Brief Introduction to Vector Spaces

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Contents

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1 Introduction

As an undergraduate student of mathematics, I realized linear algebra has really essential concepts and methods that we use in every field of mathematics. Also, not just in mathematics, every field that is somehow related to mathematics has used these concepts and methods to solve its problems. Also, It is a powerful tool that consists of beautiful theorems and problems. The notes contain brief explanations about the concepts and the beautiful theorems chosen by me.

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2 Group, Ring and Field

Before mentioning the vector spaces, we will talk about some of the mathematical structures and give brief explanations of them. The structures are **group**, **ring** and **field**. These structures are the foundations of abstract algebra. Even though they are the study field of abstract algebra, I regardfully believe that before going deeper into vector spaces, one should have basic knowledge about these structures.

2.1 Groups

The groups are the fundamental structure in abstract algebra. The groups can be seen in other algebraic structures like rings, fields, and vector spaces. Groups play an essential role in computer science, statistics, and natural sciences. They are heavily used in coding theory, cryptography, and some fields of physics and chemistry. It is so fascinating that such an abstract concept has a lot of applications in the physical world. Let's define what a group is. If we want to define a group, we need two certain things. A set and a binary operation. Let's give the exact definition:

A **group** is a pair of a set and an operation which can be satisfied the following properties:

Associativity: Let $\alpha, \beta, \gamma, \in G$ then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Closure: Let $\alpha, \beta \in G$ then $\alpha + \beta \in$ G

Identity Element: $\exists\ \epsilon \in$ G, $\forall\ \alpha \in$G $\alpha + \epsilon = \epsilon + \alpha = \alpha$

Inverse Elements: $\forall\ \alpha \in$ G, $\exists\ \alpha' \in$ G $\alpha + \alpha' = \alpha' + \alpha = \epsilon$

If these four property has been satisfied, we say that **(G,+)** is a group.
Note: G is abelian group if $\forall\ \alpha, \beta \in$ G then we have $\alpha + \beta = \beta + \alpha$.

Examples: Some well-known groups are $(\mathbb{R}, +)(\mathbb{Q}, \cdot), (\mathbb{Z}, +)$

Examples: (General Linear Group) The set of invertible matrices or $A \in \mathbb{R}^{nxn}$ where $det(A) \neq 0$ is a group with matrix multiplication. General Linear Group is not an abelian group since matrix multiplication doesn't not commute. (AB $\neq$ BA)

2.2 Rings

Rings are very important structures in algebra. In the simplest terms, a ring is a set equipped with two binary operations. A ring must be satisfied the following properties:

1.(R,+) is an abelian group.

2. $\forall\ \alpha, \beta, \gamma \in$ R, we have $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

3. $\forall\ \alpha, \beta, \gamma \in$ R, we have $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$ and $(\alpha + \beta) \cdot \gamma = (\alpha \cdot \gamma) + (\beta \cdot \gamma)$

If the above properties have been satisfied, we say that **(R,+,·)** is a ring.

Examples: Some ring examples are $(\mathbb{Z}, +, \cdot)(\mathbb{Q}, +, \cdot), (\mathbb{Z}_6, +)$

2.3 Fields

Fields are very special rings. From this point of view, we can state that every field is a ring but every ring is a not field. This is very significant to keep in mind. Since every field is a ring, we already know fields have satisfied the ring's properties. Now we will try to explain what's required for a ring to be a field.

Assume that **(F,+,·)** is a commutative ring with identity element. Then we define a set $F^* = F - \{\epsilon\}$. The element ϵ is the identity element concerning the first operations. If all elements of the F^* have an inverse then we say F is a field.

Examples: Some field examples are $(\mathbb{Z}_p, +, \cdot)$ where p is a prime integer, $(\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot)$

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3 Vector Spaces

A vector space V over the field F is a set of vectors that can be added and multiplied in such a way that the sum of two elements of V is again an element of V, and the product of an element of V and F is again an element of V. In the below notation the operation $\bigoplus$ is an operation between the elements of V, the operation $\cdot$ is an operation between the elements of F and the operation $\bigodot$ is an operation between the elements of V and the elements of F. Let's give the mathematical axioms exactly.
A Vector space satisfies the following properties:

Let V is a set, **(V,$\bigoplus$)** is an abelian group and **(F,+,·)** is a field. Also we define an operation:

$\bigodot : F \times V \to V$
$\quad (\lambda, \alpha) \to \lambda \bigodot \alpha$

VS 1 $\forall \alpha, \beta, \gamma \in$ V, we have

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

VS 2 $\forall\ \alpha \in$ V, there is a element of V, denoted by 0_V, such that

$$0_V + \alpha = \alpha + 0_V = \alpha$$

VS 3 $\forall \alpha \in$ V, $\exists \alpha' \in$ V, such that,,

$$\alpha + \alpha' = 0_V$$

VS 4 $\forall \alpha, \beta \in$ V, we have

$$\alpha + \beta = \beta + \alpha$$

VS 5 If $\lambda \in$ F , $\forall \alpha \in$ V then

$$\lambda \bigodot (\alpha \bigoplus \beta) = (\lambda \bigodot \alpha) \bigoplus (\lambda \bigodot \beta)$$

VS 6 If $\lambda_1, \lambda_2 \in$ F, $\forall \alpha \in$ V then

$$(\lambda_1 + \lambda_2) \bigodot \alpha = (\lambda_1 \bigodot \alpha) \bigoplus (\lambda_2 \bigodot \alpha)$$

VS 7 If $\lambda_1, \lambda_2 \in$ F, $\forall \alpha \in$ V then

$$(\lambda_1 \cdot \lambda_2) \bigodot \alpha = \lambda_1 \bigodot (\lambda_2 \bigodot \alpha)$$

VS 8 δ is the identity element of the second operation of (F,+,$\cdot$), then we have

$$\forall\ \alpha \in \text{V}\ \delta \bigodot \alpha = \alpha$$

If these eight properties above satisfied, we claim that V is a vector space over the field K. Also the below notation is widely used:

$$(\text{V},\bigoplus,\text{F},+,\cdot,\bigodot) \equiv V$$

The first four property states that V and $\bigoplus$ is an abelian group and on the other hand the last four property defines the operations among the elements of V and the elements of F. In general, we can say, the vector spaces can be formed by an abelian group and a field. When studying linear algebra we will be working with vector spaces.

Examples: $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}$. However, $\mathbb{Q}^n$ is not a vector space over $\mathbb{R}$. Simply, if we take $n = 4$ we can show that by $\pi \bigodot (0, 1, 1, 0) = (0, \pi, \pi, \pi)$ and $\pi \in \mathbb{R}$ and $(0, 1, 1, 1) \in \mathbb{Q}^4$. However, $(0, \pi, \pi, \pi) \notin \mathbb{Q}^4$. Thus, $\mathbb{Q}^4$ is not a vector space over $\mathbb{R}$. If we generalize this example, we can conclude $\mathbb{Q}^n$ is not a vector space over $\mathbb{R}$.

From now on we will use usual + and $\cdot$ instead of $\bigoplus$ and $\bigodot$

3.1 Vector Subspaces

Let's assume that V is a vector space over the field F, S is a subset of V. Then S has to be satisfy by the following properties to be a subspace of V:

1. $\forall\ s_1, s_2 \in$ S, we have $_1 + s_1 \in$ S

2.$\forall\ s_1 \in$ S, $\forall \lambda \in$ F, we have $\lambda \cdot_1 \in$ S

3. The identity element of V (0_V) must be in S.

3.2 Linear Combinations

Let α be a vector in the vector space V^n. If α can be expressed as:

$$\alpha = \lambda_1 \cdot \beta_1 + \lambda_2 \cdot \beta_1 + ... + \lambda_n \cdot \beta_n$$
$$(\lambda_1, ..., \lambda_n \in \text{F and } \beta_1, ..., \beta_n \in \text{V})$$

We say that the expression on the right is the linear combination of the vector α.

Examples: The vector $(5, 3) \in \mathbb{R}^2$ can be written as the linear combination of $(1, 0), (0, 1)$.
$(5, 3) = 5 \cdot (1, 0) + 3 \cdot (0, 1)$