Systems of Linear Equations

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First of all, we can make observations about linear equations. We can begin with a linear equation. $2x_1 - x_2 = 3$ is a simple linear equation. There are two unknown variables and one equation, so in this case, there might be infinitely many solutions. What I mean is that if you make some arrangements in the equation. We'll end up with $x_2 = 2x_1 - 3$. If we choose $x_2 = 5$ then $x_1 = 4$. It means for any $x_2 \in \mathbb{R}$ we can have a corresponding value for x_1. Hence, we have infinitely many solutions.

Now let's examine a system of linear equations. Consider the linear equations $2x_1 - x_2 = 3$ and $3x_1 + 2x_2 = 5$. If this is the system we're working with, then we must find a solution that works for both of them. If we solve this system, we'll get $x_1 = \frac{11}{7}$ and $x_2 = \frac{1}{7}$. Now we have just one solution and no other solutions cannot be found. (Why?) We're gonna answer this question with the help of linear algebra but now we continue our problem.

Consider this system:

$$\begin{aligned} 3x_1 + 2x_2 &= 5 \\ 6x_1 + 4x_2 &= 10 \end{aligned}$$

Now, we have infinitely many solutions. If $x_2 = 1$ then $x_1 = 1$, $x_2 = 2$ then $x_1 = \frac{1}{2}$, $x_2 = 3$ then $x_1 = -2$. Hence, for any $x_2 \in \mathbb{R}$ we can find a corresponding value for x_1.

Now we know that systems can have an **unique solution** or sometimes they have **infinitely many solutions**. Also, in the third case, sometimes they don't have any solutions. We say that kind of system is "inconsistent systems".

Systems of Linear Equations as Matrices

In mathematics, many things can be represented as matrices. For now, we will just consider the systems of linear equations.

We can represent these systems as matrix multiplication. Assume we have n variable and m equation:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{gathered}$$

If we want to write the above system as the matrix multiplication we would write it in the following way:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Example: Consider the system of linear equations

$$\begin{gathered} 4x_1 + 3x_2 - 2x_3 = 5 \\ \mathrm{x}_1 + 2x_2 + 4x_3 = 9 \\ 2\mathrm{x}_1 - x_3 = 4 \end{gathered}$$

It can be expressed as in the following matrix multiplication

$$\begin{bmatrix} 4 & 3 & -2 \\ 1 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$$

If you perform the above matrix multiplication, you will end up with the same system. (Try)

Example: Consider the system of linear equations

$$\begin{gathered} x_1 + 6x_2 - x_3 + 3x_4 = -2 \\ 5\mathrm{x}_1 + 2x_2 + x_3 = 23 \\ \mathrm{x}_1 - 4x_2 + 3x_3 + 2x_4 = 12 \\ \mathrm{x}_1 7x_2 - 12x_4 = -153 \end{gathered}$$

The matrix form of the system is:

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 5 & 2 & 1 & 0 \\ 1 & -4 & 3 & 2 \\ 1 & 7 & 0 & -12 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 23 \\ 12 \\ -153 \end{bmatrix}$$