AQA Approved

AQA
Level 2

Certificate in Further Mathematics

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Introduction

This book has been written to support the AQA Level 2 Certificate in Further Mathematics, but you may also use it independently as an introduction to Mathematics beyond GCSE. It is expected that many of the students using this book could be working with little day-to-day teacher support and with this in mind the text has been written in an interactive way and the answers are fuller than is often the case in books of this nature.

The qualification is designed for high-achieving students who have already acquired, or are expected to achieve, grades 7 to 9 in GCSE Mathematics. It is hoped that many of these students will progress to study Mathematics at Advanced Level and beyond.

Higher order mathematical skills are studied in greater depth with an emphasis on algebraic reasoning, rigorous argument and problem-solving skills. Students following this course will be well prepared to tackle a Level 3 Mathematics qualification.

The content is split into Algebra, Geometry, Calculus and Matrices, with each section containing work that stretches and challenges, and which goes beyond the Key Stage 4 Programme of Study. The topics are frequently linked together as progress is made through the book, highlighting the beauty and inter-connectedness of mathematics.

Each chapter begins with a quote, designed to engage and bring the topic to life and/or provide an alternative viewpoint. The chapters are then broken down into sub-sections, each with a short introduction followed by a number of worked examples (with solutions) covering important techniques and question styles, and finally one or more sets of exercise questions. Coloured boxes, hints and notes help to clarify some of the key points.

In addition, each chapter includes a number of activities. These are often used to introduce a new concept, or to reinforce the examples in the text. Throughout the book the emphasis is on understanding the mathematics being used rather than merely being able to perform the calculations, but the exercises do, nonetheless, provide plenty of scope for practising basic techniques.

Three symbols are used throughout the book:

! This 'warning sign' alerts you either to restrictions that need to be imposed or to possible pitfalls.

PS This indicates a problem-solving question. These questions will sometimes involve more than one topic area.

RWC This indicates a question that relates to real-world contexts.

Numerous 'Discussion points' are used throughout as prompts to help you understand the theory that has been, or is about to be, introduced. Answers to these are also included.

'Prior knowledge' boxes highlight the GCSE Mathematics, or content earlier in the book, that you should be familiar with before you tackle a topic.

'Future uses' sections explain how the mathematics covered in a chapter can be used for further study, including at later points in the book, while 'Real-world contexts' explain the applications of the mathematics covered in each chapter. Also at the end of each chapter you will find a list of learning outcomes and key points.

A short glossary of key words is provided, followed by two practice question papers. Answers to these, all exercise questions, activities and discussion points are then given at the back of the book.

It is hoped that students who use this book will develop a fascination for mathematics, be inspired and challenged by the rigorous nature of the course and be able to appreciate the power of mathematics for its own sake as well as a problem-solving tool.

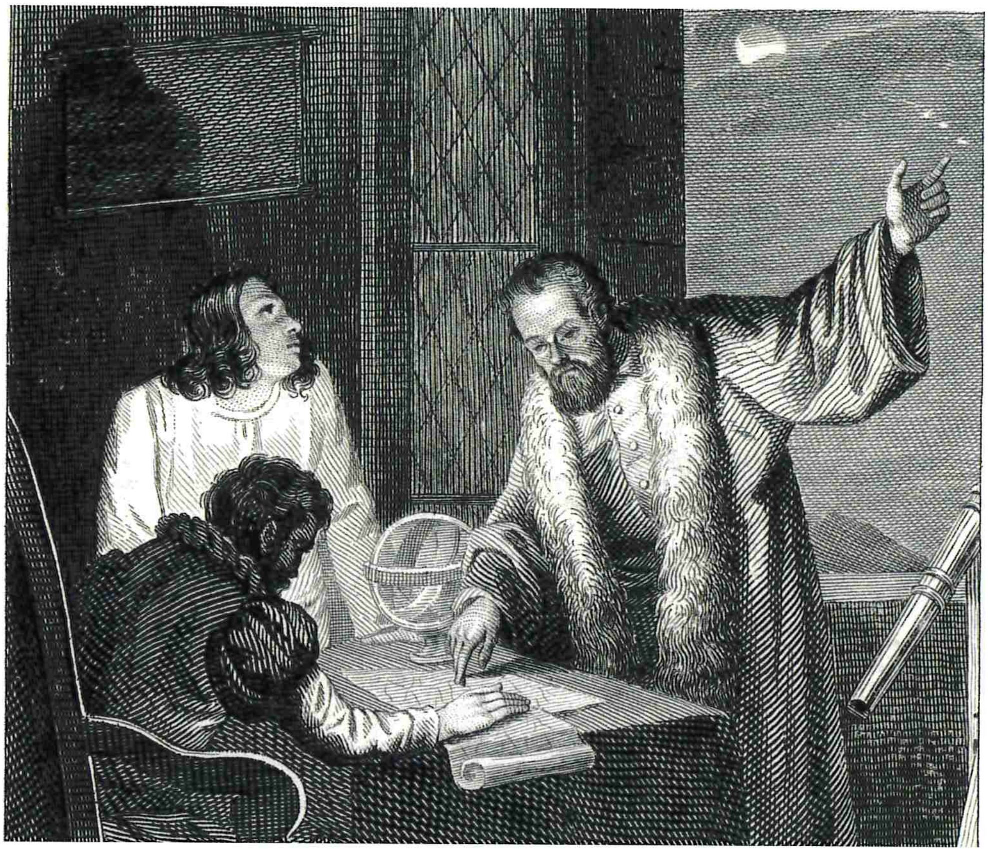
This book is supported by free narrated step-by-step examples, which you can access online at:

www.hoddereducation.co.uk/aqal2maths

Each example is numbered to correspond to the topics in the textbook. This makes it quick and easy for you to find additional guidance for each topic that will help you to approach the questions in the book.

1

Number and algebra I



The Book of Nature is written in the language of mathematics

Galileo Galilei

1 Numbers and the number system

Number will be tested implicitly throughout the course. The following examples and questions provide practice of some of the basic number skills that may be needed.

Prior knowledge

Students are expected to be familiar with all number and ratio topics from GCSE. In Exercise 1A, the GCSE specification references particularly assessed are N2, N3, N8, N12, N15, R4, R5 and R9.

Example 1.1

Simplify the ratio 3 kilometres : 840 metres.

Solution

$$\begin{aligned} 3\,\text{km} : 840\,\text{m} &= 3000\,\text{m} : 840\,\text{m} \\ &= 3000 : 840 \\ &= 300 : 84 \\ &= 100 : 28 \\ &= 25 : 7 \end{aligned}$$

Example 1.2

Work out 43% of 5680

Solution

$$
\begin{aligned}
43\% \text{ of } 5680 &= 0.43 \times 5680 \\
&= 2442.4
\end{aligned}
$$

Example 1.3

Increase 540 by 17.5%.

Solution

$$
\begin{aligned}
540 + 17.5\% \text{ of } 540 &= 540 \times 117.5\% \\
&= 540 \times 1.175 \\
&= 634.5
\end{aligned}
$$

Example 1.4

Without using a calculator, work out $\frac{9}{10} - \frac{2}{5} \div \frac{6}{7}$.

Solution

$$
\begin{aligned}
\frac{9}{10} - \frac{2}{5} \div \frac{6}{7} &= \frac{9}{10} - \frac{2}{5} \times \frac{7}{6} \\
&= \frac{9}{10} - \frac{2 \times 7}{5 \times 6} \\
&= \frac{9}{10} - \frac{1 \times 7}{5 \times 3} \\
&= \frac{9}{10} - \frac{7}{15} \\
&= \frac{27}{30} - \frac{14}{30} \\
&= \frac{13}{30}
\end{aligned}
$$

Example 1.5

Given the ratios $x : y = 5 : 3$ and $y : z = 4 : 7$, work out the ratio $x : z$ in its simplest form.

Solution

$x : y = 20 : 12$ and $y : z = 12 : 21$

so $x : y : z = 20 : 12 : 21$

so $x : z = 20 : 21$

Example 1.6

Work out, giving your answer to 3 significant figures, $\dfrac{3.76 \times 34}{78.4 \times 980}$.

Solution

$$\frac{3.76 \times 34}{78.4 \times 980} = 1.663\,890\,046 \times 10^{-3}$$

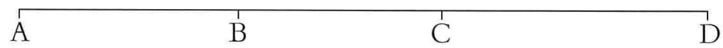
$$= 0.001\,663\,890\,046$$

$$= 0.00166 \quad (3\text{ s.f.})$$

Exercise 1A

Do not use a calculator for the questions marked *.

① ABCD is a straight line (not drawn to scale). AB = 4 cm, AC = 10 cm, AD = 22 cm.



Work out these ratios, giving your answers in their simplest form.

(i) AC : AB (ii) AB : BC (iii) AD : AB

(iv) BC : CD (v) BD : BC

> **!** If a question involving money requires an answer to be given in pounds and pence, remember to give any non-integer answers to 2 decimal places.

*② Work out

(i) 60% of £115 (ii) $33\frac{1}{3}$% of 780 (iii) 17.5% of 64 cm.

③ Work out

(i) 95% of 7540 (ii) $12\frac{1}{2}$% of 53.76 (iii) 4.2% of £150.

*④ (i) Increase 80 by 5%. (ii) Increase £240 by 75%.

(iii) Decrease £20 by 40%. (iv) Decrease 36 by $66\frac{2}{3}$%.

⑤ (i) Increase 650 by 14%. (ii) Decrease 3250 by 3.5%.

(iii) Decrease £3650 by 64%. (iv) Increase £46.30 by $5\frac{1}{2}$%.

*⑥ Work out, giving your answers as fractions in their simplest form,

(i) $\frac{3}{5} + \frac{2}{3} \times \frac{5}{6}$ (ii) $\left(\frac{1}{2}\right)^3 \div 4$ (iii) $3\frac{2}{5} - \frac{3}{4}$.

⑦ (i) Work out, giving your answer to 3 significant figures, $52.7 \div 4.93$

(ii) Work out, giving your answer to 2 significant figures, $5.9 - 0.53 \times 1.8$

(iii) Work out, giving your answer to 1 significant figure, $0.23 \times 0.14 + 0.09^2$

(iv) Work out, giving your answer to 2 decimal places, $\dfrac{19 + 36}{144 - 52}$.

⑧ A bag contains blue, green and white beads.

The ratio of blue beads to green beads is 4 : 3.

The ratio of green beads to white beads is 2 : 7.

Work out the smallest possible number of beads in the bag.

⑨ 55% of teachers in a school are female. The other 36 teachers are male. Work out the number of teachers in the school.

Prior knowledge

Students are expected to be familiar with all aspects of GCSE algebra. Exercise 1B particularly assesses GCSE specification reference A4.

2 Simplifying expressions

When you are asked to *simplify* an algebraic expression you need to write it in its most compact form. This will involve techniques such as collecting like terms, removing brackets, factorising and finding a common denominator (if the expression includes fractions).

Example 1.7

Simplify this expression.

$3a + 4b - 2c + a - 3b - c$

Solution

$$\begin{aligned} \text{Expression} &= 3a + a + 4b - 3b - 2c - c \qquad \text{(collecting like terms)} \\ &= 4a + b - 3c \end{aligned}$$

Example 1.8

Simplify this expression.

$2(3x - 4y) - 3(x + 2y)$

A common error in questions like this is to forget to multiply all terms in the second bracket by -3.

Solution

$$\begin{aligned} \text{Expression} &= 6x - 8y - 3x - 6y \qquad \text{(removing the brackets)} \\ &= 3x - 14y \end{aligned}$$

$\boxed{-3 \times 2y = -6y}$

Example 1.9

Simplify this expression.

$3x^2yz \times 2xy^3$

Solution

$$\begin{aligned} \text{Expression} &= (3 \times 2) \times (x^2 \times x) \times (y \times y^3) \times z \qquad \text{(collecting like terms)} \\ &= 6x^3y^4z \end{aligned}$$

Example 1.10

Simplify this expression.

$$\frac{12a^3b^2c^2}{8ab^5c}$$

Solution

Divide the numerator and denominator by their highest common factor.

$$\begin{aligned} \frac{12a^3b^2c^2}{8ab^5c} &= \frac{3a^{3-1}c^{2-1}}{2b^{5-2}} \\ &= \frac{3a^2c}{2b^3} \end{aligned}$$

Note

It is not necessary to include the intermediate step shown here.

Example 1.11

Factorise this expression.

$3a^2b + 6ab^2$

Discussion point

→ Explain what the word *factorise* means.

Solution

First write the highest common factor of the two terms, and then work on the contents of the brackets.

$3a^2b + 6ab^2 = 3ab(a + 2b)$

$3a^2b = 3ab \times a$ and $6ab^2 = 3ab \times 2b$

Example 1.12

Simplify this expression.

$$\frac{2x^2}{3yz} \div \frac{4xy^2}{5z^2}$$

Solution

$$\begin{aligned}\text{Expression} &= \frac{2x^2}{3yz} \times \frac{5z^2}{4xy^2}\\ &= \frac{10x^2z^2}{12xy^3z}\\ &= \frac{5xz}{6y^3}\end{aligned}$$

Example 1.13

Write as a single fraction

$$\frac{x}{4t} - \frac{2y}{5t} + \frac{z}{2t}.$$

Solution

$$\begin{aligned}\frac{x}{4t} - \frac{2y}{5t} + \frac{z}{2t} &= \frac{5x}{20t} - \frac{8y}{20t} + \frac{10z}{20t}\\ &= \frac{5x - 8y + 10z}{20t}\end{aligned}$$

$20t$ is the lowest common multiple of $4t$, $5t$ and $2t$.

Exercise 1B

① Simplify the following expressions.

(i) $12a + 3b - 7c - 2a - 4b + 5c$

(ii) $4x - 5y + 3z + 2x + 2y - 7z$

(iii) $3(5x - y) + 4(x + 2y)$

(iv) $2(p + 5q) - (p - 4q)$

(v) $x(x + 3) - x(x - 2)$

(vi) $a(2a + 3) + 3(3a - 4)$

(vii) $3p(q - p) - 3q(p - q)$

(viii) $5f(g + 2h) - 5g(h - f)$

② Factorise the following expressions by taking out the highest common factor.

(i) $8 + 10x^2$ (ii) $6ab + 8bc$

(iii) $2a^2 + 4ab$ (iv) $pq^3 + p^3q$

(v) $3x^2y + 6xy^4$ (vi) $6p^3q - 4p^2q^2 + 2pq^3$

(vii) $15lm^2 - 9l^3m^3 + 12l^2m^4$ (viii) $84a^5b^4 - 96a^4b^5$

③ Simplify the following expressions and factorise the results.

(i) $4(3x + 2y) + 8(x - 3y)$ (ii) $x(x - 2) - x(x - 8) + 6$

(iii) $x(y + z) - y(x + z)$ (iv) $p(2q - r) + r(p - 2q)$

(v) $k(l + m + n) - km$ (vi) $a(a - 2) - a(a + 4) + 2(a - 4)$

(vii) $3x(x + y) - 3y(x - 2y)$ (viii) $a(a - 2) - a(a - 4) + 8$

④ Simplify the following expressions as much as possible.

(i) $2a^2b \times 5ab^3$ (ii) $6p^3q \times 2q^3r$

(iii) $lm \times mn \times np$ (iv) $3r^3 \times 6s^2 \times 2rs$

(v) $ab \times 2bc \times 4cd \times 8de$ (vi) $3xy^2 \times 4yz^2 \times 5x^2z$

(vii) $2ab^3 \times 6a^4 \times 7b^6$ (viii) $6p^2q^3r \times 7pq^5r^4$

⑤ Simplify the following fractions as much as possible.

(i) $\dfrac{4a^2b}{2ab}$ (ii) $\dfrac{p^2}{q} \times \dfrac{q^2}{p}$

(iii) $\dfrac{8a}{3b^2} \times \dfrac{6b^3}{4a^2}$ (iv) $\dfrac{3ab}{2c^2} \times \dfrac{4cd}{6a^2}$

(v) $\dfrac{8xy^3z^2}{12yz}$ (vi) $\dfrac{3a^2}{9b^3} \div \dfrac{2a^4}{15b}$

(vii) $\dfrac{5p^3q}{8rs^2} \div \dfrac{15pq^5}{28r^4}$

⑥ Write the following expressions as single fractions.

(i) $\dfrac{2a}{3} + \dfrac{a}{4}$ (ii) $\dfrac{2x}{5} - \dfrac{x}{2} + \dfrac{3x}{4}$

(iii) $\dfrac{4p}{3} - \dfrac{3p}{4}$ (iv) $\dfrac{2s}{5} - \dfrac{s}{3} + \dfrac{4s}{15}$

(v) $\dfrac{3b}{8} - \dfrac{b}{6} + \dfrac{5b}{24}$ (vi) $\dfrac{3a}{b} - \dfrac{2a}{3b}$

(vii) $\dfrac{5}{2p} - \dfrac{3}{2q}$ (viii) $\dfrac{2x}{3y} - \dfrac{3x}{2y}$

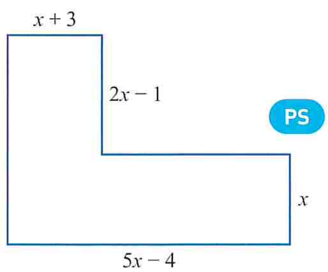


Figure 1.1

PS ⑦ The angles of the hexagon in Figure 1.1 are all 90° or 270°.

Its side lengths are given in terms of x.

(i) Work out its perimeter in terms of x.

(ii) Work out its area in terms of x.

Give your answers in simplified form.

PS ⑧ The rectangle in Figure 1.2 has length $5x + 2$ and width $3x - 1$.

Squares of side x are removed from each corner of the rectangle.

(i) Write down a simplified expression for the perimeter of the new shape.

(ii) Write down a simplified expression for the area of the new shape.

The new shape is the net of an open cuboid.

(iii) Write down an expression for the volume of the cuboid.

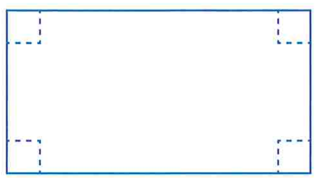


Figure 1.2

Discussion points

- ➜ What is an equation?
- ➜ What does solving an equation mean?

3 Solving linear equations

Linear equations can be solved in different ways. The final answer will always be the same regardless of the method used. The only rule to remember is that whatever is done to one side is also done to the other side. The following examples illustrate this. With practice you would probably omit some of the working.

Prior knowledge

Students are expected to be familiar with solving linear equations. Exercise 1C particularly assesses GCSE specification references A17 and A21.

Example 1.14

Solve this equation.

$3(3x - 17) = 2(x - 1)$

Solution

Multiply out the brackets	$\Rightarrow$	$9x - 51$	$= 2x - 2$
Subtract $2x$ from both sides	$\Rightarrow$	$9x - 51 - 2x$	$= 2x - 2 - 2x$
Tidy up	$\Rightarrow$	$7x - 51$	$= -2$
Add 51 to both sides	$\Rightarrow$	$7x - 51 + 51$	$= -2 + 51$
Tidy up	$\Rightarrow$	$7x$	$= 49$
Divide both sides by 7	$\Rightarrow$	x	$= 7$

Example 1.15

Solve this equation.

$\frac{1}{2}(x + 8) = 2x + \frac{1}{3}(4x - 5)$

Solution

Start by clearing the fractions by multiplying both sides by 6 (the least common multiple of 2 and 3).

Multiply both sides by 6	$\Rightarrow$	$6 \times \frac{1}{2}(x + 8) = 6 \times 2x + 6 \times \frac{1}{3}(4x - 5)$
Tidy up	$\Rightarrow$	$3(x + 8) = 12x + 2(4x - 5)$
Multiply out the brackets	$\Rightarrow$	$3x + 24 = 12x + 8x - 10$
Tidy up	$\Rightarrow$	$3x + 24 = 20x - 10$
Subtract $3x$ from both sides	$\Rightarrow$	$24 = 17x - 10$
Add 10 to both sides	$\Rightarrow$	$34 = 17x$
Divide both sides by 17	$\Rightarrow$	$x = 2$

Discussion point

- ➜ Why have the letter and the number swapped sides on the last line?

Solving linear equations

Sometimes you will need to set up the equation as well as solve it. When you are doing this, make sure that you define any variables you introduce.

Example 1.16

In a triangle, the largest angle is nine times as big as the smallest. The third angle is 60°.

(i) Write this information in the form of an equation.

(ii) Solve the equation to work out the sizes of the three angles.

Solution

(i) Let s = the smallest angle in degrees

So $9s$ = the largest angle

The sum of all three angles is 180°

$s + 9s + 60 = 180$

(ii) Solving $\Rightarrow 10s = 120$

$\Rightarrow \quad s = 12$

The largest angle is then $9 \times 12 = 108$

So the angles are 12°, 60° and 108°

Exercise 1C

① Solve the following equations.

(i) $2x - 3 = x + 4$

(ii) $5a + 3 = 2a - 3$

(iii) $2(x + 5) = 14$

(iv) $7(2y - 5) = -7$

(v) $5(2c - 8) = 2(3c - 10)$

(vi) $3(p + 2) = 4(p - 1)$

(vii) $3(2x - 1) = 6(x + 2) + 3x$

(viii) $\frac{x}{3} + 7 = 5$

(ix) $\frac{5y - 2}{11} = 3$

(x) $\frac{k}{2} + \frac{k}{3} = 35$

(xi) $\frac{2t}{3} - \frac{3t}{5} = 4$

(xii) $\frac{5p - 4}{6} - \frac{2p + 3}{2} = 7$

(xiii) $p + \frac{1}{3}(p + 1) + \frac{1}{4}(p + 2) = \frac{5}{6}$

② The length, l metres, of a field is 80 m greater than the width. The perimeter is 600 m.

(i) Write this information in the form of an equation in l.

(ii) Work out the area of the field.

PS ③ Ben and Chris are twins and their brother Stephen is four years younger. The total of their three ages is 17 years.

(i) Write this information in the form of an equation in s, Stephen's age in years.

(ii) What are all their ages?

PS ④ In a multiple-choice examination of 20 questions, four marks are given for each correct answer and one mark is deducted for each wrong answer. There is no penalty for not attempting a question. A candidate attempts a questions and gets c correct.

(i) Write down, and simplify, an expression for the candidate's total mark in terms of a and c.

(ii) A candidate attempts three-quarters of the questions and scores 40. Write down, and solve, an equation for the number of correct answers.

PS ⑤ Chris is three times as old as his son, Joe, and in 12 years' time he will be twice as old as him.

(i) Given that Joe is j years old now, write an expression for Chris' age in 12 years' time.

(ii) Write down, and solve, an equation in j.

PS ⑥ A square has sides of length $2a$ metres, and a rectangle has length $3a$ metres and width 3 metres.

(i) Write down, in terms of a, the perimeter of the square.

(ii) Write down, in terms of a, the perimeter of the rectangle.

(iii) The perimeters of the square and the rectangle are equal. Work out the value of a.

PS ⑦ The sum of five consecutive numbers is equal to 105. Let m represent the middle number.

(i) Write down the five numbers in terms of m.

(ii) Form an equation in m and solve it.

(iii) What are the five consecutive numbers?

PS ⑧ One rectangle has a length of $(x + 2)$ cm and a breadth of 2 cm. Another rectangle, of equal area, has a length of 5 cm and a width of $(x - 3)$ cm.

(i) Write down an equation in x and solve it.

(ii) What is the area of each of the rectangles?

Discussion point

→ A large ice cream costs 40p more than a small one. Two large ice creams cost the same as three small ones. What is the cost of each size of ice cream?

→ This is an example of the type of question that you might find in a puzzle book or the puzzle section of a newspaper or magazine. How would you set about solving it?

Discussion point

→ You may think that the following question appears to be very similar to the one on the left. What happens when you try to solve it?

→ A large ice cream costs 40p more than a small one. Five small ice creams plus three large ones cost 80p less than three small ice creams plus five large ones. What is the cost of each size of ice cream?

4 Algebra and number

Some algebra questions will involve using number skills.

Example 1.17

a is 75% of b and $b : c = 3 : 2$

Show that $8a = 9c$.

Solution

a is 75% of b

$$a = \frac{75}{100}b$$

$$a = \frac{3}{4}b \quad ①$$

$b : c = 3 : 2$

$$\frac{b}{c} = \frac{3}{2}$$

$$b = \frac{3}{2}c \quad ②$$

Substitute ② in ①

$$a = \frac{3}{4} \times \frac{3}{2}c$$

$$a = \frac{9}{8}c$$

$$8a = 9c$$

Example 1.18

Write an expression for x increased by 13%.

Solution

$$x \text{ increased by } 13\% = x + \frac{13}{100}x$$

$$= 1.13x$$

Example 1.19

$p : q = 4 : 5$

Work out $p + 2q : 4q$, giving your answer in its simplest form.

Solution

Thinking in terms of parts:

p is 4 parts, q is 5 parts

$p + 2q$ is $4 + 2 \times 5 = 14$ parts

$4q$ is 20 parts

$p + 2q : 4q = 14 : 20$

$= 7 : 10$

An alternative solution is:

$$p : q = 4 : 5 \quad \Rightarrow \quad \frac{p}{q} = \frac{4}{5}$$

$$\Rightarrow \quad p = \frac{4}{5}q$$

$$\begin{aligned} p + 2q : 4q &= \frac{4}{5}q + 2q : 4q \\ &= \frac{14}{5}q : 4q \\ &= \frac{14}{5} : 4 \\ &= 7 : 10 \end{aligned}$$

Exercise 1D

① Write expressions for the following, giving your answers in their simplest form.

(i) 30% of b (ii) y% of 450 (iii) c% of d

② 60% of p = 40% of q

Work out p as a percentage of q.

③ Write expressions for the following, giving your answers in their simplest form.

(i) a increased by 20% (ii) b increased by 5%

(iii) k decreased by 35% (iv) m decreased by 2%

④ a increased by 80% is equal to b increased by 50%.

Show that $\frac{b}{a} = 1.2$

PS ⑤ p increased by 25% is equal to q decreased by 25%.

Work out p as a percentage of q.

PS ⑥ $x : y = 2 : 3$ and $y : z = 4 : 9$

Work out $x : y : z$, giving your answer in its simplest form.

PS ⑦ $a : b = 5 : 2$

(i) Write a in terms of b.

(ii) Work out $2a + b : b$, giving your answer in its simplest form.

(iii) Work out $7a - 5b : 4a$, giving your answer in its simplest form.

PS ⑧ $m : n = 3 : 8$ and r is 20% of n.

Work out $m : r$.

PS ⑨ y is 20% greater than x.

w is 20% less than y.

Work out the ratio $w : x$ in its simplest form.

PS ⑩ p is m% greater than q.

p is m% less than r.

Work out the ratio $r : q$ in terms of m.

PS ⑪ The ratio of boys to girls in a room is 3 : 7

16 boys enter and 6 girls leave. The ratio is now 4 : 5

How many boys and how many girls are now in the room?

5 Expanding brackets

Prior knowledge

Students are expected to be familiar with multiplication of two or three linear expressions. Exercise 1E particularly assesses GCSE specification reference A4h.

Discussion point

➜ Why is $(x + 5)(2x - 3)$ a quadratic expression?

An expression of the form $ax^2 + bx + c$ (where the coefficient of x is non-zero) is a quadratic in x.

For example,

$x^2 + 3$

a^2 (a quadratic expression in a),

$2y^2 - 3y + 5$ (a quadratic expression in y).

Example 1.20

Expand $(x + 5)(2x - 3)$.

Solution

$$\begin{aligned}(x+5)(2x-3) &= x(2x-3) + 5(2x-3)\\ &= 2x^2 - 3x + 10x - 15\\ &= 2x^2 + 7x - 15\end{aligned}$$

This method has multiplied everything in the second bracket by each term in the first bracket. An alternative way of setting this out is used in the next example.

Example 1.21

Expand $(3x - 5)^2$.

Solution

$$(3x-5)^2 = (3x-5)(3x-5)$$

Write the square as the product of two brackets so you don't forget the middle term.

$$\begin{array}{r} 3x - 5\\ \times\ 3x - 5\\ \hline -15x + 25\\ \underline{9x^2 - 15x\qquad}\\ 9x^2 - 30x + 25 \end{array}$$

Multiply the top line by −5.

Multiply the top line by $3x$.

Add the two products.

Example 1.22 Multiply $(x^3 + 2x - 4)$ by $(x^2 - x + 3)$.

Solution

$$(x^3 + 2x - 4)(x^2 - x + 3)$$
$$= x^3(x^2 - x + 3) + 2x(x^2 - x + 3) - 4(x^2 - x + 3)$$
$$= x^5 - x^4 + 3x^3 + 2x^3 - 2x^2 + 6x - 4x^2 + 4x - 12$$
$$= x^5 - x^4 + 5x^3 - 6x^2 + 10x - 12$$

Example 1.23 Expand and simplify $(a - 2)^3$.

Solution

$(a - 2)^3 = (a - 2)(a - 2)^2$

First, work out $(a - 2)^2$

$$\begin{aligned}(a-2)(a-2) &= a(a-2) - 2(a-2)\\ &= a^2 - 2a - 2a + 4\\ &= a^2 - 4a + 4\end{aligned}$$

Then multiply this by $(a - 2)$

$$\begin{aligned}(a-2)^3 &= (a-2)(a^2 - 4a + 4)\\ &= a(a^2 - 4a + 4) - 2(a^2 - 4a + 4)\\ &= a^3 - 4a^2 + 4a - 2a^2 + 8a - 8\\ &= a^3 - 6a^2 + 12a - 8\end{aligned}$$

Exercise 1E

1. Expand the following expressions.

   (i) $(x + 5)(x + 4)$ (ii) $(x + 3)(x + 1)$
   (iii) $(a + 5)(2a - 1)$ (iv) $(2p + 3)(3p - 2)$
   (v) $(x + 3)^2$ (vi) $(2x + 3)(2x - 3)$
   (vii) $(2 - 3m)(m - 4)$ (viii) $(6 + 5t)(2 - t)$
   (ix) $(4 - 3x)^2$ (x) $(m - 3n)^2$

2. (i) Multiply $(x^3 - x^2 + x - 2)$ by $(x^2 + 1)$
   (ii) Multiply $(x^4 - 2x^2 + 3)$ by $(x^2 + 2x - 1)$
   (iii) Multiply $(2x^3 - 3x + 5)$ by $(x^2 - 2x + 1)$
   (iv) Multiply $(x^5 + x^4 + x^3 + x^2 + x + 1)$ by $(x - 1)$
   (v) Expand $(x + 2)(x - 1)(x + 3)$
   (vi) Expand $(2x + 1)(x - 2)(x + 4)$
   (vii) Expand and simplify $(x + 1)^3$
   (viii) Expand and simplify $(p - 5)^3$
   (ix) Expand and simplify $(2a + 3)^3$
   (x) Simplify $(2x^2 - 1)(x + 2) - 4(x + 2)^2$
   (xi) Simplify $(x^2 - 1)(x + 1) - (x^2 + 1)(x - 1)$

Hint: Expand the first two sets of brackets first.

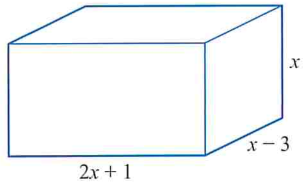


Figure 1.3

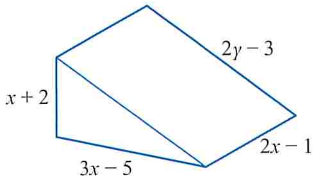


Figure 1.4

3. The cuboid in Figure 1.3 has length $(2x+1)$, width $(x-3)$, and height x.
   (i) Work out its volume.
   (ii) Work out its surface area.
   Leave your answers in expanded and simplified form.
4. The prism in Figure 1.4 has three rectangular faces, and two congruent right-angled triangular faces. It has length $(3x-5)$ and height $(x+2)$. Its slant height is $(2y-3)$ and its width is $(2x-1)$.
   (i) Work out its volume.
   (ii) Work out its surface area.
   Leave your answers in expanded and simplified form.
5. (i) Expand and simplify $(a+b)^2$.
   (ii) Hence expand and simplify $(a+b)^3$.
   (iii) Hence expand and simplify $(a+b)^4$.
   (iv) Hence expand and simplify $(a+b)^5$.
6. Use your answers to question 5 to write down the expansions of
   (i) $(x+6)^3$
   (ii) $(p-2)^3$
   (iii) $(2y+1)^4$
   (iv) $(x-3)^4$
   (v) $(3w-4)^5$.

In the expansion of $(a+b)^3$ replace a with x, and b with 6.

ACTIVITY 1.1

Using your answers to Exercise 1E, question 5, make predictions about the simplified expansion of $(a+b)^6$.

(i) How many terms will there be in the simplified expansion?
(ii) What will be the coefficient of the a^6 term?
(iii) What will be the coefficient of the a^5b term in the simplified expansion?
(iv) Can you make any other predictions?

6 The binomial expansion

This section deals with the expansion of $(a+b)^n$ where n is a positive integer.

You already know
$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

Repeatedly multiplying by $(a+b)$ gives
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Each time you multiply by $(a + b)$, the simplified expansion gains an extra term.

So the simplified expansion of $(a + b)^n$ will have $n + 1$ terms.

Each term will be of the form $Pa^q b^r$, where the indices are non-negative integers whose sum is n.

The value of P is found in Pascal's triangle.

Pascal's triangle

Consider the coefficients of the expansions of $(a + b)^n$ and $(a + b)^{n+1}$.

For example:

$$\begin{aligned}(a+b)^5 &= (a+b)(a+b)^4 \\ &= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &= 1a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + 1ab^4 \\ &\quad + 1a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + 1b^5 \\ &= 1a^5 + (4+1)a^4b + (6+4)a^3b^2 + (4+6)a^2b^3 \\ &\quad + (1+4)ab^4 + 1b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

The coefficients of $(a + b)^5$ are the sums of adjacent coefficients of the $(a + b)^4$ expansion.

The coefficients of $(a + b)^n$ form Pascal's triangle:

$(a+b)^0$:	1
$(a+b)^1$:	1 1
$(a+b)^2$:	1 2 1
$(a+b)^3$:	1 3 3 1
$(a+b)^4$:	1 4 6 4 1

The 1 at the top of the triangle is usually referred to as the 0th row.

Figure 1.5

Each number in the triangle is the sum of the two numbers above it.

row 4 is 1 4 6 4 1

row 5 is 1 5 10 10 5 1

Figure 1.6

So $$\begin{aligned}(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Example 1.24

Expand $(2x + 3y)^3$.

Solution

In the expansion of $(a + b)^3$, replace a with $2x$ and b with $3y$.

$$\begin{aligned}(2x + 3y)^3 &= 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3 \\ &= 1 \times 8x^3 \times 1 + 3 \times 4x^2 \times 3y + 3 \times 2x \times 9y^2 + 1 \times 1 \times 27y^3 \\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3\end{aligned}$$

Example 1.25

Expand $(3 - w)^4$.

Solution

In the expansion of $(a + b)^4$, replace a with 3 and b with $-w$.

$$\begin{aligned}(3 - w)^4 &= 1(3)^4(-w)^0 + 4(3)^3(-w)^1 + 6(3)^2(-w)^2 + 4(3)^1(-w)^3 + 1(3)^0(-w)^4 \\ &= 1 \times 81 \times 1 + 4 \times 27 \times (-w) + 6 \times 9 \times w^2 + 4 \times 3 \times (-w^3) + 1 \times 1 \times w^4 \\ &= 81 - 108w + 54w^2 - 12w^3 + w^4\end{aligned}$$

Note

It is not necessary to learn the expansions of $(a + b)^n$.
Instead, just learn one of the rows of Pascal's triangle.

Example 1.26

Expand $(x + 2)^5$.

Solution

The third row of Pascal's triangle is 1, 3, 3, 1.

Every row starts and finishes with 1, so the fourth row is $1, 1 + 3, 3 + 3, 3 + 1, 1$ which simplifies to 1, 4, 6, 4, 1.

And the fifth row is $1, 1 + 4, 4 + 6, 6 + 4, 4 + 1, 1$ which is 1, 5, 10, 10, 5, 1

So $\quad (x + 2)^5 = 1x^?2^? + 5x^?2^? + 10x^?2^? + 10x^?2^? + 5x^?2^? + 1x^?2^?$

One set of indices goes up from 0 to 5, whilst the other goes down from 5 to 0.

$$\begin{aligned}\text{So} \quad (x + 2)^5 &= 1x^0 2^5 + 5x^1 2^4 + 10x^2 2^3 + 10x^3 2^2 + 5x^4 2^1 + 1x^5 2^0 \\ &= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5\end{aligned}$$

Example 1.27

Work out the first three terms in ascending powers of x in the expansion of $(2 + 5x)^6$.

Solution

Row 5 of Pascal's triangle is 1 5 10 10 5 1

The first three numbers in row 6 are therefore 1 6 15

So the first three terms in the expansion of $(2 + 5x)^6$ are

$$1 \times 2^6 \times (5x)^0 + 6 \times 2^5 \times (5x)^1 + 15 \times 2^4 \times (5x)^2$$

$$= 1 \times 64 \times 1 + 6 \times 32 \times 5x + 15 \times 16 \times 25x^2$$

$$= 64 + 960x + 6000x^2$$

! After expanding an expression of the form $(a + b)^n$, and before simplifying, check that the sum of each pair of indices is n.

Exercise 1F

① Use Pascal's triangle to expand

(i) $(1 + x)^3$ (ii) $(y + 1)^4$ (iii) $(x + y)^5$

(iv) $(5 + w)^3$ (v) $(p + 4)^4$ (vi) $(2 + m)^5$.

② Use Pascal's triangle to expand

(i) $(x - y)^3$ (ii) $(1 - 2x)^4$ (iii) $(2 - y)^5$

(iv) $(5 - 2p)^3$ (v) $(3x - 4)^4$ (vi) $(4x - 1)^5$.

③ Work out the first three terms in ascending powers of x in the expansion of $(1 + x)^6$.

④ Work out the first three terms in descending powers of x in the expansion of $(2 + x)^7$.

⑤ Work out the coefficient of the x^3 term in the expansion of $(4 - 3x)^5$.

⑥ Write down the second number in the 10th row of Pascal's triangle.

⑦ Write down the last number in the 19th row of Pascal's triangle.

⑧ Write down the third number in the 9th row of Pascal's triangle.

⑨ Expand $\left(3x^2 + \frac{1}{x}\right)^4$.

⑩ (i) Expand $(1 + 2x)^5$.

(ii) Hence write down the expansion of $(1 - 2x)^5$.

(iii) Hence simplify $(1 + 2x)^5 - (1 - 2x)^5$.

⑪ (i) Expand $(3 + w)^3$.

(ii) Hence, by replacing w with $x + 2y$, write down the expansion of $(3 + x + 2y)^3$.

PS ⑫ The simplified expansion of $(mx + y)^n$ includes the term $240x^2y^4$.

(i) Write down the value of n.

(ii) Hence work out the possible values of m.

(iii) Hence work out the coefficient of the x^4y^2 term.

PS ⑬ In the expansion of $\left(x + \frac{2}{x}\right)^6$ work out the term which is independent of x.

PS ⑭ Given that the 10th row of Pascal's triangle is

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1,

work out the coefficient of x^2 in the expansion of $\left(x - \frac{2}{x}\right)^{10}$.

REAL-WORLD CONTEXT

The binomial expansion has many applications in the real world, including in the distribution of IP addresses. It is also used by economists when making predictions for the future behaviour of markets, and similarly by meteorologists when forecasting the weather.

ACTIVITY 1.2

(i) How are the numbers 1, 11, 121, 1331, 14 641 related?

(ii) Write a formula for the sum of the numbers in the nth row of Pascal's triangle. (Assume that the top row, which contains only '1', is the 0th row.)

(iii) Write a formula for the second number in the nth row of Pascal's triangle.

(iv) Write a formula for the third number in the nth row of Pascal's triangle.

(v) Each row of Pascal's triangle reads the same backwards as forwards.

What single word describes such a property?

FUTURE USES

At A-Level, the factorial function is used to quickly generate the numbers of Pascal's triangle. Students may like to investigate the ${}_nC_r$ function on their calculator to see how it relates to this topic.

Prior knowledge

Students are expected to be familiar with the manipulation of surd expressions. Exercise 1G particularly assesses GCSE specification reference N8h.

7 Manipulating surds

Simplifying expressions containing square roots

In mathematics there are times when it is helpful to be able to manipulate square roots, rather than just find their values from a calculator.
This ensures that you are working with the exact value, not just a rounded version.

Example 1.28

Simplify the following.

(i) $\sqrt{8}$

(ii) $\sqrt{6} \times \sqrt{3}$

(iii) $\sqrt{32} - \sqrt{18}$

(iv) $(4 + \sqrt{3})(4 - \sqrt{3})$

Solution

(i)
$$\begin{aligned}\sqrt{8} &= \sqrt{2 \times 2 \times 2} \\ &= \sqrt{2} \times \sqrt{2} \times \sqrt{2} \\ &= \left(\sqrt{2}\right)^2 \times \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

(ii)
$$\begin{aligned}\sqrt{6} \times \sqrt{3} &= \sqrt{6 \times 3} \\ &= \sqrt{2 \times 3 \times 3} \\ &= \left(\sqrt{3}\right)^2 \times \sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$

(iii) $\sqrt{32} - \sqrt{18} = \sqrt{16 \times 2} - \sqrt{9 \times 2}$
$= 4\sqrt{2} - 3\sqrt{2}$
$= \sqrt{2}$

Look for square factors of 32 and 18.

16 is the largest square factor of 32.

9 is the largest square factor of 18.

(iv) $(4 + \sqrt{3})(4 - \sqrt{3}) = 16 - 4\sqrt{3} + 4\sqrt{3} - (\sqrt{3})^2$
$= 16 - 3$
$= 13$

Discussion point

➜ What is a rational number?

Notice that in part (iv) of Example 1.28 there is no square root in the answer. In the next example, all the numbers involve fractions with a square root as part of the denominator. It is easier to work with numbers if any square roots are only part of the numerator. Manipulating a number to that form is called *rationalising the denominator*.

When the numerator and the denominator of a fraction are multiplied by the same number, then the value of the fraction stays the same. For example, $\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$. We use this principle when rationalising a denominator. In the next examples, and question 3 of Exercise 1G, the denominators have only one term. In each case, multiply both the numerator and denominator by this number, and then simplify.

Example 1.29

Simplify the following by rationalising their denominators.

(i) $\frac{2}{\sqrt{3}}$ (ii) $\sqrt{\frac{3}{5}}$ (iii) $\sqrt{\frac{3}{8}}$

Solution

(i)
$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3}$$

(ii)
$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{3} \times \sqrt{5}}{(\sqrt{5})^2} = \frac{\sqrt{15}}{5}$$

(iii)
$$\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3} \times \sqrt{2}}{2(\sqrt{2})^2} = \frac{\sqrt{6}}{4}$$

Exercise 1G

Do not use a calculator for this exercise.

① Simplify the following.

(i) $\sqrt{32}$ (ii) $\sqrt{125}$

(iii) $\sqrt{5} \times \sqrt{15}$ (iv) $\sqrt{8} - \sqrt{2}$

(v) $3\sqrt{27} - 6\sqrt{3}$ (vi) $4(3 + \sqrt{2}) - 3(5 - \sqrt{2})$

(vii) $4\sqrt{32} - 3\sqrt{8}$ (viii) $5(6 - \sqrt{3}) + 2(3 + 4\sqrt{3})$

(ix) $2\sqrt{125} + 6\sqrt{5}$ (x) $3\left(2\sqrt{2} - 3\sqrt{3}\right) - 2\left(3\sqrt{2} - 5\sqrt{3}\right)$

② Simplify the following.

(i) $(\sqrt{2} - 1)^2$ (ii) $(4 - \sqrt{5})(2 + \sqrt{5})$

(iii) $(2 - \sqrt{7})(\sqrt{7} - 1)$ (iv) $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

(v) $(3 + \sqrt{2})(5 - 2\sqrt{2})$ (vi) $(\sqrt{7} - 3)(2\sqrt{7} + 3)$

(vii) $(3\sqrt{3} - 2)(2\sqrt{3} - 3)$ (viii) $(\sqrt{5} - \sqrt{3})^2$

(ix) $(5 - 3\sqrt{2})(2\sqrt{2} - 1)$ (x) $(2\sqrt{2} + 3)^2$

③ Simplify the following by rationalising their denominators.

(i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{5}{\sqrt{5}}$ (iii) $\dfrac{8}{\sqrt{6}}$

(iv) $\sqrt{\dfrac{2}{3}}$ (v) $\dfrac{2\sqrt{2}}{\sqrt{8}}$ (vi) $\sqrt{\dfrac{3}{7}}$

(vii) $\dfrac{21}{\sqrt{7}}$ (viii) $\dfrac{5}{3\sqrt{5}}$ (ix) $\dfrac{\sqrt{75}}{\sqrt{125}}$

(x) $\dfrac{8}{\sqrt{128}}$

④ Simplify the following by writing them as single fractions.

(i) $\dfrac{2}{3 - \sqrt{2}} + \dfrac{2}{3 + \sqrt{2}}$ (ii) $\dfrac{5}{2 - \sqrt{3}} - \dfrac{3}{2 + \sqrt{3}}$

(iii) $\dfrac{1}{5 - 2\sqrt{6}} + \dfrac{3}{5 + 2\sqrt{6}}$ (iv) $\dfrac{4}{4 + \sqrt{3}} - \dfrac{1}{4 - \sqrt{3}}$

⑤ (i) Use the expansion of $(a + b)^3$ to simplify $\left(3 + \sqrt{2}\right)^3$.

(ii) Use the expansion of $(a + b)^3$ to simplify $\left(2 + \sqrt{5}\right)^3$.

(iii) Use the expansion of $(a + b)^4$ to simplify $\left(2 - \sqrt{3}\right)^4$.

(iv) Use the expansion of $(a + b)^4$ to simplify $\left(1 + \sqrt{6}\right)^4$.

(v) Use the expansion of $(a + b)^5$ to simplify $\left(1 + \sqrt{5}\right)^5$.

(vi) Use the expansion of $(a + b)^5$ to simplify $\left(2 - \sqrt{5}\right)^5$.

⑥ Solve the following equations.

(i) $\sqrt{32} - v\sqrt{2} = \sqrt{8}$ (ii) $w\sqrt{18} + \sqrt{8} = \sqrt{98}$

(iii) $3\sqrt{3} + y\sqrt{12} = 2\sqrt{27}$ (iv) $x\sqrt{50} + \sqrt{18} = 5x\sqrt{8}$

7. Simplify $\left(2 + \sqrt{3}\right)^6$.

8. Solve the following equations.

(i) $\dfrac{m}{\sqrt{3}} + \dfrac{1}{\sqrt{12}} = \sqrt{3}$

(ii) $\dfrac{3n}{\sqrt{2}} - \dfrac{n+4}{\sqrt{8}} = \sqrt{18}$

(iii) $\dfrac{2x}{\sqrt{5}} = \sqrt{20} + \dfrac{x}{\sqrt{45}}$

(iv) $3\left(\dfrac{x}{\sqrt{2}} + \sqrt{8}\right) = \dfrac{5x}{\sqrt{18}} + \dfrac{3x}{\sqrt{32}}$

9. Solve $\left(x - \sqrt{2}\right)^3 = x^2\left(x - \sqrt{18}\right) + 3\sqrt{8}$.

10. The area of a square is $7 + 4\sqrt{3}$.

Given that the length of each side of the square is of the form $m + n\sqrt{3}$ (where m and n are integers), work out the perimeter.

Leave your answer in the form $p + \sqrt{q}$, where p and q are integers.

Rationalising denominators with two terms

For a denominator with two terms, the multiplier we use is the denominator with one of its signs changed.

Example 1.30

Rationalise the denominator of

$$\frac{3\sqrt{2}}{4 - \sqrt{5}}.$$

Solution

$$\begin{aligned}
\frac{3\sqrt{2}}{4 - \sqrt{5}} &= \frac{3\sqrt{2}}{4 - \sqrt{5}} \times \frac{4 + \sqrt{5}}{4 + \sqrt{5}} \\
&= \frac{12\sqrt{2} + 3\sqrt{2}\sqrt{5}}{16 + 4\sqrt{5} - 4\sqrt{5} - (\sqrt{5})^2} \\
&= \frac{12\sqrt{2} + 3\sqrt{10}}{16 - 5} \\
&= \frac{12\sqrt{2} + 3\sqrt{10}}{11}
\end{aligned}$$

Example 1.31

Write $\dfrac{2\sqrt{3} - 4}{3\sqrt{3} + 5}$ in the form $a + b\sqrt{3}$, where a and b are integers.

Solution

$$\begin{aligned}
\frac{2\sqrt{3} - 4}{3\sqrt{3} + 5} &= \frac{2\sqrt{3} - 4}{3\sqrt{3} + 5} \times \frac{3\sqrt{3} - 5}{3\sqrt{3} - 5} \\
&= \frac{6(\sqrt{3})^2 - 10\sqrt{3} - 12\sqrt{3} + 20}{9(\sqrt{3})^2 - 15\sqrt{3} + 15\sqrt{3} - 25} \\
&= \frac{18 - 22\sqrt{3} + 20}{27 - 25} \\
&= \frac{38 - 22\sqrt{3}}{2} \\
&= 19 - 11\sqrt{3}
\end{aligned}$$

Exercise 1H

Do not use a calculator for this exercise.

① Rationalise the denominators of these fractions:

(i) $\dfrac{2\sqrt{3}}{5+\sqrt{2}}$ (ii) $\dfrac{\sqrt{7}}{4-\sqrt{2}}$ (iii) $\dfrac{3\sqrt{3}}{\sqrt{3}+1}$

(iv) $\dfrac{2+\sqrt{2}}{3-\sqrt{2}}$ (v) $\dfrac{\sqrt{7}-3}{1-\sqrt{7}}$ (vi) $\dfrac{10+\sqrt{3}}{\sqrt{3}+\sqrt{2}}$.

② Write $\dfrac{3\sqrt{2}+6}{\sqrt{2}-1}$ in the form $a+b\sqrt{2}$, where a and b are integers.

③ Write $\dfrac{2\sqrt{5}}{4\sqrt{5}+9}$ in the form $c\sqrt{5}+d$, where c and d are integers.

④ Write $\dfrac{1+\sqrt{3}}{3+2\sqrt{3}}$ in the form $p+\frac{q}{r}\sqrt{3}$, where p, q and r are integers.

⑤ A rectangle has a width of $2+\sqrt{5}$ and an area of $1+\sqrt{5}$.
Work out its length.

⑥ Simplify $\dfrac{19}{\sqrt{27}-\sqrt{8}}$.

⑦ A triangle has an area of $11\sqrt{2}-2$ and a base length of $2+\sqrt{18}$.
Work out its perpendicular height.

⑧ The area of a trapezium is $4+\sqrt{27}$.
The lengths of its parallel sides are $3+\sqrt{12}$ and $2-\sqrt{3}$.
Work out the perpendicular distance between the parallel sides.

8 The product rule for counting

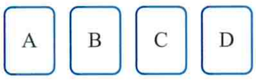


Figure 1.7

The product rule for counting is an efficient method for finding the number of combinations given a particular condition.

Consider the four cards in Figure 1.7.

In how many different ways can the four cards be arranged?

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

As there are only 24 arrangements it is possible to list them all and then count them.

However, this strategy is impractical if the number of arrangements is large.

Instead, consider the number of possibilities for each position. There are 4 letters which could be chosen first. This leaves 3 letters for the second position. So there are $4 \times 3 = 12$ ways of choosing the first two letters. For each of these, there are 2 letters to choose for the third position, leaving just 1 for the final position.

A quick way to calculate the number of arrangements is $4 \times 3 \times 2 \times 1 = 24$.

Example 1.32

Here are seven cards.

2	3	4	5	6	7	8

Figure 1.8

Using all seven cards, in how many different ways can they be arranged to form a seven-digit number?

Solution

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Calculations such as this are sometimes made easier using the factorial function on a calculator. The factorial function is defined as

$$n! = \begin{cases} 1 & n = 0 \\ n \times (n-1)! & n \in \mathbb{N} \end{cases}$$

$\mathbb{N}$ is the set of natural numbers, i.e. positive integers.

For example, $4! = 4 \times 3!$

$$= 4 \times 3 \times 2!$$

$$= 4 \times 3 \times 2 \times 1!$$

$$= 4 \times 3 \times 2 \times 1 \times 0!$$

$$= 4 \times 3 \times 2 \times 1 \times 1$$

$$= 24$$

Note: $n!$ is more simply remembered as the product of all the natural numbers up to and including n.

ACTIVITY 1.3

Find the factorial function on your calculator. Try different inputs.

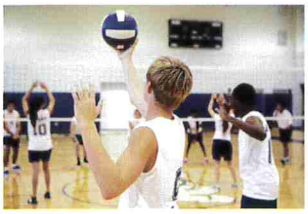
For which numbers does the factorial function not work?

Try $-5!$

Then try $(-5)!$

Why does the calculator respond differently?

Example 1.33



In a volleyball team, six players start in six different positions.

At the start of a game, how many different arrangements are possible for the six players?

Solution

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$
$$= 720$$

Note

This specification does not assess a candidate's knowledge of the factorial function.

FUTURE USES

At A-Level, the factorial function is used to extend this topic further. It is also used in algebraic expansions.

Example 1.34

A volleyball team of six players is chosen from a squad of twelve.

(i) How many different starting arrangements are possible?

(ii) How many different teams can be selected?

Solution

(i) There are 12 possibilities for the first position, 11 for the second position, 10 for the third, then 9, and so on.

$12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665\,280$

(ii) As you saw in Example 1.33, each team can be arranged in 720 different ways. In this second part of the question, the arrangement of each team is not important. So the number of different teams is

$665\,280 \div 720 = 924$

Discussion point

→ Is it possible to use the factorial function to make this calculation easier?

Example 1.35

Using each of the following digits no more than once, 3 1 7 9 8 5

(i) how many different four-digit numbers can be made?

(ii) how many **even** three-digit numbers can be made?

(iii) how many numbers less than 4000 can be made?

(iv) how many **odd** numbers greater than 500 000 can be made?

Solution

(i) $6 \times 5 \times 4 \times 3 = 360$

(ii) An even three-digit number will have an even number as the last digit. Hence, 8 must be the last digit.

So there are five possibilities for the first digit, leaving four for the second digit.

$1 \times 5 \times 4 = 20$

(iii) This question does not specify the number of digits to be used, so each must be considered separately.

Only 6 one-digit numbers are possible.

There are $6 \times 5 = 30$ two-digit numbers,

and $6 \times 5 \times 4 = 120$ three-digit numbers.

When considering the four-digit numbers, the first digit must be either a 1 or a 3. So the number of possible four-digit numbers is $2 \times 5 \times 4 \times 3 = 120$

This gives a total of $6 + 30 + 120 + 120 = 276$

(iv) As the number must be greater than 500 000 then the first digit must be a 5, or a 7, or an 8, or a 9.

If the first digit is a 5, or a 7, or a 9, then there are only four possibilities for the last digit. If 8 is the first digit, then there are five possible last digits.

5____1	5____3	5____7	5____9	
7____1	7____3	7____5	7____9	
8____1	8____3	8____5	8____7	8____9
9____1	9____3	9____5	9____7	

So there are 17 possibilities for the first and last digits.

The remaining four digits have $4 \times 3 \times 2 \times 1 = 24$ arrangements.

So there are $17 \times 24 = 408$ odd numbers greater than 500 000 which use the above digits only once.

Example 1.36

In 2001 a new system for vehicle registration plates was introduced.

Each plate starts with two letters (excluding I, Q, Z) which identify the area in which the vehicle was registered.

These are followed by two digits (except 01) which identify the age of the vehicle.

Finally, there are three more letters, excluding I and Q.

How many such registration plates are possible?

23 letters, excluding I, Q, Z

99 two-digit numbers, excluding 01

24 letters, excluding I and Q

Solution

$23 \times 23 \times 99 \times 24 \times 24 \times 24 = 723\,976\,704$

Exercise 1I

① Work out the number of arrangements of the letters ABCDE.

② Matt, Joe, Ben, Saul, Chris, Anna and Steve stand in a straight line.

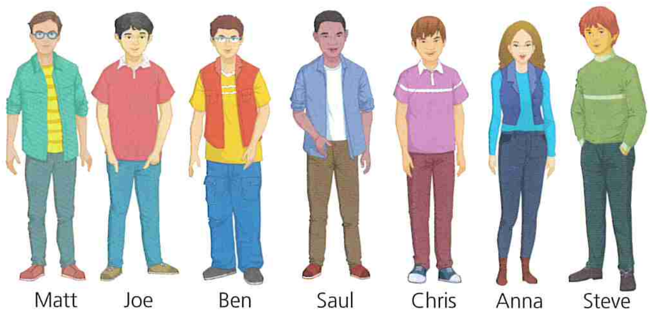


Figure 1.9

How many arrangements are possible?

The product rule for counting

③ Here are seven digits: 9 5 1 4 2 3 8

(i) If each digit can appear no more than once, how many six-digit numbers can be formed?

(ii) If each of the six digits can appear more than once, how many six-digit numbers can be formed?

④ How many **even** numbers can be made using each of the digits 2, 3, 7, 8 exactly once?

⑤ Work out the number of three-digit multiples of 5

⑥ Ten books stand next to each other on a shelf.

How many different arrangements are possible?

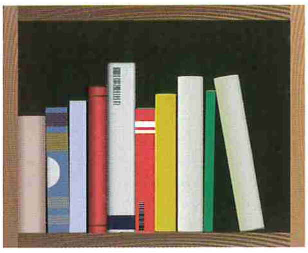


Figure 1.10

PS ⑦ A toy box has six compartments. Four different toys are to be put in the box.

(i) If each compartment can hold only one toy, how many arrangements are possible?

(ii) If each compartment can hold up to four toys, how many arrangements are possible?

PS ⑧ Figure 1.11 shows the net of a cube.

The numbers 1, 2, 3, 4, 5, 6 are to be written on the net – one number per square.

(i) How many different ways can the numbers be written on the net?

(ii) If the net is folded into a cube, the numbers on opposite faces must add to 7

In this case, how many ways can the numbers be written on the net?

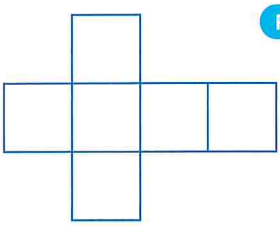


Figure 1.11

PS ⑨ A palindromic integer is a whole number which reads the same forwards and backwards.

Two examples of palindromic integers are 15 751 and 302 203

This specification will not assess candidates' knowledge of palindromes.

(i) Work out the number of three-digit palindromic integers.

(ii) Work out the number of four-digit palindromic integers.

(iii) How many palindromic integers are less than one million?

(iv) How many six-digit palindromic integers are multiples of 9?

PS ⑩ Figure 1.12 shows a network comprising five points and the links between them.

A tour starts at one point, visits each of the other points, and then returns to its starting point.

For example, ACEDBA is a tour.

Work out the number of different tours.

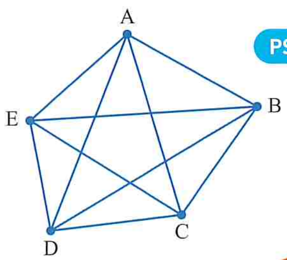


Figure 1.12

This specification will not assess candidates' knowledge of networks and tours.

PS ⑪ A network comprises eight points, with a direct link between each pair of points.

Work out the number of different tours.

(See question 10 for a description of a tour.)

PS ⑫ A five-digit number greater than 60 000 is to be made from these six cards.

1	4	8	2	7	6

Figure 1.13

Each card can be used only once.

(i) How many five-digit numbers greater than 60 000 are possible?

(ii) How many of these numbers are even?

PS (13) A network comprises n points with a direct link between each pair of points. Work out the number of different tours in terms of n.

(See question 10 for a description of a tour. Note: this would not be assessed in the Level 2 Further Maths exam.)

PS (14) (i) How many different ways can a team of 5 people be arranged.

(ii) How many different teams of 5 can be selected from a group of 13 people? (Note: the arrangement of each team is not important.)

REAL-WORLD CONTEXT

This topic is developed further in A-Level mathematics and is often referred to as combinatorics. It has many applications in the real world, not least in password security.

LEARNING OUTCOMES

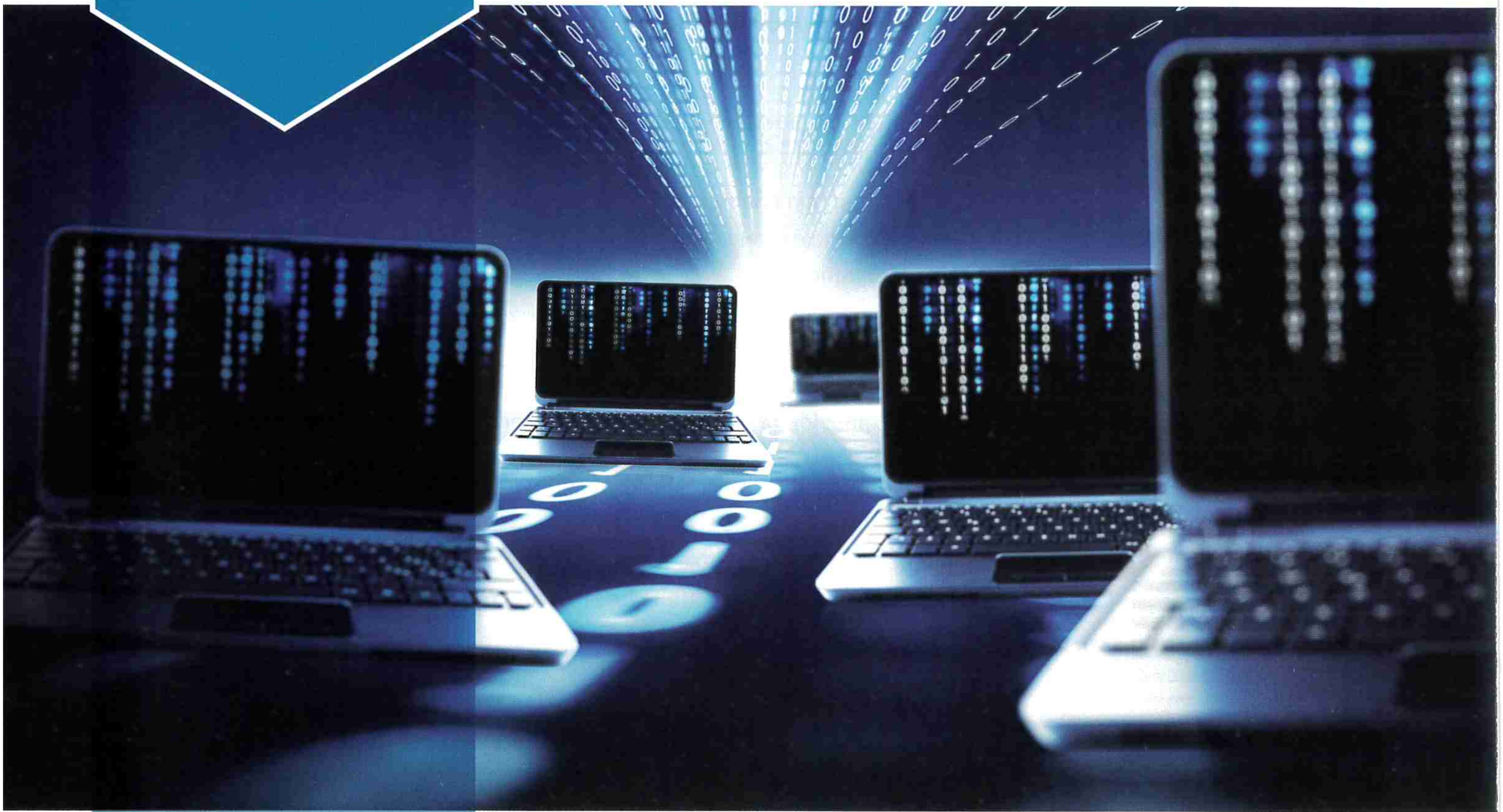
Now you have finished this chapter, you should be able to

- simplify an algebraic expression
- solve a linear equation
- solve percentage problems
- solve ratio problems
- work out the product of two (or more) algebraic expressions
- work out the expansion of an expression of the form $(a+b)^n$, where n is a positive integer
- manipulate expressions involving surds, including
  - simplifying surds
  - adding/subtracting compatible surds
  - rationalising a denominator of the form $\sqrt{a}$
  - rationalising a denominator of the form $a+\sqrt{b}$
  - rationalising a denominator of the form $\sqrt{a}+\sqrt{b}$
- efficiently find the number of combinations given a particular condition.

KEY POINTS

1. Simplify algebraic expressions by collecting like terms and/or expanding brackets.
2. Add/subtract algebraic expressions by rewriting with common denominators.
3. Simplify fractions by cancelling common factors in the numerator and denominator.
4. An expansion of $(a+b)^n$ comprises $n+1$ terms, each of the form Pa^qb^r where P is a number from Pascal's triangle, and $q+r=n$.
5. When simplifying surds
   - only like surds can be added or subtracted,
   - $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$.
6. A denominator can be rationalised as follows:
   - $\sqrt{a}$ can be rationalised by using the multiplier $\sqrt{a}$,
   - $a+\sqrt{b}$ can be rationalised by using the multiplier $a-\sqrt{b}$,
   - a two-term expression can be rationalised using a multiplier found by changing one sign.
7. When applying the product rule of counting, consider the number of options as each item is selected, and multiply them to find the number of possibilities.

2 Algebra II



If A equals success, then the formula is A equals x plus y plus z, with x being work, y play, and z keeping your mouth shut.
Albert Einstein

Prior knowledge

Students are expected to be able to identify a common factor of two or more terms. This may be just a number, a letter or both; see Chapter 1.

1 Factorising

Factorising an expression involves writing the expression as a product using brackets. Simple cases of this were seen in Chapter 1 Number and algebra 1. Here, pairs of brackets will be needed. If you have already learnt another method, and use it quickly and accurately, then you should stick with it. With practice, you may be able to factorise some of these expressions **by inspection**.

You will meet factorising again in Chapter 4.

! When factorising a quadratic with no constant term, only one bracket is required. For example, $2x^2 - 8x = 2x(x - 4)$.

Example 2.1

Factorise $xa + xb + ya + yb$.

Solution

First take out a common factor of each pair of terms.

$$\Rightarrow \quad xa + xb + ya + yb = x(a + b) + y(a + b)$$

Next notice that $(a + b)$ is now a common factor.

$$\Rightarrow \quad x(a + b) + y(a + b) = (a + b)(x + y)$$

In practice this can relate to areas of rectangles.

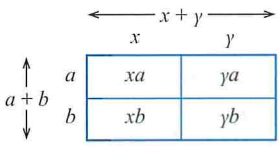


Figure 2.1

The idea illustrated in Figure 2.1 can be used to factorise a quadratic expression containing three terms, but first you must decide how to split up the term in x.

Example 2.2

Factorise $x^2 + 6x + 8$

Solution

Splitting the $6x$ as $4x + 2x$ gives

Why would you choose to split up $6x$ this way?

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + 4x + 2x + 8 \\ &= x(x + 4) + 2(x + 4) \\ &= (x + 4)(x + 2) \end{aligned}$$

Discussion point

→ Is the illustration in Figure 2.2 the only possibility?

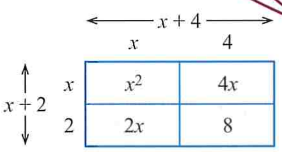


Figure 2.2

The crucial step is knowing how to split up the middle term.

To answer this question, notice that

- the numbers 4 and 2 have a sum of 6, which is the **coefficient** of x (i.e. the number multiplying x) in $x^2 + 6x + 8$
- the numbers 4 and 2 have a product of 8 which is the **constant term** in $x^2 + 6x + 8$

There is only one pair of numbers that satisfies both of these conditions.

Example 2.3

Factorise $x^2 - 7x - 18$

Solution

Pairs of numbers with a product of (-18) are:

1 and (-18)

2 and (-9)

3 and (-6)

6 and (-3)

9 and (-2)

18 and (-1)

Since the pair of numbers that you are looking for is unique, you can stop listing products when you find one that has the correct sum.

Discussion point

→ Do you get the same factors if the order in which you use the 2 and the (-9) is reversed so that you write it $x^2 - 9x + 2x - 18$?

There is only one pair, 2 and (−9), with a sum of (−7) so use these.

$$\begin{aligned} x^2 - 7x - 18 &= x^2 + 2x - 9x - 18 \\ &= x(x+2) - 9(x+2) \\ &= (x+2)(x-9) \end{aligned}$$

Notice the sign change due to the − sign in front of the 9.

Example 2.4

Factorise $x^2 - 16$

Solution

First write $x^2 - 16 = x^2 + 0x - 16$

Pairs of numbers with a product of (−16) are:

1 and (−16)

2 and (−8)

4 and (−4),…(stop here)

The only pair with a sum of 0 is 4 and (−4).

$$\begin{aligned} x^2 - 16 &= x^2 + 4x - 4x - 16 \\ &= x(x+4) - 4(x+4) \\ &= (x+4)(x-4) \end{aligned}$$

This is an example of a special case called **the difference of two squares** since you have $x^2 - 4^2 = (x+4)(x-4)$.

In general, $a^2 - b^2 = (a+b)(a-b)$.

Most people recognise this when it occurs and write down the answer straight away.

Example 2.5

Factorise $4x^2 - 9y^2$.

Solution

$$\begin{aligned} 4x^2 - 9y^2 &= (2x)^2 - (3y)^2 \\ &= (2x+3y)(2x-3y) \end{aligned}$$

Notice that this technique can be extended to any situation where the coefficients of the two terms are square numbers.

Example 2.6

Factorise fully $y^5 - 36y^3$.

Solution

The instruction to factorise fully tells you that there is likely to be more than one step involved.

Take out the highest common factor of the two terms $y^5 - 36y^3 = y^3(y^2 - 36)$.

Use the difference of two squares $y^3(y^2 - 36) = y^3(y+6)(y-6)$.

The technique for finding how to split the middle term needs modifying for examples where the expression starts with a multiple of x^2. The difference is that you now multiply the two outside numbers together to give the product you want.

Example 2.7

Factorise $2x^2 - 11x + 15$

A negative sum and a positive product means that both numbers are negative.

Solution

Here the sum is (-11) and the product is $2 \times 15 = 30$

Options to give the correct product are:

(-1) and (-30) $\quad$ (-5) and (-6)

(-2) and (-15)

(-3) and (-10)

(−5) and (−6), in either order, are the only options giving a sum of (−11).

$$\begin{aligned} 2x^2 - 11x + 15 &= 2x^2 - 5x - 6x + 15 \\ &= x(2x - 5) - 3(2x - 5) \\ &= (x - 3)(2x - 5) \end{aligned}$$

Discussion point

➜ Try this example writing $2x^2 - 11x + 15$ as $2x^2 - 6x - 5x + 15$

Example 2.8

Factorise $3x^2 - 10xy - 8y^2$.

Solution

This expression can be factorised using the same method used in the previous example.

Here the sum is (-10) and the product is $3 \times -8 = -24$

A negative product means that one number is positive and the other is negative.

Option needed is (-12) and 2

$$\begin{aligned} 3x^2 - 10xy - 8y^2 &= 3x^2 - 12xy + 2xy - 8y^2 \\ &= 3x(x - 4y) + 2y(x - 4y) \\ &= (3x + 2y)(x - 4y) \end{aligned}$$

Example 2.9

Factorise $(x + 3)^2 - 4y^2$.

Solution

This example uses the difference of two squares.

Writing the expression as $(x + 3)^2 - (2y)^2$ and factorising gives

$$[(x + 3) + (2y)]\,[(x + 3) - 2y]$$
$$= (x + 3 + 2y)(x + 3 - 2y)$$

Exercise 2A

① Factorise the following expressions.

(i) $ab - ac + db - dc$ (ii) $2xy + 2x + wy + w$
(iii) $2pq - 8p - 3rq + 12r$ (iv) $5 - 5m - 2n + 2nm$

② Factorise the following expressions.

(i) $x^2 + 5x + 6$ (ii) $y^2 - 5y + 4$ (iii) $m^2 - 8m + 16$
(iv) $m^2 - 8m + 15$ (v) $x^2 + 3x - 10$ (vi) $a^2 + 20a + 96$
(vii) $x^2 - x - 6$ (viii) $y^2 - 16y + 48$ (ix) $k^2 + 10k + 24$
(x) $k^2 - 10k - 24$

③ Each of these is a difference of two squares. Factorise them.

(i) $x^2 - 4$ (ii) $a^2 - 25$ (iii) $9 - p^2$
(iv) $x^2 - y^2$ (v) $t^2 - 64$ (vi) $4x^2 - 1$
(vii) $4x^2 - 9$ (viii) $4x^2 - y^2$ (ix) $16x^2 - 25$
(x) $9a^2 - 4b^2$

④ Factorise the following expressions.

(i) $2x^2 + 5x + 2$ (ii) $2a^2 + 11a - 21$ (iii) $15p^2 + 2p - 1$
(iv) $3x^2 + 8x - 3$ (v) $5a^2 - 9a - 2$ (vi) $2p^2 + 5p - 3$
(vii) $8x^2 + 10x - 3$ (viii) $2a^2 - 3a - 27$ (ix) $9x^2 - 30x + 25$
(x) $4x^2 + 4x - 15$

Example 2.8 shows you how to factorise an expression where there are two variables.

⑤ Factorise the following expressions.

(i) $x^2 + 3xy + 2y^2$ (ii) $x^2 + 4xy - 5y^2$ (iii) $a^2 - ab - 12b^2$
(iv) $c^2 - 11cd + 24d^2$ (v) $x^2 + 9xy + 20y^2$ (vi) $p^2 + 2pr - 15r^2$
(vii) $a^2 - 2ar - 15r^2$ (viii) $s^2 - 4st + 4t^2$ (ix) $m^2 - 5mn - 6n^2$
(x) $r^2 + 2rs - 8s^2$

⑥ Factorise the following expressions. (This question extends factorising using the difference of two squares as in Example 2.9.)

(i) $(2a + 1)^2 - a^2$ (ii) $(3x + 1)^2 - (x + 4)^2$
(iii) $(2p - 3)^2 - (p + 1)^2$ (iv) $16 - (5y - 2)^2$
(v) $(2a + 1)^2 - a^2$ (vi) $(3x + 1)^2 - (x + 4)^2$
(vii) $(2p - 3)^2 - (p + 1)^2$ (viii) $9 - (2y - 3)^2$

⑦ Factorise the following expressions.

(i) $2x^2 + 5xy + 2y^2$ (ii) $3x^2 + 5xy - 2y^2$
(iii) $5a^2 - 8ab + 3b^2$ (iv) $6c^2 + 5cd - 4d^2$
(v) $6p^2 - 37pq + 6q^2$ (vi) $7g^2 + 5gh - 2h^2$
(vii) $6h^2 - 5hk - 4k^2$ (viii) $8w^2 - 6wx + x^2$

⑧ Factorise fully the following expressions.

(i) $x^3 - 4x$ (ii) $a^4 - 16a^2$
(iii) $9y^3 - y^5$ (iv) $2x^3 - 2x$
(v) $4p^4 - 9p^2$ (vi) $100x - x^3$
(vii) $18c^3 - 2c$ (viii) $8x^3 - 50xy^2$

ACTIVITY 2.1

(i) Work out 9^2 and $(a^2)^2$.

Remember that $(a^p)^q = a^{pq}$.

(ii) Show that $a^4 - 81$ is the difference of two squares.
(iii) Factorise fully $a^4 - 81$

ACTIVITY 2.2

(i) Factorise $10x^2 + 11x + 3$

(ii) Factorise $10(p+q)^2 + 11(p+q) + 3$

2 Rearranging formulae

C is called the **subject** of the formula.

The circumference of a circle is given by $C = 2\pi r$ where r is the radius. An equation such as this is often called a formula.

In some cases, you want to calculate r directly from C. You want r to be the subject of the formula.

Example 2.10

Make r the subject of $C = 2\pi r$.

! Notice how the new subject should be on its own on the left-hand side of the new formula and must not appear on the right-hand side.

Solution

Divide both sides by $2\pi \Rightarrow \dfrac{C}{2\pi} = r$

$\Rightarrow r = \dfrac{C}{2\pi}$

Example 2.11

Make x the subject of this formula.

$h = \sqrt{(x^2 + y^2)}$

A square root is assumed to be positive unless $\pm$ is added in front of it.

Discussion point

→ What would you do with the $\pm$ sign in the case where h is the hypotenuse of a right-angled triangle with x and y as the other two sides?

Solution

Square both sides $\Rightarrow$ $h^2 = x^2 + y^2$

Subtract y^2 from both sides $\Rightarrow$ $h^2 - y^2 = x^2$

Make the x^2 term the subject $\Rightarrow$ $x^2 = h^2 - y^2$

Take the square root of both sides $\Rightarrow$ $x = \pm\sqrt{(h^2 - y^2)}$

Example 2.12

Make a the subject of this formula.

$v = u + at$

Solution

Subtract u from both sides $\Rightarrow$ $v - u = at$

Divide both sides by t $\Rightarrow$ $\dfrac{v-u}{t} = a$

Write the answer with a on the left-hand side $\Rightarrow$ $a = \dfrac{v-u}{t}$

Exercise 2B

In this exercise all the equations refer to formulae used in mathematics. How many of them do you recognise?

① Make

(i) u (ii) t

the subject of $v = u + at$.

② Make b the subject of $A = \frac{1}{2}bh$.

③ Make l the subject of $P = 2(l + b)$.

④ Make r the subject of $A = \pi r^2$.

⑤ Make c the subject of $A = \frac{1}{2}(b + c)h$.

⑥ Make h the subject of $A = \pi r^2 + 2\pi rh$.

⑦ Make l the subject of $T = \frac{\lambda e}{l}$.

⑧ Make

(i) u (ii) a

the subject of $s = ut + \frac{1}{2}at^2$.

⑨ Make x the subject of $v^2 = \omega^2(a^2 - x^2)$.

The following examples show how to rearrange a formula when the letter that is to be the subject appears more than once.

Example 2.13

Make t the subject of this formula.

$at = 3(t + 2)$

Solution

Expand the brackets $\Rightarrow$ $at = 3t + 6$

Collect all the terms in t on one side $\Rightarrow$ $at - 3t = 6$

Factorise $\Rightarrow$ $t(a - 3) = 6$

Divide both sides by $(a - 3)$ $\Rightarrow$ $t = \frac{6}{a - 3}$

The brackets are not needed in the denominator.

Example 2.14

Make x the subject of this formula.

$$y = \frac{x + 2}{1 + 3x}$$

Solution

Multiply both sides by $(1 + 3x)$ $\Rightarrow$ $y(1 + 3x) = x + 2$

Expand the brackets $\Rightarrow$ $y + 3xy = x + 2$

Collect all the terms in x on one side and all the other terms on the other side $\Rightarrow$ $3xy - x = 2 - y$

Factorise $\Rightarrow$ $x(3y - 1) = 2 - y$

Divide both sides by $(3y - 1)$ $\Rightarrow$ $x = \frac{2 - y}{3y - 1}$

Exercise 2C

1. Make m the subject of $3m = x(m + 2)$.
2. Make y the subject of $5y - 2x = xy$.
3. Make b the subject of $4(a + b) = 3(a - b)$.
4. Make h the subject of $S = 2\pi r^2 + 2\pi rh$.
5. Make x the subject of $y = \dfrac{x+1}{2+x}$.
6. Make c the subject of $d(2 + c) = 1 - 3c$.
7. (i) Make t the subject of $x = \dfrac{t}{t-3}$.

   (ii) Hence, or otherwise, work out the value of t when $x = 3$
8. (i) Make p the subject of $r = \dfrac{3p+2}{2p+3}$.

   (ii) Hence, or otherwise, work out the value of p when $r = -1$

ACTIVITY 2.3

(i) (a) Show that $(x + 3)^2 = x^2 + 6x + 9$

(b) Hence make x the subject of $y = x^2 + 6x + 9$

(ii) (a) Show that $(x - 5)^2 + 4 = x^2 - 10x + 29$

(b) Hence make x the subject of $p = x^2 - 10x + 29$

3 Simplifying algebraic fractions

Prior knowledge

Students should aim to be able to cancel fractions, find the least common multiple of two numbers and be able to identify a least common denominator of two or more fractions.

Discussion points

→ What is a fraction in arithmetic?

→ What about in algebra?

Fractions in algebra obey the same rules as fractions in arithmetic.

These cover two pairs of operations: × and ÷, and + and −.

Discussion points

→ When can you cancel fractions in arithmetic?

→ What about in algebra?

→ What is a factor in arithmetic?

→ What about in algebra?

Example 2.15

Simplify the following.

(i) $\dfrac{18}{24}$

(ii) $\dfrac{2x+2}{3x+3}$

(iii) $\dfrac{a^2-a-6}{a^2-8a+15}$

Discussion points

➜ Look at this calculation for (ii).

$$\frac{2\cancel{x}+2}{3\cancel{x}+3}=\frac{4}{6}=\frac{2}{3}$$

Why is it wrong?

➜ Look at this calculation for (iii).

$$\frac{{}^{1}\cancel{a^2}-\cancel{a}^{1}-6}{{}_{1}\cancel{a^2}-8\cancel{a}_{1}+15}$$

$$=-\frac{6}{8}=-\frac{3}{4}$$

Why is it wrong?

Solution

(i) $\dfrac{18}{24}=\dfrac{{}^{1}\cancel{6}\times 3}{{}_{1}\cancel{6}\times 4}=\dfrac{3}{4}$

(ii) $\dfrac{2x+2}{3x+3}=\dfrac{2\,\cancel{(x+1)}^{1}}{3\cancel{(x+1)}_{1}}=\dfrac{2}{3}$

(iii) $\dfrac{a^2-a-6}{a^2-8a+15}=\dfrac{\cancel{(a-3)}^{1}(a+2)}{\cancel{(a-3)}_{1}(a-5)}=\dfrac{a+2}{a-5}$

Example 2.16

Simplify the following.

(i) $\dfrac{2}{3}\times\dfrac{9}{14}$

(ii) $\dfrac{3}{4}\div\dfrac{9}{16}$

(iii) $\dfrac{3a^2b}{2c}\times\dfrac{4c^3}{9ab}$

(iv) $\dfrac{4n^2-9}{n+1}\div\dfrac{2n+3}{n^2-1}$

Discussion point

➜ Look at this calculation for (iv).

$$\frac{\cancel{4n^2}^{2n}-\cancel{9}^{3}}{\cancel{n}_{1}+1}\times\frac{\cancel{n^2}^{n}-1}{\cancel{2n}_{1}+\cancel{3}_{1}}$$

$$=\frac{(2n-3)(n-1)}{4}$$

Why is it wrong?

Solution

(i) $\dfrac{{}^{1}\cancel{2}}{{}_{1}\cancel{3}}\times\dfrac{\cancel{9}^{3}}{\cancel{14}_{7}}=\dfrac{1\times 3}{1\times 7}=\dfrac{3}{7}$

(ii) $\dfrac{3}{4}\div\dfrac{9}{16}=\dfrac{\cancel{3}^{1}}{\cancel{4}_{1}}\times\dfrac{\cancel{16}^{4}}{\cancel{9}_{3}}=\dfrac{4}{3}$

(iii) $\dfrac{{}^{1}\cancel{3}a^2\cancel{b}}{{}_{1}\cancel{2}\cancel{c}}\times\dfrac{{}_{2}\cancel{4}c^{\cancel{3}2}}{{}_{3}\cancel{9}\cancel{a}b}=\dfrac{2ac^2}{3}$

(iv) $\dfrac{4n^2-9}{n+1}\div\dfrac{2n+3}{n^2-1}=\dfrac{\cancel{(2n+3)}^{1}(2n-3)}{\cancel{(n+1)}_{1}}\times\dfrac{\cancel{(n+1)}^{1}(n-1)}{\cancel{(2n+3)}_{1}}$

$=(2n-3)(n-1)$

Example 2.17

Simplify the following.

(i) $\dfrac{2}{3}+\dfrac{3}{4}$

(ii) $\dfrac{5x}{6}+\dfrac{x}{4}$

(iii) $\dfrac{2}{(x+1)}+\dfrac{5}{(x-1)}$

(iv) $\dfrac{a}{a^2-1}-\dfrac{2}{a+1}$

Solution

(i) $\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$

(ii) $\frac{5x}{6}+\frac{x}{4}=\frac{10x}{12}+\frac{3x}{12}=\frac{13x}{12}$

Take care to ensure that the common denominator is the least common multiple of the original denominators.

(iii)
$$\begin{aligned}\frac{2}{(x+1)}+\frac{5}{(x-1)} &= \frac{2(x-1)}{(x+1)(x-1)}+\frac{5(x+1)}{(x+1)(x-1)}\\ &= \frac{2x-2+5x+5}{(x+1)(x-1)}\\ &= \frac{7x+3}{(x+1)(x-1)}\end{aligned}$$

(iv)
$$\begin{aligned}\frac{a}{a^2-1}-\frac{2}{a+1} &= \frac{a}{(a-1)(a+1)}-\frac{2}{a+1}\\ &= \frac{a}{(a-1)(a+1)}-\frac{2(a-1)}{(a-1)(a+1)}\\ &= \frac{a-2a+2}{(a-1)(a+1)}\\ &= \frac{2-a}{(a-1)(a+1)}\end{aligned}$$

Exercise 2D

1. Simplify the following.

   (i) $\frac{2(x+3)}{4x+12}$ (ii) $\frac{4x-8}{(x-2)(x+8)}$ (iii) $\frac{3(x+y)}{x^2-y^2}$

   (iv) $\frac{6x^2y^3}{9xy^4}$ (v) $\frac{2p}{6p-2p^2}$ (vi) $\frac{4ab^3}{10a^3b}$

2. Simplify the following.

   (i) $\frac{x^2-4x+3}{2x-6}$ (ii) $\frac{x^2+xy}{x^2-y^2}$ (iii) $\frac{a+2}{a^2-a-6}$

   (iv) $\frac{3x^2+15x}{10x+2x^2}$ (v) $\frac{9x^2-1}{9x+3}$ (vi) $\frac{3x^2+3xy}{6xy+6y^2}$

3. Simplify the following.

   (i) $\frac{3a}{b^2}\times\frac{b^3}{6a}$ (ii) $\frac{xy-y^2}{y}\times\frac{x}{x-y}$

   (iii) $\frac{x^2-y^2}{y}\times\frac{x}{x-y}$ (iv) $\frac{x+1}{2x}\div\frac{4x^2-4}{x^2}$

   (v) $\frac{3a^2+a-2}{2}\div\frac{6a^2-a-2}{8a+4}$ (vi) $\frac{2p^2-pq-q^2}{3p+3q}\div\frac{2p^2-3pq+q^2}{2p+2q}$

4. Simplify the following.

   (i) $\frac{x^2-4x+4}{x^2-2x}\times\frac{x-2}{x^2-4}$ (ii) $\frac{2x-1}{x+1}\div\frac{2x^2-x-1}{x^2+3x+2}$

   (iii) $\frac{4p^2+12}{p-3}\times\frac{p^2-9}{p^2+3}$ (iv) $\frac{3x^2-9}{x+2}\div\frac{x^2-6x+9}{x^2+x-2}$

   (v) $\frac{3a^2-12}{5a^2-4a-1}\times\frac{5a+1}{(a-2)^2}$ (vi) $\frac{2t}{t^2+1}\div\frac{4t^2}{t^4-1}$

5 Simplify the following.

(i) $\frac{3a}{5} - \frac{a}{4}$ (ii) $\frac{5}{3a} - \frac{4}{a}$

(iii) $\frac{2}{(m+n)} - \frac{1}{(m-n)}$ (iv) $\frac{4}{p-2} - \frac{3}{2p+1}$

(v) $\frac{1}{2(x-1)} + \frac{2}{(x+4)}$ (vi) $\frac{1}{2(a-1)} + \frac{2}{3(a+4)}$

6 Simplify the following.

(i) $\frac{2}{a^2+a} + \frac{3}{a^2-a}$ (ii) $\frac{2x}{x-y} + \frac{2y}{y-x}$

(iii) $\frac{p}{p^2-1} - \frac{1}{p+1}$ (iv) $\frac{a-b}{a+b} + \frac{a+b}{a-b}$

(v) $\frac{4}{x^2-4} - \frac{3}{x+2}$ (vi) $\frac{7}{5(x-2)} - \frac{2}{x+4}$

7 Simplify the following.

(i) $\frac{1}{x+1} - \frac{2}{x+2} + \frac{3}{x+3}$ (ii) $\frac{3}{x+1} - \frac{2}{x-2} + \frac{4}{x+3}$

(iii) $\frac{x+2}{(x+1)^2} - \frac{1}{x}$

8 Simplify the following.

(i) $\frac{4t}{t^2+2t+1} + \frac{3}{t+1}$ (ii) $\frac{1}{y^2-x^2} + \frac{3}{y+x}$

(iii) $1 + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$

4 Solving linear equations involving fractions

Prior knowledge

Students are expected to be familiar with the basic rules for the mathematical operations +, −, × and ÷ which are used in Chapter 1.

Example 2.18

Solve the following.

$$\frac{x+2}{6} = \frac{x-6}{2}$$

Solution

The LCM of 6 and 2 is 6, so multiply by 6.

$$^{1}\cancel{6} \times \frac{(x+2)}{\cancel{6}_{1}} = {}^{3}\cancel{6} \times \frac{(x-6)}{\cancel{2}_{1}}$$

$$\begin{aligned} \Rightarrow \quad x + 2 &= 3x - 18 \\ \Rightarrow \quad 20 &= 2x \\ \Rightarrow \quad x &= 10 \end{aligned}$$

Discussion point

→ When you multiply a fraction by an integer, you only multiply its numerator (top line). Why?

Example 2.19

Solve the following.

$$\frac{x+2}{6} + 3 = \frac{x}{5}$$

Discussion point

→ Look at this version of the first stage of the solution for Example 2.19.

$$30 \times \frac{(x+2)}{6} + 3 = 30 \times \frac{x}{5}$$

Why is it wrong?

Solution

The LCM of 6 and 5 is 30, so multiply by 30.

$$\cancel{30}^{5} \times \frac{(x+2)}{\cancel{6}_1} + 30 \times 3 = \cancel{30}^{6} \times \frac{x}{\cancel{5}_1}$$

$$\Rightarrow \quad 5x + 10 + 90 = 6x$$

$$\Rightarrow \quad x = 100$$

Exercise 2E

Solve the following equations.

① $x - \frac{x}{5} = \frac{2}{3}$

② $\frac{2}{a} - \frac{3}{4a} = 2$

③ $\frac{x-4}{6} = \frac{x+2}{3}$

④ $\frac{2-3x}{6} = \frac{2}{3}$

⑤ $\frac{3p+2}{2} - \frac{p-1}{5} = 3$

⑥ $\frac{3(x-2)}{2} - \frac{x-5}{4} = 2$

⑦ $x + 1 - \frac{3(x-2)}{2} = 7$

⑧ $\frac{3(t+4)}{8} + 2 = \frac{2t}{3}$

5 Completing the square

When considering a quadratic expression it will sometimes be useful to write it to include the term $(x + a)^2$ or $(x - a)^2$, where a is a constant. Some uses of this approach will be seen later in sections on quadratic equations and quadratic graphs.

You will meet completing the square again in Chapter 4.

Example 2.20

Work out the values of p and q such that $x^2 - 6x + 2 = (x - p)^2 + q$.

Solution

Expand the bracket $\quad x^2 - 6x + 2 = x^2 - 2px + p^2 + q$.

$$(x-p)^2 = (x-p)(x-p)$$
$$= x^2 - px - px + p^2$$
$$= x^2 - 2px + p^2$$

Equate coefficients of x means making equal the number of x on each side of the identity.

Equate coefficients of x $\quad -6 = -2p$

$$3 = p$$

Equate constants $\quad 2 = p^2 + q$

$$2 = 9 + q$$

$$-7 = q$$

$$p = 3 \text{ and } q = -7$$

Example 2.21

Work out the values of a, b and c such that $2x^2 + bx + 5 = a(x - 3)^2 + c$.

Solution

Expand the bracket $\quad 2x^2 + bx + 5 = a(x^2 - 6x + 9) + c$

$$= ax^2 - 6ax + 9a + c$$

Equate coefficients of x^2 $\quad 2 = a$

Equate coefficients of x $\quad b = -6a$

$$b = -12$$

Equate constants $\quad 5 = 9a + c$

$$5 = 18 + c$$

$$-13 = c$$

$$a = 2,\ b = -12 \text{ and } c = -13$$

Example 2.22

Work out the values of a, b and c such that $3x^2 + 5x - 1 = a(x + b)^2 + c$.

Solution

Expand the bracket $\quad 3x^2 + 5x - 1 = a(x^2 + 2bx + b^2) + c$

$$= ax^2 + 2abx + ab^2 + c$$

Equate coefficients of x^2 $\quad 3 = a$

Equate coefficients of x $\quad 5 = 2ab$

$$5 = 6b$$

$$\frac{5}{6} = b$$

Equate constants $\quad -1 = ab^2 + c$

$$-1 = 3 \times \left(\frac{5}{6}\right)^2 + c$$

$$-1 = 3 \times \frac{25}{36} + c$$

$$-1 = \frac{25}{12} + c$$

$$-\frac{37}{12} = c$$

$$a = 3,\ b = \frac{5}{6} \text{ and } c = -\frac{37}{12}$$

Note

Comparing coefficients is a useful technique which can be applied to any polynomial.

An alternative method when rewriting a quadratic in the form $a(x + b)^2 + c$, is to use the technique given in Chapter 4.

Exercise 2F

① Work out the values of a and b such that $x^2 + 8x + 10 = (x + a)^2 + b$.

② Work out the values of c and d such that $x^2 - cx + 7 = (x - 1)^2 + d$.

③ Work out the values of p and q such that $x^2 - 12x - 4 = (x - p)^2 + q$.

④ Work out the values of a and b such that $x^2 + 5x - 2 = (x + a)^2 + b$.

⑤ Work out the values of p and q such that $5 + 4x - x^2 = p - (x - q)^2$.

⑥ Work out the values of c and d such that $2 - x - x^2 = c - (x + d)^2$.

⑦ Work out the values of a, b and c such that $2x^2 + bx + 5 = a(x + 2)^2 + c$.

⑧ Work out the values of a, b and c such that $5x^2 + 30x + 10 = a(x + b)^2 + c$.

⑨ Work out the values of p, q and r such that $3x^2 - 12x + 14 = p(x + q)^2 + r$.

⑩ Work out the values of a, b and c such that $3x^2 - bx + 1 = a(x - 4)^2 + c$.

⑪ Work out the values of a, b and c such that $6 + bx - 2x^2 = c - a(x - 1)^2$.

⑫ Work out the values of p, q and r such that $5 - 12x - 2x^2 = p - q(x + r)^2$.

⑬ (i) Work out the values of a and b such that $x^2 - 8x + 20 = (x - a)^2 + b$.

(ii) Hence make x the subject of $y = x^2 - 8x + 20$

⑭ (i) Work out the values of p, q and r such that $3x^2 + 6x + 1 = p(x + q)^2 + r$.

(ii) Hence make x the subject of $y = 3x^2 + 6x + 1$

FUTURE USES

The ability to manipulate algebraic expressions and solve equations is fundamental to much of Pure Mathematics. Factorisation and completing the square will be revisited in Chapter 4 where you will be solving quadratic equations and inequalities.

You will use the skills learnt in this chapter extensively throughout this book. If you choose to study Mathematics at a higher level you will find the techniques introduced here invaluable.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- factorise an algebraic expression using no more than two brackets
- rearrange a formula to make a different letter the subject
  - when the new subject occurs once
  - when the new subject appears more than once
- simplify algebraic fractions connected by any of the symbols +, −, ×, or ÷
- solve linear equations containing algebraic fractions
- write a quadratic expression in the form $a(x + b)^2 + c$.

KEY POINTS

1 When factorising a quadratic expression you need to write it as a product using brackets.

2 When changing the subject of an equation the new subject should be on its own on the left-hand side.

3 When simplifying an algebraic fraction involving addition or subtraction you need to find a common denominator.

4 When solving an equation involving fractions you start by multiplying through by the least common multiple of all the denominators to eliminate the fractions.

5 Quadratic expressions can be written in the form $a(x + b)^2 + c$.

3 Algebra III

Others have done it before me. I can, too.

Corporal John Faunce (American soldier)

1 Function notation

Here is a flow chart.

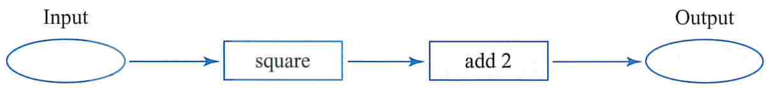


Figure 3.1

For an input of 5, $5 \rightarrow 25 \rightarrow 27$ the output is 27

For an input of -2, $-2 \rightarrow 4 \rightarrow 6$ the output is 6

For an input of x, $x \rightarrow x^2 \rightarrow x^2 + 2$ the output is $x^2 + 2$

This leads to the use of function notation $f(x) = x^2 + 2$

For an input of 5,

$$\begin{aligned} f(5) &= 5^2 + 2 \\ &= 25 + 2 \\ &= 27 \end{aligned}$$

For an input of -2,

$$\begin{aligned} f(-2) &= (-2)^2 + 2 \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

A **function** must have a unique output for every input.

Consequently $f(x) = \pm x^2$ is not a function.

Discussion point

➜ Which of the following are functions?

(i) $f(x) = (\pm x)^2$

(ii) $f(x) = (1 \pm x)^2$

Example 3.1

$f(x) = 10 - 4x$ and $g(x) = x^3$

(i) Evaluate $f(-1)$ and $g\left(\frac{1}{2}\right)$.

(ii) Write down an expression for $f(3x)$.

(iii) Solve $g(x) = -64$

Solution

(i)
$$f(-1) = 10 - 4(-1) = 10 + 4 = 14$$
$$g\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

(ii)
$$f(3x) = 10 - 4(3x) = 10 - 12x$$

(iii)
$$g(x) = -64$$
$$x^3 = -64$$
$$x = \sqrt[3]{-64} = -4$$

Exercise 3A

① $f(x) = 2x - 1$ and $g(x) = x^2 + 2x$

Work out the value of

(i) $f(-4)$ (ii) $f(0.6)$ (iii) $g(3)$

(iv) $g(-1)$ (v) $f(0)$ (vi) $g(0)$.

② $f(x) = 3x^2$ and $g(x) = \frac{6}{x}$

Work out the value of

(i) $f(2)$ (ii) $f(-5)$ (iii) $g(2)$

(iv) $g(-1.5)$ (v) $g\left(\frac{1}{2}\right)$ (vi) $g\left(-\frac{2}{3}\right)$.

③ $f(x) = (2x - 1)^2$ and $g(x) = 2x + 1$. Work out the value of

(i) $f(0)$ (ii) $g(-2)$ (iii) $f(0.5)$

(iv) $f\left(-\frac{1}{4}\right)$ (v) $g\left(-\frac{1}{2}\right)$ (vi) $g(1.6)$.

④ $f(x) = 8 - 3x$ and $g(x) = 4(x + 3)$. Solve

(i) $f(x) = 0$ (ii) $g(x) = 20$ (iii) $f(x) = g(x)$.

⑤ $h(x) = 3x - 2$

Write down expressions, giving answers in the simplest form, for

(i) $h(2x)$ (ii) $h(x + 1)$ (iii) $h(x^2)$.

⑥ $f(x) = (x - 1)^2$

Write down expressions, giving answers in the simplest form, for

(i) $f(x^2)$ (ii) $[f(x)]^2$ (iii) $(f(x + 1))^2$.

⑦ $f(x) = x^2 + 5x - 1$

Write down expressions, giving answers in the simplest form, for

(i) $f(3x)$ (ii) $f(x - 2)$.

⑧ $g(x) = \dfrac{x + 6}{2x}$

(i) Work out the value of $g(3)$. (ii) Solve $g(x) = 3$.

(iii) Solve $g(2x) = 1$.

2 Domain and range of a function

Discussion point

➜ Why is it not possible for $x = 0$ to be in the domain of $g(x) = \dfrac{1}{x}$?

! Take care to check whether the domain uses the symbols $>$ and $<$ or $\geqslant$ and $\leqslant$. If the domain includes the option of equality then so must the range. If the domain excludes a certain point or points, then look for exclusions in the range.

Domain of f(x)

The set of input values is the domain of $f(x)$.

When a function is defined it will include a domain. If a domain is not stated it can be assumed that the domain is all real values of x.

$f(x) = 2x - 3$	$x > 1$	The domain is $x > 1$
$g(x) = \dfrac{1}{x}$	$x \neq 0$	The domain is all real values of x apart from 0
$h(x) = x^2$		The domain is all real values of x.

Range of f(x)

The set of output values is the range of $f(x)$.

The range is usually dependent upon the domain (an exception is $f(x) = 2$).

The range is given as a set of $f(x)$ values as shown in the following examples.

Example 3.2

$f(x) = x^2$ for all real values of x.

Write down the range of $f(x)$.

Solution

The value of x^2 is positive or zero for all real values of x.

Range is $f(x) \geqslant 0$.

Note

Domain and range can be seen on a sketch graph of $f(x) = x^2$.

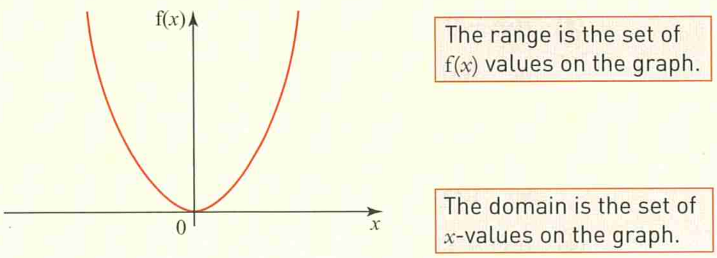


Figure 3.2

ACTIVITY 3.1

Sketch the graph of $g(x) = \dfrac{1}{x}$ for $x \neq 0$. Use the graph to identify the range of $g(x)$.

Example 3.3

$f(x) = 6 - 4x \qquad -2 \leqslant x \leqslant 3$

Write down the range of $f(x)$.

Solution

A sketch of $f(x) = 6 - 4x$ for the given domain is

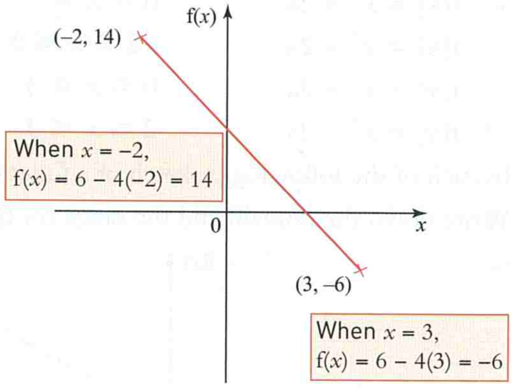


Figure 3.3

Range is $-6 \leqslant f(x) \leqslant 14$.

Exercise 3B

① Write down the range of $f(x)$ in each of the following.

(i) $f(x) = 3x \qquad x < 2$

(ii) $f(x) = x + 4 \qquad x \geqslant 1$

(iii) $f(x) = 2x + 4 \qquad x \geqslant -1$

(iv) $f(x) = 10 - x \qquad x \leqslant 4$

② Write down the range of $f(x)$ in each of the following.

(i) $f(x) = 2x \qquad 1 \leqslant x \leqslant 5$

(ii) $f(x) = x - 3 \qquad 0 < x < 10$

(iii) $f(x) = 5 - 2x \qquad x \geqslant -3$

(iv) $f(x) = 3 - 4x \qquad -2 \leqslant x < 3$

③ Write down the range of $f(x)$ in each of the following.

(i) $f(x) = \dfrac{x + 5}{2} \qquad 0 \leqslant x \leqslant 5$

(ii) $f(x) = \dfrac{2x - 3}{4} \qquad -2 \leqslant x \leqslant 2$

(iii) $f(x) = \dfrac{3 - 2x}{3} \qquad -3 \leqslant x \leqslant 5$

(iv) $f(x) = \dfrac{1 - 3x}{2} \qquad -3 \leqslant x \leqslant 5$

④ Write down the range of $f(x)$ in each of the following.

(i) $f(x) = x^2 \qquad -2 \leqslant x < 2$

(ii) $f(x) = x^2 \qquad 0 < x < 4$

(iii) $f(x) = x^3 \qquad x \geqslant 0$

(iv) $f(x) = x^3 \qquad -1 \leqslant x \leqslant 3$

⑤ Write down the range of f(x) in each of the following.

(i) $f(x) = 2x^2 - 3$ $\quad 0 \leqslant x \leqslant 4$

(ii) $f(x) = 3x^2 - 2$ $\quad 0 \leqslant x \leqslant 4$

(iii) $f(x) = 3 - 2x^2$ $\quad -1 \leqslant x \leqslant 2$

(iv) $f(x) = 2 - 3x^2$ $\quad -1 \leqslant x \leqslant 2$

⑥ Write down the range of f(x) in each of the following.

(i) $f(x) = x^2 + 2x$ $\quad 0 \leqslant x \leqslant 3$

(ii) $f(x) = x^2 + 2x$ $\quad -2 \leqslant x \leqslant 3$

(iii) $f(x) = x^2 - 2x$ $\quad 0 \leqslant x \leqslant 3$

(iv) $f(x) = x^2 - 2x$ $\quad 2 \leqslant x \leqslant 3$

⑦ In each of the following, a sketch of a function, f(x), is shown. Write down the domain and the range for f(x).

(i)

(ii)

(iii)

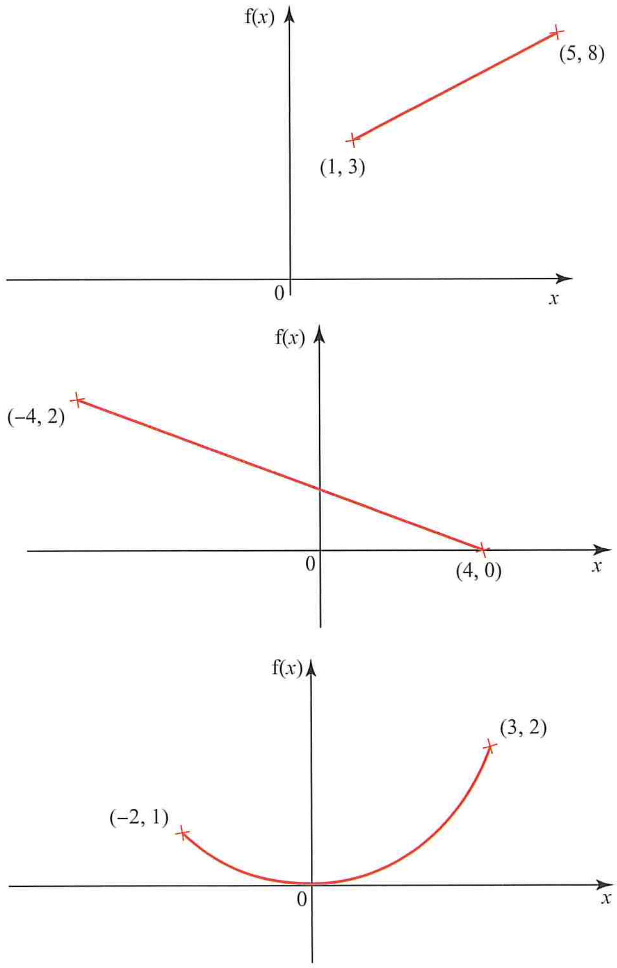


Figure 3.4

⑧ Sketch the graph of each of the following functions and write down the corresponding range.

(i) $f(x) = 3x - 2$ for $-1 \leqslant x \leqslant 3$

(ii) $f(x) = 2 - 3x$ for $-1 \leqslant x \leqslant 2$

(iii) $f(x) = x^2 + 3$ for $-2 \leqslant x \leqslant 2$

(iv) $f(x) = 4 - x^2$ for $-2 \leqslant x \leqslant 3$

> ❗ Unlike when adding or multiplying two functions, the order in which you apply the functions is critical.

3 Composite functions

A **composite function** occurs when two or more functions act in succession.

$fg(x)$ means that the function next to x, which is g in this case, is the one that is applied first. Using $f(x) = x^2$ and $g(x) = x + 2$, it is often easier to think of this in words initially, rather than symbols.

$fg(x) = f[g(x)]$ which means that you first apply the function g to x.
$g(x) = (x + 2)$ so you now have $f(x + 2)$.

In this example the function f tells you to square what you have in the brackets, giving $(x + 2)^2$ so $fg(x) = (x + 2)^2$.

Similarly $gf(x) = x^2 + 2$

Discussion point

→ Using the functions above, are there any values of x for which $fg(x) = gf(x)$?

Combining functions using addition, multiplication or division

Using the functions above, if $f(x) = x^2$ and $g(x) = x + 2$, then

$f(x) + g(x) = x^2 + (x + 2) = x^2 + x + 2$

$f(x)g(x) = f(x) \times g(x) = x^2(x + 2) = x^3 + 2x^2$

$\dfrac{f(x)}{g(x)} = f(x) \div g(x) = \dfrac{x^2}{x + 2}.$

Example 3.4

Express $(5x + 6)^4$ in the form $fg(x)$, stating the expressions corresponding to $f(x)$ and $g(x)$.

Solution

Starting with x gives the flow chart $x \rightarrow 5x + 6 \rightarrow (5x + 6)^4$

This is the same as $x \rightarrow g(x) \rightarrow fg(x)$ ← g must be applied first.

giving $g(x) = 5x + 6$, $f(x) = x^4$

so $fg(x) = (5x + 6)^4$.

Example 3.5

Given that $f(x) = 2x - 3$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$, write the following as functions of x.

(i) $fg(x)$ (ii) $gf(x)$

(iii) $fgh(x)$ (iv) $f^2(x)$

Solution

(i) $$\begin{aligned} fg(x) &= f[g(x)] \\ &= f(x^2) \\ &= 2x^2 - 3 \end{aligned}$$

(ii) $$\begin{aligned} gf(x) &= g[f(x)] \\ &= g(2x - 3) \\ &= (2x - 3)^2 \end{aligned}$$

(iii) $$\begin{aligned} fgh(x) &= fg[h(x)] \\ &= f\left[g\left(\frac{1}{x}\right)\right] \\ &= f\left(\frac{1}{x^2}\right) \\ &= \frac{2}{x^2} - 3 \end{aligned}$$

(iv) $$\begin{aligned} f^2(x) &= f[f(x)] \\ &= f(2x - 3) \\ &= 2(2x - 3) - 3 \\ &= 4x - 6 - 3 \\ &= 4x - 9 \end{aligned}$$

ACTIVITY 3.2

For the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$, use your calculator to evaluate f(4), g(4), fg(4) and gf(4) and say what you notice.
What happens when you replace 4 by −4?

Exercise 3C

1. Express $(3 + x)^2$ in the form $fg(x)$, stating the expressions corresponding to $f(x)$ and $g(x)$.
2. (i) Given that $f(x) = 2x - 1$ and $g(x) = x^3$, write
   - (a) $fg(x)$ and
   - (b) $gf(x)$.

   (ii) Work out the values of
   - (a) $fg(2)$
   - (b) $gf(2)$
   - (c) $fg(-3)$
   - (d) $gf(-3)$.
3. (i) Given that $f(x) = x^2$, $g(x) = \frac{1}{x}$ and $h(x) = 1 - x$, write the following as functions of x.
   - (a) $fg(x)$
   - (b) $fh(x)$
   - (c) $gf(x)$
   - (d) $hf(x)$

   (ii) What do you notice?
4. The function $f(x) = (1 - x)^3$ is a composite function. Define $g(x)$ and $h(x)$ such that $f(x) = gh(x)$.

⑤ The function $f(x)$ is a composition of two functions $gh(x)$. Define $g(x)$ and $h(x)$, where $f(x)$ is

(i) $\dfrac{3}{x-2}$ (ii) $\dfrac{x-2}{3}$

(iii) $(3x-1)^2$ (iv) 2^{3x-1}.

⑥ The function $f(x)$ is a composition of two functions $uv(x)$. Define $u(x)$ and $v(x)$, where $f(x)$ is

(i) $\sin 2x$ (ii) $\cos\dfrac{x}{2}$

(iii) $\tan(x - 30°)$ (iv) $\sin^2 x$.

⑦ The function $f(x)$ is a composition of three functions $pqr(x)$.
Define $p(x)$, $q(x)$ and $r(x)$, where $f(x)$ is

(i) $3(x-2)^4$

(ii) $\dfrac{2x+3}{4}$.

Although there is theoretically no limit to the number of functions that you will be asked to combine, in an algebraic question it will usually be only two or three.

However, in a practical situation you are often required to perform a number of tasks in the correct order, as in the question below.

⑧ Daniel and Amanda are on holiday when their car gets a puncture and they need to change the wheel. Tasks here are not listed in the correct order.

(a) Remove the wheel with the punctured tyre

(b) Jack up the car

(c) Open the boot

(d) Put the jack and the wheel with the punctured tyre in the boot

(e) Get the jack and the new wheel from the boot

(f) Put the replacement wheel on the car

(g) Close the boot

There is a fixed order for performing these tasks. Treating each task as a function, write down the composite function for replacing the wheel. (Remember that the function performed first will appear at the end of the list).

4 Graphs of functions

Drawing or plotting a graph

If asked to draw a graph you should use graph paper. The axes should be numbered. The graph should be drawn passing through the points which have either been given or calculated.

On the next page is a drawing of the graph of $y = 2x + 1$ for values of x from -2 to 4. In this case the coordinates of the points have been calculated.

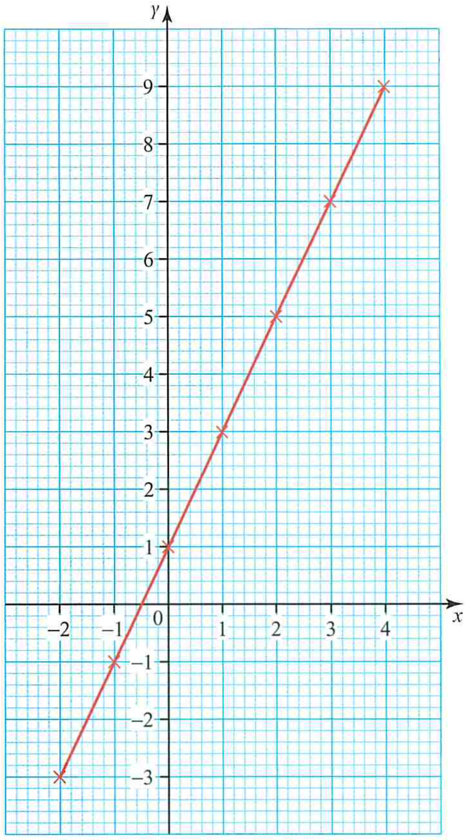


Figure 3.5

Note

In some cases you may meet the notation $f(x) = 2x + 1$ and then be told to draw the graph of $y = f(x)$. This is exactly the same instruction as 'Draw $y = 2x + 1$'.

Sketching a graph

If asked to sketch a graph you should *not* use graph paper. Axes should be drawn and only certain numbers need to be marked on the axes (e.g. points where the graph crosses the axes).

The correct shape of the graph should be shown and it should be in the correct position relative to the axes.

This means that the main features of the graph are shown although there is no requirement to plot points accurately.

Here is a sketch of the graph of $y = 2x + 1$

No domain (x value) was given so axes should be drawn to show all four quadrants.

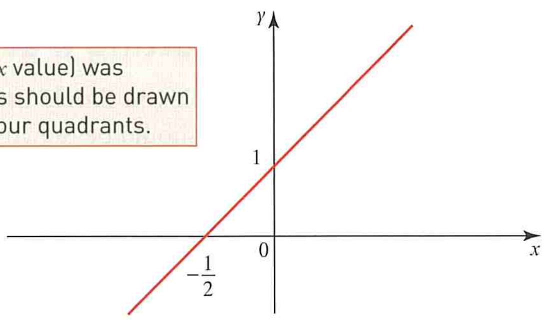


Figure 3.6

5 Graphs of linear functions

The gradient of a line

Discussion point

➔ What information do you need to have in order to fix the position of a line?

In mathematics the word *linear* refers to a straight line. The slope of a line is measured by its *gradient* and the letter m is often used to represent this.

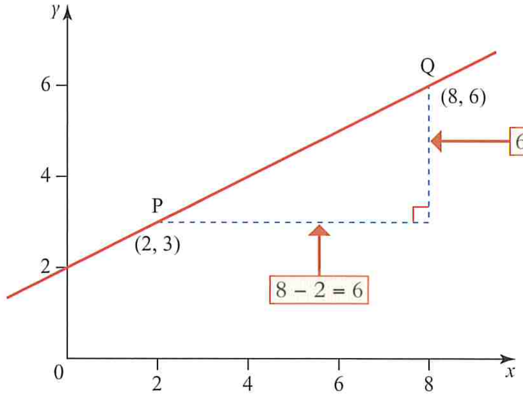


Figure 3.7

$$\text{gradient} = \frac{\text{change in } y\text{-coordinate from P to Q}}{\text{change in } x\text{-coordinate from P to Q}}$$

In Figure 3.7, gradient $= \dfrac{6-3}{8-2} = \dfrac{3}{6} = \dfrac{1}{2}$.

ACTIVITY 3.3

On each line in Figure 3.8, take any two points and use them to calculate the gradient of the line.

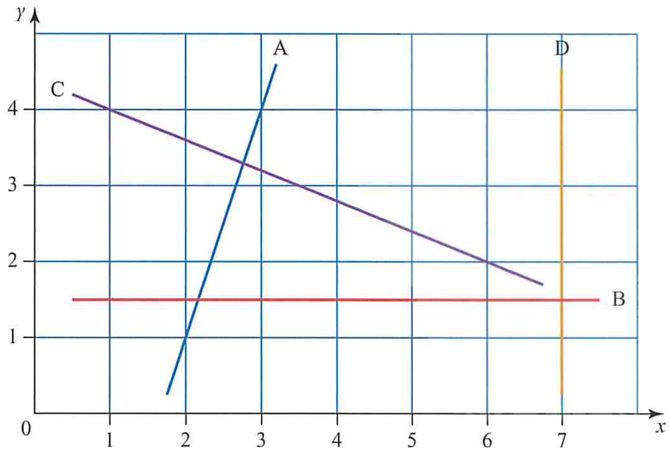


Figure 3.8

Discussion point

➔ Does it matter which point you call (x_1, y_1) and which (x_2, y_2)?

You can generalise the previous activity to find the gradient m of the line joining (x_1, y_1) to (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

You can easily tell by looking at a line if its gradient is positive, negative, zero or infinite.

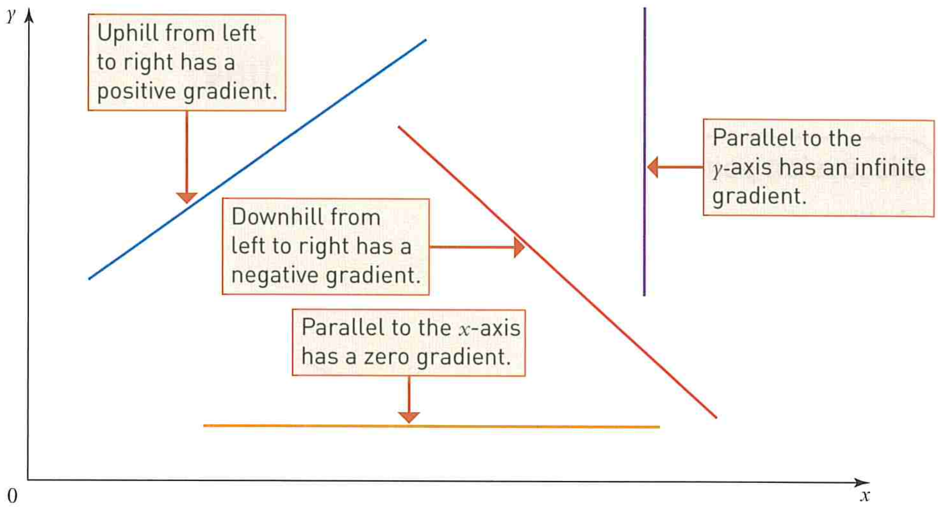


Figure 3.9

The equation of a straight line

Example 3.6

Work out the equation of the straight line with gradient 2 through the point with coordinates (0, 1).

Solution

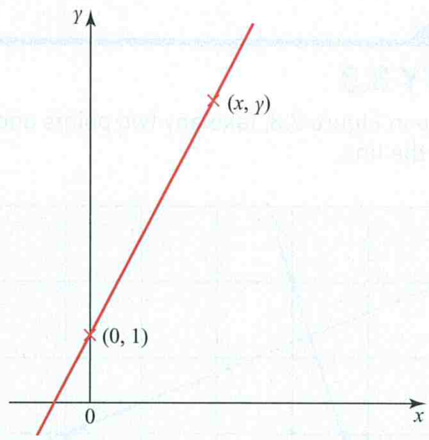


Figure 3.10

Take a general point (x, y) on the line, as shown in Figure 3.10, together with the point (0, 1) that you are given. The gradient of the line joining (0, 1) to (x, y) is given by

$$\text{gradient} = \frac{y-1}{x-0} = \frac{y-1}{x}.$$

Since you are given that the gradient of the line is 2,

$$\frac{y-1}{x} = 2 \quad \Rightarrow \quad y = 2x + 1$$

Since (x, y) is a general point on the line, this holds for any point on the line and is therefore the equation of the line.

This example can be generalised to give the result that the equation of the line with gradient m cutting the y-axis at the point $(0, c)$ is

$$\frac{y - c}{x - 0} = m$$

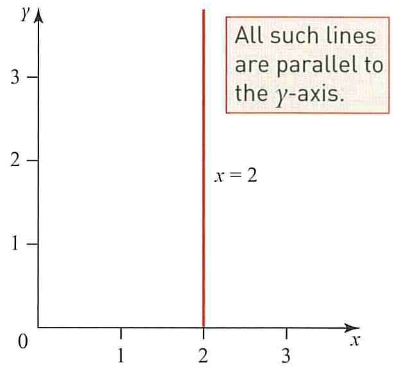
$$\Rightarrow \quad y = mx + c.$$

This is a well-known standard form for the equation of a straight line.

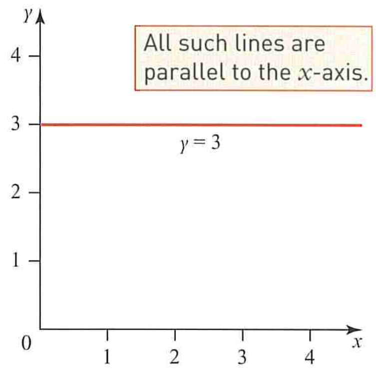
Drawing or sketching a line given its equation

There are several standard forms for the equation of a straight line. When you need to draw or sketch a line, look at its equation and see if it fits one of these.

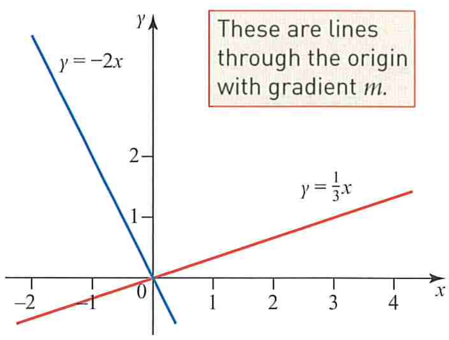
Equations of the form $x = a$



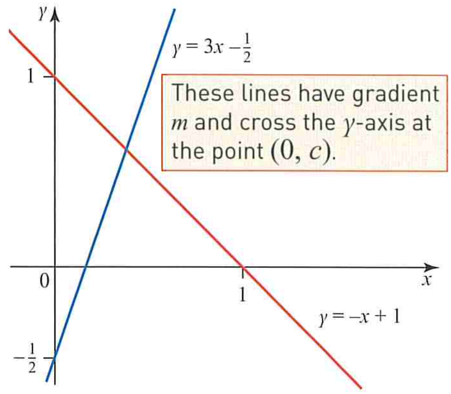
Equations of the form $y = b$



Equations of the form $y = mx$



Equations of the form $y = mx + c$



Equations of the form $px + qy + r = 0$

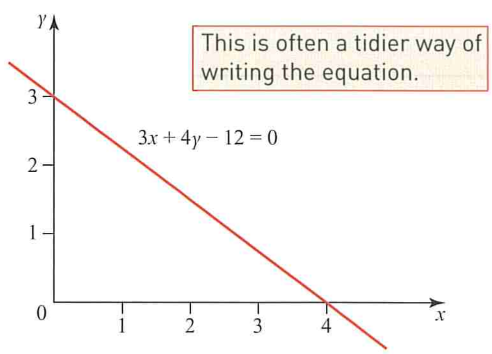


Figure 3.11

Graphs of linear functions

Graphs of equations in the form $px + qy + r = 0$ will usually be sketched by finding the coordinates of the points where the line crosses the x- and y-axes.

Discussion point

(i) Rearrange the equation $3x + 4y - 12 = 0$ into the form $\frac{x}{a} + \frac{y}{b} = 1$.

(ii) What are the values of a and b?

(iii) What do these numbers represent?

Example 3.7

Sketch the lines $y = -2$, $y = 3x - 2$ and $x + 3y - 9 = 0$ on the same axes.

Solution

The line $y = -2$ is parallel to the x-axis and passes through $(0, -2)$.

The line $y = 3x - 2$ has gradient 3 and passes through $(0, -2)$.

To sketch the line $x + 3y - 9 = 0$ find two points on it.

$x = 0 \quad \Rightarrow \quad 3y - 9 = 0 \quad \Rightarrow \quad y = 3 \qquad (0, 3)$ is on the line

$y = 0 \quad \Rightarrow \quad x - 9 = 0 \quad \Rightarrow \quad x = 9 \qquad (9, 0)$ is on the line

Figure 3.12 shows the three lines.

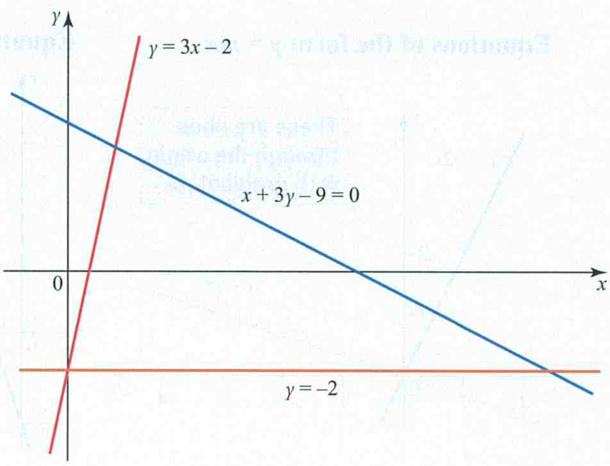


Figure 3.12

Exercise 3D

① For each of the following pairs of points, A and B, calculate the gradient of the line AB.

(i)	A(4, 3)	B(8, 11)	(ii)	A(3, 4)	B(0, 13)
(iii)	A(5, 3)	B(10, –8)	(iv)	A(–6, –14)	B(1, 7)
(v)	A(6, 0)	B(8, 15)	(vi)	A(–2, –4)	B(3, 9)
(vii)	A(–3, –6)	B(2, –7)	(viii)	A(4, 7)	B(7, –4)

In the following questions mark the coordinates of all points of intersection with the axes.

② Sketch these lines.

(i) $x = 5$ (ii) $y = -3$ (iii) $x = 0$ (iv) $y = 0$

③ Sketch these lines.

(i) $y = 4x$ (ii) $y = -3x$ (iii) $y = 4 + x$ (iv) $y = -3 + x$

④ Sketch these lines.

(i) $y = 2x + 3$ (ii) $y = 2x - 3$ (iii) $y = 2 + 3x$ (iv) $y = 2 - 3x$

⑤ Sketch these lines.

(i) $y = \frac{1}{2}x - 1$ (ii) $y = \frac{1}{3}x + \frac{2}{3}$

(iii) $y = 2 - \frac{1}{2}x$ (iv) $y = 3 - \frac{2x}{3}$

⑥ Sketch these lines.

(i) $x + 2y = 5$ (ii) $3x - y = 4$

(iii) $2x + y = 0$ (iv) $x - 2y = 0$

⑦ Sketch these lines

(i) $x + y - 1 = 0$ (ii) $2x + y - 4 = 0$

(iii) $x - 3y + 6 = 0$ (iv) $y - 3x + 9 = 0$

⑧ Sketch these lines.

(i) $\frac{x}{2} - \frac{y}{3} - 1 = 0$ (ii) $\frac{x}{3} - \frac{y}{2} - 1 = 0$

(iii) $\frac{3x}{2} - \frac{2y}{3} - 1 = 0$ (iv) $\frac{2x}{3} - \frac{3y}{2} - 1 = 0$

⑨ A printer quotes the cost £C of printing n business cards as $C = 60 + 0.06n$.

(i) Work out the cost of printing

(a) 500 cards

(b) 5000 cards

and the cost per card in each case.

(ii) The cost is made up of a fixed cost for setting up the printer and a cost per card printed. State the cost of each of these.

(iii) Sketch the graph of C against n.

If you have access to a graphic calculator, you can use it to check your results. Alternatively, check your answers with a free online graphing resource.

Prior knowledge

Students should be confident when manipulating algebraic expressions, including algebraic fractions, as covered in Chapter 2.

6 Finding the equation of a line

The simplest way of finding the equation of a straight line depends on what information you have been given.

Given the gradient, m, and the point of intersection $(0, c)$ with the y-axis

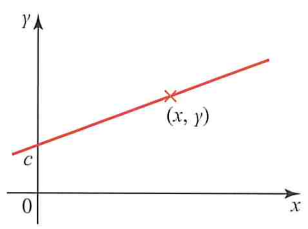


Figure 3.13

The gradient of the line joining $(0, c)$ to (x, y) is given by

$$m = \frac{y - c}{x - 0}$$

$$\Rightarrow y = mx + c$$

Given the gradient, m, and the coordinates (x_1, y_1) of a general point on the line

Take the general point (x, y) on the line, as shown in Figure 3.14.

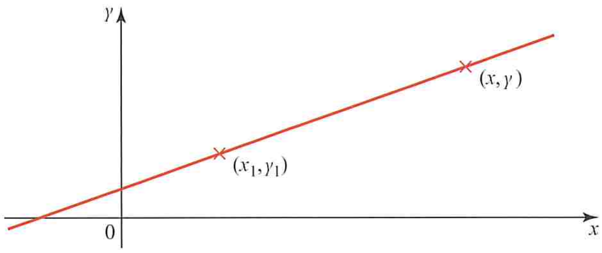


Figure 3.14

The gradient m of the line joining (x_1, y_1) to (x, y) is given by

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow y - y_1 = m(x - x_1).$$

This is a standard result, and one you will find very useful.

Example 3.8

Work out the equation of the line with gradient 2 which passes through the point $(-1, 3)$.

Solution

Using $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 3 = 2(x - (-1))$$
$$\Rightarrow y - 3 = 2x + 2$$
$$\Rightarrow y = 2x + 5$$

In the formula

$$y - y_1 = m(x - x_1)$$

two positions of the point (x_1, y_1) lead to results you have met already.

(x_1, y_1) is at $(0, 0)$

(x_1, y_1) is at $(0, c)$

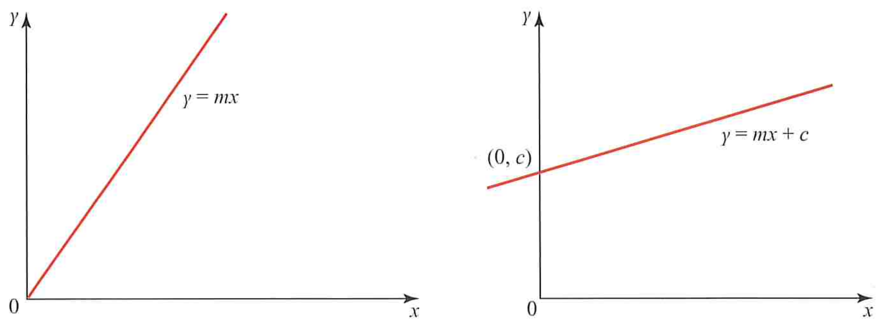


Figure 3.15

Given two points (x_1, y_1) and (x_2, y_2)

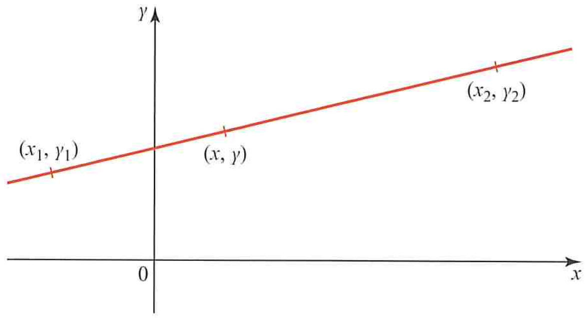


Figure 3.16

The two points are used to work out the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This value is then substituted in the equation.

$$y - y_1 = m(x - x_1).$$

This gives

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Rearranging this gives

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \text{or}$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 3.9

Work out the equation of the line joining (–1, 4) to (2, –3).

Solution

Let (x_1, y_1) be (–1, 4) and (x_2, y_2) be (2, –3).

Substituting these values in $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

$$\begin{aligned}
\text{gives} \quad \frac{y-4}{(-3)-4} &= \frac{x-(-1)}{2-(-1)} \\
\Rightarrow \quad \frac{y-4}{(-7)} &= \frac{x+1}{3} \\
\Rightarrow \quad 3(y-4) &= (-7)(x+1) \\
\Rightarrow \quad 7x + 3y - 5 &= 0
\end{aligned}$$

Applying the different techniques

Example 3.10

Work out the equations of the lines **(a)** to **(c)** in Figure 3.17.

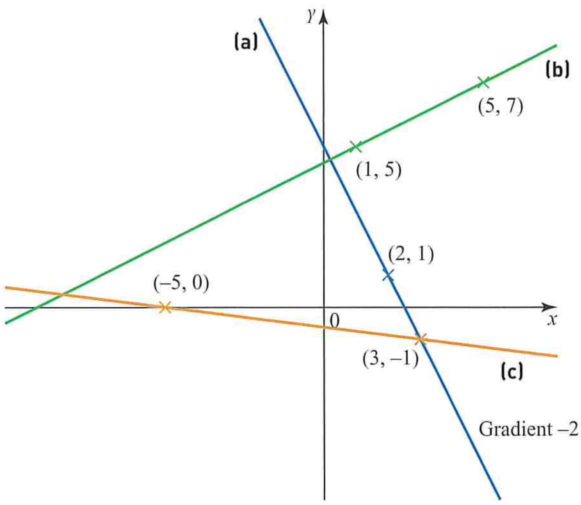


Figure 3.17

Solution

Line **(a)** has a gradient of –2 and passes through the point (2, 1).

Using $y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 2)$

$\Rightarrow y - 1 = -2x + 4$

$\Rightarrow y = -2x + 5$

This may also be written as $2x + y = 5$ or $2x + y - 5 = 0$

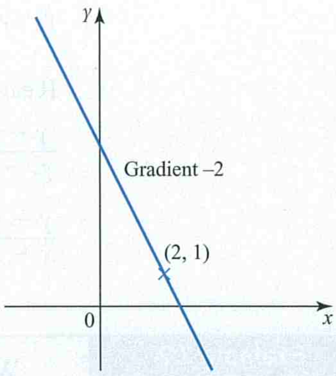


Figure 3.18

Line **(b)** passes through the points (1, 5) and (5, 7).

Using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient

$m = \dfrac{7 - 5}{5 - 1} = 0.5$

Using $y - y_1 = m(x - x_1)$ and the point (1, 5)

$y - 5 = 0.5(x - 1)$

$\Rightarrow y - 5 = 0.5x - 0.5$

$\Rightarrow y = 0.5x + 4.5$

Avoiding a decimal in the answer this could also be given as $2y = x + 9$ or $x - 2y + 9 = 0$.

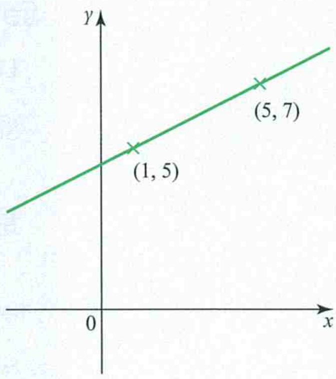


Figure 3.19

Line **(c)** passes through the points $(-5, 0)$ and $(3, -1)$.

Let (x_1, y_1) be $(-5, 0)$ and (x_2, y_2) be $(3, -1)$.

Substituting these values in

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

gives $\dfrac{y - 0}{-1 - 0} = \dfrac{x - (-5)}{3 - (-5)}$

$$\Rightarrow \frac{y}{-1} = \frac{x + 5}{8}$$

$$\Rightarrow 8y = -x - 5$$

$$\Rightarrow x + 8y + 5 = 0$$

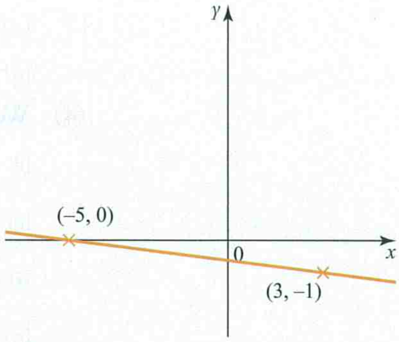


Figure 3.20

Exercise 3E

① Work out the equations of the lines **(i)** – **(v)** in this diagram.

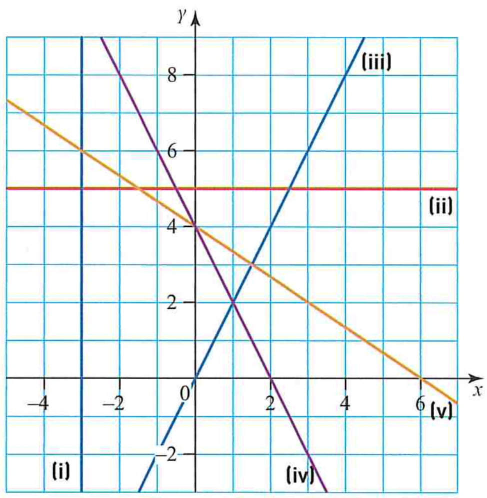


Figure 3.21

② Work out the equations of the lines **(i)** – **(v)** in this diagram.

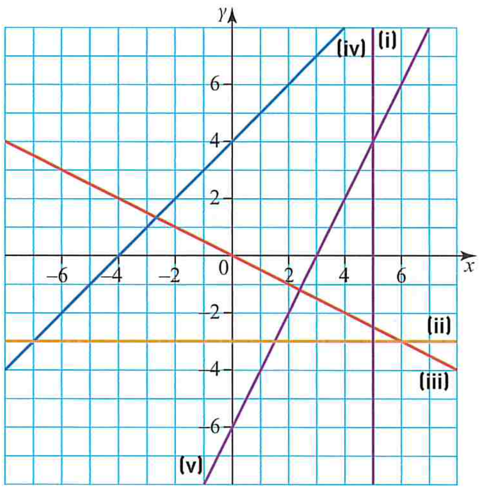


Figure 3.22

③ Work out the equations of these lines.

(i) Gradient 3 and passing through $(2, -1)$

(ii) Gradient 2 and passing through $(0, 0)$

(iii) Gradient 3 and passing through $(2, -7)$

(iv) Gradient 4 and passing through $(4, 0)$

④ Work out the equations of these lines.

(i) Gradient $\frac{1}{3}$ and passing through $(3, 1)$

(ii) Gradient $\frac{2}{5}$ and passing through $(-4, -10)$

(iii) Gradient $-\frac{3}{2}$ and passing through $(1, -2)$

(iv) Gradient $-\frac{1}{2}$ and passing through $(0, 6)$

⑤ Work out the equation of the line AB in each of these cases.

(i) A$(2, 0)$ B$(3, 1)$

(ii) A$(3, -1)$ B$(0, 4)$

(iii) A$(2, -3)$ B$(3, -2)$

⑥ Work out the equation of the line AB in each of these cases.

(i) A$(-1, 3)$ B$(4, 0)$

(ii) A$(3, -5)$ B$(10, -6)$

(iii) A$(-1, -2)$ B$(-4, -8)$

⑦ A taxi journey costs £2 plus 80 pence per mile. Use £C to represent the total cost of the journey and m miles to represent the total distance travelled.

(i) Write down an equation giving C in terms of m.

(ii) How much would a journey of 4 miles cost?

(iii) How far could I travel if I only had £10?

⑧ A junior school is ordering exercise books for their students and is working on the assumption that most students will only use 8 books during the year, but they want to order an additional 100. Let N represent the number of books to be ordered and let s be the number of students enrolled for the year.

(i) Write down an equation giving N in terms of s.

(ii) The exercise books cost £1.50 each. If there are 240 students that year, what would be the total cost of the books?

(iii) The school budget for exercise books is only £3000. How could the order be amended?

Discussion point

➜ What is the difference between a quadratic function and a quadratic equation?

7 Graphs of quadratic functions

ACTIVITY 3.4

Copy and complete the table of values and draw the graph of $y = x^2 - 5$ for values of x from -3 to 4.

x	-3	-2	-1	0	1	2	3	4
y	4			-5		-1	4	

ACTIVITY 3.5

Copy and complete the table of values and draw the graph of $y = 4x - x^2$ for values of x from -2 to 6.

x	-2	-1	0	1	2	3	4	5	6
y	-12	-5			4				-12

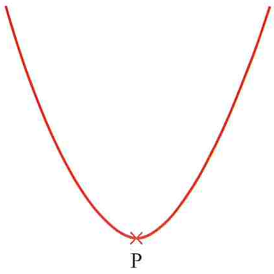


Figure 3.23

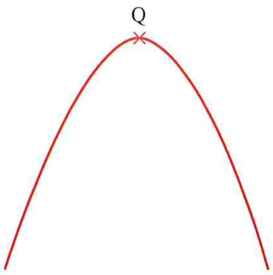


Figure 3.24

The shape of the graph of $y = ax^2 + bx + c$ is a parabola.

The sign of the coefficient of x^2 determines the direction of the curve.

$a > 0$

P is the lowest point on the graph in Figure 3.23.

P is the vertex.

$a < 0$

Q is the highest point on the graph in Figure 3.24.

Q is the vertex.

Symmetry

Quadratic graphs have a line of symmetry when drawn using an appropriate domain.

Here is the graph of $y = x^2 - 2x - 3$ for domain $-2 \leqslant x \leqslant 4$

The vertex is $(1, -4)$.

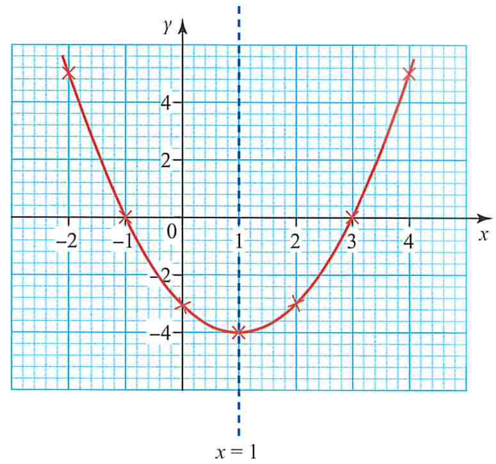


Figure 3.25

The line of symmetry has equation $x = 1$ and passes through the vertex.

Here is a sketch of the graph of $y = 9 - x^2$ for domain $-4 \leqslant x \leqslant 4$

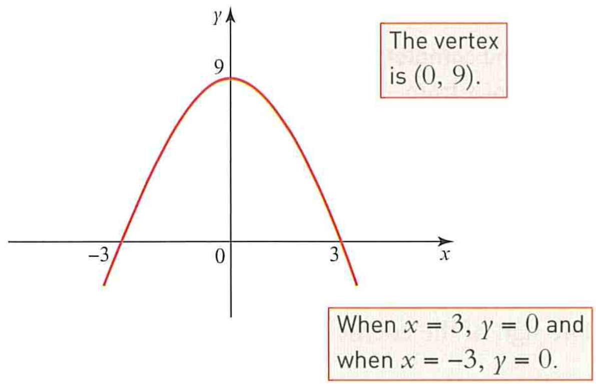


Figure 3.26

Prior knowledge

Students are expected to be familiar with the technique of completing the square, used to find the vertex and the line of symmetry of a quadratic graph, from GCSE. This is used in the following example.

This technique is covered in more detail in the next chapter.

The line of symmetry is the y-axis and passes through the vertex.

Example 3.11

For the graph of $y = x^2 + 6x + 11$, state

(i) the vertex

(ii) the equation of the line of symmetry

(iii) the coordinates of the point where the graph intersects the y-axis.

Sketch the graph.

Solution

Write the quadratic expression in the form $(x + a)^2 + b$.

$$x^2 + 6x + 11 \equiv (x + a)^2 + b$$
$$\equiv x^2 + 2ax + a^2 + b$$

Equate coefficients of x $\quad 6 = 2a$

$$3 = a$$

Equate constants $\quad 11 = a^2 + b$

$$11 = 9 + b$$
$$2 = b$$
$$y = x^2 + 6x + 11$$
$$= (x + 3)^2 + 2$$

Equate coefficients of x means making equal the number of x on each side of the identity.

$(x + 3)^2$ is always positive or zero.

The least value of $(x + 3)^2 + 2$ is 2 and this occurs when $x = -3$.

(i) The vertex is $(-3, 2)$.

(ii) The equation of the line of symmetry is $x = -3$.

(iii) When $x = 0$, $y = 11$, coordinates of y-axis intercept are $(0, 11)$.

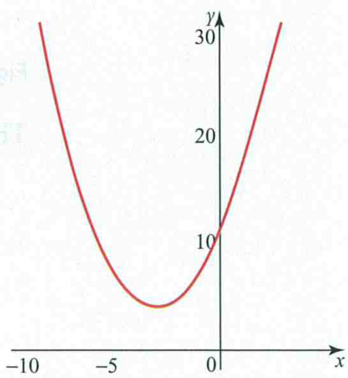


Figure 3.27

Exercise 3F

① Choose an equation from the list below to fit each of the quadratic curves **(i)** and **(ii)**.

$y = x^2 - 2x + 4$

$y = 5 - x^2$

$y = 3x - x^2$

$y = x^2 - 2x - 3$

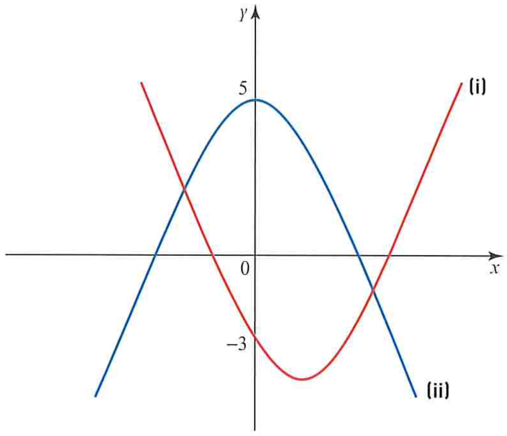


Figure 3.28

② Choose an equation from the list below to fit each of the quadratic curves **(i)** and **(ii)**.

$y = x^2 + 3x + 4$

$y = 4 - 7x - 2x^2$

$y = x^2 + 2x$

$y = 4x - x^2$

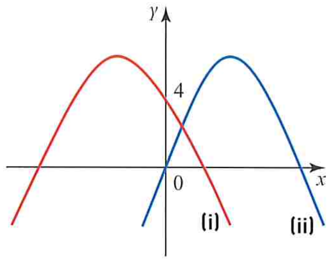


Figure 3.29

③ (i) For the graph of $y = x^2 + 2x + 3$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

④ (i) For the graph of $y = x^2 - 4x + 5$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

⑤ (i) For the graph of $y = x^2 - 6x + 7$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Sketch the graph.

⑥ (i) For the graph of $y = x^2 - 3x - 4$, work out

(a) the vertex

(b) the equation of the line of symmetry

(c) the coordinates of the point where the graph intersects the y-axis.

(ii) Use trial and improvement to work out the coordinates of the points where the curve intersects the x-axis.

(iii) Sketch the graph.

7. (i) Show that $x^2 - 6x + 10 = (x - 3)^2 + 1$.
   (ii) Write down the vertex of the graph of $y = x^2 - 6x + 10$.
   (iii) Write down the equation of the line of symmetry of $y = x^2 - 6x + 10$.
   (iv) Write down the coordinates of the points where the graph of $y = x^2 - 6x + 10$ intersects the coordinate axes.
   (v) Sketch the graph of $y = x^2 - 6x + 10$.
8. (i) Show that $6 - 4x - x^2 = 10 - (x + 2)^2$.
   (ii) Write down the vertex of the graph of $y = 6 - 4x - x^2$.
   (iii) Write down the equation of the line of symmetry of $y = 6 - 4x - x^2$.
   (iv) Write down the coordinates of the points where the graph of $y = 6 - 4x - x^2$ intersects the y-axis.
   (v) Sketch the graph of $y = 6 - 4x - x^2$.
9. A quadratic curve has its vertex at $(3, -7)$ and passes through $(0, 2)$. Work out the equation of the curve.
10. A car hire company is negotiating the price of n identical new cars with the manufacturer and offers a total price £P, where $P = 20\,000n - 200n^2$.
    (i) What is the likely cost of one car?
    (ii) Work out the cost of an order for
    (a) 5 cars (b) 20 cars (c) 50 cars.
    (iii) Work out the average cost per car when the number of cars ordered is
    (a) 5 cars (b) 20 cars (c) 50 cars.
    (iv) Suggest why the manufacturer may not want to sell 50 cars at once.

8 Inverse functions

Finding the inverse function

Consider the function f where $f(x) = x + 2$ for $x \in \{1, 2, 3\}$.

Under this function the domain $\{1, 2, 3\}$ maps onto the image set $\{3, 4, 5\}$.

It is possible to reverse this mapping and find the inverse function where each element of this image set $\{3, 4, 5\}$ is mapped back onto the corresponding member of the set $\{1, 2, 3\}$, i.e. the inverse of the function 'undoes' the effect of the function.

In the example above, where the function is given by $f(x) = x + 2$, the inverse function is given by $f^{-1}(x) = x - 2$

Example 3.12

Work out $f^{-1}(x)$ when $f(x) = 2x + 3$

Solution

When $f(x) = 2x + 3$ the steps used to build up the function are

$$x \xrightarrow{\times 2} 2x \xrightarrow{+3} 2x + 3 = f(x)$$

Reversing this gives

$$x \xleftarrow{\div 2} 2x \xleftarrow{-3} 2x + 3 = f(x)$$

i.e. the reverse steps are subtract 3 then divide by 2.

This shows that when $f(x) = 2x + 3$, then $f^{-1}(x) = \dfrac{x-3}{2}$

An alternative method for finding the inverse

Writing the function in the previous example as $y = 2x + 3$.

Step 1: Interchange x and y to give $x = 2y + 3$

Step 2: Make y the subject of $x = 2y + 3$

Subtract 3 $\quad x - 3 = 2y$

Divide by 2 $\quad y = \dfrac{x-3}{2}$

EXTENSION

Note

The graph of a function and its inverse is extension material and will not be assessed.

The graph of a function and its inverse

Below is a sketch of the graphs of $f(x) = 2x + 3$ and its inverse, using the same scale on both axes.

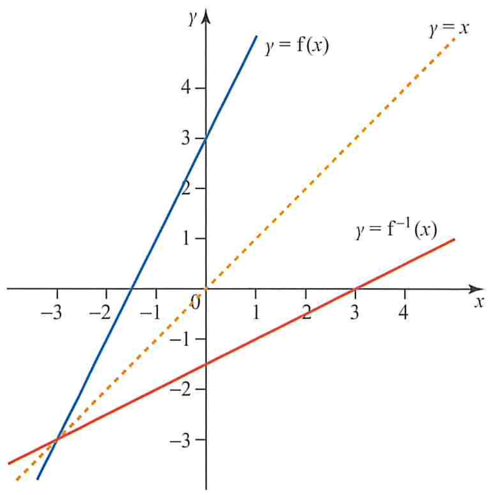


Figure 3.30

You can see from the graph that the function and its inverse are reflections of each other in the line $y = x$.

A one-to-one function is one which has exactly one value of x for each value of y and one value of y for each value of x.

A function will only have an inverse if it is a one-to-one function in the domain given (the domain is the set of values of x).

For example, $y = x^2$ for $x \geqslant 0$ has an inverse $y = \sqrt{x}$ as shown below.

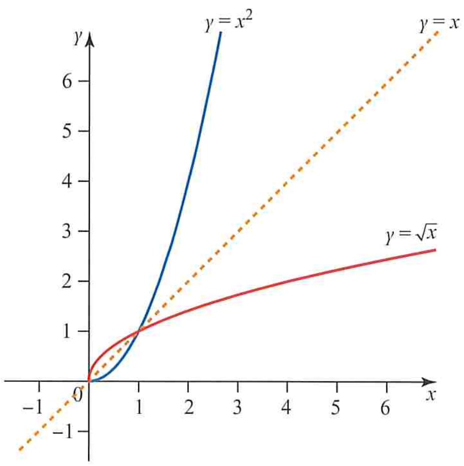


Figure 3.31

However, the function $y = x^2$ for all values of x does not have a single inverse, since $y = \pm\sqrt{x}$ is not a function. (A function must have a unique value of y for each value of x.)

Summary

To find the inverse of a function:

- write the function in the form $y = f(x)$
- interchange x and y to give $x = f(y)$
- rearrange to make y the subject.

Example 3.13

(i) Draw the graph of the function $f(x) = x^2 - 2$ for $x \geqslant 0$ using the same scale on both axes.

(ii) Use algebra to work out the inverse function.

(iii) Using your graph from part (i) as a guide, sketch the graphs of $y = x$, $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Mark the points of intersection with the axes.

Solution

(i)

x	0	1	2	3
x^2	0	1	4	9
$f(x) = x^2 - 2$	−2	−1	2	7

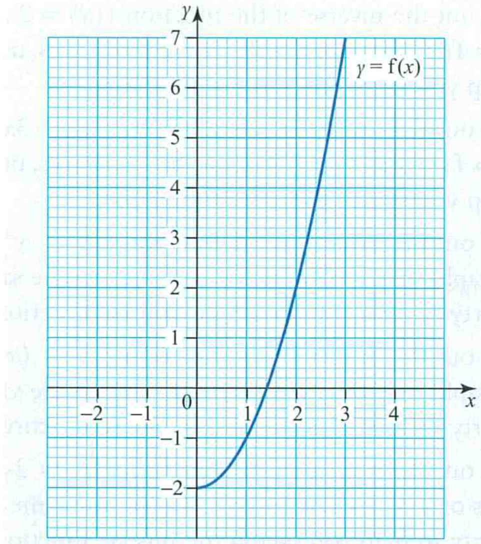


Figure 3.32

(ii) Write the function as $y = x^2 - 2$.

Interchange x and y to give $x = y^2 - 2$.

Make y the subject: $y^2 = x + 2$

$$y = \sqrt{x+2}.$$

(iii) The graph of $y = \mathrm{f}^{-1}(x)$ is the reflection of the graph of $y = \mathrm{f}(x)$ in the line $y = x$.

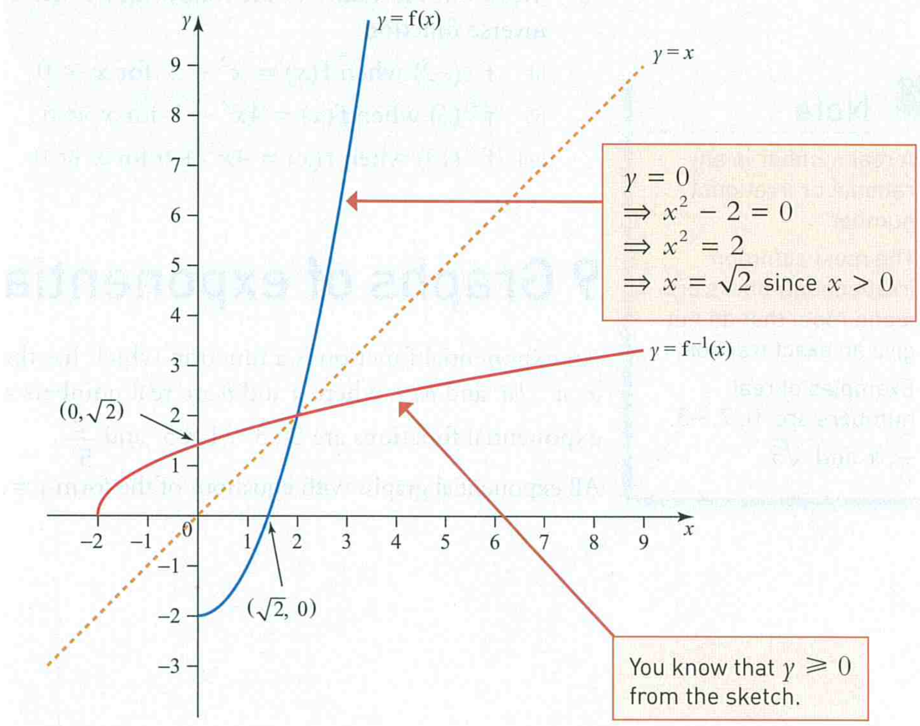


Figure 3.33

Discussion points

- In the Oxford English dictionary, the mathematical definition of 'inverse' is 'a reciprocal quantity' and the mathematical definition of 'reciprocal' is 'the quantity obtained by dividing the number one by the given quantity'.
- Comment on these definitions in the light of what you have just learnt.

Exercise 3G

① Work out the inverse of the function $f(x) = 2x - 3$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

② Work out the inverse of the function $f(x) = 3x - 2$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

③ Work out the inverse of the function $f(x) = x^2 - 4$ for $x \geqslant 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

④ Work out the inverse of the function $f(x) = (x - 2)^2$ for $x \geqslant 2$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑤ Work out the inverse of the function $f(x) = 2\sqrt{x}$ for $x \geqslant 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑥ Work out the inverse of the function $f(x) = \frac{1}{x}$ for $x > 0$ and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, using the reflection property to help you sketch the inverse function.

⑦ Work out the values of the following by first finding an expression for the inverse function.

(i) $f^{-1}(4)$ when $f(x) = 5x - 1$

(ii) $f^{-1}(3)$ when $f(x) = 2 + \frac{1}{x}$ for $x > 0$

(iii) $f^{-1}(9)$ when $f(x) = x^2$ for $x \geqslant 0$

⑧ Work out the values of the following by first finding an expression for the inverse function.

(i) $f^{-1}(-2)$ when $f(x) = x^2 - 3$ for $x \geqslant 0$

(ii) $f^{-1}(3)$ when $f(x) = 4x^2 - 1$ for $x \geqslant 0$

(iii) $f^{-1}(13)$ when $f(x) = 4x^2 + 9$ for $x \geqslant 0$

9 Graphs of exponential functions

An exponential function is a function which has the variable as a power, such as a^x, a^{-x}, ba^x and ba^{-x}, where a and b are real numbers and $a > 0$. Some examples of exponential functions are 2^x, 3^{-x}, 4×5^x and $\frac{2}{5^x}$.

All exponential graphs with equations of the form $y = a^x$ pass through the point (0, 1).

Note

A real number is any rational or irrational number.

The most common irrational numbers are π and roots that do not give an exact fraction.

Examples of real numbers are: $0, 7, -3, \frac{2}{3}, \pi$ and $\sqrt{5}$

Example 3.14

Plot the graphs of $y = 2^x$ and $y = 2^{-x}$ on the same axes for $-2 \leqslant x \leqslant 2$.

Solution

x	-2	-1	0	1	2
$y = 2^x$	0.25	0.5	1	2	4
$y = 2^{-x}$	4	2	1	0.5	0.25

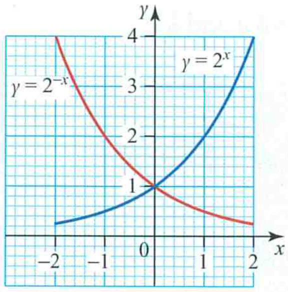


Figure 3.34

Note

Both of these graphs pass through the point (0, 1). This is true for all graphs of the form $y = a^x$ and $y = a^{-x}$, where a is a positive constant.

The effect of replacing x by $-x$ is to reflect the graph in the y-axis as shown in Figure 3.34.

Example 3.15

(i) Sketch, on the same axes, the graphs of $y = 2^x$, $y = 2 + x$ and $y = 2 - x$.

(ii) State the number of roots of these equations.

(a) $2^x = 2 + x$ (b) $2^x = 2 - x$

Solution

(i) $y = 2^x$ is as in the previous example.

$y = 2 + x$ is a straight line through (0, 2) with a gradient of 1.

$y = 2 - x$ is a straight line through (0, 2) with a gradient of (-1).

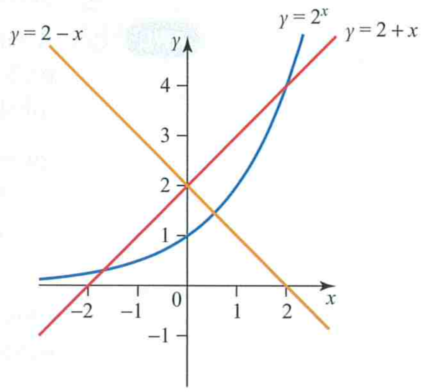


Figure 3.35

(ii) (a) There are two points of intersection of the graphs $y = 2^x$ and $y = 2 + x$, therefore there are two roots of the equation $2^x = 2 + x$.

(b) There is one point of intersection of the graphs $y = 2^x$ and $y = 2 - x$, therefore there is one root of the equation $2^x = 2 - x$.

Real life application of exponential functions (using differentiation, which is introduced in Chapter 8)

One important application of the exponential function is in the theory of radioactive decay. It has been shown that the rate of decay of a radioactive material is proportional to the amount of material present.

Example 3.16

The points (0, 0.5) and (3, 0.004) are on the curve

$y = ab^{-x}$.

Work out the values of a and b.

Solution

When $x = 0$, $y = 0.5$, so $0.5 = ab^0$

As $b^0 = 1$, $a = 0.5$

When $x = 3$, $y = 0.004$, so $0.004 = 0.5b^{-3}$

$0.008 = b^{-3}$

$125 = b^3$

$b = 5$

Exercise 3H

① Sketch the graphs of $y = 2^x$ and $y = 3^x$ on the same axes.

② Sketch the graphs of $y = 3^x$ and $y = 3^{-x}$ on the same axes.

③ Sketch the graphs of $y = 2^x$, $y = 2^x - 1$ and $y = 2^x + 1$ on the same axes.

④ Sketch the graphs of $y = 3^{-x}$, $y = 3^{-x} - 1$ and $y = 3^{-x} + 1$ on the same axes.

RWC ⑤ A virus is spreading among the inhabitants of a remote island and it will be 5 days before an antidote can be shipped there. Initially there are 100 inhabitants on the island. The number of people surviving after t hours is given by $N = 100 \times 10^{-\frac{t}{1000}}$.

(i) Calculate how many people are surviving after 24 hours.

(ii) Calculate how many people are surviving at the beginning of the fifth day when the vaccine is delivered.

Once the vaccine has been administered, it is a further 24 hours before it begins to be effective.

(iii) How many more people die before the vaccine begins to take effect?

⑥ Use any graphing software at your disposal to draw the graphs of $y = 2 \times 3^x$, $y = 3 \times 2^x$ and $y = 6^x$ on the same axes and write down what you notice.

⑦ (i) Use any graphing software at your disposal to draw the graphs of $y = 2 \times 4^x$, $y = 4 \times 2^x$ and $y = 8^x$ on the same axes.

(ii) In what ways is the result the same as in the previous question and in what ways is it different?

⑧ Use any graphing software at your disposal to draw the graphs of $y = 2^x + 2^{-x}$ and $y = 3^x + 3^{-x}$ on the same axes and write down what you notice.

10 Graphs of functions with up to three parts to their domains

A function may be defined with more than one part to its domain.

Here are two examples.

$$\begin{aligned} f(x) &= x + 1 & -2 \leqslant x < 1 \\ &= 2 & 1 \leqslant x \leqslant 4 \end{aligned}$$

The domain of f(x) is $-2 \leqslant x \leqslant 4$.

$$\begin{aligned} g(x) &= 3 & 0 \leqslant x < 2 \\ &= 7 - 2x & 2 \leqslant x < 5 \\ &= 3x + 12 & 5 \leqslant x \leqslant 6 \end{aligned}$$

The domain of g(x) is $0 \leqslant x \leqslant 6$.

Example 3.17

Draw the graph of $y = f(x)$ where

$$\begin{aligned} f(x) &= 4 & -4 \leqslant x < -2 \\ &= x^2 & -2 \leqslant x < 2 \\ &= 8 - 2x & 2 \leqslant x \leqslant 4 \end{aligned}$$

Solution

The graph is drawn for each part of the given domain.

$y = 4$ is a horizontal line.

$y = x^2$ is a quadratic curve.

$y = 8 - 2x$ is a straight line with gradient -2.

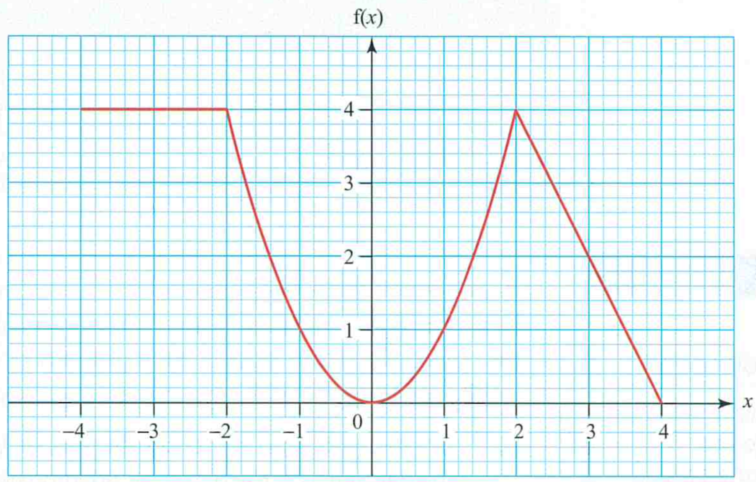


Figure 3.36

Example 3.18

Here is the graph of $y = \mathrm{f}(x)$.

(i) Define $\mathrm{f}(x)$, stating clearly the domain for each part.

(ii) State the range of $\mathrm{f}(x)$.

Figure 3.37

Solution

(i) For $0 \leqslant x < 2$ gradient $= \dfrac{2-0}{2-0}$

$= 1$

Line passes through (0, 0) so $y = x$

For $2 \leqslant x < 4$ horizontal line so $y = 2$

For $4 \leqslant x \leqslant 8$ gradient $= \dfrac{2-0}{4-8}$

$= \dfrac{2}{-4}$

$= -\dfrac{1}{2}$

Line passes through (8, 0) so $y - 0 = -\dfrac{1}{2}(x - 8)$

$y = -\dfrac{1}{2}x + 4$

$$\begin{aligned} \mathrm{f}(x) &= x & 0 \leqslant x < 2 \\ &= 2 & 2 \leqslant x < 4 \\ &= -\tfrac{1}{2}x + 4 & 4 \leqslant x \leqslant 8 \end{aligned}$$

No x value should be included more than once in the domain.

(ii) The range is obtained by looking at the y values on the graph.

$0 \leqslant \mathrm{f}(x) \leqslant 2$

Example 3.19

Prior knowledge

You will have studied speed–time and distance–time graphs from GCSE. These provide a number of examples where there are up to three parts to their domain.

A car sets off from rest and accelerates uniformly for 5 seconds, after which it has reached a speed of $12\,\mathrm{m\,s^{-1}}$. After travelling for a further minute at that speed the driver notices the lights ahead change to red so decelerates uniformly for 5 seconds, coming to rest at the lights.

(i) Draw a graph to represent the journey, plotting time t seconds on the horizontal axis for $0 \leqslant t \leqslant 80$ and speed $v\,\mathrm{m\,s^{-1}}$ on the vertical axis for $0 \leqslant v \leqslant 15$.

(ii) Work out the acceleration for each of the three stages of the journey.

(iii) Work out the total distance travelled.

Solution

(i) While accelerating, the graph is represented by a straight line from (0, 0) to (5, 12). The part of the journey at constant speed is represented by a straight line from (5, 12) to (65, 12) since a minute is 60 seconds. Finally, the car comes to rest after a further 5 seconds, so join the point (65, 12) to (70, 0). Here is the graph of $v = f(t)$.

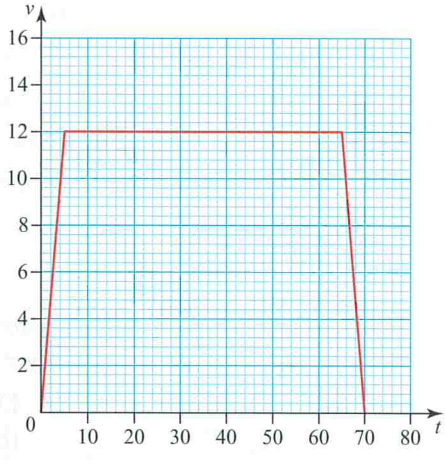


Figure 3.38

(ii) Acceleration = (change in speed) ÷ (change in time).

Stage 1: Acceleration $= \dfrac{12 - 0}{5 - 0} = 2.4\,\text{m}\,\text{s}^{-2}$

Stage 2: Constant speed so acceleration = 0

Stage 3: Acceleration $= \dfrac{0 - 12}{5 - 0} = -2.4\,\text{m}\,\text{s}^{-2}$

(iii) The distance travelled is represented by the area under a velocity–time graph.

In this case the shape of the graph is a trapezium, so

Area = (half the sum of parallel sides) × (distance between them)

$$= \left(\frac{60 + 70}{2}\right) \times 12 = 780$$

The distance travelled is 780 metres.

> Alternatively, the distance can be calculated as the sum of three separate areas:
>
> triangle + rectangle + triangle.

Exercise 3I

① Draw the graph of $y = f(x)$ where

$$f(x) = \begin{cases} 2 & -2 \leqslant x < 1 \\ 2x & 1 \leqslant x \leqslant 3 \end{cases}$$

② Draw the graph of $y = f(x)$ where

$$f(x) = \begin{cases} x^2 & 0 \leqslant x < 3 \\ 9 & 3 \leqslant x \leqslant 5 \end{cases}$$

③ Draw the graph of $y = g(x)$ where

$$g(x) = \begin{cases} x + 3 & -3 \leqslant x < 0 \\ 3 - x & 0 \leqslant x \leqslant 3 \end{cases}$$

④ Draw the graph of $y = f(x)$ where

$$f(x) = \begin{cases} 3x - 1 & -2 \leqslant x < 1 \\ 3 - x & 1 \leqslant x < 4 \\ -1 & 4 \leqslant x \leqslant 6 \end{cases}$$

⑤ Draw the graph of $y = \mathrm{f}(x)$ where

$$\begin{aligned} \mathrm{f}(x) &= 2x & -2 \leqslant x < 0 \\ &= \frac{1}{2}x & 0 \leqslant x < 2 \\ &= 5 - 2x & 2 \leqslant x \leqslant 4 \end{aligned}$$

⑥ Draw the graph of $y = \mathrm{g}(x)$ where

$$\begin{aligned} \mathrm{g}(x) &= -4 & -3 \leqslant x < -2 \\ &= -x^2 & -2 \leqslant x < 2 \\ &= 3x - 10 & 2 \leqslant x \leqslant 4 \end{aligned}$$

⑦ Figure 3.39 shows the graph of $y = \mathrm{f}(x)$.

(i) Define $\mathrm{f}(x)$, stating clearly the domain for each part.

(ii) State the range of $\mathrm{f}(x)$.

(iii) Solve $\mathrm{f}(x) = 5$.

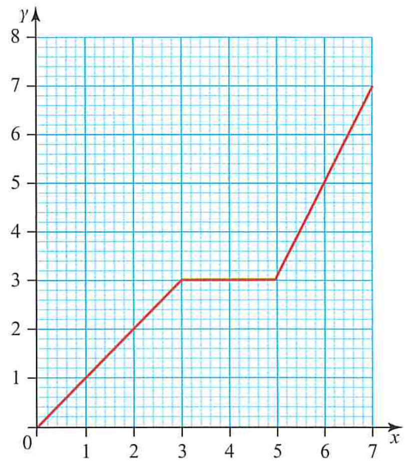


Figure 3.39

⑧ Figure 3.40 shows the graph of $y = \mathrm{f}(x)$.

(i) Define $\mathrm{f}(x)$, stating clearly the domain for each part.

(ii) State the range of $\mathrm{f}(x)$.

(iii) Solve $\mathrm{f}(x) = 3$.

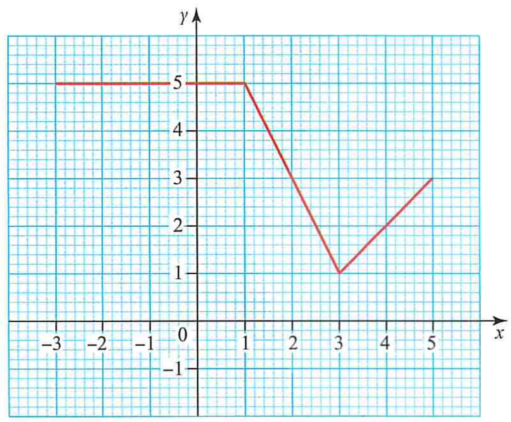


Figure 3.40

⑨ The graph of $y = \mathrm{f}(x)$ is shown in Figure 3.41.

(i) Define $\mathrm{f}(x)$, stating clearly the domain for each part.

(ii) Work out the area between the graph and the x-axis.

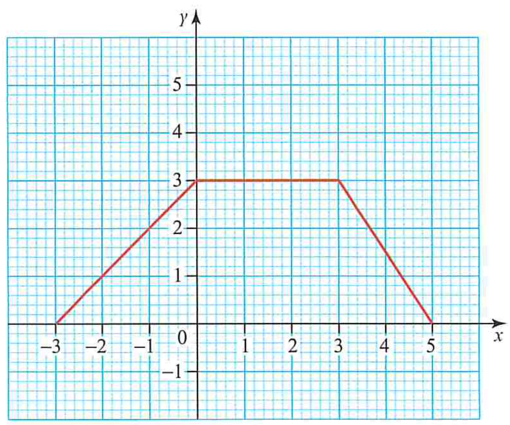


Figure 3.41

⑩ The graph of $y = g(x)$ is shown in Figure 3.42.

(i) Define $g(x)$, stating clearly the domain for each part.

(ii) Work out the area between the graph and the x-axis.

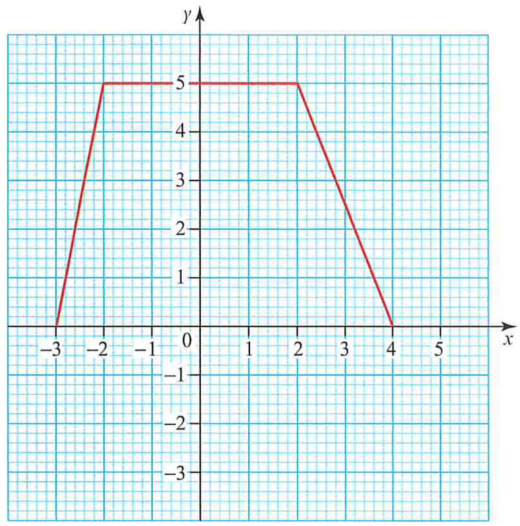


Figure 3.42

⑪ The distance–time graph in Figure 3.43 shows the relationship between distance travelled and time for a student who leaves home at 8:15 a.m., walks to the bus stop and then catches the bus to school.

(i) Describe what is happening between O and A.

(ii) Why do you think the distance doesn't change between A and B?

(iii) Think about the part of the graph from B to C. What does this tell you about the bus journey? How realistic is that?

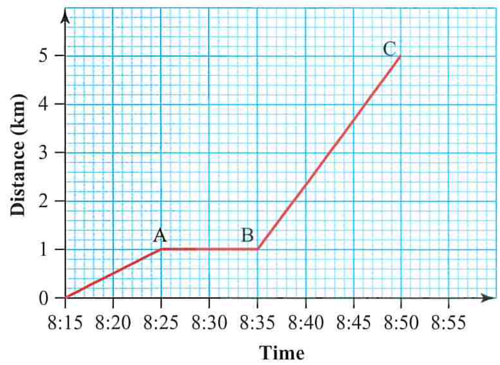


Figure 3.43

FUTURE USES

- You will use the techniques introduced here both in the next chapter and throughout your study of A-Level Pure Mathematics, but particularly in the section on graphs and their transformations.
- The section on graphs of inverse functions is not part of the Level 2 Further Mathematics specification, but you will meet it again at A-Level.
- Exponential functions also have applications in Mechanics and Physics in connection with problems on growth and decay.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- state the domain and range of a function on graph paper
- draw the graph of a function on graph paper
- sketch a graph – not using graph paper
- find the equation of a line given either the coordinates of two points on the line or the gradient of the line and the coordinates of one point
- recognise the shape of a quadratic graph from its equation, determining whether it has a maximum or minimum turning point
- recognise the graphs of the basic exponential functions $y = a^x$ and $y = a^{-x}$ and functions associated with them
- interpret graphs which relate to practical or physical situations.

KEY POINTS

1 A function maps an input, x, to an output, $f(x)$.

2 The set of input values is the domain of $f(x)$.
The set of output values is the range of $f(x)$.

3 A composite function occurs when two or more functions act in succession.

4 When asked to draw a graph, use graph paper.

5 When asked to sketch a graph, do not use graph paper.

6 The gradient of the straight line joining the points (x_1, y_1) and (x_2, y_2) is given by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

7 The equation of a straight line may take any of these forms.

- Line parallel to the y-axis: $x = a$
- Line parallel to the x-axis: $y = b$
- Line through the origin with gradient m: $y = mx$
- Line through $(0, c)$ with gradient m: $y = mx + c$
- Line through (x_1, y_1) with gradient m: $y - y_1 = m(x - x_1)$
- Line through (x_1, y_1) and (x_2, y_2):
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \text{ or } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

8 The shape of a quadratic graph is a parabola. It may be either $\cup$ shaped (when the coefficient of the squared term is positive) or $\cap$ shaped (when it is negative).

9 For any one-to-one function $f(x)$ there is an inverse function $f^{-1}(x)$.

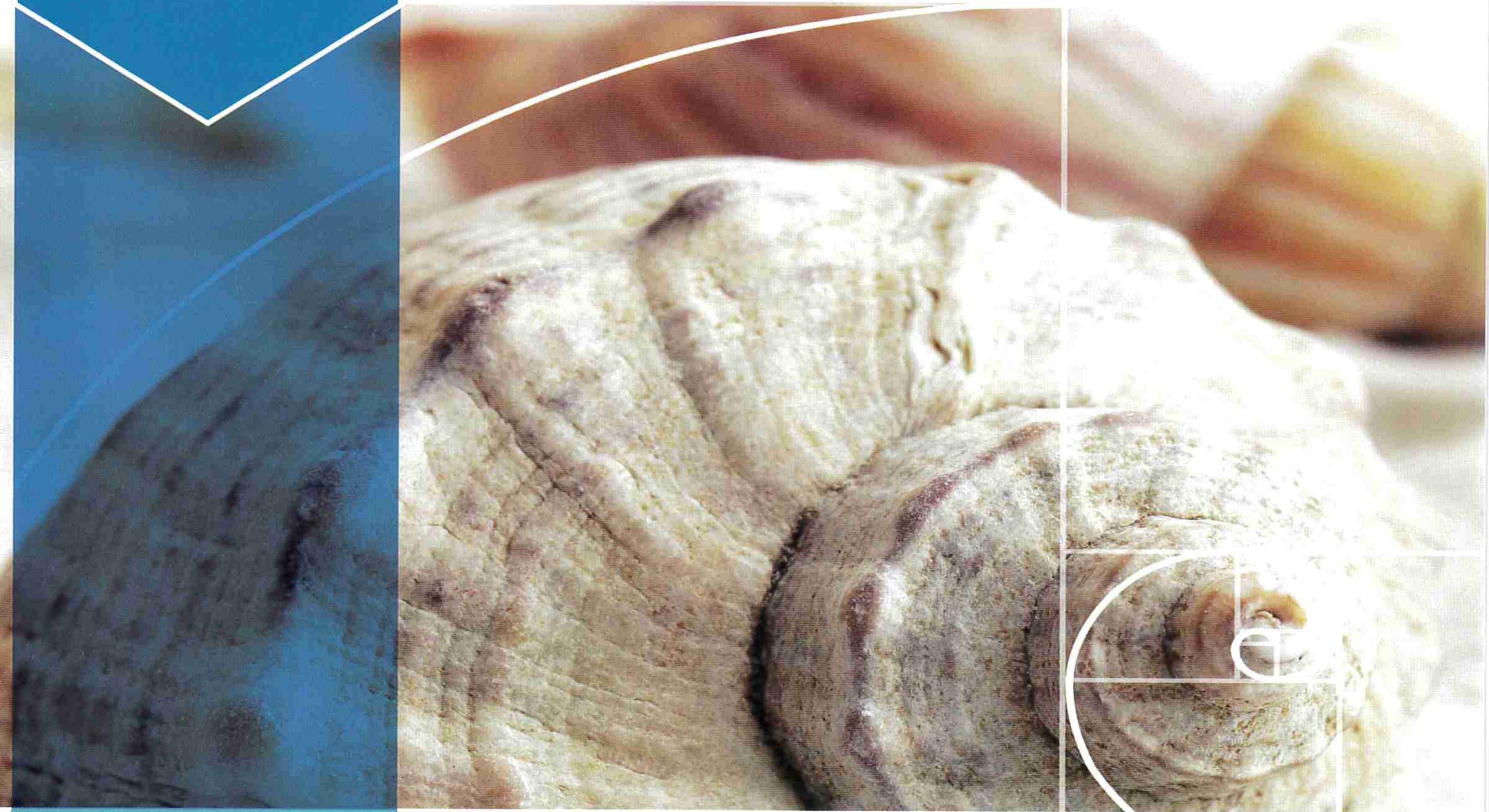
10 The inverse of a function 'undoes' the effect of the function.

11 The graphs of a function and its inverse are reflections of each other in the line $y = x$, provided that the same scale is used on both axes.

12 An exponential function is a function of the form $y = a^x$ or $y = a^{-x}$ where $a > 0$.

13 Functions may be defined with more than one part to their domain. In this case consider each part separately.

4 Algebra IV



'Obvious' is the most dangerous word in mathematics

E. T. Bell

1 Quadratic equations

Solving a quadratic equation by factorising

When solving an equation by factorisation it is essential that all non-zero terms are moved to one side of the equation, leaving zero on the other side. This is due to a unique property of zero: when the product of two (or more) numbers or expressions is zero, then at least one of the numbers or expressions must be zero. No other number has such a property.

Example 4.1 Solve $x^2 = 4x + 21$

Solution

$$x^2 = 4x + 21$$

$$\Rightarrow \quad x^2 - 4x - 21 = 0$$

$$\Rightarrow \quad (x + 3)(x - 7) = 0$$

$$\Rightarrow \quad x + 3 = 0 \quad \text{or} \quad x - 7 = 0$$

$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = 7$$

Note

The *solution* of the equation is the *pair* of values $x = -3$ or $x = 7$.

The *roots* of the equation are the *individual* values $x = -3$ and $x = 7$.

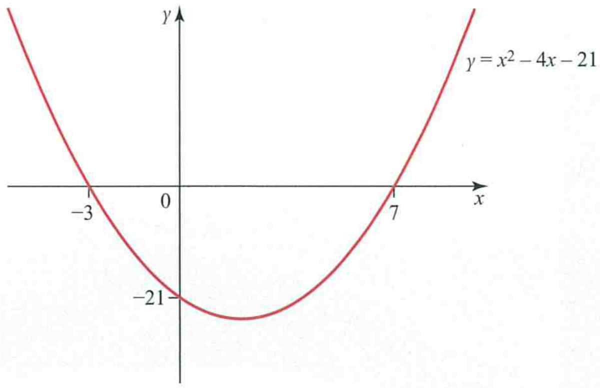


Figure 4.1

! Before solving a quadratic equation by factorisation, ensure that all non-zero terms are on one side of the equation only.

Example 4.2

Solve $8x^2 + 10x = 3$

Solution

First, move the 3 to the left-hand side of the equation, leaving zero on the other side.

$$8x^2 + 10x = 3$$
$$\Rightarrow \quad 8x^2 + 10x - 3 = 0$$
$$\Rightarrow \quad 8x^2 + 12x - 2x - 3 = 0$$
$$\Rightarrow \quad 4x(2x + 3) - 1(2x + 3) = 0$$
$$\Rightarrow \quad (2x + 3)(4x - 1) = 0$$
$$\Rightarrow \quad 2x + 3 = 0 \quad \text{or} \quad 4x - 1 = 0$$
$$\Rightarrow \quad x = -\frac{3}{2} \quad \text{or} \quad x = \frac{1}{4}$$

Find two numbers whose sum is 10 and product is −24 (10 is the coefficient of x, and −24 is the product of the constant term and the coefficient of x^2).

The numbers required are 12 and −2.

Sometimes a quadratic equation will not factorise.

In this case, you must complete the square or use the quadratic formula.

Solving a quadratic equation by completing the square

Example 4.3

Solve $x^2 - 8x + 3 = 0$

Solution

Subtract the constant term from both sides of the equation.

$$\Rightarrow \quad x^2 - 8x = -3$$

Add 16 to both sides of the equation $\Rightarrow \quad x^2 - 8x + 16 = -3 + 16$

Factorise the left-hand side $\Rightarrow \quad (x - 4)^2 = 13$

Square root both sides $\Rightarrow \quad x - 4 = \pm\sqrt{13}$

Add 4 to both sides $\Rightarrow \quad x = 4 + \sqrt{13} \quad \text{or} \quad x = 4 - \sqrt{13}$

$$\Rightarrow \quad x = 7.6055\ldots \text{ or } x = 0.3944\ldots$$

Consider the coefficient of x (-8).
Halve it (-4).
Then square the half (16).
We then add this to both sides of the equation.
This process is called 'completing the square'.

If the square has been completed correctly, the left-hand side will always factorise to the form $(x \pm p)^2$.

If the coefficient of the squared term is not 1, then first divide the equation by the coefficient.

Example 4.4

Solve $2x^2 + 3x - 7 = 0$

Solution

$$
\begin{aligned}
2x^2 + 3x - 7 &= 0 \\
\Rightarrow \quad x^2 + \frac{3}{2}x - \frac{7}{2} &= 0 \\
\Rightarrow \quad x^2 + \frac{3}{2}x &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 - \frac{9}{16} &= \frac{7}{2} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 &= \frac{56}{16} + \frac{9}{16} \\
\Rightarrow \quad \left(x + \frac{3}{4}\right)^2 &= \frac{65}{16} \\
\Rightarrow \quad x + \frac{3}{4} &= \pm\sqrt{\frac{65}{16}} \\
\Rightarrow \quad x &= -\frac{3}{4} \pm \frac{\sqrt{65}}{4}
\end{aligned}
$$

Generalisation

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
\Rightarrow \quad x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} &= -\frac{c}{a} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\
\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\Rightarrow \quad x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
\Rightarrow \quad x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
\Rightarrow \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Discussion points

- If $b^2 - 4ac$ is zero, how are the answers affected?
- If $b^2 - 4ac$ is negative, how are the answers affected?

The result $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the *quadratic formula*. It can be used to solve any quadratic equation.

The $\pm$ sign indicates that there are two possible roots. One root is found by using the + sign, and the other by using the − sign.

Figure 4.2 shows a parabola – the shape of a quadratic curve.

The dotted line is the line of symmetry.

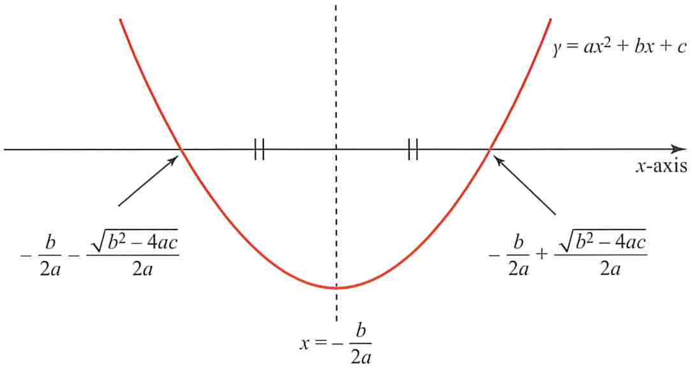


Figure 4.2

Solving a quadratic equation using the quadratic formula

Example 4.5

Use the quadratic formula to solve $3x^2 - 4x - 2 = 0$.

Solution

Comparing

$$3x^2 - 4x - 2 = 0$$

with

$$ax^2 + bx + c = 0$$

gives

$$a = 3, \quad b = -4, \quad c = -2$$

Using these values in the formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times -2}}{2 \times 3}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{6}$$

$$x = \frac{4 \pm \sqrt{40}}{6}$$

In a non-calculator paper, the answer could be simplified and left in an exact form as

$$x = \frac{2 \pm \sqrt{10}}{3}$$

Alternatively, in a calculator paper, approximate roots could be calculated as

$x = 1.72, \quad x = -0.39 \qquad$ (rounded to 2 d.p.)

Example 4.6

The length of a carpet is 1 m greater than its width. Its area is 9 m^2.

Work out the dimensions of the carpet to the nearest centimetre.

Solution

Figure 4.3

Let the length be x metres, so the width is $(x-1)$ metres.

length × width = area

$$\text{so} \quad x(x-1) = 9$$
$$\Rightarrow \quad x^2 - x = 9$$
$$\Rightarrow \quad x^2 - x - 9 = 0 \text{ (collecting all terms on the left-hand side)}$$

Substituting $a = 1$, $b = -1$, $c = -9$ into the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{gives } x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-9)}}{2 \times 1}$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{37}}{2}$$

$$\Rightarrow \quad x = 3.541\ldots \quad \text{or} \quad x = -2.541\ldots$$

Clearly a negative answer is not feasible, so the dimensions are

length = 3.54 m

width = 2.54 m (to the nearest cm).

Example 4.7

Solve

$$\frac{5}{a+1} - \frac{2a}{a^2-1} = \frac{1}{2}$$

Solution

First factorise $(a^2 - 1)$ as $(a+1)(a-1)$.

$$\frac{5}{a+1} - \frac{2a}{(a+1)(a-1)} = \frac{1}{2}$$

Multiply each term by $2(a+1)(a-1)$.

$$\Rightarrow \quad 2\,\cancel{(a+1)}^{1}(a-1) \times \frac{5}{\cancel{(a+1)}_{1}} - 2\,\cancel{(a+1)}^{1}\,\cancel{(a-1)}^{1} \times \frac{2a}{\cancel{(a+1)}_{1}\cancel{(a-1)}_{1}}$$
$$= \cancel{2}^{1}(a+1)(a-1) \times \frac{1}{\cancel{2}_{1}}$$

$$\Rightarrow \quad 10(a-1) - 4a = (a+1)(a-1)$$
$$\Rightarrow \quad 10a - 10 - 4a = a^2 - 1$$
$$\Rightarrow \quad 0 = a^2 - 6a + 9$$
$$\Rightarrow \quad 0 = (a-3)(a-3)$$
$$\Rightarrow \quad a = 3 \text{ (repeated root)}$$

Chapter 4 Algebra IV

Exercise 4A

① Solve the following equations by factorising.

(i) $x^2 - 8x + 12 = 0$ (ii) $m^2 - 4m + 4 = 0$

(iii) $p^2 - 2p - 15 = 0$ (iv) $a^2 + 11a + 18 = 0$

(v) $2x^2 + 5x + 2 = 0$ (vi) $4x^2 + 3x - 7 = 0$

(vii) $15t^2 + 2t - 1 = 0$ (viii) $24r^2 + 19r + 2 = 0$

(ix) $3x^2 + 8x = 3$ (x) $3p^2 = 14p - 8$

② Solve the following equations

(a) by completing the square

(b) by using the quadratic formula.

Give your answers correct to 2 decimal places.

(i) $x^2 - 2x - 10 = 0$ (ii) $x^2 + 3x - 6 = 0$

(iii) $x^2 + x - 8 = 0$ (iv) $2x^2 + x - 8 = 0$

(v) $2x^2 + 2x - 9 = 0$ (vi) $x^2 + x = 10$

(vii) $x^2 = 4x + 1$ (viii) $2x^2 - 8x + 5 = 0$

③ Solve the following equations by using the quadratic formula. Give your answers correct to 2 decimal places.

(i) $3x^2 + 5x = -1$ (ii) $4x^2 = -9x - 3$

(iii) $2x^2 + 11x = 4$ (iv) $4x^2 + 4 = 9x$

(v) $5x^2 + 1 = 10x$ (vi) $-9 - 11x = 3x^2$

④ The sides of a right-angled triangle, in centimetres, are x, $2x - 2$, and $x + 2$, where $x + 2$ is the hypotenuse. Use Pythagoras' theorem to work out their lengths.

PS ⑤ A rectangular lawn measures 8 m by 10 m and is surrounded by a path of uniform width x m. The total area of the path is 63 m². Work out the value of x.

PS ⑥ The difference between two positive numbers is 2 and the difference between their squares is 40. Work out the two numbers.

⑦ The formula $h = 15t - 5t^2$ gives the height h metres of a ball, t seconds after it is thrown up into the air.

(i) Work out the times when the height is 10 m.

(ii) After how long does the ball hit the ground?

⑧ The area of this triangle is 68 cm².

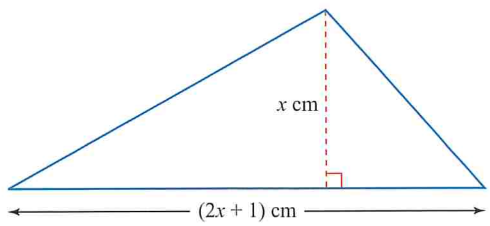


Figure 4.4

(i) Show that x satisfies the equation $2x^2 + x - 136 = 0$.

(ii) Solve the equation to work out the length of the base of the triangle.

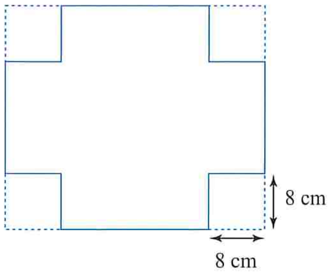


Figure 4.5

⑨ Boxes are made by cutting squares of side 8 cm from the corners of rectangular sheets of cardboard and then folding the remaining card. The sheets of cardboard are 6 cm longer than they are wide.

(i) For a sheet of cardboard with width x cm, write expressions, in terms of x, for

(a) the length of the sheet

(b) the length of the finished box

(c) the width of the finished box.

(ii) Show that the volume of the box is $8x^2 - 208x + 1280$ cm^3.

(iii) Work out the dimensions of the sheet of cardboard needed to make a box with a volume of 1728 cm^3.

⑩ Solve the following equations.

(i) $\dfrac{1}{x} = 3 - \dfrac{2}{x+1}$

(ii) $\dfrac{2}{3x-1} + \dfrac{1}{x+8} = \dfrac{1}{2}$

(iii) $\dfrac{2}{a} - \dfrac{5}{2a-1} = 0$

(iv) $\dfrac{6}{p-2} + \dfrac{6}{p+1} = 1$

⑪ Solve the following equations.

(i) $\dfrac{1}{p} + p + 1 = \dfrac{13}{3}$

(ii) $1 + \dfrac{1}{x-1} = \dfrac{2x}{x+1}$

(iii) $\dfrac{6r}{r+1} - \dfrac{5}{r+3} = 3$

⑫ Solve the following equations.

(i) $\dfrac{a+4}{2a-3} = \dfrac{3(a+7)}{4(a+2)}$

(ii) $\dfrac{4x-13}{2x+1} = \dfrac{5x-23}{x+5}$

(iii) $\dfrac{2x+7}{x+7} = \dfrac{5x+13}{3-x}$

PS ⑬ A formula used in physics is $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ where f is the focal length of a mirror, u is the distance of the object from the mirror, and v is the distance of the image from the mirror. For a mirror with focal length 20 cm, work out the distance of the object from the mirror when the image is twice as far away from the mirror as is the object.

PS ⑭ Anna has used the quadratic formula to solve a quadratic equation.

She correctly calculated the answers as $x = \dfrac{4 \pm \sqrt{124}}{6}$.

Write down an equation that Anna might have solved.

2 Simultaneous equations in two unknowns

The equations you have met so far have only involved one unknown.

For example, $2x + 2 = x - 5$ or $a^2 - 3a + 2 = 0$

Figure 4.6 shows the line $x + y = 4$.

Discussion point

➜ When an equation involves two unknowns, for example $x + y = 4$, how many possible pairs of values are there for x and y?

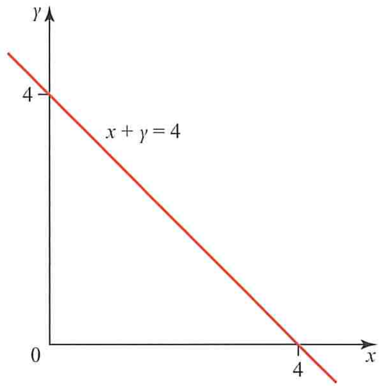


Figure 4.6

The coordinates of every point on that line give a pair of possible values for x and y. If the line $y = 2x + 1$ is included, as in Figure 4.7, the two lines can be seen to intersect at a single point.

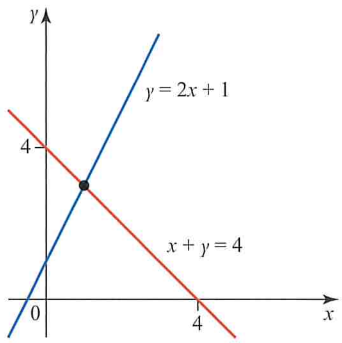


Figure 4.7

The coordinates of this point (1, 3) are the solution ($x = 1$, $y = 3$) to the *simultaneous equations*

$$x + y = 4$$

and $$y = 2x + 1.$$

There are several ways of solving simultaneous equations. You have just seen one method, that of drawing graphs. This is valid, but it has two drawbacks

(i) it is tedious

(ii) it may not be very accurate, particularly if the solution does not have integer values.

Solving simultaneous equations by substitution

Example 4.8

Solve the simultaneous equations

$$x + y = 4$$

$$y = 2x + 1$$

by substitution.

This method is particularly suitable when one of the unknowns is already written as the subject of one of the equations.

Solution

Take the expression for y from the second equation and substitute it into the first. This gives

$$x + (2x + 1) = 4$$

$$\Rightarrow \quad 3x = 3$$

$$\Rightarrow \quad x = 1$$

Then substitute $x = 1$ into one of the original equations, e.g. $y = 2x + 1$ giving $y = 2 \times 1 + 1 = 3$

So the solution is $x = 1$, $y = 3$ as indicated by the graphs.

Example 4.9

Figure 4.8 shows the graphs of $y = x^2 + x$ and $2x + y = 4$.

Solve the simultaneous equations

$$y = x^2 + x$$

and $\quad 2x + y = 4$

using the method of substitution.

This method is also suitable when one of the equations represents a curve.

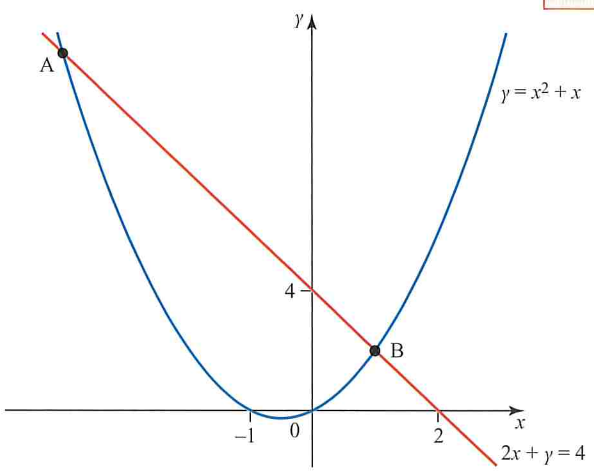


Figure 4.8

! Notice from Figure 4.8 that there are two points of intersection, A and B, so expect the solution to be two pairs of values for x and y.

Discussion point

→ Having found the values of x in the example, the values of y were found by substituting into the equation of the line. Why was it advisable to use the linear equation rather than the quadratic?

Solution

$$y = x^2 + x \qquad ①$$
$$2x + y = 4 \qquad ②$$

Substitute for y from equation ① into equation ②.

$$2x + (x^2 + x) = 4$$
$$\Rightarrow \quad x^2 + 3x - 4 = 0$$
$$\Rightarrow \quad (x + 4)(x - 1) = 0$$
$$\Rightarrow \quad x = -4 \text{ or } x = 1$$

Substituting into $2x + y = 4$

Always substitute back into the linear equation.

$x = -4 \Rightarrow -8 + y = 4 \Rightarrow y = 12$

$x = 1 \Rightarrow 2 + y = 4 \Rightarrow y = 2$

The solution is $x = -4$, $y = 12$ (point A) and $x = 1$, $y = 2$ (point B).

Check your solution also fits equation ①.

! The solution must always be given as pairs of values. It is wrong to write $x = -4$ or 1, $y = 12$ or 2, since not all pairs of values are possible.

Solving linear simultaneous equations by elimination

When both equations are linear and written in the same form, it may be preferable to use a process referred to as elimination.

Example 4.10

Solve the simultaneous equations

$$2x + y = 8 \qquad ①$$
$$5x + 2y = 21 \qquad ②$$

Solution

Notice that multiplying equation ① by 2 gives another equation containing $2y$.

$$5x + 2y = 21 \qquad \text{equation ②}$$
$$4x + 2y = 16 \qquad 2 \times \text{equation ①}$$

Subtracting $\Rightarrow$ $x \quad = 5$

Substitute $x = 5$ into equation ①

$$10 + y = 8 \quad \Rightarrow \quad y = -2$$

The solution is $x = 5$, $y = -2$

Sometimes you need to manipulate both equations to eliminate one of the unknowns, as in the following example.

Example 4.11

Solve the simultaneous equations

$$2x + 3y = -1 \qquad ①$$

$$3x - 2y = 18 \qquad ②$$

Solution

It is equally easy to eliminate x or y. It is up to you to choose which. The following method eliminates y.

$$4x + 6y = -2 \qquad 2 \times \text{equation } ①$$

$$9x - 6y = 54 \qquad 3 \times \text{equation } ②$$

Adding $\Rightarrow$ $13x = 52$

$\Rightarrow$ $x = 4$

Substitute $x = 4$ into equation ①

$8 + 3y = -1 \quad \Rightarrow \quad y = -3$

The solution is $x = 4$, $y = -3$

Discussion point

→ In Example 4.10 the equations were subtracted; in Example 4.11 they were added. How do you decide whether to add or subtract?

Simultaneous equations may arise in everyday problems.

Example 4.12

Tracey is buying fruit for a picnic.

Five apples and four pears cost exactly £2.20.

Two apples and six pears also cost exactly £2.20.

(i) Write this information as a pair of simultaneous equations.

(ii) Solve your equations to work out the cost of each type of fruit.

Solution

Let a pence be the cost of an apple and p pence be the cost of a pear.

Make sure you introduce your unknowns.

The cost of each piece of fruit will be a number of pence, so writing £2.20 as 220 pence avoids working with decimals.

(i) $5a + 4p = 220$ ①

$2a + 6p = 220$ ②

(ii) $\Rightarrow$ $15a + 12p = 660$ $\quad 3 \times$ equation ①

$4a + 12p = 440$ $\quad 2 \times$ equation ②

Subtracting $11a = 220$

$\Rightarrow$ $a = 20$

Substitute $a = 20$ into equation ①

$100 + 4p = 220$

$\Rightarrow$ $p = 30$

An apple costs 20 pence and a pear costs 30 pence.

Example 4.13

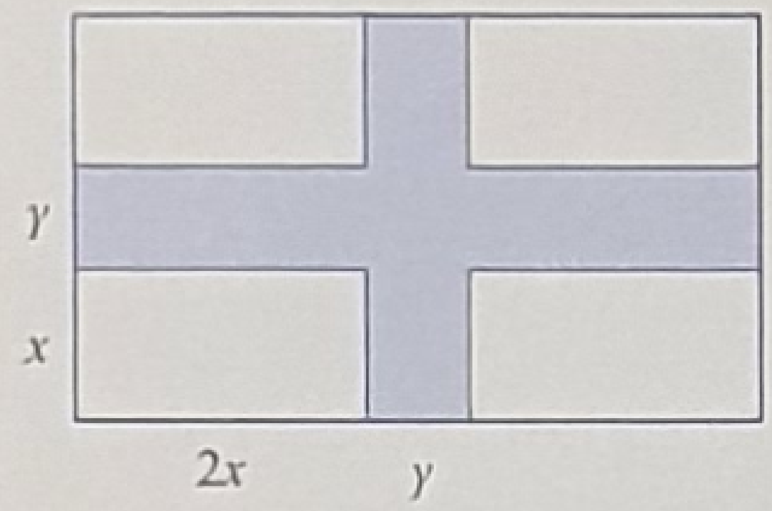


Figure 4.9

A flag consists of a blue cross on a white background. Each white rectangle measures $2x$ cm by x cm, and the cross is y cm wide.

(i) Work out the total area of the flag in terms of x and y.

(ii) Show that the area of the cross is $6xy + y^2$.

(iii) The total area of the flag is 4500 cm^2 and the area of the cross is 1300 cm^2. Work out the values of x and y.

Solution

(i) length $= 4x + y$ and width $= 2x + y$

so area $= (4x + y)(2x + y)$

$= 8x^2 + 6xy + y^2$

(ii) Each white rectangle has an area of $2x \times x = 2x^2$.

$\therefore$ area of cross $= 8x^2 + 6xy + y^2 - (4 \times 2x^2)$

$= 6xy + y^2$

(iii) $8x^2 + 6xy + y^2 = 4500$

$6xy + y^2 = 1300$

Subtracting $8x^2 = 3200$

$\Rightarrow$ $x^2 = 400$

$\Rightarrow$ $x = 20$ (positive answer only)

Substitute $x = 20$ into $6xy + y^2 = 1300$

$120y + y^2 = 1300$

$\Rightarrow$ $y^2 + 120y - 1300 = 0$

$\Rightarrow$ $(y + 130)(y - 10) = 0$

$\Rightarrow$ $y = -130$ (reject since y is a length and so cannot be negative) or $y = 10$

$\Rightarrow$ $x = 20$ and $y = 10$

Exercise 4B

① Solve the following pairs of simultaneous equations using the substitution method.

(i) $y = x - 3$, $3x + 2y = 19$

(ii) $y = 2x - 9$, $4x - y = 17$

(iii) $y = 11 - 2x$, $2x + 5y = 37$

(iv) $y = 3x + 3$, $x - 2y = 4$

(v) $y = 7 - 2x$, $2x + 3y = 15$

(vi) $y = 3x - 5$, $x + 3y = -20$

② Solve the following pairs of simultaneous equations using the elimination method.

(i) $3x + 2y = 12$, $4x - y = 5$

(ii) $3x - 2y = 6$, $5x + 6y = 38$

(iii) $3x + 2y = 22$, $4x - 3y = 18$

(iv) $5x + 4y = 11$ (v) $4x + 5y = 33$ (vi) $4x - 3y = 2$

$2x + 3y = 9$ $3x + 2y = 16$ $5x - 7y = 9$

③ Solve the following pairs of simultaneous equations.

(i) $x + y = 5$ (ii) $x - y + 1 = 0$ (iii) $x^2 + xy = 8$

$x^2 + y^2 = 17$ $3x^2 - 4y = 0$ $x - y = 6$

(iv) $2x - y + 3 = 0$ (v) $x = 2y$ (vi) $x + 2y = -3$

$y^2 - 5x^2 = 20$ $x^2 - y^2 + xy = 20$ $x^2 - 2x + 3y^2 = 11$

PS ④ For each of the following situations, form a pair of simultaneous equations and solve them to answer the question.

(i) Three chews and four lollipops cost 72p. Five chews and two lollipops cost 64p. Work out the cost of a chew and the cost of a lollipop.

(ii) A taxi firm charges a fixed amount plus an extra fee per mile. A journey of five miles costs £5.00 and a journey of seven miles costs £6.60. How much does a journey of two miles cost?

(iii) Three packets of crisps and two packets of nuts cost £1.45. Two packets of crisps and five packets of nuts cost £2.25. How much does one packet of crisps and four packets of nuts cost?

(iv) Two adults and one child paid £37.50 to go to the theatre. The cost for one adult and three children was also £37.50. How much does it cost for two adults and five children?

PS ⑤ The diagram shows the circle $x^2 + y^2 = 25$ and the line $x + y = 7$. Work out the coordinates of A and B.

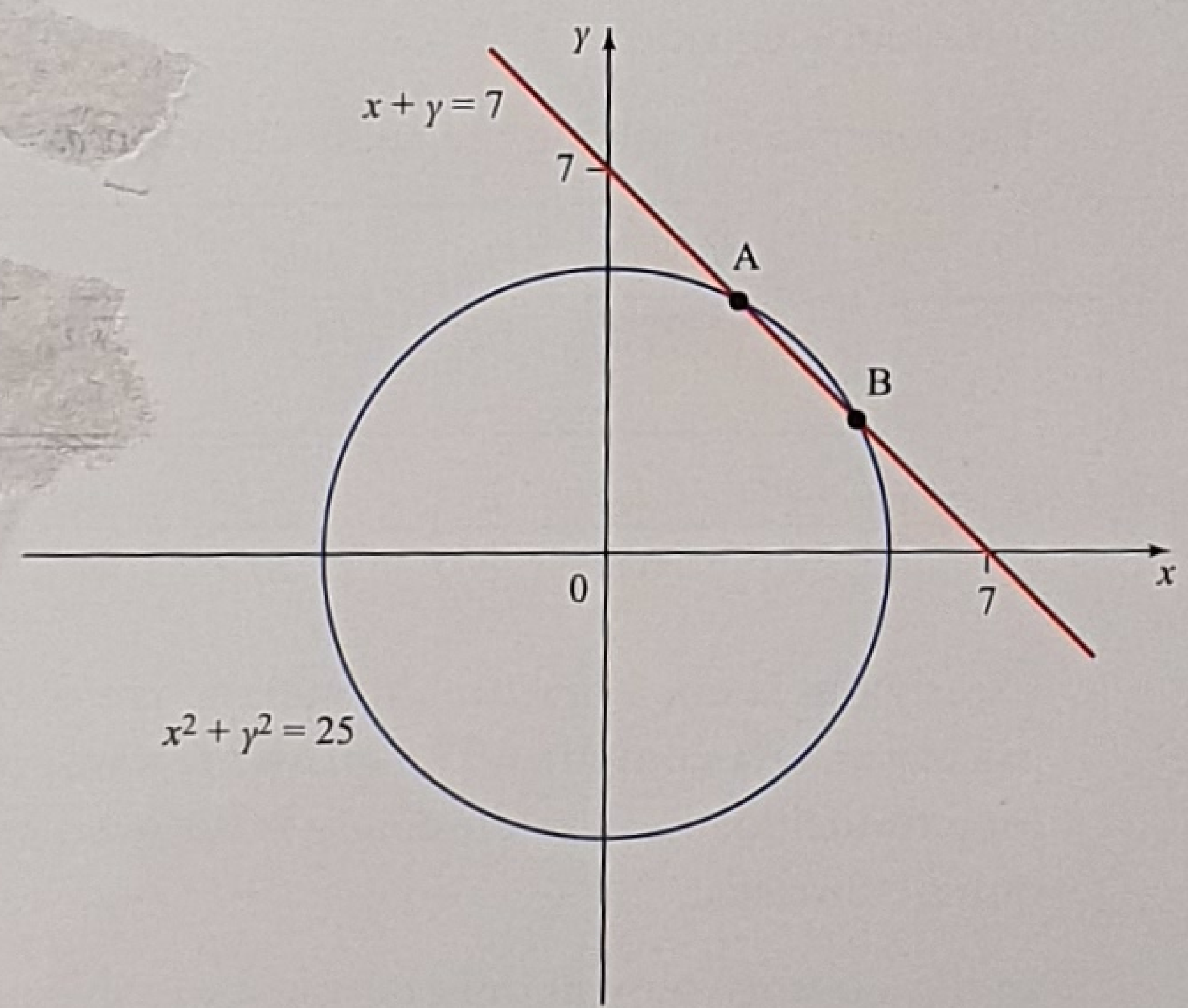


Figure 4.10

⑥ The sum of two numbers is 10. The product is –96

Work out the two numbers.

PS ⑦ (i) Work out the point of intersection of the circle $x^2 + y^2 = 8$ and the straight line $y - x = 4$

(ii) There is only one point of intersection of the circle and the line in part (i). Which of these diagrams is a sketch of the two graphs?
Give a reason for your choice.

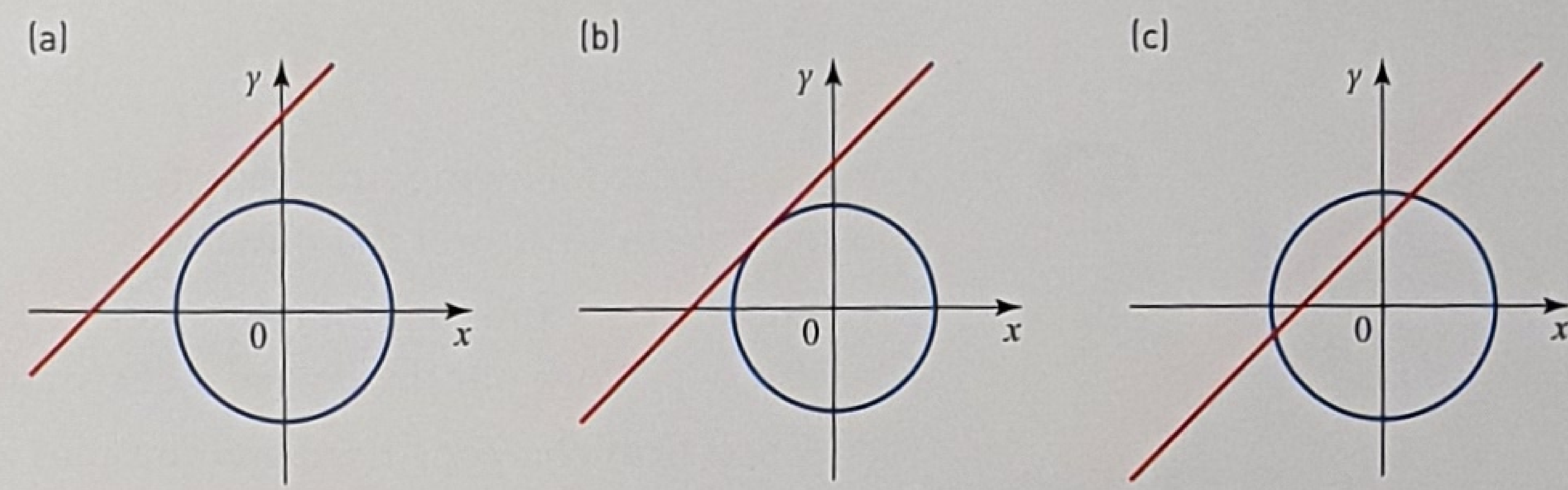


Figure 4.11

3 The factor theorem

Prior knowledge

Students should be familiar with function notation from their GCSE work.

The highest power in a quadratic is 2. Cubic expressions go up to 3, quartics to 4, quintics to 5, and so on. Such expressions are collectively referred to as polynomials. The degree of a polynomial is its highest power.

Note: a polynomial does not have negative or non-integer powers.

Just like the quadratic formula, there are formulae for solving cubic equations and quartic equations.

The formula for solving the cubic equation $ax^3 + bx^2 + cx + d = 0$ is

Students should not attempt to learn this formula. It is included here for interest only.

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Clearly this is not a practical formula to use without a pre-programmed calculator, or a computer. The quartic formula is even more complicated, and long to include here. Interestingly, it has been proved to be impossible to w quintic formula.

In this course, you will only be asked to solve polyno be reduced to linear and/or quadratic factors.

FUTURE USES

- Some calculators can solve complicated equations. Su use the Newton-Raphson method, which is a technique Mathematics at A-Level will learn. The above formula so the use of imaginary numbers even if the final answers a imaginary. Students of A-Level Further Mathematics will imaginary numbers (square roots of negative numbers).

Solving polynomial equations first involves use of the factor theorem.

Look at this quadratic equation.

$$x^2 - 5x - 6 = 0$$

Factorising $\Rightarrow$ $(x-6)(x+1) = 0$

$\Rightarrow$ $(x-6) = 0$ or $(x+1) = 0$

$\Rightarrow$ $x = 6$ or $x = -1$

Discussion points

- ➜ What happens if you substitute $x = 6$ into $x^2 - 5x - 6$?
- ➜ What about $x = -1$?

The factor theorem states this result in a general form.

If $(x - a)$ is a factor of the polynomial $f(x)$, then

- $f(a) = 0$
- $x = a$ is a root of the equation $f(x) = 0$.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Example 4.14

Given that

$f(x) = x^3 + 2x^2 - x - 2$

(i) find $f(1)$, $f(-1)$, $f(2)$, $f(-2)$

(ii) and hence factorise $x^3 + 2x^2 - x - 2$

Solution

(i) $f(1) = 1 + 2 - 1 - 2$
$= 0$ $\Rightarrow$ $(x - 1)$ is a factor

$f(-1) = (-1)^3 + 2(-1)^2 - (-1) - 2$
$= -1 + 2 + 1 - 2$
$= 0$ $\Rightarrow$ $(x + 1)$ is a factor

$f(2) = 8 + 8 - 2 - 2$
$= 12$ $\Rightarrow$ $(x - 2)$ is not a factor

$f(-2) = (-2)^3 + 2(-2)^2 - (-2) - 2$
$= -8 + 8 + 2 - 2$
$= 0$ $\Rightarrow$ $(x + 2)$ is a factor

(ii) Hence $x^3 + 2x^2 - x - 2 = k(x - 1)(x + 1)(x + 2)$
where k is a constant.
The coefficient of x^3 is 1, so k must be 1.
$f(x) = (x - 1)(x + 1)(x + 2)$

Example 4.15

Given that

$f(x) = x^3 + 3x^2 - x - 3$

(i) show that $(x + 1)$ is a factor of $f(x)$

(ii) suggest other values of x you should try when looking for another factor

(iii) solve the equation $f(x) = 0$

Solution

(i) $f(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3$

$= -1 + 3 + 1 - 3$

$= 0$

$\therefore$ $(x + 1)$ is a factor of $f(x)$

(ii) Any other linear factor will be of the form $(x - a)$ where a is a factor of the constant term (-3).

This means that the only other values of x which are worth trying are 1, 3 and -3.

(iii) $f(1) = 1 + 3 - 1 - 3$

$= 0 \quad \Rightarrow \quad (x - 1)$ is a factor

$f(3) = 27 + 27 - 3 - 3$

$= 48$

$f(-3) = -27 + 27 + 3 - 3$

$= 0 \quad \Rightarrow \quad (x + 3)$ is a factor

As $f(x)$ is a cubic, then there are no more than three roots.

$x = -1, x = 1, x = -3$

Sometimes you may only be able to find one linear factor for the cubic and, in this case, you then need to use long division.

Example 4.16

Given that

$f(x) = x^3 - x^2 - 3x - 1$

(i) show that $(x + 1)$ is a factor

(ii) factorise $f(x)$

(iii) solve $f(x) = 0$

Discussion points

- What happens when you try $x = 1$?
- Is there any other value you should try?

Solution

(i) $f(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1$

$= -1 - 1 + 3 - 1$

$= 0$

$\Rightarrow \quad (x + 1)$ is a factor of $x^3 - x^2 - 3x - 1$

(ii) Since $(x + 1)$ is a factor, then divide $f(x)$ by $(x + 1)$.

$$
\begin{array}{r}
x^2 - 2x - 1 \\
x + 1 \overline{\big) x^3 - x^2 - 3x - 1} \\
\underline{x^3 + x^2} \quad\quad\quad\quad \\
-2x^2 - 3x \quad\quad \\
\underline{-2x^2 - 2x} \quad\quad \\
-x - 1 \\
\underline{-x - 1} \\
0
\end{array}
$$

$x^3 + x^2$ is $x^2 \times (x + 1)$

$-2x^2 - 2x$ is $-2x \times (x + 1)$

$-x - 1$ is $-1 \times (x + 1)$

So $f(x) = (x + 1)(x^2 - 2x - 1)$

$x^2 - 2x - 1$ cannot be factorised, so $f(x)$ is now fully factorised.

(iii) $f(x) = 0 \Rightarrow (x+1)(x^2 - 2x - 1) = 0$

$\Rightarrow$ either $x = -1$ or $x^2 - 2x - 1 = 0$

Using the quadratic formula on $x^2 - 2x - 1 = 0$

gives

$$x = \frac{2 \pm \sqrt{4 - (4 \times 1 \times (-1))}}{2}$$
$$= \frac{2 \pm \sqrt{8}}{2}$$
$$= 2.414 \text{ or } -0.414$$

The complete solution is $x = -1, -0.414$ or 2.414 (to 3 d.p.)

! When using long division it is advisable to keep the terms in columns. This may mean that an extra zero term should be included to help with this.

Example 4.17

Given that $(x+2)$ is a factor of $x^3 - 5x - 2$, work out a quadratic factor.

Solution

Method 1

$$\begin{array}{r}
x^2 - 2x - 1 \\
x + 2 \overline{\smash{\big)}\, x^3 + 0x^2 - 5x - 2} \\
\underline{x^3 + 2x^2} \\
-2x^2 - 5x \\
\underline{-2x^2 - 4x} \\
-x - 2 \\
\underline{-x - 2} \\
0
\end{array}$$

$\Rightarrow \quad x^2 - 2x - 1$ is also a factor.

Method 2

As $(x+2)$ is a factor, $x^3 - 5x - 2 \equiv (x+2)(ax^2 + bx + c)$

You could compare coefficients as shown in Chapter 2.

However, it is clear that a must be 1 and c must be -1, so it is simpler to immediately write:

$x^3 - 5x - 2 \equiv (x+2)(x^2 + bx - 1)$

and then compare coefficients of x^2 or x to find b:

$0 = 2 + b$ or $-5 = 2b - 1$ both of which give $b = -2$,

so $x^3 - 5x - 2 \equiv (x+2)(x^2 - 2x - 1)$

The factor theorem

An extension to the factor theorem includes simplified fractional roots.

$$f\left(\frac{b}{a}\right) = 0 \quad \Leftrightarrow \quad (ax - b) \text{ is a factor of } f(x)$$

Example 4.18

(i) Show that $(x + 1)$ and $(3x - 2)$ are factors of

$3x^4 + 4x^3 - 16x^2 - 7x + 10$

(ii) Hence solve $3x^4 + 4x^3 - 16x^2 - 7x + 10 = 0$

Solution

(i) Let $f(x) = 3x^4 + 4x^3 - 16x^2 - 7x + 10$

$$\begin{aligned} f(-1) &= 3 \times (-1)^4 + 4 \times (-1)^3 - 16 \times (-1)^2 - 7 \times (-1) + 10 \\ &= 3 - 4 - 16 + 7 + 10 \\ &= 0 \end{aligned}$$

$\Rightarrow \quad (x + 1)$ is a factor of $3x^4 + 4x^3 - 16x^2 - 7x + 10$

$$\begin{aligned} f\left(\frac{2}{3}\right) &= 3 \times \left(\frac{2}{3}\right)^4 + 4 \times \left(\frac{2}{3}\right)^3 - 16 \times \left(\frac{2}{3}\right)^2 - 7 \times \frac{2}{3} + 10 \\ &= \frac{16}{27} + \frac{32}{27} - \frac{64}{9} - \frac{14}{3} + 10 \\ &= 0 \end{aligned}$$

$\Rightarrow \quad (3x - 2)$ is also a factor of $3x^4 + 4x^3 - 16x^2 - 7x + 10$

(ii) $(x + 1)(3x - 2) = 3x^2 + x - 2$

$$\begin{array}{r} x^2 + x - 5 \\ 3x^2 + x - 2 \overline{\big) 3x^4 + 4x^3 - 16x^2 - 7x + 10} \\ \underline{3x^4 + x^3 - 2x^2} \\ 3x^3 - 14x^2 - 7x \\ \underline{3x^3 + x^2 - 2x} \\ -15x^2 - 5x + 10 \\ \underline{-15x^2 - 5x + 10} \\ 0 \end{array}$$

$\Rightarrow \quad f(x) = (x + 1)(3x - 2)(x^2 + x - 5)$

This equals zero when

$$x + 1 = 0, \quad 3x - 2 = 0, \quad x^2 + x - 5 = 0$$

$$\Rightarrow x = -1 \quad \text{or} \quad x = \frac{2}{3} \qquad \text{or} \quad x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -5}}{2 \times 1}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{21}}{2}$$

Exercise 4C

① Determine whether the following linear functions are factors of the given polynomials.

(i) $x^3 - 8x + 7$ $\quad (x - 1)$

(ii) $x^3 + x^2 - 4x - 5$ $\quad (x + 2)$

(iii) $x^4 - 6x^2 + 10x - 12$ $\quad (x - 2)$

(iv) $x^5 + 32$ $\quad (x + 2)$

(v) $2x^4 - x^3 - 20$ $\quad (x + 2)$

(vi) $x^3 - ax^2 + a^2x - a^3$ $\quad (x - a)$

Exam questions are unlikely to require students to find five different linear factors of an expression. However, students will be expected to feel comfortable working with polynomials of such a high degree.

② Factorise the following functions as a product of linear factors.

(i) $x^3 - 3x^2 - x + 3$

(ii) $x^3 - 7x - 6$

(iii) $x^3 - x^2 - 2x$

(iv) $x^3 - 2x^2 - 13x - 10$

(v) $x^3 - x^2 - 14x + 24$

(vi) $x^4 - 3x^3 - 11x^2 + 3x + 10$

(vii) $x^4 - 4x^3 + 6x^2 - 4x + 1$

(viii) $x^4 - 13x^2 + 36$

(ix) $x^5 - 4x^4 - 17x^3 + 24x^2 + 36x$

(x) $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120$

③ Solve the following equations.

(i) $x^3 - 2x^2 - 5x + 6 = 0$

(ii) $x^3 + 3x^2 - 6x - 8 = 0$

(iii) $x^3 - 2x^2 - 21x - 18 = 0$

(iv) $x^4 + 3x^3 - 5x^2 - 3x + 4 = 0$

(v) $2x^3 + x^2 - 7x + 4 = 0$

(vi) $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120 = 0$

④ $f(x) = x^3 + 2x^2 + ax - 76$

Given that $(x - 4)$ is a factor of $f(x)$, work out the value of a.

⑤ $f(x) = x^3 + px^2 + qx + 6$

(i) Given that $(x - 1)$ is a factor of $f(x)$, write an equation in p and q.

(ii) Given also that $(x + 3)$ is a factor of $f(x)$, write another equation in p and q.

(iii) Solve your simultaneous equations to work out the values of p and q.

PS ⑥ (i) Work out the value of k for which $x = 2$ is a root of $x^3 + kx + 6 = 0$

(ii) Work out the other roots when k takes this value.

PS ⑦ The diagram shows an open cuboid tank whose base is a square of side x metres and whose volume is $8\ \text{m}^3$.

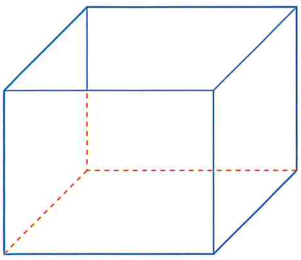


Figure 4.12

(i) Write down an expression in terms of x for the height of the tank.

(ii) Show that the surface area of the tank is $\left(x^2 + \dfrac{32}{x}\right)\ \text{m}^2$.

(iii) Given that the surface area is $24\ \text{m}^2$, show that

$x^3 - 24x + 32 = 0$

(iv) Solve $x^3 - 24x + 32 = 0$ to work out the possible values of x.

PS ⑧ $(x - 1)$ and $(5x + 2)$ are factors of $5x^4 + px^3 - 10x^2 + qx + 2$

Solve $5x^4 + px^3 - 10x^2 + qx + 2 = 0$

4 Linear inequalities

Discussion point

- The radius of the Earth's orbit around the Sun is approximately 1.5×10^8 km; that of Mars is about 2.3×10^8 km. The Earth takes 365 days for one orbit and Mars takes 687 days.
- Given that the distance from Earth to Mars is x km, what can you say about x?

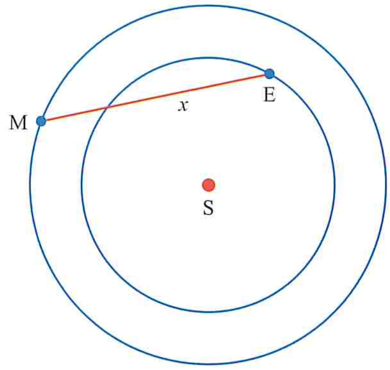


Figure 4.13

Discussion point

- What is the difference between an equation and an inequality?

Some inequalities appear as a combination of one, or more, expressions which must be solved in order to work out a range of possible values for the variable.

Example 4.19

Solve $2y + 6 < 5y + 12$

Discussion points

- In what ways is solving an inequality like solving an equation?
- In what ways is it different?
- Explain, with examples, why you need to reverse the inequality sign when you multiply or divide an inequality by a negative number.

Solution

Method 1

		$2y + 6 < 5y + 12$
Subtract $2y$	$\Rightarrow$	$6 < 3y + 12$
Subtract 12	$\Rightarrow$	$-6 < 3y$
Divide by 3	$\Rightarrow$	$-2 < y$
Make y the subject	$\Rightarrow$	$y > -2$

Method 2

		$2y + 6 < 5y + 12$
Subtract $5y$	$\Rightarrow$	$-3y + 6 < 12$
Subtract 6	$\Rightarrow$	$-3y < 6$
Divide by (-3)	$\Rightarrow$	$y > -2$

Example 4.20

Solve the inequality $5 < 3x - 1 \leqslant 17$

Solution

Add 1 throughout	$\Rightarrow$	$6 < 3x \leqslant 18$
Divide by 3	$\Rightarrow$	$2 < x \leqslant 6$

The following example, although algebraic, is solved using knowledge of number operations and number facts.

Example 4.21

(i) Given that $-2 < x < 5$, work out an inequality for x^2.

(ii) Given that $1 \leqslant a \leqslant 4$ and $-3 \leqslant b \leqslant 2$, work out an inequality for $a - b$.

Solution

(i) Squaring a negative number results in a positive number. Also $(-2)^2 < 5^2$.

So $\quad 0 \leqslant x^2 < 25$

(ii) The least value of $a - b$ will occur when a takes its least value and b takes its greatest value.

Least value $= 1 - 2$

$= -1$

The greatest value of $a - b$ will occur when a takes its greatest value and b takes its least value.

Greatest value $= 4 - (-3)$

$= 4 + 3$

$= 7$

So, $\quad -1 \leqslant a - b \leqslant 7$

Exercise 4D

① Solve the following inequalities.

(i) $2x - 3 < 7$ (ii) $5 + 3x \geqslant 11$

(iii) $6y + 1 \leqslant 4y + 9$ (iv) $y - 4 > 3y - 12$

(v) $4x + 1 \geqslant 3x - 2$ (vi) $b - 3 \leqslant 5b + 9$

(vii) $\dfrac{x + 5}{2} > 1$ (viii) $\dfrac{2x - 3}{3} < 7$

(ix) $\dfrac{5 - 3x}{4} \leqslant 5$ (x) $\dfrac{2 - 4x}{3} \geqslant 6$

(xi) $4 \leqslant 5x - 6 \leqslant 14$ (xii) $11 \leqslant 3x + 5 \leqslant 20$

(xiii) $5 < 7 - 2x < 13$ (xiv) $5 > 9 - 4x > 1$

② Given that $0 \leqslant p \leqslant 3$ and $2 \leqslant q \leqslant 5$, work out the inequality for $p - q$.

③ Given that $-2 < x < 4$ and $1 < y < 3$, work out the inequality for $x + y$.

④ Given that $1 \leqslant a \leqslant 6$ and $-3 \leqslant b \leqslant 3$, work out inequalities for

(i) $a + b$ (ii) $a - b$.

⑤ Given that $-3 \leqslant a \leqslant 0$ and $-1 \leqslant b \leqslant 10$, work out inequalities for

(i) $a + b$ (ii) $a - b$ (iii) $2a + 3b$.

PS ⑥ Given that $x > 2$ and $y < 0$, decide whether each of the following statements are ALWAYS TRUE, SOMETIMES TRUE or NEVER TRUE.

(i) $4x > 8$ (ii) $2y > 0$ (iii) $x + y < 2$

(iv) $x^2 > 10$ (v) $y^2 < 0$ (vi) $x - y > 2$

PS ⑦ Given that $0 < x < 1$ and $y > 0$, decide whether each of the following statements are ALWAYS TRUE, SOMETIMES TRUE or NEVER TRUE.

(i) $\dfrac{1}{x} > 1$ (ii) $\dfrac{y}{x} > 0$ (iii) $x + y < 0$

(iv) $xy > 4$ (v) $x^2 > 1$ (vi) $x - y < 0$

PS ⑧ The square and rectangle have dimensions in centimetres.

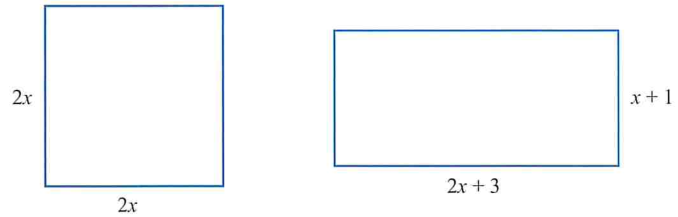


Figure 4.14

The perimeter of the square is greater than the perimeter of the rectangle. Which values of x satisfy this condition?

5 Quadratic inequalities

The quadratic inequalities in this section all involve quadratic expressions that factorise. This means that you can either find a solution by sketching the appropriate graph, or you can use line segments to reduce the quadratic inequality to two simultaneous linear inequalities.

Example 4.22

Solve

(i) $x^2 - 2x - 3 < 0$

(ii) $x^2 - 2x - 3 \geqslant 0$

Solution

Method 1

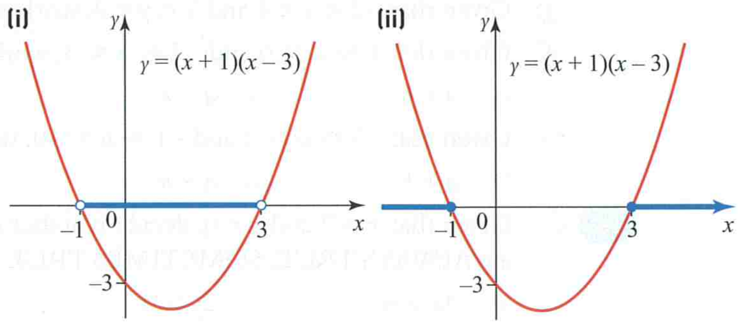
$x^2 - 2x - 3 = (x + 1)(x - 3)$

So the graph of $y = x^2 - 2x - 3$ crosses the x-axis when $x = -1$ and $x = 3$.

Look at the two graphs in Figure 4.15.

Here the end points *are not* included in the solution, so you draw open circles: ○.

Here the end points *are* included in the solution, so you draw solid circles: ●.



Notice how the solution is in two parts when there are two line segments.

Figure 4.15

(i) You want the values of x for which $y < 0$, that is where the curve is below the x-axis. The solution is $-1 < x < 3$

(ii) You want the values of x for which $y \geqslant 0$, that is where the curve crosses or is above the x-axis. The solution is $x \leqslant -1$ or $x \geqslant 3$

An alternative method identifies the values of x for which each of the factors is zero and considers the sign of each factor in the intervals between these critical values.

Method 2

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
sign of $(x+1)$	$-$	0	$+$	$+$	$+$
sign of $(x-3)$	$-$	$-$	$-$	0	$+$
sign of $(x+1)(x-3)$	$(-) \times (-) = +$	$(0) \times (-) = 0$	$(+) \times (-) = -$	$(+) \times (0) = 0$	$(+) \times (+) = +$

Don't forget to reverse the sign if you multiply (or divide) throughout any inequality by a negative value.

Short method for solving $ax^2 + bx + c < 0$ (or $>$ or $\leqslant$ or $\geqslant$)

Step 1: Solve $ax^2 + bx + c = 0$ to get the critical values p and q ($p \leqslant q$)

Step 2: If $a < 0$ then multiply throughout by -1

Step 3: If the inequality is $<$ (or $\leqslant$) the solution is $p < x < q$ (or $p \leqslant x \leqslant q$)

Step 4: If the inequality is $>$ (or $\geqslant$) the solution is $x < p$ or $x > q$ (or $x \leqslant p$ or $x \geqslant q$)

Example 4.23

Solve $2x + x^2 > 3$

Solution

$$2x + x^2 > 3$$

$\Rightarrow \quad x^2 + 2x - 3 > 0$

$\Rightarrow \quad (x-1)(x+3) > 0$

$\Rightarrow$ the critical values are -3 and 1

Select the outer regions because the inequality sign is $>$.

$\therefore$ the solution is $x < -3$ or $x > 1$

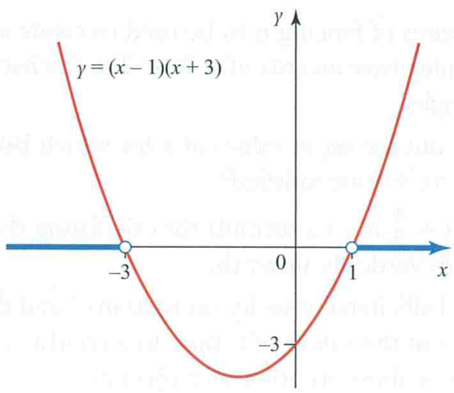


Figure 4.16

Exercise 4E

1. Solve the following inequalities.

   (i) $x^2 - 6x + 5 > 0$
   (ii) $a^2 + 3a - 4 \leqslant 0$
   (iii) $2y^2 + y - 3 < 0$
   (iv) $4 - y^2 \geqslant 0$
   (v) $x^2 - 4x + 4 > 0$
   (vi) $p^2 - 3p \leqslant -2$
   (vii) $(a + 2)(a - 1) > 4$
   (viii) $8 - 2a \geqslant a^2$
   (ix) $3y^2 + 2y - 1 > 0$
   (x) $y^2 \geqslant 4y + 5$

2. For which values of x are the following graphs below the x-axis?

   (i) $y = x^2 - x - 6$
   (ii) $y = x^2 + 2x - 8$
   (iii) $y = x^2 + 6x + 8$
   (iv) $y = x^2 - 5x + 6$
   (v) $y = 2x^2 - x - 1$
   (vi) $y = -x^2 - x + 6$
   (vii) $y = 21 + 4x - x^2$
   (viii) $y = 5x + 2 - 3x^2$

PS 3. The square and rectangle have dimensions in centimetres.

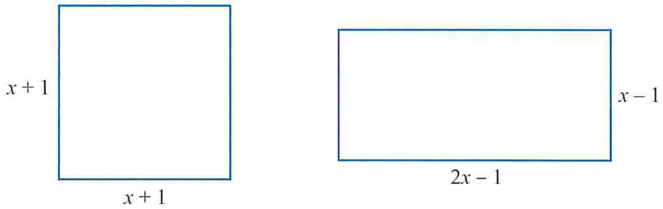


Figure 4.17

The area of the square is less than the area of the rectangle.
For which values of x is this condition satisfied?

PS 4. The triangle and parallelogram have dimensions in metres.

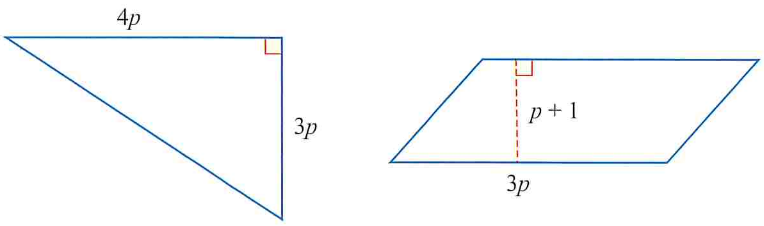


Figure 4.18

The area of the triangle is less than the area of the parallelogram.
For which values of p is this condition satisfied?

PS 5. 20 metres of fencing is to be used to create a rectangular sheep pen. If the pen must have an area of at least $21\,\text{m}^2$, what are the possible lengths of the rectangle?

PS 6. Work out the set of values of x for which **both** $2(x + 4) \geqslant 13$ **and** $x^2 < 5x + 6$ are satisfied?

PS 7. $s = ut + \frac{1}{2}at^2$ is a formula for calculating the height (s) of a ball when thrown vertically upwards.

   If the ball's initial velocity (u) is $20\,\text{m}\,\text{s}^{-1}$ and the acceleration (a) is $-10\,\text{m}\,\text{s}^{-2}$, work out the values of t (time in seconds) for which the ball is more than 18.75 m above its point of projection.

PS ⑧ The graphs of $y = 4 - x^2$ and $y = x^2 - 2x - 3$ have been plotted on the same axes.

For which values of x is the graph of $y = 4 - x^2$ above the graph of $y = x^2 - 2x - 3$?

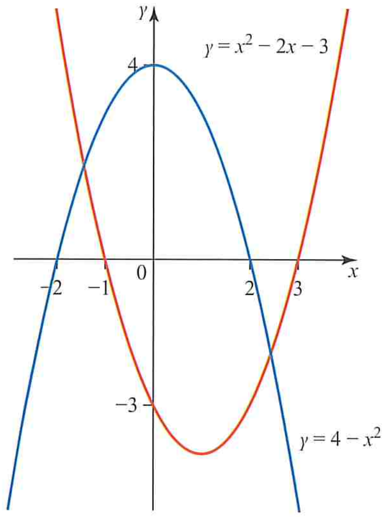


Figure 4.19

6 Indices

The three index laws are summarised here.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{m \times n}$$

These laws apply for **all** values of m and n.

ACTIVITY 4.1

Use the law $a^m \times a^n = a^{m+n}$ to answer these.

(i) Write $a^3 \times a^0$ as a single power of a.

Hence state the value of a^0.

(ii) Write $a^2 \times a^{-2}$ as a single power of a.

Hence write a^{-2} in the form $\frac{1}{a^p}$ where p is a positive integer.

(iii) Write $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ as a single power of a.

Hence copy and complete the statement $a^{\frac{1}{2}}$ is the .... root of a.

(iv) Write $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}$ as a single power of a.

Hence copy and complete the statement $a^{\frac{1}{3}}$ is the .... root of a.

The following facts should be known.

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example 4.24

Express as single powers of x.

(i) $\dfrac{x^3 \times x^2}{x^9}$ (ii) $\sqrt[3]{x^5 \div x^3}$

(iii) $\sqrt{\dfrac{x^{\frac{3}{2}} \times x^{\frac{1}{2}}}{(x^3)^2}}$

Solution

(i) $\dfrac{x^3 \times x^2}{x^9} = \dfrac{x^{3+2}}{x^9}$

$= \dfrac{x^5}{x^9}$

$= x^{5-9}$

$= x^{-4}$

(ii) $\sqrt[3]{x^5 \div x^3} = \sqrt[3]{x^{5-3}}$

$= \sqrt[3]{x^2}$

$= (x^2)^{\frac{1}{3}}$

$= x^{2\times\frac{1}{3}}$

$= x^{\frac{2}{3}}$

(iii) $\sqrt{\dfrac{x^{\frac{3}{2}} \times x^{\frac{1}{2}}}{(x^3)^2}} = \sqrt{\dfrac{x^{\frac{3}{2}+\frac{1}{2}}}{x^{3\times 2}}}$

$= \sqrt{\dfrac{x^2}{x^6}}$

$= \sqrt{x^{2-6}}$

$= \sqrt{x^{-4}}$

$= (x^{-4})^{\frac{1}{2}}$

$= x^{-4\times\frac{1}{2}}$

$= x^{-2}$

Example 4.25

Solve

(i) $x^{\frac{3}{2}} = 8$

(ii) $x^{-\frac{1}{3}} = 10$

Solution

(i) **Method 1**

Square both sides $\left(x^{\frac{3}{2}}\right)^2 = 8^2$

$x^3 = 64$

Cube root both sides $x = \sqrt[3]{64}$

$x = 4$

Method 2

Take power $\frac{2}{3}$ of both sides $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 8^{\frac{2}{3}}$

$$x^1 = (8^2)^{\frac{1}{3}}$$

$$x = 64^{\frac{1}{3}}$$

$$x = \sqrt[3]{64}$$

$$x = 4$$

(ii) **Method 1**

Take the reciprocal of both sides $\left(x^{-\frac{1}{3}}\right)^{-1} = 10^{-1}$

The reciprocal of a^{-m} is a^m → $x^{\frac{1}{3}} = \frac{1}{10}$

Cube both sides $x = \left(\frac{1}{10}\right)^3$

$$x = \frac{1}{1000}$$

Method 2

Cube both sides $\left(x^{-\frac{1}{3}}\right)^3 = 10^3$

$$x^{-1} = 1000$$

Take the reciprocal of both sides $x = 1000^{-1}$ ← The reciprocal of x^{-1} is x

$$x = \frac{1}{1000}$$

Disguised quadratic equations

Some equations, which initially look difficult to solve, can be transformed into quadratic equations which can be solved by factorising or by use of the formula.

An example of such an equation would be $x^4 = 5x^2 - 4$

As it is a quartic equation, the initial reaction is to rearrange it to equal zero, and then search for factors.

However, on closer inspection we notice that there are no odd powers. The substitution $y = x^2$ can be used to transform the equation.

As $x^4 = \left(x^2\right)^2 = y^2$ then the equation can be rewritten as $y^2 = 5y - 4$, which can be solved easily.

$$y^2 - 5y + 4 = 0$$

$$\Rightarrow \quad (y-4)(y-1) = 0$$

$$\Rightarrow \quad y = 4 \text{ or } y = 1$$

but $y = x^2$ $\quad \therefore \quad x^2 = 4 \text{ or } x^2 = 1$

$$\Rightarrow \quad x = \pm 2 \text{ or } x = \pm 1$$

Disguised quadratics are sometimes difficult to spot. However they are easier to spot if the index laws are learnt and practised thoroughly.

Example 4.26

Solve $4^x = 2^x + 56$

> **!** A common error is to treat 4^x as 2×2^x but this is wrong.
> Instead, 4^x is $2^x \times 2^x$.

Solution

Remembering the index law $\left(a^m\right)^n = a^{mn}$, you can rewrite 4^x.

$$4^x = \left(2^2\right)^x = 2^{2x} = \left(2^x\right)^2$$

So a substitution of $y = 2^x$ should make the question easier to answer.

$$\begin{aligned}
&& 4^x &= 2^x + 56 \\
&\Rightarrow & \left(2^x\right)^2 &= 2^x + 56 \\
\text{Let } y = 2^x & \therefore & y^2 &= y + 56 \\
&\Rightarrow & y^2 - y - 56 &= 0 \\
&\Rightarrow & (y-8)(y+7) &= 0 \\
&\Rightarrow & y = 8 &\text{ or } y = -7 \\
\text{but } y = 2^x & \therefore & 2^x = 8 &\text{ or } 2^x = -7
\end{aligned}$$

but $2^x > 0$ for all values of x

$$\begin{aligned}
&\therefore & 2^x &= 8 \\
&\Rightarrow & x &= 3 \qquad \text{because } 2^3 = 8
\end{aligned}$$

Discussion points

- Which powers, and which roots, can be negative, and which ones can't?
- Can an exponential function be negative?

> **!** Always check your answers by substituting them back into the original equation. This is good advice when solving any equation, but is particularly good advice in questions such as these.
>
> To avoid rogue answers occurring, remember:
>
> A negative value of x^2 cannot produce a valid solution; but a negative x^3 can.
>
> $\sqrt{x}$ can never be negative; but $\sqrt[3]{x}$ can be negative.
>
> 2^x can never be negative.

Exercise 4F

① Write as single powers of x.

(i) $\dfrac{x \times x^6}{x^3}$

(ii) $x^2 \times x^3 \div x^8$

(iii) $\sqrt{x^7 \div x^2}$

(iv) $x^{\frac{1}{2}} \times x^{\frac{3}{2}} \times x^4$

(v) $\left(x^{\frac{1}{3}}\right)^{12}$

(vi) $\sqrt[3]{x^7 \times x^{-1}}$

(vii) $\sqrt{\dfrac{x^{\frac{5}{2}} \times x^{\frac{1}{2}}}{x^5}}$

(viii) $\left(\sqrt{x}\right)^{10} \div x^{-5}$

(ix) $\sqrt[4]{\dfrac{x^5 \times x^8}{x^3 \times x^2}}$

② Solve, giving solutions as exact values.

(i) $x^{\frac{1}{2}} = 3$

(ii) $x^{\frac{1}{3}} = -2$

(iii) $x^{\frac{1}{2}} = \dfrac{1}{3}$

(iv) $x^{-\frac{1}{2}} = 2$

(v) $x^{-\frac{1}{3}} = 4$

(vi) $x^{-\frac{1}{3}} = 2$

(vii) $x^{\frac{2}{3}} = 4$

(viii) $x^{-\frac{1}{2}} = \dfrac{1}{3}$

(ix) $x^{-2} = \dfrac{9}{25}$

(x) $x^{\frac{3}{2}} = 27$

(xi) $x^{-\frac{1}{3}} = \dfrac{2}{3}$

(xii) $x^{-4} = 10\,000$

③ Expand the following.

(i) $x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$

(ii) $x^{-2}(x^3 - x^2)$

(iii) $x^{\frac{2}{3}}\left(x^{\frac{1}{3}} - x^{\frac{7}{3}}\right)$

(iv) $x^{-3}(x^{-2} + x^{-1})$

(v) $x^{-\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{7}{2}}\right)$

(vi) $x^{\frac{1}{3}}\left(x^{\frac{8}{3}} - x^{-\frac{1}{3}}\right)$

④ Solve these quadratic equations in $\sqrt{x}$.

(i) $\left(\sqrt{x}\right)^2 - 3\sqrt{x} + 2 = 0$

(ii) $2\left(\sqrt{x}\right)^2 - 5\sqrt{x} + 3 = 0$

(iii) $x - \sqrt{x} - 6 = 0$

(iv) $\sqrt{x} - 6 + \dfrac{5}{\sqrt{x}} = 0$

(v) $2\sqrt{x} + 3 = \dfrac{2}{\sqrt{x}}$

(vi) $\dfrac{3}{\sqrt{x}} + \dfrac{2}{x} = 5$

⑤ Solve these equations in x^2.

(i) $(x^2)^2 - 13x^2 + 36 = 0$ (ii) $x^4 - 15x^2 - 16 = 0$

(iii) $2x^4 - 7x^2 - 4 = 0$ (iv) $4x^4 - x^2 = 18$

(v) $2x^2 = 17 + \frac{9}{x^2}$

PS ⑥ Solve these disguised quadratics.

(i) $x^6 - 7x^3 - 8 = 0$ (ii) $x^{\frac{1}{3}} = 1 + 2x^{-\frac{1}{3}}$

(iii) $(2^x)^2 - 6 \times 2^x + 8 = 0$ (iv) $2^{2x} = 12 \times 2^x - 32$

(v) $9^x = 3^x + 6$ (vi) $4^x = 2^{x+1} + 48$

PS ⑦ Solve the simultaneous equations $5^x + 2^y = 33$ and $5^x - 2^y = 17$.

PS ⑧ Solve these equations.

(i) $2^x = 16$ (ii) $2^{-x} = 8$

(iii) $4^x = 32$ (iv) $9^x = 27$

(v) $27^x = 9^{x+2}$

Hint: Rewrite 27 as 3^3 and 9 as 3^2.

PS ⑨ Solve these index equations.

Hint: Use $x^0 = 1$ and $1^x = 1$.

(i) $(x + 1)^x = 1$ (ii) $(x - 4)^{x+2} = 1$

(iii) $(x - 3)^{x+2} = 1$ (iv) $(x^2 - 5x + 5)^{x+4} = 1$

(v) $(x^2 - 4x + 5)^{x^2-2x-8} = 1$

ACTIVITY 4.2

Create an equation of the form $(ax^2 + bx + c)^{px^2+qx+r} = 1$ which is true for six different values of x.

7 Algebraic proof

Any of the algebraic skills covered in previous sections may be needed in proofs.

When constructing a proof, avoid writing the required result as the first line of working. This is a common error. Instead, start with a given expression and gradually change it using algebraic processes, or start with a known fact which, when combined with other known facts, proves the required result.

Example 4.27

Prove that $2a^3 - a^2(2a - 9)$ is a square number when a is an integer.

Solution

Expand and simplify $\quad 2a^3 - a^2(2a - 9) = 2a^3 - 2a^3 + 9a^2$

$$= 9a^2$$

$$= (3a)^2$$

As a is an integer then $3a$ is also an integer.

$\therefore\ 2a^3 - a^2(2a - 9)$ is a square number when a is an integer.

The final line of any proof should be a repeat of the required statement in the question.

Example 4.28

(i) Express $x^2 - 8x + 18$ in the form $(x - a)^2 + b$ where a and b are integers.

(ii) Hence, prove that $x^2 - 8x + 18$ is always positive.

Solution

(i) $x^2 - 8x + 18 \equiv (x - a)^2 + b$

$x^2 - 8x + 18 \equiv x^2 - 2ax + a^2 + b$

Equate coefficients of x $\quad -8 = -2a$

$4 = a$

Equate constants $\quad 18 = a^2 + b$

$18 = 16 + b$

$2 = b$

An alternative is to complete the square of $x^2 - 8x$

i.e. $x^2 - 8x + 18$

$= (x - 4)^2 - 4^2 + 18$

$= (x - 4)^2 + 2$

$x^2 - 8x + 18 \equiv (x - 4)^2 + 2$

(ii) You know that $\quad (x - 4)^2 \geqslant 0$

When a number is squared the answer can never be negative.

$\Rightarrow\ (x - 4)^2 + 2 \geqslant 2$

$\Rightarrow\ (x - 4)^2 + 2 > 0$

$\Rightarrow\ x^2 - 8x + 18 > 0$

Example 4.29

c and d are positive integers such that $c > d$.

$$f(x) = \frac{2c + cx}{2d + dx} \qquad x \neq -2$$

Prove that $f(x) > 1$

Solution

Factorise the numerator and denominator $\quad f(x) = \dfrac{c(2 + x)}{d(2 + x)}$

Cancel $(2 + x)$ $\quad f(x) = \dfrac{c}{d}$

But $c > d$ (and d is positive) $\Rightarrow\ \dfrac{c}{d} > 1 \quad \therefore\ \ f(x) > 1$

Exercise 4G

1. Prove that $2(m + 7) - 2(5 + m)$ is always a positive integer.
2. Prove that $5(c - 3) + 3(c + 7)$ is always even when c is a positive integer.
3. Prove that $(y + 6)(y + 3) - y^2$ is a multiple of 9 when y is a positive integer.
4. $f(n) = n^2$ for all positive integer values of n.
   (i) Show that $f(n + 1) = n^2 + 2n + 1$.
   (ii) Prove that $f(n + 1) + f(n - 1)$ is always even.
   (iii) Prove that $f(n + 1) - f(n - 1)$ is always a multiple of 4.
5. (i) Express $x^2 + 2x + 5$ in the form $(x + a)^2 + b$ where a and b are integers.
   (ii) Hence, prove that $x^2 + 2x + 5$ is always positive.
6. PS Prove that $y^2 - 10y + 26 > 0$ for all values of y.
7. PS Prove that $9m^2(3m - 1) + (3m)^2$ is a cube number when m is a positive integer.
8. PS Prove that $\dfrac{6p - 18}{2p - 6}$ is always a positive integer when $p \neq 3$
9. PS a is a positive number, b is a negative number.

   $a \neq -b$

   Prove that $\dfrac{a^2 + ab}{ab + b^2}$ is negative.
10. PS $f(x) = x^2 + 2x$.

    Prove that $f(4x) = kx(2x + 1)$ where k is an integer.

FUTURE USES

Students who go on to study mathematics at A-Level will learn a variety of proof techniques, including proof by contradiction and proof by induction.

8 Sequences

Here are the first few terms of a sequence.

4 10 16 22 28 ...

The first term is 4. Each subsequent term can be obtained by adding 6 to the previous term.

Here is another sequence

1 4 9 16 25 ...

This is the set of square numbers.

The nth term of a sequence is an expression in terms of n.

To find a particular term, a value for n is substituted into the expression.

Example 4.30

Work out the first three terms and the tenth term of the sequence given by

nth term $= n^2 + \dfrac{30}{n}$.

Solution

$$1\text{st term} = 1^2 + \frac{30}{1} = 31$$

$$2\text{nd term} = 2^2 + \frac{30}{2} = 19$$

$$3\text{rd term} = 3^2 + \frac{30}{3} = 19$$

$$10\text{th term} = 10^2 + \frac{30}{10} = 103$$

Example 4.31

The nth term of a sequence is $n^2 - 2n$.

Work out which two consecutive terms have a sum of 179

Solution

$$n\text{th term} = n^2 - 2n$$

$$\Rightarrow \quad (n+1)\text{th term} = (n+1)^2 - 2(n+1)$$

$$\therefore \quad \text{sum of two consecutive terms} = n^2 - 2n + (n+1)^2 - 2(n+1) = 179$$

$$\Rightarrow \quad n^2 - 2n + n^2 + 2n + 1 - 2n - 2 = 179$$

$$\Rightarrow \quad 2n^2 - 2n - 180 = 0$$

$$\Rightarrow \quad n^2 - n - 90 = 0$$

$$\Rightarrow \quad (n-10)(n+9) = 0$$

$$\Rightarrow \quad n = 10 \quad \text{or} \quad n = -9$$

but n is a number of terms and so must be positive $\therefore\ n = 10$

$\therefore$ the required terms are the 10th term and the 11th term

$$10\text{th term} = 10^2 - 2 \times 10 = 80$$

$$11\text{th term} = 11^2 - 2 \times 11 = 99$$

Linear sequences

The nth term of a linear sequence will be of the form $an + b$ where a and b are constants. Such sequences are sometimes called arithmetic sequences.

Example 4.32

Work out the nth term of the linear sequence

3 12 21 30 39 ...

Solution

Method 1

$$n\text{th term} = an + b$$

When $n = 1$, the term is 3 $\quad 3 = a(1) + b$

$$3 = a + b \qquad ①$$

When $n = 2$, the term is 12 $\quad 12 = a(2) + b$

$12 = 2a + b \quad$ ②

Subtracting ① from ② $\quad 9 = a$

Substituting in ① $\quad 3 = 9 + b$

$-6 = b$

nth term $= 9n - 6$

Method 2

nth term $= an + b$

The common difference between the terms is $\quad 12 - 3 = 9$

This will be the coefficient of n $\quad a = 9$

When $n = 1$ the term is 3 $\quad 3 = 9(1) + b$

$3 = 9 + b$

$-6 = b$

nth term $= 9n - 6$

Example 4.33

Work out the nth term of the linear sequence

15 13 11 9 …

> **Note**
>
> For a linear sequence which decreases, the common difference is negative.

Solution

The common difference is $13 - 15 = -2$

Find a number C such that $-2 \times 1 + C = 15$

C must be 17

So the nth term is $-2n + 17$

(which can also be written as $17 - 2n$)

Exercise 4H

① Work out an expression for the nth term for each of the following linear sequences.

(i) 10 14 18 22 26 …
(ii) 2 9 16 23 …
(iii) -5 -3 -1 1 3 …
(iv) 0 25 50 75 …
(v) -11 -3 5 13 …
(vi) 3 3.5 4 4.5 5 …
(vii) 40 30 20 10 0 -10 …
(viii) 7 4 1 -2 -5 …
(ix) 1 $\frac{1}{2}$ 0 $-\frac{1}{2}$ -1 …
(x) -4 -5.5 -7 -8.5 -10 …

② (i) Work out the 100th term of the linear sequence -5 1 7 13 …
(ii) Work out the 50th term of the linear sequence 35 28 21 14 …
(iii) Work out the 200th term of the linear sequence -1 -10 -19 -28 …

PS ③ Here is a linear sequence.

3 5.5 8 10.5 13 …

Work out the value of the first term of the sequence that is greater than 250.

PS ④ Here are two linear sequences.

Sequence A 7 5 3 1 −1 …

Sequence B 3 0 −3 −6 …

Prove that the sum of the nth terms of the two sequences is $15 - 5n$.

PS ⑤ Here is a linear sequence.

$p \quad p + 2q \quad p + 4q \quad p + 6q \quad \ldots$

The third term of the sequence is 20

The fourth term of the sequence is 56

(i) Work out the values of p and q.

(ii) Work out an expression for the nth term of the sequence.

PS ⑥ The nth term of a sequence is $7n - 3$

Explain why 44 is not a term in the sequence.

PS ⑦ The nth term of a sequence is n^3.

Prove that the difference between consecutive terms is never a multiple of 3

PS ⑧ The nth term of a sequence is $n^2 - 40n + 405$.

Prove that every term is positive.

Quadratic sequences

The nth term of a quadratic sequence is of the form $an^2 + bn + c$ where a, b and c are constants.

One method for finding the nth term of a quadratic sequence is to work out the difference between the differences.

Consider three consecutive terms:

$$(n-1)\text{th term} = a(n-1)^2 + b(n-1) + c = an^2 + bn + c - 2an + a - b$$

$$n\text{th term} = an^2 + bn + c$$

$$(n+1)\text{th term} = a(n+1)^2 + b(n+1) + c = an^2 + bn + c + 2an + a + b.$$

The difference between the nth term and the $(n-1)$th term is $2an - a + b$.

The difference between the $(n+1)$th term and the nth term is $2an + a + b$.

The difference between consecutive differences is then $2a$.

So the coefficient of n^2 (a) is always half of the second difference.

Example 4.34

Work out the nth term of the quadratic sequence 3 6 13 24 39 …

Solution

nth term $= an^2 + bn + c$

Work out the second differences.

For a quadratic sequence, all the second differences will be the same

3		6		13		24		39
	3		7		11		15	
		4		4		4		

The coefficient a is half of the second difference.

In this case a = half of 4 = 2.

Then compare the sequence $2n^2$ with the original sequence.

Original sequence	=	3	6	13	24	39
$2n^2$	=	2	8	18	32	50

What needs to be added to each of the $2n^2$ sequence to make the original sequence?

1 −2 −5 −8 −11

Notice that this sequence is linear and its nth term is $-3n + 4$

The required quadratic sequence is the sum of the two sequences $2n^2$ and $-3n + 4$

nth term $= 2n^2 - 3n + 4$

Another method for finding the nth term of a quadratic sequence is shown in the final section of this chapter, in Example 4.38.

Exercise 4I

1. Work out the nth term for each of the following quadratic sequences.
   (i) 4 9 16 25 36 …
   (ii) 0 6 14 24 36 …
   (iii) 4 13 24 37 52 …
   (iv) 8 21 40 65 96 …
   (v) 4 13 26 43 64 …
   (vi) −4 −4 0 8 20 …
   (vii) 11 10 7 2 −5 …
   (viii) 98 92 82 68 50 …
2. (i) Work out the nth term of the linear sequence
   1 5 9 13 17 …
   (ii) Hence work out the nth term of the quadratic sequence
   1 25 81 169 289 …
   Give your answer in the form $an^2 + bn + c$.
3. (i) Work out the nth term of the quadratic sequence
   2 7 14 23 34 …
   (ii) Hence work out the nth term of the quadratic sequence
   5 10 17 26 37 …
   Give your answer in the form $an^2 + bn + c$.
4. (i) Work out the nth term of the quadratic sequence
   −5 −6 −5 −2 3 …

(ii) Hence work out the nth term of the quadratic sequence

$-15 \quad -18 \quad -15 \quad -6 \quad 9 \quad \ldots$

Give your answer in the form $an^2 + bn + c$.

(iii) Hence work out the nth term of the quadratic sequence

$0 \quad -3 \quad 0 \quad 9 \quad 24 \quad \ldots$

Give your answer in the form $an^2 + bn + c$.

5 A sequence starts with 2 and then follows the rule 'double the previous term and add 3' to generate subsequent terms. The first three terms are 2, 7, 17

(i) Calculate the 5th term.

(ii) How many terms will be even? Explain your answer.

6 The 1st, 3rd and 5th terms of a quadratic sequence are 11, 15 and 27 respectively.

Work out the nth term of the sequence.

7 The 2nd, 3rd and 4th terms of a quadratic sequence are -4, -1, and 4 respectively.

Work out the nth term of the sequence.

ACTIVITY 4.3

Draw n points on the edge of a circle – not equally spaced.
Join every pair of points with a straight line in such a way that no more than two lines intersect at any position inside the circle. You may have to move one (or more) points to avoid this happening.
Count the number of regions inside the circle for each value of n.

For $n = 1$ the number of regions is 1.
For $n = 2$ the number of regions is 2.
For $n = 3$ the number of regions is 4.
For $n = 4$ the number of regions is 8.
For $n = 5$ the number of regions is 16.

The sequence seems obvious!
Make a prediction for $n = 6$ and check it!

As the mathematician Eric Bell commented during the early twentieth century,
'Obvious is the most dangerous word in mathematics'.

9 Limiting value of a sequence

ACTIVITY 4.4

A sequence is given by nth term $= \dfrac{3n}{n+1}$.

(i) Work out the first 15 terms of the sequence (giving your answers to 3 decimal places where necessary).

(ii) Work out the 20th, 30th, 40th, 50th, 100th, 200th and 500th terms of the sequence.

(iii) Explain what is happening to the terms as n increases.

Limiting value of a sequence

To find the limiting value of a sequence, consider the nth term as $n \to \infty$ (n becomes very large).

Example 4.35

The nth term of a sequence is $\frac{2n-1}{3n+2}$.

Prove that the limiting value of the sequence as $n \to \infty$ is $\frac{2}{3}$.

Solution

Divide numerator and denominator by n.

$$\frac{2n-1}{3n+2} = \frac{\frac{2n}{n} - \frac{1}{n}}{\frac{3n}{n} + \frac{2}{n}} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$$

but as n gets bigger (approaching infinity) then $\frac{1}{n}$ and $\frac{2}{n}$ both get smaller, (they both approach zero).

∞ is the symbol for infinity.

This is written like this: as $n \to \infty$ then $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$

So as $n \to \infty$ then $\frac{2 - \frac{1}{n}}{3 + \frac{2}{n}} \to \frac{2-0}{3+0} = \frac{2}{3}$

So the limiting value of $\frac{2n-1}{3n+2}$ is $\frac{2}{3}$

! Do not write $\frac{1}{n} = 0$ as this is wrong. Zero is the **limit** as n approaches infinity but, as n can never get to infinity, then $\frac{1}{n}$ can never get to zero.

Likewise, do not write nth term $= \frac{2}{3}$, as it doesn't. $\frac{2}{3}$ is the limit.

Exercise 4J

① The nth term of a sequence is $\frac{n+1}{2n+1}$.

(i) Work out the first three terms of the sequence.

(ii) Work out the position of the term that has value 0.52.

② The nth term of a sequence is $\frac{4n-1}{2n-5}$.

(i) Work out the position of the term that has value 2.36.

(ii) Show that 1 is not a term in the sequence.

③ The nth terms of sequences are shown.

Work out the limiting value of each sequence as $n \to \infty$.

(i) $\frac{2n}{n+1}$ (ii) $\frac{n+2}{n+3}$ (iii) $\frac{n}{3n-1}$

(iv) $\frac{2n-1}{4n+1}$ (v) $\frac{3n}{4n+1}$ (vi) $\frac{1-n}{n+4}$

(vii) $\frac{2n}{3-4n}$ (viii) $\frac{2-6n}{5-2n}$

④ The nth term of a sequence is $\frac{5n+1}{2n+1}$.

Prove that the limiting value of the sequence as $n \to \infty$ is $\frac{5}{2}$.

⑤ The nth term of a sequence is $\frac{10-6n}{8n-3}$.

Prove that the limiting value of the sequence as $n \to \infty$ is -0.75

PS ⑥ The nth term of a sequence is $2 + \frac{7}{an+b}$, $a \neq 0$

Work out its limiting value.

PS ⑦ The nth term of a sequence is $\frac{an+3}{cn-1}$.

The 1st term of the sequence is 11, and its limiting value as $n \to \infty$ is 4

Work out the value of the 2nd term of the sequence.

PS ⑧ The nth term of a sequence is $\frac{2n^2+3n-4}{7n^2-n+2}$.

Work out its limiting value.

(Hint: divide the numerator and denominator by n^2.)

Prior knowledge

Students should already have a good understanding of simultaneous equations in two unknowns.

10 Simultaneous equations in three unknowns

Three equations are required when solving simultaneous equations in three unknowns.

Step 1: Eliminate one unknown by combining a pair of equations.

Step 2: Eliminate the same unknown by combining another pair of equations.

Step 3: Solve the resulting pair of equations using the method described in section 4.2.

Step 4: Substitute these two unknowns into one of the original equations to work out the value of the third unknown.

Example 4.36

Solve the three equations

$3x + 2y - 3z = -13$, $2x - 3y + 4z = 24$ and $4x - 5y + 2z = 22$

Solution

$3x + 2y - 3z = -13$ ①

$2x - 3y + 4z = 24$ ②

$4x - 5y + 2z = 22$ ③

Step 1:		$12x + 8y - 12z = -52$	$4 \times$ equation ①
		$6x - 9y + 12z = 72$	$3 \times$ equation ②
	Adding:	$18x - y = 20$	④
Step 2:		$8x - 10y + 4z = 44$	$2 \times$ equation ③
		$2x - 3y + 4z = 24$	equation ②
	Subtracting:	$6x - 7y = 20$	⑤

Step 3:

$$\begin{aligned} 18x - y &= 20 && \text{equation } ④ \\ 18x - 21y &= 60 && 3 \times \text{equation } ⑤ \end{aligned}$$

Subtracting:
$$\begin{aligned} 20y &= -40 \\ y &= -2 \end{aligned}$$

Substitute $y = -2$ into equation ④:
$$\begin{aligned} 18x - (-2) &= 20 \\ 18x &= 18 \\ x &= 1 \end{aligned}$$

Step 4: Substitute $x = 1$ and $y = -2$ into equation ①:

$$\begin{aligned} 3 \times 1 + 2 \times -2 - 3z &= -13 \\ -3z &= -12 \\ z &= 4 \end{aligned}$$

$$\therefore \quad x = 1, \quad y = -2 \quad \text{and} \quad z = 4$$

Example 4.37

Solve the three equations

$z - 8 = 2x + 3y$, $3x - 21 = 2y - 2z$ and $2x + y = 3z - 4$

Solution

$$\begin{aligned} z - 8 &= 2x + 3y && ① \\ 3x - 21 &= 2y - 2z && ② \\ 2x + y &= 3z - 4 && ③ \end{aligned}$$

Step 1: Rearrange equation ①: $z = 2x + 3y + 8$

Substitute for z in equation ②:
$$\begin{aligned} 3x - 21 &= 2y - 2(2x + 3y + 8) \\ 3x - 21 &= 2y - 4x - 6y - 16 \\ 7x + 4y &= 5 && ④ \end{aligned}$$

Step 2: Substitute for z in equation ③:
$$\begin{aligned} 2x + y &= 3(2x + 3y + 8) - 4 \\ 2x + y &= 6x + 9y + 24 - 4 \\ -4x - 8y &= 20 \\ x + 2y &= -5 && ⑤ \end{aligned}$$

Step 3: Rearrange equation ⑤: $x = -2y - 5$

Substitute for x in equation ④:
$$\begin{aligned} 7(-2y - 5) + 4y &= 5 \\ -14y - 35 + 4y &= 5 \\ -10y &= 40 \\ y &= -4 \end{aligned}$$

Substitute $y = -4$ into equation ⑤:
$$\begin{aligned} x + 2 \times -4 &= -5 \\ x - 8 &= -5 \\ x &= 3 \end{aligned}$$

Step 4: Substitute $x = 3$ and $y = -4$ into $z = 2x + 3y + 8$

$$z = 2 \times 3 + 3 \times -4 + 8$$

$$z = 6 - 12 + 8$$

$$z = 2$$

$\therefore \quad x = 3, \quad y = -4 \quad \text{and} \quad z = 2$

Example 4.38

0, 3, 10 are the first three terms of a sequence with nth term $= an^2 + bn + c$.

(i) By substituting $n = 1, 2$ and 3, set up three simultaneous equations in a, b and c.

(ii) Solve your simultaneous equations to find an expression for the nth term.

Solution

(i) $a + b + c = 0$ ①

$4a + 2b + c = 3$ ②

$9a + 3b + c = 10$ ③

(ii) ② − ① $\quad 3a + b = 3$ ④

③ − ② $\quad 5a + b = 7$ ⑤

⑤ − ④ $\quad 2a = 4$

$a = 2$

Substitute $a = 2$ into ④ $\quad 6 + b = 3$

$b = -3$

Substitute $a = 2$ and $b = -3$ into ① $\quad 2 + (-3) + c = 0$

$c = 1$

The nth term is $2n^2 - 3n + 1$

Exercise 4K

① (i) Solve, by eliminating z:

$$2x + 3y + z = 12$$
$$3x + 2y + z = 13$$
$$4x - 5y + z = 8$$

(ii) Solve, by eliminating y:

$$3x + y - 2z = 4$$
$$5x - y + 3z = 22$$
$$2x + y + 4z = 13$$

(iii) Solve, by eliminating x:

$$2x - 3y + 4z = -7$$
$$2x + 2y - 3z = 19$$
$$2x - 5y + 2z = -3$$

② (i) Solve $x + 2y + 3z = 13$
$2x + 3y - z = -7$
$3x - y + 2z = 18$

(ii) Solve $2x - y + 2z = 16$
$3x + 2y - z = 5$
$x + 4y - 3z = -13$

(iii) Solve $4x + 2y - z = 29$
$-x + 3y + 2z = -16$
$2x + y - 3z = 22$

(iv) Solve $5x + 3y + 2z = 8$
$7x - 4y + 4z = 20$
$3x + 2y - 2z = 3$

③ (i) Solve $2a - 5b + 2c = -36$, $3a + 4b - 3c = 10$ and $4a - 3b + 4c = -44$

(ii) Solve $3p + 5q - 2r = 13$, $4p - 2q + 5r = -25$ and $-2p + 3q - 7r = 25$

(iii) Solve $2\alpha + 3\beta + 2\gamma = 1$, $4\alpha - 2\beta - 5\gamma = 15$ and $-5\alpha + 4\beta + \gamma = -42$

④ (i) Solve $x + 3y = 2z - 23$, $2y = 3x - 4z + 2$ and $5z - 2x = 3y + 37$

(ii) Solve $2x = y + z + 13$, $5y - x = 2z - 16$ and $3z + 2y = 11 - x$.

(iii) Solve $y + 5 = 3x$, $2x - z = 7$ and $4y = 3z + 13$

PS ⑤ $x = 3$, $y = 5$, $z = 1$ is the solution to the simultaneous equations
$ax + by + cz = 28$
$az - cy = 2bx + 10$
$bz = ax + 3cy + 26$

Work out the values of a, b and c.

PS ⑥ 7, 9, 13 are the first three terms of a sequence with nth term $= an^2 + bn + c$.

(i) By substituting $n = 1, 2$ and 3 set up three simultaneous equations in a, b and c.

(ii) Solve your simultaneous equations and hence write down an expression for the nth term.

PS ⑦ 9, 16 and 42 are the 2nd, 3rd and 5th terms of a quadratic sequence. Work out the nth term of the sequence.

PS ⑧ If (x, y, z) are the general coordinates of a point in a 3-dimensional coordinate system, then each of the equations

$5x - 2y + z = 3$

$z = x + y$

$2x + 6y = 5z$

represents a plane. Solve the equations and hence write down the coordinates of the point at which the three planes meet.

Students of Level 2 Further Mathematics will not need to have any understanding of plane geometry. This question is for interest only.

ACTIVITY 4.5

The following questions are beyond the specification for Level 2 Further Mathematics.
However, students could use a 3-D graph plotter, along with three pieces of card, to discover the answers.

(i) Under what circumstances would two planes never meet?

(ii) If two planes meet, how many points lie on both planes?

The general equation for a plane is $ax + by + cz = d$ where a, b, c and d are constants.

(iii) Write an equation for a plane which is parallel to the plane $3x + 2y - z = 5$

(Hint: Use a 3-D graph plotter and see what happens as each coefficient is changed.)

(iv) Do three non-parallel planes always share a common point?

(v) Is it possible to have more than one common point on three different planes?

REAL-WORLD CONTEXT

Simultaneous equations and inequalities involving multiple variables are used when solving real-life problems. Given a variety of constraints, cost can be minimised and profit maximised, along with other optimisation situations. Such problems can be solved using linear programming techniques which are studied in A-Level Further Mathematics.

FUTURE USES

Students who go on to study Further Mathematics at A-Level will learn more about plane geometry.

LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- solve quadratic equations
  - by factorising
  - by completing the square
  - using the quadratic formula
  - by drawing a graph
- solve simultaneous equations
  - in two unknowns
  - in three unknowns
  - by plotting their graphs
- use the factor theorem
  - to factorise a polynomial
  - to solve a polynomial equation
- solve inequalities
  - linear
  - quadratic
- use the index laws
- prove mathematical statements algebraically
- find and use the nth term of sequences
  - linear
  - quadratic
- find the limit of a sequence with an nth term of the form $\frac{an + b}{cn + d}$.

KEY POINTS

1 When factorising quadratics of the form $ax^2 + bx + c$, find two numbers with a sum of b and product ac. Then split the coefficient of x into these two numbers.

2 When completing the square on a quadratic of the form $ax^2 + bx + c$, take a out as a factor, or divide both sides of the equation by a.

3 If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4 $f(a) = 0 \quad \Leftrightarrow \quad (x - a)$ is a factor of $f(x)$

5 When solving an inequality, treat it as an equation, but remember to reverse the sign if the inequality is multiplied (or divided) throughout by a negative value.

6 When solving a quadratic inequality, first take all terms to one side.

7 The index laws are:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$

8 Some useful results are:

- $a^0 = 1$
- $a^{-m} = \dfrac{1}{a^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

9 In a proof question, show every step clearly.

10 A linear sequence has an nth term of the form $an + b$.

11 A quadratic sequence has an nth term of the form $an^2 + bn + c$.

12 A sequence with an nth term of the form $\dfrac{an + b}{cn + d}$ has a limit of $\dfrac{a}{c}$ as n approaches infinity.

5 Coordinate geometry

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

Albert Einstein

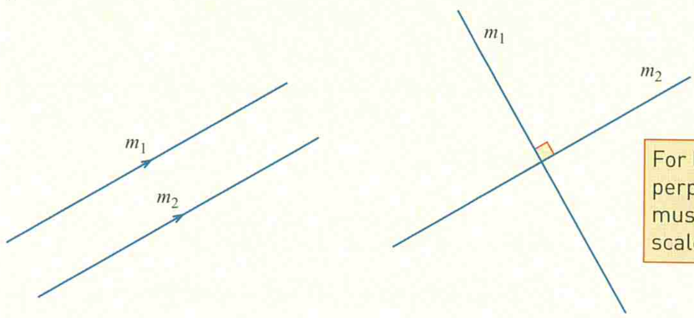
1 Parallel and perpendicular lines

Prior knowledge

In Chapter 3 we used this fact:

The line joining (x_1, y_1) to (x_2, y_2) has gradient m, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

If you know the gradients m_1 and m_2 of two lines, you can tell at once if they are parallel or perpendicular.



For lines to look perpendicular you must use the same scale on both axes.

Figure 5.1

Parallel lines: $m_1 = m_2$ Perpendicular lines: $m_1 m_2 = -1$