points is wrong.

They won't intersect if they are parallel.

Exercise 5C (page 132)

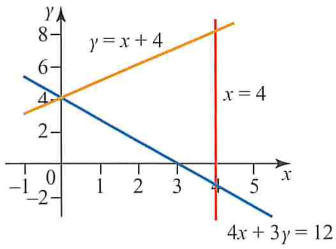
1 (i) $x = 1, y = 0$

(ii) $x = -1, y = 4$

2 (i) $x = 3, y = 2$

(ii) $x = \frac{1}{2}, y = -2$

3 (i)



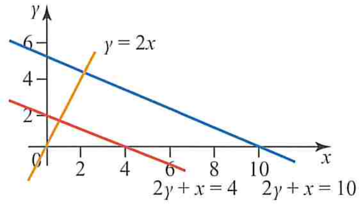
(ii) (4, 8) at the intersection of $x = 4$ and $y = x + 4$

$\left(4, -1\frac{1}{3}\right)$ at the intersection of $x = 4$ and $4x + 3y = 12$

(0, 4) at the intersection of $y = x + 4$ and $4x + 3y = 12$

(iii) $18\frac{2}{3}$ units^2

4 (i)



The lines look parallel. They have the same gradient.

(ii) This line looks perpendicular to the first two lines. The first two lines have a gradient of $-\frac{1}{2}$ and the third line has a gradient of +2. The product of the gradients is −1 so they are perpendicular.

(iii) $\left(\frac{4}{5}, 1\frac{3}{5}\right)$ at the intersection of $y = 2x$ and $2y + x = 4$

(2, 4) at the intersection of $y = 2x$ and $2y + x = 10$

5 (i) AB = AC = $3\sqrt{2}$; BC = 6

(ii) AB: $y = x + 3$; AC: $y = 3 - x$; BC: $x = 3$

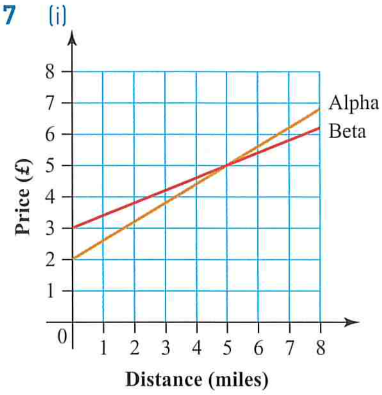
(iii) Isosceles

6 (i) AB: 3; BC: $-\frac{1}{3}$; CD: 3; DA: $-\frac{1}{3}$. The opposite sides are parallel and adjacent sides are perpendicular

(ii) AB = BC = $\sqrt{10}$

(iii) A square.

7 (i)

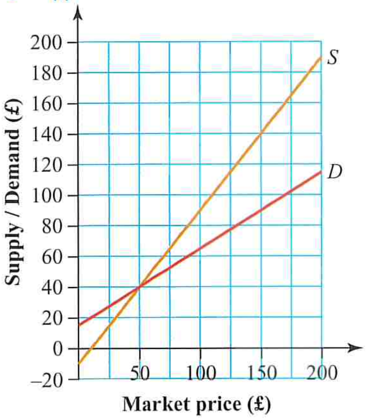


(ii) A: $y = 2 + 0.6x$
B: $y = 3 + 0.4x$

(iii) Beta

(iv) 5 miles

8 (i)



(ii) Equilibrium price = £50; Number bought and sold = 40

Activity 5.2 (page 134)

AE $= x_2 - x_1$ BE $= y_2 - y_1$

$$\frac{AC}{AB} = \frac{p}{p+q}$$

Triangles ACD and ABE are similar.

$$\frac{AD}{AE} = \frac{AC}{AB} \text{ so } \frac{AD}{x_2 - x_1} = \frac{p}{p+q}$$

$$AD = \frac{p}{p+q}(x_2 - x_1)$$

x-coordinate of C is

$$x_1 + \frac{p}{p+q}(x_2 - x_1)$$

$$= \frac{(p+q)x_1 + p(x_2 - x_1)}{p+q}$$

$$= \frac{(px_1 + qx_1 + px_2 - px_1)}{p+q}$$

$$= \frac{qx_1 + px_2}{p+q}$$

Also

$$\frac{CD}{BE} = \frac{AC}{AB} \text{ so } \frac{CD}{y_2 - y_1} = \frac{p}{p+q}$$

$$CD = \frac{p}{p+q}(y_2 - y_1)$$

y-coordinate of C is

$$y_1 + \frac{p}{p+q}(y_2 - y_1)$$

$$= \frac{(p+q)y_1 + p(y_2 - y_1)}{p+q}$$

$$= \frac{(py_1 + qy_1 + py_2 - py_1)}{p+q}$$

$$= \frac{qy_1 + py_2}{p+q}$$

Discussion points (page 135)

1 : 2

1 : 4

Exercise 5D (page 135)

1 (i) (5, 12)

(ii) (9, 1)

(iii) (4, 1)

(iv) $\left(\frac{7}{5}, 17\right)$

(v) (−10, −11)

2 (i) (14, 8)

(ii) (−2, 9)

(iii) (1, 3)

(iv) $\left(-1, -\frac{7}{5}\right)$

(v) (7, 3)

3 (i) 5 : 4

(ii) (7, −8)

4 $\left(-\frac{3}{2}, 2\right)$

5 $\left(\frac{13}{4}, 4\right)$

6 (i) (a) 10 cm by 8 cm

(b) 6 cm by 4 cm.

(ii) 2500 : 1

7 (i) 58 cm

(ii) Height = 52 cm, width = 86 cm

8 (i) $4\sqrt{2}$

(ii) A′B′ = 2, B′C′ = 2, C′A′ = $2\sqrt{2}$

(iii) Ratio is 1 : 2

(iv) Ratio is 1 : 4

Exercise 5E (page 140)

1 (i) $(x-1)^2 + (y-2)^2 = 9$

(ii) $(x-4)^2 + (y+3)^2 = 16$

(iii) $(x-1)^2 + y^2 = 25$

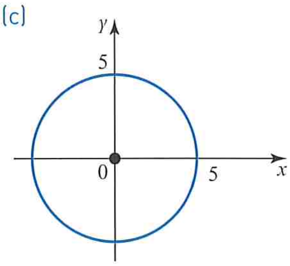
(iv) $(x+2)^2 + (y+2)^2 = 4$

(v) $(x+4)^2 + (y-3)^2 = 1$

2 (i) (a) (0, 0)

(b) 5

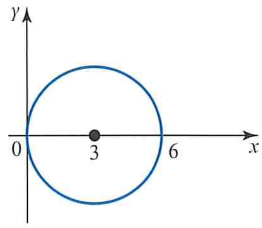
(c)



(ii) (a) (3, 0)

(b) 3

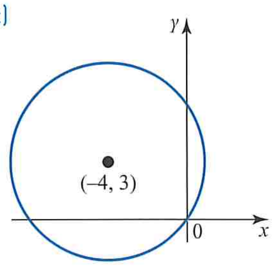
(c)



(iii) (a) (−4, 3)

(b) 5

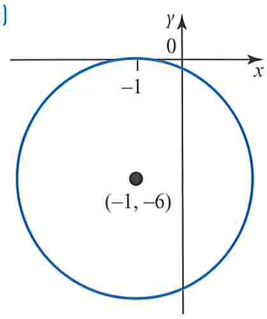
(c)



(iv) (a) (−1, −6)

(b) 6

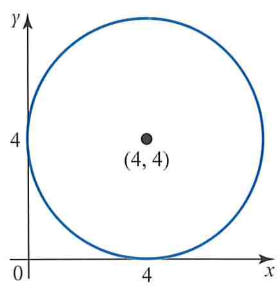
(c)



(v) (a) (4, 4)

(b) 4

(c)



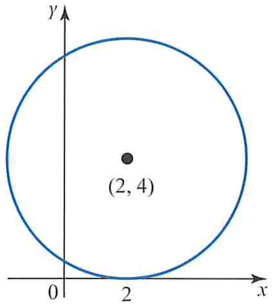
3 $(x-2)^2 + (y+3)^2 = 5$

4 (i) $(3, 1)$

(ii) $\sqrt{26}$

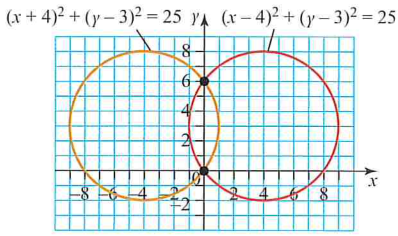
(iii) $(x-3)^2 + (y-1)^2 = 26$

5 Centre $(2, 4)$, radius 4



6 Rewriting the equation gives an apparent radius of $\sqrt{(-14)}$ units

7



Equations are

$(x-4)^2 + (y-3)^2 = 25$ and

$(x+4)^2 + (y-3)^2 = 25$

8 (i) $\sqrt{50}$

(ii) $(x-11)^2 + (y-8)^2 = 50$

(iii) $(16, 13)$

Discussion point (page 141)

In triangles CRM and CSM

CR = CS (radii)

CM is common to both triangles

Angle CMR = angle CMS = 90°

Triangles are congruent (2 sides and a non-included right angle)

Hence RM = MS

Discussion point (page 142)

In triangles TAC and TBC

AC = BC (radii)

TC is common to both triangles

Angle TAC = angle TBC = 90°

Triangles are congruent (2 sides and a non-included right angle)

Hence TA = TB

Exercise 5F (page 144)

1 $\frac{1}{5}$

2 $(5, 0)$

3 $-\frac{3}{2}$

4 $y = -x + 7$

5 $y = \frac{3}{4}x - \frac{25}{4}$

6 $(x-2)^2 + (y-3)^2 = 25$

7 (i) $(x-2)^2 + (y-1)^2 = 25$

(ii) $4y = 3x - 27$

(iii) $x = -3$, $(-3, 1)$

(iv) 10

8 (ii) $x = -5$

(iii) $(-5, -10)$

Chapter 6

Activity 6.1 (page 149)

1 1 4 9 16 25

36 49 64 81 100

121 144 169 196

225 256 289 324

361 400 441 484

529 576 625

$10^2 = 6^2 + 8^2$

$15^2 = 9^2 + 12^2$

$20^2 = 12^2 + 16^2$

$25^2 = 15^2 + 20^2$

$13^2 = 5^2 + 12^2$

$17^2 = 8^2 + 15^2$

$25^2 = 7^2 + 24^2$

Right-angled triangles can be made with sides the lengths of the numbers used.

Exercise 6A (page 152)

1 $x = 28°$ $y = 25°$

2 $x = 113°$

3 $x = 62°$ $y = 48°$

4 $x = 118°$ $y = 18°$

5 $x = 57.5°$

6 $x = 19°$

7 $x = 38°$

8 $c = 42°$

9 $x = 36°$ $y = 18°$

Exercise 6B (page 157)

These solutions may not be unique.

1 Angle ABC = 90 (angle in a semi-circle)

$x + y + 90 = 180$ (angle sum of triangle)

$x = 90 - y$

2 Angle CBF = x (alternate angles)

$x + 2x + y = 180$ (angle sum of triangle)

$3x + y = 180$

3 Angle YAB = b (angles in the same segment)

angle AYB = 90 (angle in a semi-circle)

$a + b + 90 = 180$ (angle sum of triangle)

$a + b = 90$

4 Angle CDB = a (base angles of isosceles triangle)

angle ABF = a (corresponding angles)

angle ABC = a (alternate angles)

angle ABC = angle ABF

5 Angle PTC = 90 (tangent is perpendicular to radius)

angle PCT + 90 + $2y$ = 180 (angle sum of triangle)

angle PCT = $90 - 2y$

angle TMN = $45 - y$ (angle at circumference is half angle at centre)

6 Angle ACB = angle DBA (alternate segment theorem)

angle ACB = angle BAD (base angles of isosceles triangle)

angle DBA = angle BAD

Triangle ABD is isosceles as base angles are equal

7 Angle EDG = $180 - y$ (opposite angles of cyclic quadrilateral)

angle CDG = $130 - y$

angle CGD = $130 - y$ (base angles of isosceles triangle)

$x + 130 - y + 130 - y = 180$
(angle sum of triangle)
$x + 260 - 2y = 180$
$x = 2y - 80$

8 Reflex angle PCR $= 2y$
(reflex angle at centre is double angle at circumference)
angle PCR $= 360 - 2y$
(angles at a point)
Angle CRQ $= x$
(since PQ = QR)
$x + x + y + 360 - 2y = 360$
(angle sum of quadrilateral)
$2x = y$

Discussion point (page 159)

No: since they are defined using the sides of a right-angled triangle they are restricted to
$0 < \theta < 90°$.

Activity 6.2 (page 159)

(i) Around 0.89
(ii) Depends on (i)
(iii) Draw a larger triangle.

Discussion point (page 160)

You need at least 3 decimal places:

$\tan^{-1} 0.714 = 35.5°$, but

$\tan^{-1} 0.71 = 35.4°$.

Discussion point (page 161)

The best function would be $\tan\theta$, since this does not use the value of h that you calculated earlier.

Exercise 6C (page 162)

1 (i) 11.2 cm
(ii) 7.7 cm
(iii) 12.1 cm
(iv) 15.1 cm
(v) 6.8 cm
(vi) 7.7 cm

2 (i) 30.6°
(ii) 50.4°
(iii) 55.7°
(iv) 41.4°
(v) 45.0°
(vi) 64.2°

3 (i) 63.6°
(ii) 14.9 cm
(iii) 9.1 cm

4 4.5 m

5 78.2 m

6 282.7 m

7 33.7°

8 (i) 119 km
(ii) 33°
(iii) 333 km h^{-1}

Discussion point (page 164)

The results would be unchanged

Exercise 6D (page 165)

1 (i) $5 + \sqrt{3}$
(ii) $3 + 2\sqrt{3}$
(iii) 5
(iv) $\frac{9}{2}$

2 $\cos 30° = \frac{y}{6\sqrt{3}}$

$\frac{\sqrt{3}}{2} = \frac{y}{6\sqrt{3}}$

$\frac{\sqrt{3}}{2} \times 6\sqrt{3} = y$

$9 = y$

3 $\sin 45° = \frac{\sqrt{8} + \sqrt{2}}{p}$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{8} + \sqrt{2}}{p}$

$p = \sqrt{2}\left(\sqrt{8} + \sqrt{2}\right)$

$p = \sqrt{16} + 2$

$p = 4 + 2$

$p = 6$

4 $(9 + 6\sqrt{3})$ cm^2

5 $(2\sqrt{6})$ cm

6 45°

7 (i) 100 (ii) 1559 (iii) 136

8 (i) $10\sqrt{3}$ m (ii) BC = 10 m; AB = 20 m

Discussion points (page 170)

Undefined means that you cannot find a value for it. When $\theta = 90°$, $x = 0$ and $\cos\theta = 0$, so neither definition works since you cannot divide by zero. $\tan\theta$ is also undefined for $\theta = 90° \pm$ any multiple of 180°.

Discussion point (page 170)

It is a line that is very close to the shape of the curve for large values of x or y.

Discussion points (page 170)

The period is 180° since it repeats itself every 180°.

For $-90° \leqslant \theta \leqslant 0°$, rotate the part of the curve for $0° \leqslant \theta \leqslant 90°$ through 180° about the origin. This gives one complete branch of the curve.

Translating this branch through multiples of 180° to the right or left gives the rest of the curve.

Discussion point (page 171)

The equation will have infinitely many roots since the curve continues to oscillate and the line $y = 0.5$ crosses it infinitely many times.

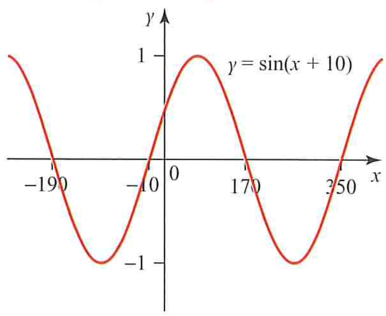
Discussion points (page 171)

$293.6° = -66.4° + 360°$

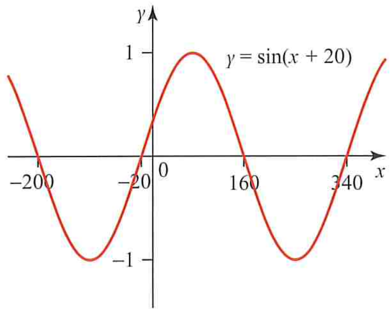
Change the sign of the principal angle (i.e. swap a + for a −, or vice versa) and add 360°.

Alternatively, subtract the principal angle from 360°.

Activity 6.3 (page 173)



- The graph of $y = \sin(x + 10)$ is a translation of the graph of $y = \sin x$ by 10° to the left.



- The graph of $y = \sin(x + 20)$ is a translation of the graph of $y = \sin x$ by 20° to the left.
- The graph of $y = \sin(x + 90)$ is the same as the graph of $y = \cos x$.
- $\cos x \equiv \sin(x + 90)$
  Note: 90 could be replaced with $90 + 360n$, where n is an integer, e.g. 450 or −270
- $\sin x \equiv \cos(x - 90)$
  Note: −90 could be replaced with $-90 + 360n$, where n is an integer, e.g. +270 or −450

Exercise 6E (page 173)

1 (i) 60°, 300°
(ii) 45°, 225°
(iii) 60°, 120°
(iv) 210°, 330°
(v) 90°, 270°
(vi) 101.3°, 281.3°
(vii) 0°, 180°, 360°
(viii) 122.7°, 237.3°
(ix) 90°

2 (i) $\theta = 48.2°$ or $-48.2°$
(ii) $\theta = 45.6°$ or $134.4°$
(iii) $\theta = 69.4°$ or $-110.6°$
(iv) $\theta = -56.4°$ or $-123.6°$
(v) $\theta = 113.6°$ or $-113.6°$
(vi) $\theta = -29.1°$ or $150.9°$

3 (i) $\theta = 60°, 120°, 240°$ or $300°$
(ii) $\theta = 45°, 135°, 225°$ or $315°$
(iii) $\theta = 45°, 135°, 225°$ or $315°$
 (a) $\theta = -330°, -210°, -90°, 30°, 150°$ or $270°$
 (b) $\theta = -300°, -180°, -60°, 60°, 180°$ or $300°$
 (c) $\theta = -333.4°, -225°, -153.4°, -45°, 26.6°, 135°, 206.6°$ or $315°$

4 (i) $(2x - 1)(x + 1)$
(ii) $x = 0.5$ or -1
(iii) (a) $\theta = -330°, -210°, -90°, 30°, 150°$ or $270°$
 (b) $\theta = -300°, -180°, -60°, 60°, 180°$ or $300°$
 (c) $\theta = -333.4°, -225°, -153.4°, -45°, 26.6°, 135°, 206.6°$ or $315°$

5 (i) $x = -180°, -108.4°, 0°, 71.6°$ or $180°$
(ii) $x = -135°, -45°, 45°$ or $135°$
(iii) $x = -180°, -70.5°, 70.5°$ or $180°$
(iv) $x = -150°, -30°$ or $90°$

6 (i) $x = -300°, -120°, 60°$ or $240°$
(ii) $x = -330°, -210°, 30°$ or $150°$
(iii) $x = -315°, -45°, 45°$ or $315°$
(iv) $x = -300°, -240°, 60°$ or $120°$
(v) $x = -315°, -180°, -135°, 0°, 45°, 180°$ or $225°$
(vi) $x = -330°, -30°, 30°$ or $330°$

7 0°, 60°, 300° or 360°

8 (i) 0
(ii) −150°, −141.8°, −38.2° or −30°

Exercise 6F (page 176)

1 (i) (a) $2\sin^2\theta - \sin\theta - 1 = 0$
 (b) $\theta = 90°, 210°$ or $330°$
(ii) (a) $\cos^2\theta - \cos\theta - 2 = 0$
 (b) $\theta = 180°$
(iii) (a) $2\cos^2\theta + \cos\theta - 1 = 0$
 (b) $\theta = 60°, 180°$ or $300°$
(iv) (a) $\sin^2\theta - \sin\theta = 0$
 (b) $\theta = 0°, 90°, 180°$ or $360°$
(v) (a) $2\sin^2\theta + \sin\theta - 1 = 0$
 (b) $\theta = 30°, 150°$ or $270°$

2 (i) (a) $\cos^2\theta + 2\cos\theta - 2 = 0$
 (b) $\theta = 42.9°$
(ii) (a) $\sin^2\theta + \sin\theta - 1 = 0$
 (b) $\theta = 38.2°$ or $141.8°$
(iii) (a) $\cos^2\theta + 3\cos\theta - 1 = 0$
 (b) $\theta = 72.4°$

3 (i) $\tan\theta = 2$
(ii) $\theta = 63.4°$

4 (i) $\theta = 153.4°$ or $333.4°$
(ii) $\theta = 0°, 30°, 180°, 330°$ or $360°$
(iii) $\theta = 14.5°$ or $165.5°$

5 (i) $\sin^2 x$
(ii) $(1 - \sin^2 x)\sin x$
(iii) $2 - 3\sin x - 2\sin^2 x$

6 $3\sin^2 x + 6\sin x - 6\sin x + 3\cos^2 x$
$= 3\sin^2 x + 3\cos^2 x$
$= 3(\sin^2 x + \cos^2 x)$
$= 3 \times 1$
$= 3$

7 (i) $\tan x\sqrt{1 - \sin^2 x}$
$\equiv \tan x\sqrt{\cos^2 x}$
$\equiv \tan x \cos x$
$\equiv \dfrac{\sin x}{\cos x}\cos x$
$\equiv \sin x$

(ii) $\dfrac{1 - \cos^2 x}{1 - \sin^2 x} \equiv \dfrac{\sin^2 x}{\cos^2 x}$
$\equiv \tan^2 x$

(iii) $(1 + \sin x)(1 - \sin x)$
$\equiv 1 - \sin x + \sin x - \sin^2 x$
$\equiv 1 - \sin^2 x$
$\equiv \cos^2 x$

(iv) $\dfrac{2\sin x\cos x}{\tan x}$
$\equiv \dfrac{2\sin x\cos x}{\frac{\sin x}{\cos x}}$
$\equiv \dfrac{2\sin x\cos^2 x}{\sin x}$
$\equiv 2\cos^2 x$
$\equiv 2(1 - \sin^2 x)$
$\equiv 2 - 2\sin^2 x$

8 $x = 26.6°, 108.4°, 206.6°, 288.4°$

Chapter 7

Exercise 7A (page 183)

1 (i) $9.85\,\text{cm}^2$
(ii) $19.5\,\text{cm}^2$

(iii) 15.2 cm^2
(iv) 20.5 cm^2

2 127 cm^2

3 (i) 23.8 cm^2
(ii) 5.56 cm
(iii) 126 cm^2

4 (i) 308 cm^2
(ii) 325
(iii) There is likely to be a lot of wastage when tiles are cut for the edges, so he will need more tiles.

5 173 cm^2

6 $\sqrt{8}$ cm or $2\sqrt{2}$ cm or 2.83 cm

7 3.04 cm

8 53.1°

Discussion point (page 184)

It is easier to solve an equation involving fractions if the unknown quantity is in the numerator.

Activity 7.1 (page 186)

$$\frac{\sin z}{6} = \frac{\sin 78°}{8} \Rightarrow z = 47.2°$$

or $z = 132.8°$, but 132.8° is too large to fit into a triangle where one of the other angles is 78°.

Exercise 7B (page 186)

1 (i) 4.61 m
(ii) 11.0 cm
(iii) 5.57 cm
(iv) 7.52 cm

2 (i) 57.7°
(ii) 16.5°
(iii) 103°
(Reject 76.7° since the angle in the diagram is obtuse.)

3 37.9°

4 3.79 km

5 3.59 cm, 3.59 cm, 10.7 cm, 10.7 cm

6 Anna (4.15 km < 4.18 km) – alternatively compare triangle areas

Exercise 7C (page 190)

1 (i) 6.40 cm
(ii) 8.76 cm
(iii) 13.3 cm

2 (i) 41.4°
(ii) 107°
(iii) 90°

3 9.14 cm, 12.3 cm

4 (i) 10 cm
(ii) 112°

5 55.8°

6 13.8 cm

7 19.0 cm^2

8 10.0 km

9 8.54 cm or 4.57 cm

Discussion point (page 192)

If angle A was 118°, then the angles A and B add up to more than 180°.

Alternatively, side BC is shorter than side AC, so angle A must be less than angle B.

Exercise 7D (page 193)

1 12.2 cm

2 6.12 km

3 (i) 26.5 m
(ii) 19.4 m

4 (i) 57.1°, 57.1°, 122.9°, 122.9°
(ii) 14.5 cm

5 (i) 10.2 km
(ii) 117°

6 (i) 29.9 km
(ii) 12.9 km h^{-1}

7 (i) BD = 2.05 m, EG = 2.07 m
(ii) DE = 4.53 m

8 4.77 km

Discussion point (page 194)

The shortest route is along an arc of a 'great circle', i.e. a circle whose centre is the centre of the earth.

Discussion point (page 196)

One possible example is a ramp used for disabled access to a building.

Discussion point (page 197)

The shelves of a bookcase are parallel; the side of a filing cabinet meets the floor in a line.

Activity 7.2 (page 201)

There is an infinite set of such integers, e.g.

$8^2 + 9^2 + 12^2 = 17^2$ and

$12^2 + 16^2 + 21^2 = 29^2$

Exercise 7E (page 203)

1 (i) 14.1 cm
(ii) 17.3 cm
(iii) 35.3°

2 (i) 3 cm
(ii) 72.1°
(iii) 76.0°

3 (i) 18.4°
(ii) 13 cm
(iii) 17.1°
(iv) Halfway along

4 (i) 75 m
(ii) 67.5 m
(iii) 42°

5 (i) 33.4 m
(ii) 66.7 m
(iii) 115.6 m
(iv) 22.8°

6 (i) 28.3 cm
(ii) 42.4 cm
(iii) 40.6 cm

7 (i) 41.8°
(ii) 219 m
(iii) 186 m
(iv) 51.7°

8 (i) 1.57 m
(ii) 1.547 m
(iii) 3.05 m

9 (i) 15 m
(ii) 16.4 m
(iii) 65.4°
(iv) 69.9°

10 (i) 5.20 cm
(ii) 5.20 cm
(iii) 54.7°
(iv) 16.9 cm

11 (i) 5.20 cm
(ii) 22.0 cm^2
(iii) 35.3°

12 (i) 8.49 cm
(ii) 54.7°
(iii) 125 cm^2
(iv) $200\sqrt{3}$ cm^2

13 37.4°

14 $10.3\,\text{cm}$

15 $29.6°$

Chapter 8

Activity 8.1 (page 209)

Taking $R_1 = (2, 4)$, $R_2 = (2.5, 6.25)$, $R_3 = (2.9, 8.41)$, $R_4 = (2.99, 8.9401)$ and $R_5 = (2.999, 8.994\,001)$ gives the gradient sequence 5, 5.5, 5.9, 5.99, 5.999.
Again the sequence seems to converge to 6.

Activity 8.2 (page 210)

(i) The gradient of the chord is $4 + h$. The gradient of the tangent is 4.

(ii) The gradient of the chord is $-2 + h$. The gradient of the tangent is -2.

(iii) The gradient of the chord is $-6 + h$. The gradient of the tangent is -6.
In each case the gradient of the tangent is twice the value of the x-coordinate.

Activity 8.3 (page 211)

Let P be the point (x, x^4) and Q be the point $((x + h), (x + h)^4)$

The gradient of the chord PQ is given by

$$\frac{QR}{PR} = \frac{(x+h)^4 - x^4}{(x+h) - x}$$

$$= \frac{x^4 + 4x^3h + 6x^2h^2 + 4h^3x + h^4 - x^4}{h}$$

$$= \frac{4x^3h + 6x^2h^2 + 4h^2x + h^4}{h}$$

$$= 4x^3 + 6x^2h + 4xh^2 + h^3$$

As Q gets closer to P, $h \to 0$ showing that the gradient of the tangent at (x, x^4) is $4x^3$.

Activity 8.4 (page 212)

In all cases the graphs have a vertical displacement from the origin by an amount equal to the constant term, for all values of x.

Exercise 8A (page 215)

1 (i) $\frac{dy}{dx} = 4x^3$
(ii) $\frac{dy}{dx} = 6x^2$
(iii) $\frac{dy}{dx} = 10x$
(iv) $\frac{dy}{dx} = 63x^8$
(v) $\frac{dy}{dx} = -18x^5$
(vi) $\frac{dy}{dx} = 0$
(vii) $\frac{dy}{dx} = 10$
(viii) $\frac{dy}{dx} = \frac{3}{4}x^2$
(ix) $\frac{dy}{dx} = 2\pi$
(x) $\frac{dy}{dx} = 2\pi x$

2 (i) $\frac{dy}{dx} = 10x^4 + 8x$
(ii) $\frac{dy}{dx} = 12x^3 + 8$
(iii) $\frac{dy}{dx} = 3x^2$
(iv) $\frac{dy}{dx} = 1 - 15x^2$
(v) $\frac{dy}{dx} = 12x^2 + 2$
(vi) $\frac{dy}{dx} = 2$
(vii) $\frac{dy}{dx} = 15x^4$

3 (i) $\frac{dy}{dx} = 15x^4 + 16x^3 - 6x$
(ii) $\frac{dy}{dx} = 5x^4 + 36x^2 + 3$
(iii) $\frac{dy}{dx} = 3x^2 + 84x - 5$

4 (i) $-4x^{-5}$
(ii) $-6x^{-3}$
(iii) $6x - 4x^{-2}$
(iv) $-6x^{-4}$
(v) $2x - 2x^{-3}$
(vi) $-6x^{-3} - 6x^{-4}$

5 (i) $6x - \frac{6}{x^4}$
(ii) $2x - \frac{2}{x^3}$
(iii) $9x^2 - \frac{9}{x^4}$
(iv) $-\frac{2}{x^2} + \frac{6}{x^3}$
(v) $-\frac{1}{2x^2} + \frac{2}{3x^3}$
(vi) $-\frac{2}{3x^2} + \frac{3}{2x^3}$

6 (i) $y = 18x^2$
(ii) $36x$

7 (i) $30t\ \text{cm}\,\text{s}^{-1}$
(ii) $130\,\text{m}^2$

8 (i) $y = \frac{4}{3}\pi(2x)^3$
$= \frac{4}{3}\pi \times 8x^3$
$= \frac{32}{3}\pi x^3$
(ii) 128π

Exercise 8B (page 216)

1 (i) $3x^2 + 2$
(ii) $18x^2 - 16x$
(iii) $2x + 5$
(iv) $2x + 7$
(v) $12x^2 + 4x^3 - 5x^4$
(vi) $2x - 3$

2 (i) $\frac{5x^4 + 3x^2}{4}$
(ii) $5x^4 + 1$
(iii) $16x^3$
(iv) $6x - 5$
(v) $2x + 1$
(vi) $4x^3$

3 (i) $3x^2 - 2x$
(ii) $6x - 2$

4 54

5 1

6 -7

7 10

8 $\frac{15}{16}$

Exercise 8C (page 219)

1 (i) $\frac{dy}{dx} = 5 - 2x$
(ii) -1
(iii) $x + y - 9 = 0$
(iv) $x - y + 3 = 0$

2 (i) (a) $y = 4$
(b) $x = 2$

(ii) $9x + y = 27$
(iii) $y = 0$

3 (i) $(1, 0)$
(ii) $y = 2x - 2$
(iii) $x + 2y - 1 = 0$
(iv) $Q(0, -2)$, $R(0, \frac{1}{2})$; $1\frac{1}{4}$ units^2

4 (i) $\frac{dy}{dx} = 3x^2 - 6x + 4$
(ii) (a) $y = 4x - 3$
(b) $x + 4y - 22 = 0$
(iii) $x = -1, x = 3$

5 (i) $x + y = 5$
(ii) $x + y = 1$

6 (i) $2p - q = 16$
(ii) $p = 12$
(iii) $(-2, 24)$
(iv) $x - 12y + 96 = 0$

7 (i) $y = 3x - 5$
(ii) $\left(\frac{1}{3}, -1\frac{2}{9}\right)$

8 (i) $2x + y - 15 = 0$
(ii) $x - 2y = 0$
(iii) The normal

9 (i) Substituting $x = 0$
$y = 0 - 0 + 0 = 0$
Substituting $x = 1$
$y = 1.5 - 3.5 + 2 = 0$
(ii) At $(0, 0)$ the tangent is $y = 2x$ and the normal is $x + 2y = 0$.
At $(1, 0)$ the tangent is $x + 2y - 1 = 0$ and the normal is $2x - y - 2 = 0$.
(iii) a rectangle.

10 (i) $\frac{dy}{dx} = 2x - \frac{2}{x^2}$; $(1, 3)$
(ii) Tangent: $y = 3$; normal: $x = 1$
(iii) Tangent: $y = 3.5x - 2$; normal: $2x + 7y = 39$

Discussion point (page 222)

(i) $x + y + 2 = 0$ and $x + y - 2 = 0$
(ii) They are parallel
(iii) $y = x$ and $y = x$
(iv) They are the same line

Exercise 8D (page 222)

1 (i) $x > 0$
(ii) All x values
(iii) $x > -1$
(iv) $x > \frac{3}{2}$
(v) $x > -\frac{2}{3}$
(vi) $x > -2$
(vii) $x < 0$ or $x > \frac{4}{3}$
(viii) $x < -5$ or $x > 1$
(ix) $x < -1$ or $x > 3$

2 (i) $x < 0$
(ii) $x < 3$
(iii) $x < -1$
(iv) $x > 2$
(v) All x values
(vi) $x < -\frac{1}{2}$
(vii) $-2 < x < 0$
(viii) $-3 < x < 4$
(ix) $x < -3$ or $x > 3$

3 $\frac{dy}{dx} = x^2 + 4x + 7 = (x + 2)^2 + 3$
$(x + 2)^2 \geqslant 0$ for all x values.
Adding 3 means always positive, so increasing function.

4 $\frac{dy}{dx} = 3x^2 - 12x + 27$
$= 3(x^2 - 4x + 9)$
$= 3((x - 2)^2 + 5)$
$= 3(x - 2)^2 + 15$
$3(x - 2)^2 \geqslant 0$ for all x values.
Adding 15 means always positive, so increasing function.

5 $x > 1$

6 $\frac{dy}{dx} = -2 - 3x^2$
$-3x^2 \leqslant 0$ for all x values.
Subtracting -2 means always negative, so decreasing function.

7 $\frac{dy}{dx} = -\frac{1}{x^2}$ which is negative for all $x \neq 0$ so the function is decreasing.

8 (i) (a) $x < -1$ and $x > 1$
(b) $-1 < x < 0$ and $0 < x < 1$
(ii) (a) All $x \neq 0$
(b) Never
(iii) (a) $-1 < x < 0$ and $x > 1$
(b) $x < -1$ and $0 < x < 1$
(iv) (a) $x > 0$ (b) $x < 0$

9 (i) $10\,000\,\text{cm}^3$
(ii) $1000t\,\text{cm}^3$
(iii) $1000\,\text{cm}^3\,\text{s}^{-1}$
(iv) Radius $= 10\left(\sqrt[3]{\frac{(3t)}{(4\pi)}}\right)$

Exercise 8E (page 225)

1 (i) $\frac{dy}{dx} = 9x^2 + 3$
$\frac{d^2y}{dx^2} = 18x$
(ii) $\frac{dy}{dx} = 5x^4$
$\frac{d^2y}{dx^2} = 20x^3$
(iii) $\frac{dy}{dx} = 3 - 20x^3$
$\frac{d^2y}{dx^2} = -60x^2$

2 (i) $\frac{dy}{dx} = 4x^3 - 4x + 5$
$\frac{d^2y}{dx^2} = 12x^2 - 4$
(ii) $\frac{dy}{dx} = 6x^2 + 3$
$\frac{d^2y}{dx^2} = 12x$
(iii) $\frac{dy}{dx} = 3x^2 - 4x$
$\frac{d^2y}{dx^2} = 6x - 4$

3 (i) $\frac{dy}{dx} = 4x + 3$; $\frac{d^2y}{dx^2} = 4$
(ii) $\frac{dy}{dx} = 8x - 4$; $\frac{d^2y}{dx^2} = 8$
(iii) $\frac{dy}{dx} = 11 - 12x$
$\frac{d^2y}{dx^2} = -12$

4 (i) $\frac{dy}{dx} = 9x^2 - 24x + 21$
$\frac{d^2y}{dx^2} = 18x - 24$

(ii) $\dfrac{dy}{dx} = 8x^3 - 12x^2 + 4x$

$\dfrac{d^2y}{dx^2} = 24x^2 - 24x + 4$

(iii) $\dfrac{dy}{dx} = 45x^4 + 24x^3 + 3x^2$

$\dfrac{d^2y}{dx^2} = 180x^3 + 72x^2 + 6x$

5 (i) $y = 13 - x$

(ii) $P = x(13 - x)$

(iii) $\dfrac{dy}{dx} = -1$; $\dfrac{dP}{dx} = 13 - 2x$

(iv) -2

6 (i) $\dfrac{dy}{dx} = 9x^2 - 4x - 6$

$\dfrac{d^2y}{dx^2} = 18x - 4$

(ii) $7, -1, 22$

(iii) $-22, 14, 32$

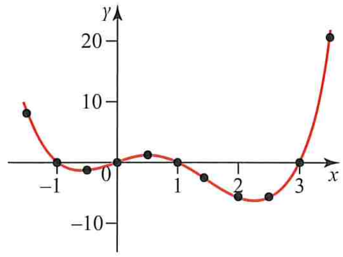
7 (i) $\dfrac{ds}{dt} = u + at$

$v(12) = u + 12a$

(ii) $\dfrac{d^2s}{dt^2} = a$

Activity 8.5 (page 226)

(i)



(ii) Decreasing function initially and goes from positive to negative values of y, then gradient is zero before the function increases. Passes through (0, 0) with a positive gradient, has another point with zero gradient before decreasing again and crossing the x-axis to negative values of y. Turns again to go through the x-axis for the 4th time.

Activity 8.6 (page 226)

When $x = 0°$ the gradient is zero. It then decreases through negative values reaching its most negative value when $x = 90°$. It increases to zero when $x = 180°$ and continues to increase through positive values until it is greatest when $x = 270°$. The gradient then decreases to zero when $x = 360°$.

Discussion point (page 228)

There are no more values when $\dfrac{dy}{dx} = 0$, so there are no more turning points. As x increases beyond the point where $x = 2$, $\dfrac{dy}{dx}$ takes positive values and so the curve will cross the x-axis again. To the left of $x = -2$ the gradient is always negative, giving a further point of intersection with the x-axis.

Discussion point (page 228)

(i) The curve crosses the x-axis when $x^3 - 12x + 3 = 0$. This does not factorise, so the values of x cannot be found easily.

(ii) Only when the equation obtained when $y = 0$ factorises.

Exercise 8F (page 234)

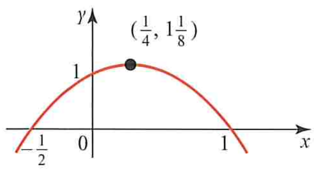
1 (i) (a) $\dfrac{dy}{dx} = 1 - 4x$; $x = \dfrac{1}{4}$

(b) $\dfrac{d^2y}{dx^2} = -4$

(c) Max

(d) $y = 1\frac{1}{8}$

(e)



(ii) (a) $\dfrac{dy}{dx} = 12 + 6x - 6x^2$;
$x = -1, x = 2$

(b) $\dfrac{d^2y}{dx^2} = 6 - 12x$.

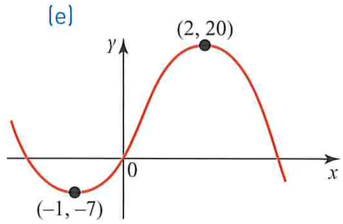
When $x = -1$, $\dfrac{d^2y}{dx^2} = 18$

When $x = 2$, $\dfrac{d^2y}{dx^2} = -18$

(c) Min when $x = -1$, max when $x = 2$

(d) $x = -1, y = -7$;
$x = 2, y = 20$

(e)



(iii) (a) $\dfrac{dy}{dx} = 3x^2 - 8x$;
$x = 0, x = 2\frac{2}{3}$

(b) $\dfrac{d^2y}{dx^2} = 6x - 8$.

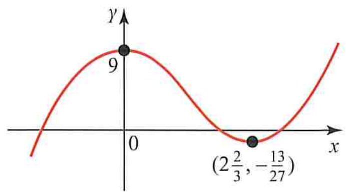
When $x = 0$, $\dfrac{d^2y}{dx^2} = -8$

When $x = 2\frac{2}{3}$, $\dfrac{d^2y}{dx^2} = 8$

(c) Max when $x = 0$, min when $x = 2\frac{2}{3}$

(d) $x = 0, y = 9$; $x = 2\frac{2}{3}$, $y = -\dfrac{13}{27}$

(e)



(iv) (a) $\dfrac{dy}{dx} = 3x^2 - 4x + 1$;

$x = \dfrac{1}{3}$; $x = 1$

(b) $\dfrac{d^2y}{dx^2} = 6x - 4$

When $x = \dfrac{1}{3}$, $\dfrac{d^2y}{dx^2} = -2$

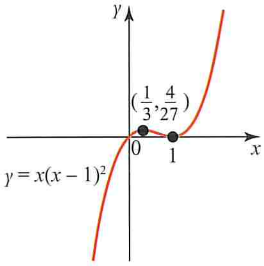
When $x = 1$, $\dfrac{d^2y}{dx^2} = 2$

(c) Max when $x = \dfrac{1}{3}$;
min when $x = 1$

(d) $x = \frac{1}{3}$, $y = \frac{4}{27}$;

$x = 1$, $y = 0$

(e)



(v) (a) $\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$

$x = 0, \frac{1}{2}$ and 1

(b) $\frac{d^2y}{dx^2} = 12x^2 - 12x + 2$

When $x = 0$, $\frac{d^2y}{dx^2} = 2$

When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -1$

When $x = 1$, $\frac{d^2y}{dx^2} = 2$

(c) Min when $x = 0$

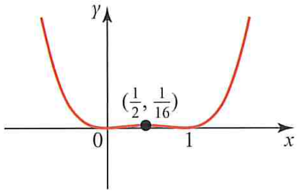
Max when $x = \frac{1}{2}$

Min when $x = 1$

(d) $x = 0$, $y = 0$

$x = \frac{1}{2}$, $y = \frac{1}{16}$

$x = 1$, $y = 0$



(vi) (a) $\frac{dy}{dx} = 3x^2 - 48$

$x = -4, x = 4$

(b) $\frac{d^2y}{dx^2} = 6x$

When $x = -4$, $\frac{d^2y}{dx^2} = -24$

When $x = 4$, $\frac{d^2y}{dx^2} = 24$

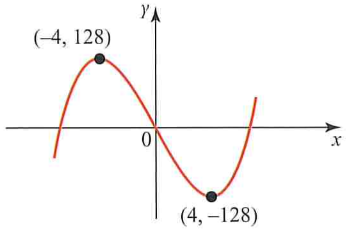
(c) Max when $x = -4$,

min when $x = 4$

(d) $x = -4$, $y = 128$;

$x = 4$, $y = -128$

(e)



(vii) (a) $\frac{dy}{dx} = 3x^2 + 12x - 36$

$x = -6, x = 2$

(b) $\frac{d^2y}{dx^2} = 6x + 12$

When $x = -6$, $\frac{d^2y}{dx^2} = -24$

When $x = 2$, $\frac{d^2y}{dx^2} = 24$

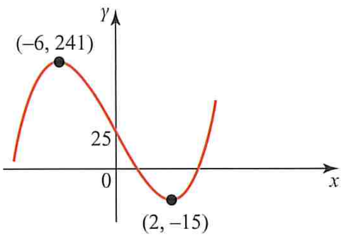
(c) Max when $x = -6$,

min when $x = 2$

(d) $x = -6$, $y = 241$

$x = 2$, $y = -15$

(e)



(viii) (a) $\frac{dy}{dx} = 6x^2 - 30x + 24$

$x = 1, x = 4$

(b) $\frac{d^2y}{dx^2} = 12x - 30$

When $x = 1$, $\frac{d^2y}{dx^2} = -18$

When $x = 4$, $\frac{d^2y}{dx^2} = 18$

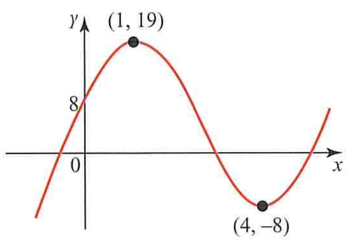
(c) Max when $x = 1$,

min when $x = 4$

(d) $x = 1$, $y = 19$

$x = 4$, $y = -8$

(e)

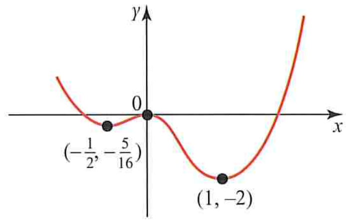


2 (i) $p = 4, q = -3$

(ii) $y = 1\frac{1}{3}, x = \frac{2}{3}$

3 (i) Min at $\left(-\frac{1}{2}, -\frac{5}{16}\right)$, max at $(0, 0)$, min at $(1, -2)$

(ii)



4 (i) $y = \frac{x^2}{9} - \frac{2x}{3} + 2$

5 (i) $q = 12 - p$

(ii) $S = 2p^2 - 24p + 144$

(iii) 72

6 (i) $80a - 2a^2$

(ii) $a = 20$, $b = 20$

(iii) 800

7 (i) $P = x(10 - x)^2$

$= 100x - 20x^2 + x^3$

(ii) $x = 10$ or $x = 3\frac{1}{3}$

(iii) $x = 3\frac{1}{3}$

(iv) $x = 10 \Rightarrow y = 0$ so not a positive number.

8 (i) 2790 metres

(ii) 1220 metres

9 (i) $A = 2x^2 + \frac{864}{x}$ cm²

(ii) $x = 6$ and $y = 6$

$\Rightarrow V = 216$ cm²

$\frac{dA}{dx} = 4x - \frac{864}{x^2}$

$\Rightarrow \frac{d^2A}{dx^2} = 4 + \frac{1728}{x^3}$

> 0 when $x = 6$

so A is a minimum

Chapter 9

Activity 9.1 (page 239)

$\mathbf{AB} = \begin{bmatrix} 14 & -12 \\ 0 & -10 \end{bmatrix}$ and

$\mathbf{BA} = \begin{bmatrix} -4 & 6 \\ 18 & 8 \end{bmatrix}$

Notice that the products **AB** and **BA** are not the same.

Discussion point (page 239)

There is an infinite number of such pairs of matrices.

If either of the matrices (or their product) is a scalar multiple of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then they can be multiplied in either order and the result will be the same, e.g.

$$\mathbf{P} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 4 & -10 \\ -2 & 6 \end{bmatrix}$$

But there are many other cases, e.g.

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Exercise 9A (page 240)

1 (i) $\begin{bmatrix} 8 & 12 \\ 4 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 0 \\ -6 & -10 \end{bmatrix}$

(iii) $\begin{bmatrix} 23 \\ 10 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

(v) $\begin{bmatrix} 0 \\ 11 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 \\ -7 \end{bmatrix}$

(vii) $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$

(viii) $\begin{bmatrix} -4 & -6 \\ 7 & 10 \end{bmatrix}$

(ix) $\begin{bmatrix} -12 & 4 \\ 15 & -7 \end{bmatrix}$

(x) $\begin{bmatrix} -18 & -2 \\ 3 & -1 \end{bmatrix}$

(xi) $\begin{bmatrix} 0 & 0 \\ -11 & -14 \end{bmatrix}$

(xii) $\begin{bmatrix} 0 & 0 \\ -3 & -5 \end{bmatrix}$

(xiii) $\begin{bmatrix} 3 & -7 \\ 3 & -3 \end{bmatrix}$

(xiv) $\begin{bmatrix} 0 & 0 \\ -3 & 11 \end{bmatrix}$

2 (i) -1 (ii) 6
(iii) 0.5 (iv) 1.5
(v) 7 (vi) -2

3 (i) $x = 4$ $y = 2$
(ii) $x = -1$ $y = -2$
(iii) $x = 3$ $y = -5$
(iv) $x = -3$ $y = 7$

4 (i) $5x + 3y = 1$ and $2x - y = -4$
(ii) $x = -1$ $y = 2$

5 (i) $a = 3$ $b = 5$
(ii) $a = -2$ $b = 4$
(iii) $a = 1$ $b = -5$
(iv) $a = 2$ $b = 2$

6 $a = 2$ $b = -5$
$c = -1$ $d = 3$

7 $k = 27$

8 $p = 3, q = -2, r = -5$

9 (i) $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

(ii) $\mathbf{M} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Discussion point (page 241)

M does not exist if the values of a, b, c and d are such that $ad - bc = 0$

Activity 9.2 (page 242)

(i) (3, 2) (−1, 5) (6, 0) (−3, −4) and (x, y)
No transformation has occurred

(ii) (2, −1) (−4, −3) (0, −4) (−5, 1) and $(x, -y)$
Reflection in the x-axis

Exercise 9B (page 243)

1 (20, 8) **2** (9, −16)
3 (13, −1) **4** −2
5 9
6 $a = -3$ $b = 3$
7 $c = 1$ $d = -2$

8 $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 5
10 (5, 2)
11 $a = -2, b = 1, c = 8$
12 (i) (2, 4)
(ii) $y = 3x$

Exercise 9C (page 247)

1 (i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

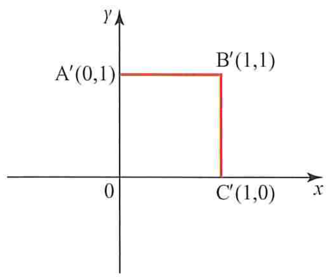
(vi) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(viii) $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

(ix) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

2



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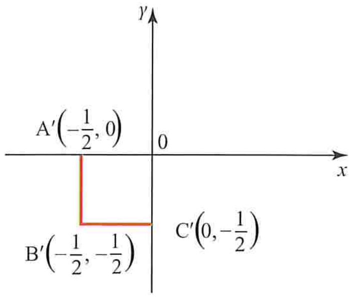
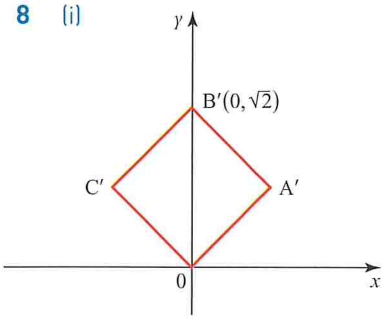


4 (i) Reflection in the y-axis
(ii) Enlargment, scale factor 5, centre (0, 0)
(iii) Reflection in the line $y = x$
(iv) Rotation of 90° anticlockwise about (0, 0)
(v) Rotation of 180° anticlockwise (or clockwise) about (0, 0) OR Enlargement, scale factor −1, centre (0, 0)
(vi) Reflection in the x-axis
(vii) Enlargement, scale factor $1\frac{1}{2}$, centre (0, 0)
(viii) Reflection in the line $y = -x$

5 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6 16 sq units **7** 8 or −8

8 (i)



(ii) $A' = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$ and

$C' = \left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$

(iii) $\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$

Discussion point (page 249)

The following pairs of transformations will produce the same result regardless of the order in which they are applied:

- An enlargement and any other transformation.
- A rotation followed by another rotation.
- A reflection in the x-axis and a reflection in the y-axis.
- A reflection in $y = x$ and a reflection in $y = -x$.

Exercise 9D (page 249)

1 (i) $\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$
(ii) −4

2 (i) $\begin{bmatrix} 3 & -1 \\ 13 & -7 \end{bmatrix}$
(ii) (−7, −41)

3 $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4 $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

5 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

6 (i) Reflection in the y-axis
(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(iii) Two successive reflections in the y-axis takes you back to the original position.

7 (i) Rotation through 90°, centre (0, 0)
(ii) Rotation through 180°, centre (0, 0)
(iii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(iv) Rotation through 270°, centre (0, 0)
(v) A rotation through 180°, centre (0, 0) followed by a rotation through 90°, centre (0, 0) is equivalent to a rotation through 270°, centre (0, 0).
(vi) **ED** is a rotation through 90°, centre (0, 0) followed by a rotation through 180°, centre (0, 0).
This is also equivalent to a rotation through 270°, centre (0, 0), the same as **DE**.

8 $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$

$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ represents an enlargement, centre O, scale factor −6; $k = -6$

9 (i) $\begin{bmatrix} -4 & 5 \\ 5 & -1 \end{bmatrix}$
(ii) (1, −2)

10 Any three 2 × 2 matrices will satisfy the identity **(AB)C** = **A(BC)**.
This is the associative law.
Matrix multiplication is associative.
However, this specification does not assess a candidate's knowledge of the associative law.

Practice questions 1 (page 253)

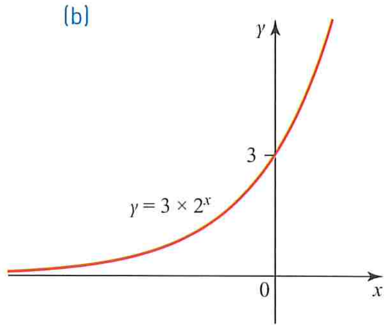
1 $p = 4\frac{1}{2}$

2 $\dfrac{2}{x + 8}$

3 $\left(\dfrac{25\pi}{4} - 6\right)$ cm^2

4 5000

5 (a) $a = 3$, $b = 2$
(b)



6 $A\hat{C}B = D\hat{B}C$ (alternate angles)

$C\hat{A}B = D\hat{B}C$ (alternate segmen theorem)

$\therefore A\hat{C}B = C\hat{A}B$

$\therefore$ triangle ABC is isosceles

7 $x^2 + y^2 + 2x - 22y + 22 = 0$ or

$(x+1)^2 + (y-11)^2 = 10^2$

8 (a) 0

(b)

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 7\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) - 1$$

$$= \frac{1}{8} + \frac{1}{8} - \frac{7}{4} + \frac{5}{2} - 1$$

$$= 0$$

$\therefore (2x - 1)$ is a factor of $f(x)$

(c) $x = 1,\ x = \frac{1}{2},$

$x = -1 \pm \sqrt{2}$

9 $P = (-2, 4),\ Q = (4, 2)$

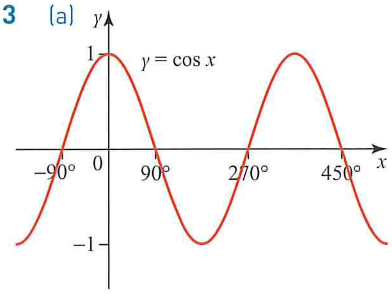
10 $\dfrac{5\sqrt{3} - 7}{2}$

11 (a) $x^{\frac{3}{2}}$

(b) $x = 15\frac{5}{8}$ or $\frac{125}{8}$

12 $p = 12,\quad q = 8$

13 (a)



(b) $x = -30°,\ x = 30°,$

$x = 330°,\ x = 390°$

14 nth term $= \dfrac{3 + \frac{2}{n}}{\frac{1}{n} - 6}$

as $n \to \infty$, then $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$

$\therefore$ nth term $\to \dfrac{3 + 0}{0 - 6} = -\dfrac{1}{2}$

15 $y = -4x - 24$

16 (a) $a + b + c = -2$

$25a + 5b + c = 2$

$49a + 7b + c = 16$

(b) nth term $= n^2 - 5n + 2$

17 (a) $(y - 1)(y - 2)$

(b) $y = 1,\quad y = 2$

(c) $x = 1,\quad x = 8$

18 (a) $f^{-1}(x) = \sqrt{x - 6},\ x \geqslant 6$

(b) $x = 1$

19 (a) $n = 7$

(b) $a = 2$

Practice questions 2 (page 256)

1 $y = (x - 3)^2$

2 $\dfrac{9y^5}{10x^4}$ or $0.9x^{-4}y^5$

3 $2x^4 - x^3 - 14x^2 + 41x - 28$

4 $3x - 2y = -15$

5 (a) $12x^2 + 2x - 7$

(b) 23

6 18

7 2 : 5

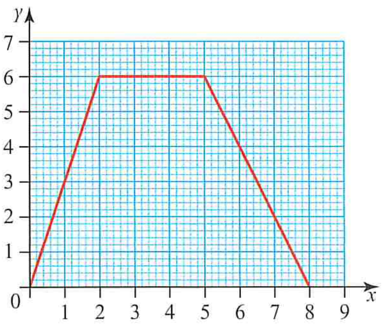
8 (a) AC = 35.5 km (3 s.f.)

(b) 093° (nearest degree)

Note: An answer of 094° is incorrect and probably the result of premature or incorrect rounding.

9 $Q = (-4.2, 4.6)$

10



11 (a) $A = w(w - y) + xy$

$= w^2 - wy + xy$

$= w^2 + xy - wy$

Note: There are several different methods of proving the required formula.

(b) $y = \dfrac{A - w^2}{x - w}$

12 $x = 54°,\quad y = 18°$

13 (a) 7.84 cm (3 s.f.) or $\dfrac{\sqrt{246}}{2}$ cm

(b) 63.2° (3 s.f.)

(c) 70.3° (3 s.f.)

14 (a) Area

$= \frac{1}{2} \times 3x \times 4x \times \sin 150°$

$= 3x^2$

(b) $0 < x < 6$

15 (a) $\begin{pmatrix} 0 & 1 \\ -6 & -3 \end{pmatrix}$

(b) $P = \left(-1\frac{1}{2},\ 2\right)$

16 $\sin x \tan x \equiv \sin x \times \dfrac{\sin x}{\cos x}$

$$\equiv \frac{\sin^2 x}{\cos x}$$

$$\equiv \frac{1 - \cos^2 x}{\cos x}$$

$$\equiv \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$$

$$\equiv \frac{1}{\cos x} - \cos x$$

17 (a) $h = \dfrac{1000}{x^2}$

(b) $A = x^2 + 4hx$

$$= x^2 + 4\left(\frac{1000}{x^2}\right)x$$

$$= x^2 + \frac{4000}{x}$$

(c) 476 cm^2 (3 s.f.)

(d) $\dfrac{d^2A}{dx^2} = 2 + \dfrac{8000}{x^3}$

at $x = 10(2)^{\frac{1}{3}}$, $\dfrac{8000}{x^3} > 0$

$\therefore \dfrac{d^2A}{dx^2} > 0$

$\therefore$ A is a minimum.

18 273 cm^2 (3 s.f.)

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