Matrices

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1 Matrices

We talked about vector spaces and their properties. Now we will introduce a new structure, "matrices". Matrices are very useful structures in mathematics. With matrices and matrix operations, we can represent and form many things like linear mappings, geometric shapes like quadratic surfaces and conic sections, systems of linear equations, systems of differential equations, etc.

In a really primitive sense, matrices can be thought of as columns vectors that are put side by side or row vectors that are put one after the other.

Definition: Let m, n be positive integers bigger than 1. M_n^m be a matrix space over some field K. Then an element of M_n^m can be written as:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

The matrix above has n columns and m rows. Then we will say A is a $m \times n$ matrix. Most of the time we will be dealing with real-valued matrices. That means a_{ij} where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ will be real numbers.

1.1 Matrix Addition

Let $M^{m\times n}$(M_n^m has the same meaning, it is just a different notation) be a matrix space over field $\mathbb{R}$. The sum of two matrices $A, B \in M^{m\times n}$ defined as:

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Example: Assume A = $\begin{bmatrix} 4 & 5 & 9 \\ 3 & 3 & 1 \end{bmatrix}$, and B = $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix}$, then the summation of these two matrices is $A + B = \begin{bmatrix} 4 & 5 & 9 \\ 3 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 10 \\ 5 & 8 & 9 \end{bmatrix}$.

Hence, we can say what we do in matrix addition is doing an element-wise summation operation. There are a few points we should pay attention to. If we want to add a matrix to another, we need matrices that have the same number of columns and rows. Unless we can't do the operation.

1.2 Matrix Multiplication

In matrix multiplication, rules are different. Let $M^{m\times n}$ and $K^{k\times l}$ be matrix spaces where $n = k$. Assume $A \in M^{m\times n}$ and $B \in K^{k\times l}$. Now we have $A/cdotB = C$. First, let's look at the columns and rows of the resulted matrix C. After that we will define the matrix multiplication

$$\underset{m\times n}{A} \cdot \underset{k\times l}{B} = \underset{m\times l}{C}$$

Remember, we have $n = k$. If we don't have that property we wouldn't be able to do the matrix multiplication.That's an important property of this operation. Let's define $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$.. Then we will define the elements of C.

$$A \cdot B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \cdots & b_{kl} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1l} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{ml} \end{bmatrix}$$

At first sight, it will look complicated but it has a very easy rule. Don't worry you will get it. (Remember $n = k$)

$$\begin{gathered} c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} = \sum_{i=1}^{n} a_{1i}b_{i1} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} = \sum_{i=1}^{n} a_{1i}b_{i2} \\ c_{13} = a_{11}b_{13} + a_{12}b_{23} + \cdots + a_{1n}b_{n3} = \sum_{i=1}^{n} a_{1i}b_{i3} \\ \vdots \\ c_{1l} = a_{11}b_{1l} + a_{12}b_{2l} + \cdots + a_{1n}b_{nl} = \sum_{i=1}^{n} a_{1i}b_{il} \end{gathered}$$

Just now, we defined the first row. Can you see the pattern? You will see it soon. If we keep following the rule above we will end up with a matrix below.

$$C_{m\times l} = \begin{bmatrix} c_{11} & \cdots & c_{1l} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{ml} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{1i}b_{il} \\ \sum_{i=1}^{n} a_{2i}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{2i}b_{il} \\ \vdots & \cdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \sum_{i=1}^{n} a_{mi}b_{i2} & \cdots & \sum_{i=1}^{n} a_{mi}b_{il} \end{bmatrix}$$

We know if A and B have the same number of columns and rows, then $A + B = B + A$. (You can check). In other words, matrix addition is a commutative operation. It commutes.

While doing matrix multiplication we should be more careful. Matrix multiplication doesn't commute. Assume C is a $n \times m$ and D is a $m \times k$ matrix where $n \neq k$. Then we can do matrix multiplication since C has m columns and D has m rows. However, we can only perform this operation as $C \cdot D$. If you realize that $D \cdot C$ is not defined. Because this time D has k columns and C has n rows. Since $n \neq k$ we cannot have that operation.

There is a second possibility that we have two square matrices. Square matrices are matrices that have the same number of columns and rows. Let $A, B \in \mathbb{R}^{n\times n}$ be two square matrices. Then we can talk about both $A \cdot B$ and $B \cdot A$. Even, in some cases, we can see that $A \cdot B = B \cdot A$ but that doesn't mean matrix multiplication is a commutative operation.

Example:

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 1 & 7 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + 2\cdot 1 + 1\cdot 3 & 3\cdot 6 + 2\cdot 7 + 1\cdot 8 \\ 1\cdot 2 + 5\cdot 1 + 7\cdot 3 & 1\cdot 6 + 5\cdot 7 + 7\cdot 8 \\ 3\cdot 2 + 1\cdot 1 + 2\cdot 3 & 3\cdot 6 + 1\cdot 7 + 2\cdot 8 \end{bmatrix} = \begin{bmatrix} 11 & 40 \\ 28 & 108 \\ 13 & 109 \end{bmatrix}$$

I believe the rule of matrix multiplication is now more clear with the example above.