Paris Saclay/Paris Evry University

Derivatives

Rough volatility models - Project

Students:
Ouassim Sebbar
Issame Sarroukh
Vannarath Vicheth

Professor:
Sergio Pulido

Contents

1 Estimation of the parameter H

1.1

The variance under Lifter Heston model verify the following equation:

$$Vt = g_0(t) + \sum_{i=1}^{n} c_i U_t^i$$
$$dU_t^i = (-x_i U_t^i - \lambda V_t) + \nu \sqrt{V_t} dW_t$$

With the function $g_0(t)$ is The function $g_0(t)$ is linked to the initial forward variance $\zeta_0(t)$

$$\zeta_0(t) + \lambda \int_0^t e^{-(t-s)x_i} \zeta_0(s) ds$$

and $c_i = \frac{\eta_i^{1-\alpha} - \eta_{i-1}^{1-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)}$ and $\frac{(1-\alpha)}{(2-\alpha)} \frac{\eta_i^{1-\alpha} - \eta_{i-1}^{2-\alpha}}{\eta_i^{1-\alpha} - \eta_{i-1}^{1-\alpha}}$, for n even and $r > 1$, the geometric partition is considered with $\zeta_i = r^{i-\frac{n}{2}}$ for $i = 0, 1, ..., n$, in this framework, $c_i = \frac{r^{1-\alpha}-1}{\Gamma(\alpha)\Gamma(2-\alpha)} r^{(1-\alpha)(i-1-\frac{n}{2})}$ and $x_i = \frac{(1-\alpha)}{(2-\alpha)} \frac{r^{2-\alpha}-1}{r^{1-\alpha}-1} r^{i-1-\frac{n}{2}}$.

The variance of a Lifted Heston model is simulated using the implicit-explicit Euler scheme, and the discretised schema becomes:

$$\hat{V}_{t_k} = g_0(t_k) + \sum_{i=0}^{n} c_i \hat{U}_{t_k}^i$$
$$\hat{U}_{t_{k+1}}^i = \frac{1}{1 + x_i \Delta t} (\hat{U}_{t_k}^i - \lambda \hat{V}_{t_k} \Delta t + \nu \sqrt{max(0, \hat{V}_{t_k})} (W_{t_{k+1}} - W_{t_k})$$

With $\hat{U}_0^i = 0$, $\Delta t = t_k - t_{k-1}$, (W_{t_k}) simulations of a Brownian Motion on a uniform partition (t_k) of the time horizon $[0, T]$

$$g_0(t) = V_0 + \lambda\theta \sum_{i=0}^{n} c_i \int_0^t e^{-x_i(t-s)} ds$$

The variance of the Lifted Heston model is simulated with $n = 20$ (which corresponds to the number of factors) and the parametres for the model are fixed as such: $V_0 = 0.05$, $\lambda = 0.3$, $\theta = 0.05$, $\nu = 0.1$, $\alpha = H + 0.5 = 0.6$, $r_{20} = 2.5$

The time horizon is $T = 1$ with $m = 10^5$ steps which correspond to $\Delta t = 10^{-5}$ for a uniform partition.

Based on the discretization method cited above, the Lifted Heston volatility is then obtained.

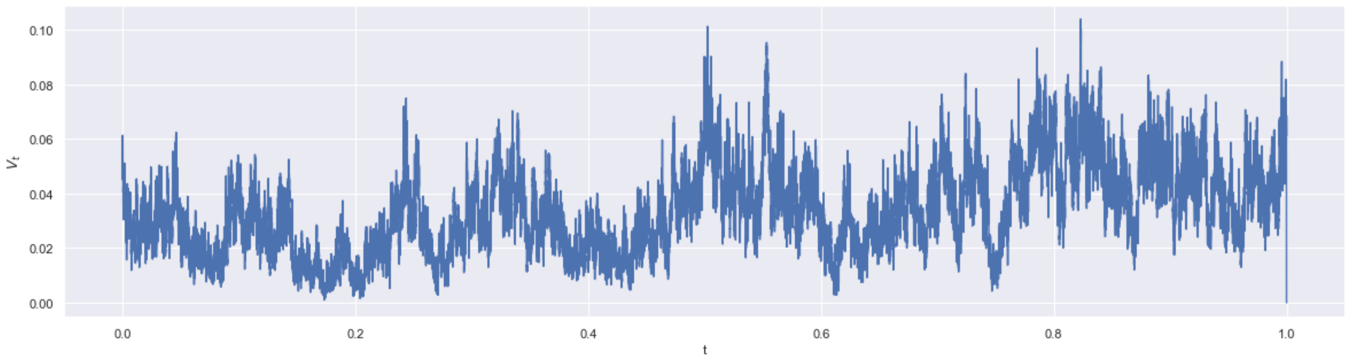

Figure 1: Evolution of the volatility under Lifted Heston model

1.2

One important parameter to determine is H, the parameter is determined by calculating the moment of the increments, which equates to funding $\mathbb{E}[|V_{t+\Delta} - V_{t+\Delta}|^q|]$, the moment is equation to $K_q \Delta^{qH}$ and the emperical value of the quantity is:

$$m(q, \Delta) = \frac{1}{M_\Delta} \sum_{i=1}^{M_\Delta} |V_{t_i+\Delta} - V_{t_i}|^q$$

For $q = 0.5, 1, 1.5, 2$ and $\Delta = 1, ..., 10$

$log(m(q, \Delta)) \sim \zeta_q log(\Delta) + C_q$, the best fit ζ_q as a function of q is determined using a linear regression model using Sklearn Python Library. The value of H is then estimated by linear fit on ζ_q.

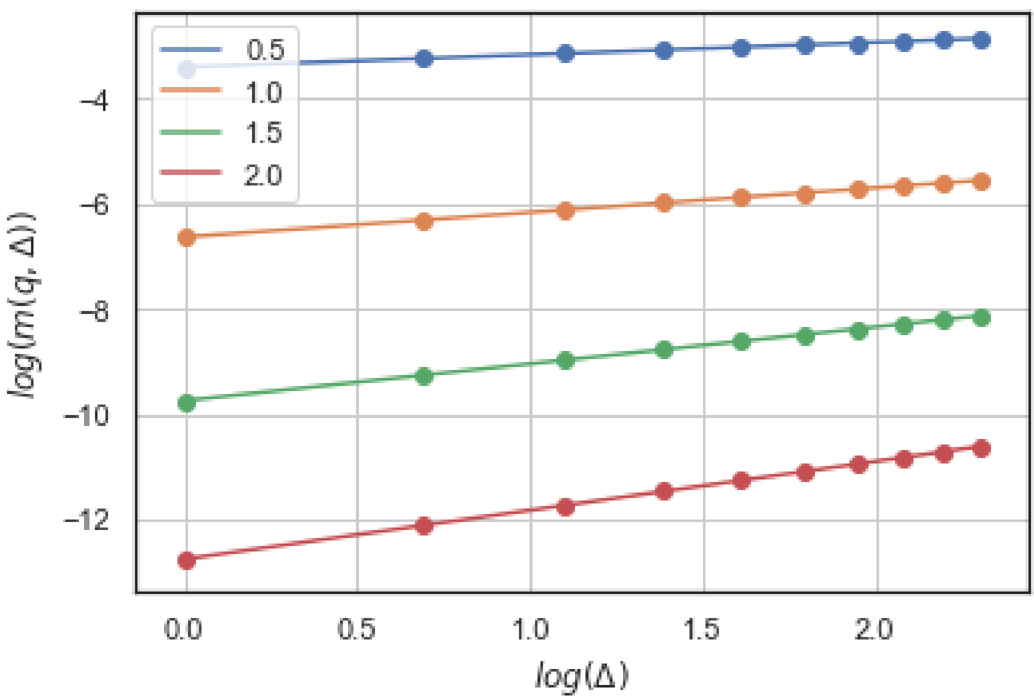

Figure 2: Evolution of the q-moment of the volatility under the Lifted Heston model

The figures are clearly linear of $m(q, \Delta)$ for each q and the linear model fits perfectly the value obtained by discretization. The same can be said for ζ_q as a function of q.

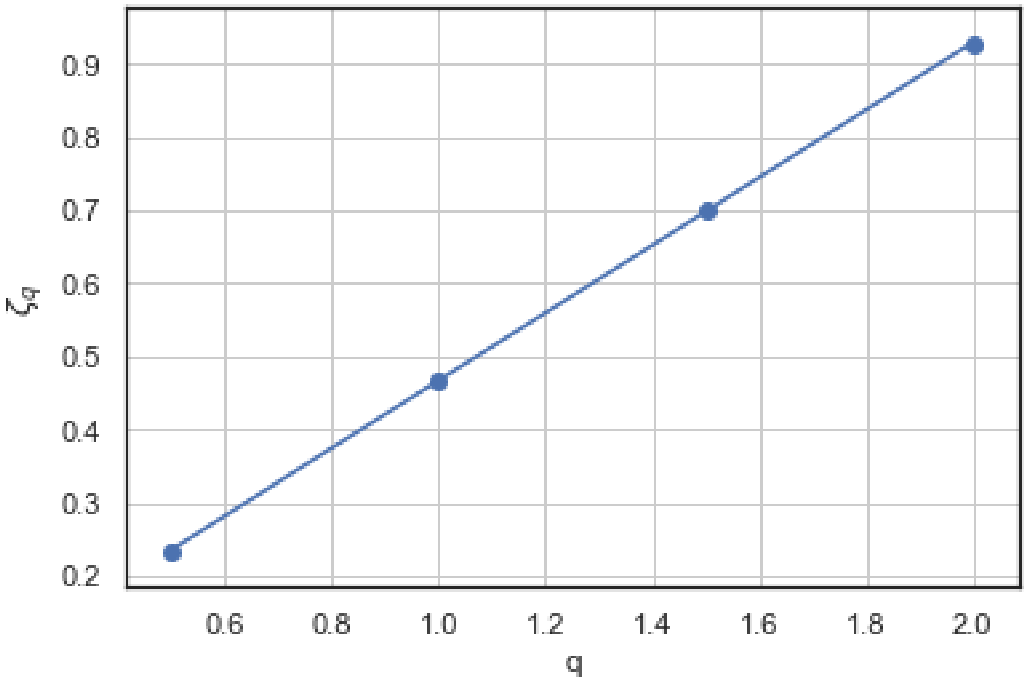

Figure 3: Evolution of ζ_q as a function of q

$H = 0.46295633$.

1.3

The estimation is repeated this time for the sampled path $W_{il\Delta t}$ for $l \in \{1, ..., 10\}$, H is determined of each of the paths.

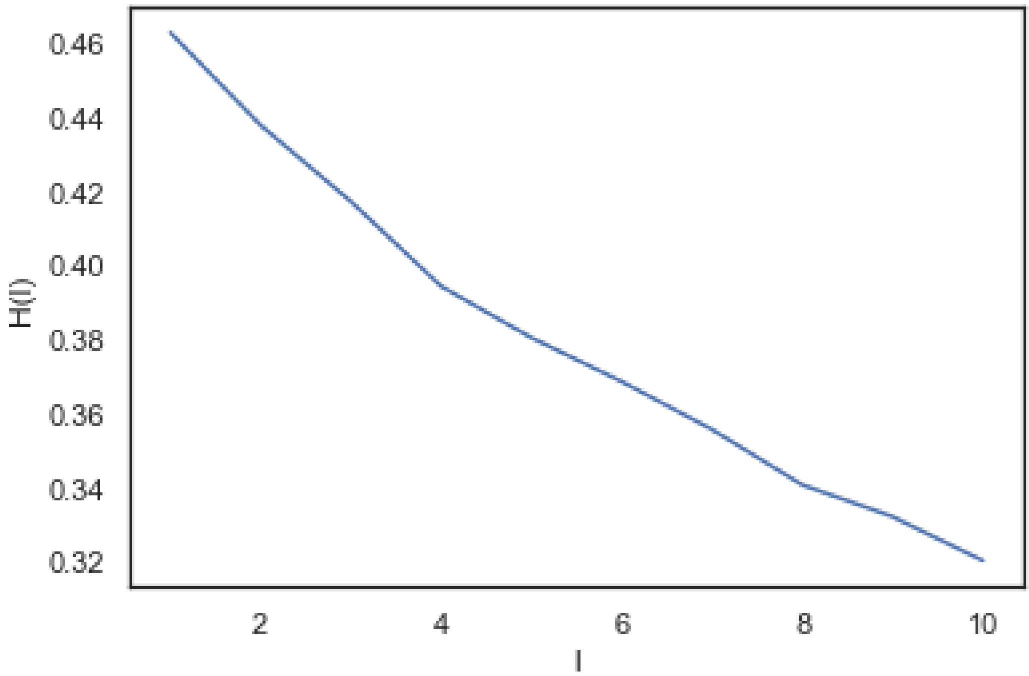

Figure 4: Evolution of H as a function of l

H decreases with l, and the volatility under the Lifted Heston model does not have a fixed H even if the value of H is taken equal to 0.1, when the step increases H becomes closer to to intended value as discussed in the paper.

1.4

In what follows, a classical Brownien motion is considered. As $W_{t_{i+1}} - W_{t_i} \sim \sqrt{\Delta t}\mathcal{N}(0,1)$, it is sufficient to simulate m independent normal random variable Z_i. $W_{t_i} = \sqrt{\Delta t}\sum_{i=1}^{n} Z_i$. The simulation of W_t for the same step as in the Volatility of the Lifted Heston model:

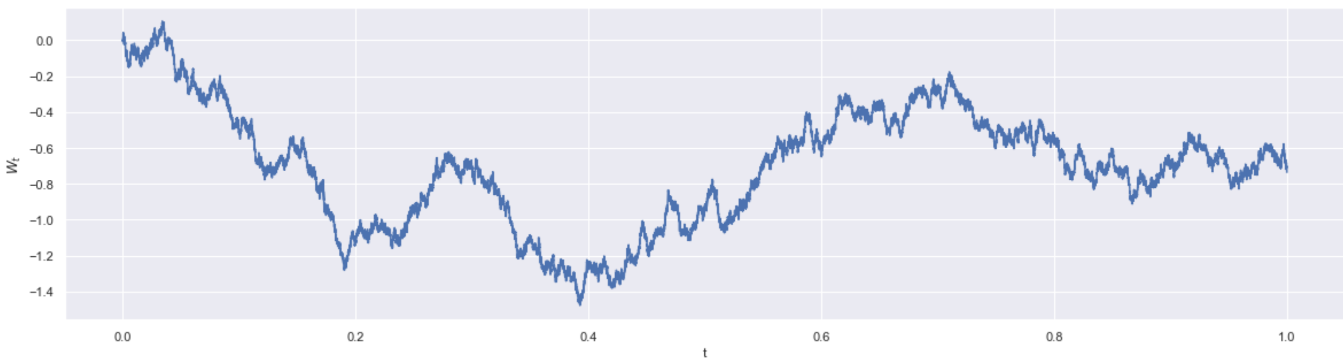

Figure 5: Evolution of the classical Brownien motion W_t

As for volatility under the Lifted Heston model, the plot for the evolution of $m(q, \Delta)$ for $q = 0.5, 1, 1.5, 2$ is:

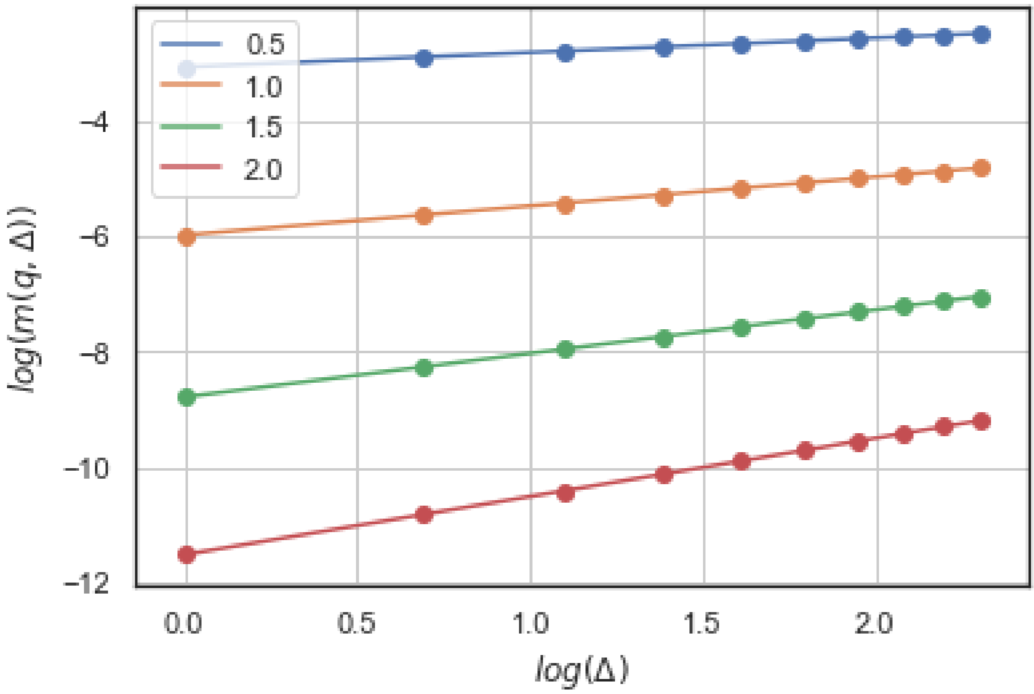

Figure 6: Evolution of q-moment of the Brownien motion

And ζ_q varies as follows:

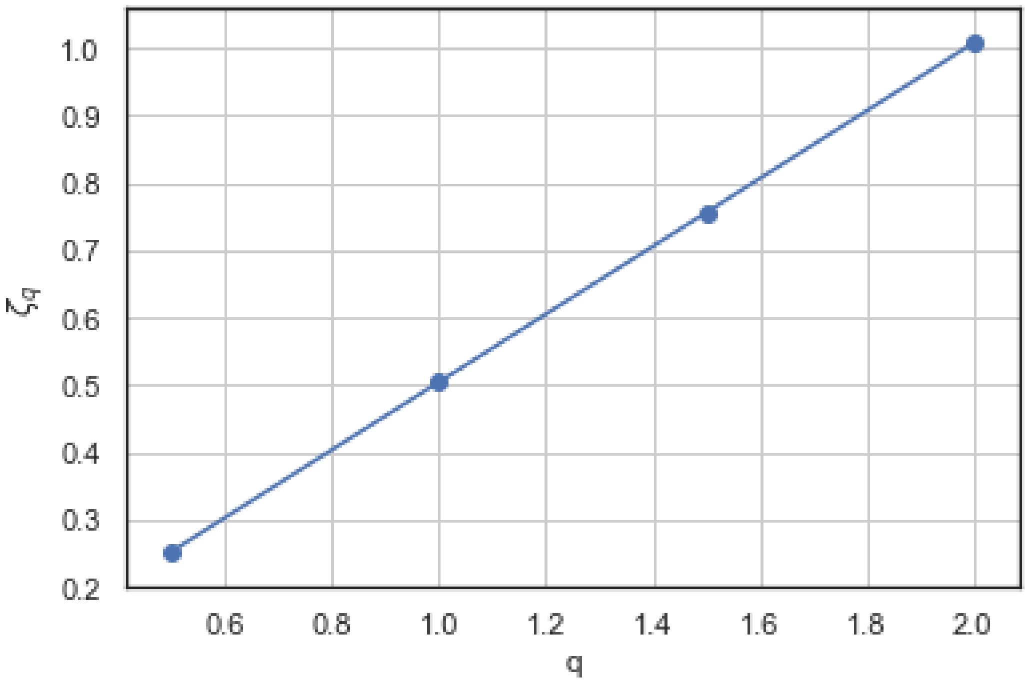

Figure 7: Evolution of ζ_q as a function of q

The value of 0.50366547 which is close to the theoretical value $H = 0.5$. By repeating the estimation as indicated for the volatility under the Lifted Heston model, H remains near its theoretical value $H = 0.5$.

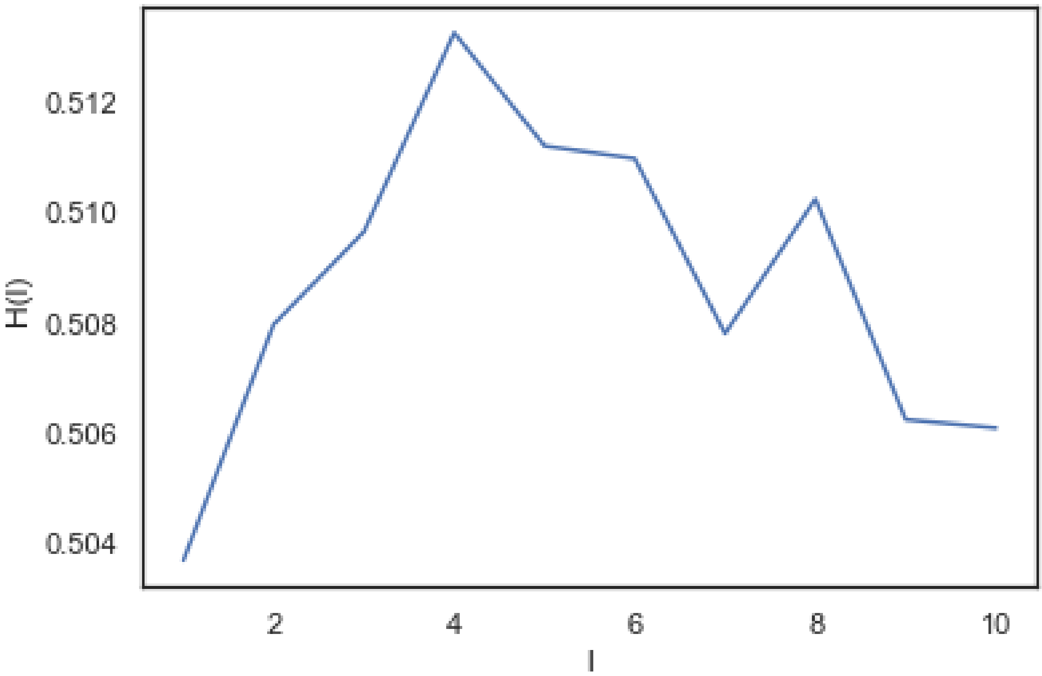

Figure 8: Evolution of H as a function of l

1.5

In this section, a Factional Brownien motion is simulated, which is a Gaussian process W^H with mean 0 and covariance $\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$ with a Hurst index $H \in]0,1[$.

To simulate such stochastic process, the interval $[0, T]$ is partitioned with step Δt, a variance-covariance matrix $\Sigma = (\frac{1}{2}\{|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}\})_{i,j}$ of the Gaussian vector $(W_{t_i}^H)_{i=1,....,N}$ is computed and the Cholsky factorization is calculated $\Sigma = C^T C$, $Z = (Z_i)_N$ a column vector for N independent normal random variables $\mathcal{N}(0,1)$ are simulated. The $W_{t_i}^H = (C^T Z)_i$ for $i = 1, ..., N$ and $W_t^H = 0$.

H is taken equal to 0.1 and $m = 10^3 (\Delta t = 10^{-3})$, the simution return the following plot:

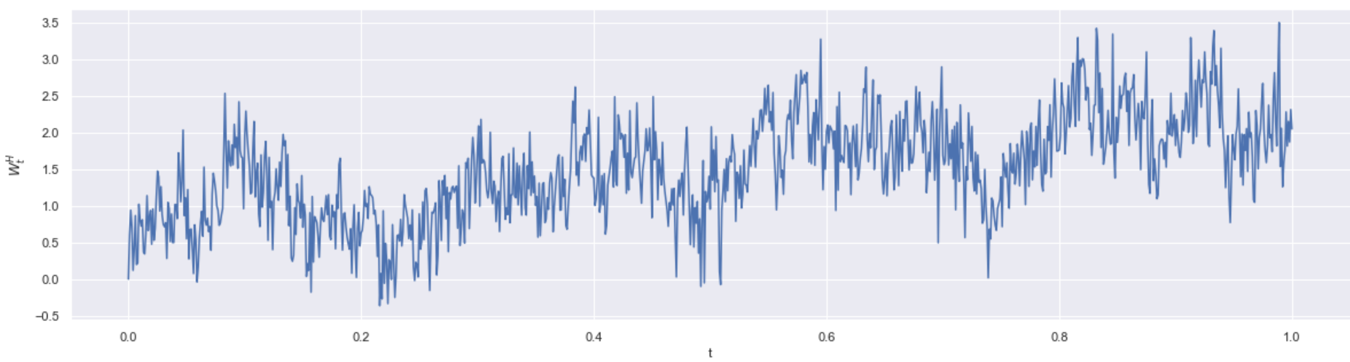

Figure 9: Evolution of the fractional Brownien motion W_t

Following the same steps as for the models studied above, the evolution of $m(q, \Delta)$ is:

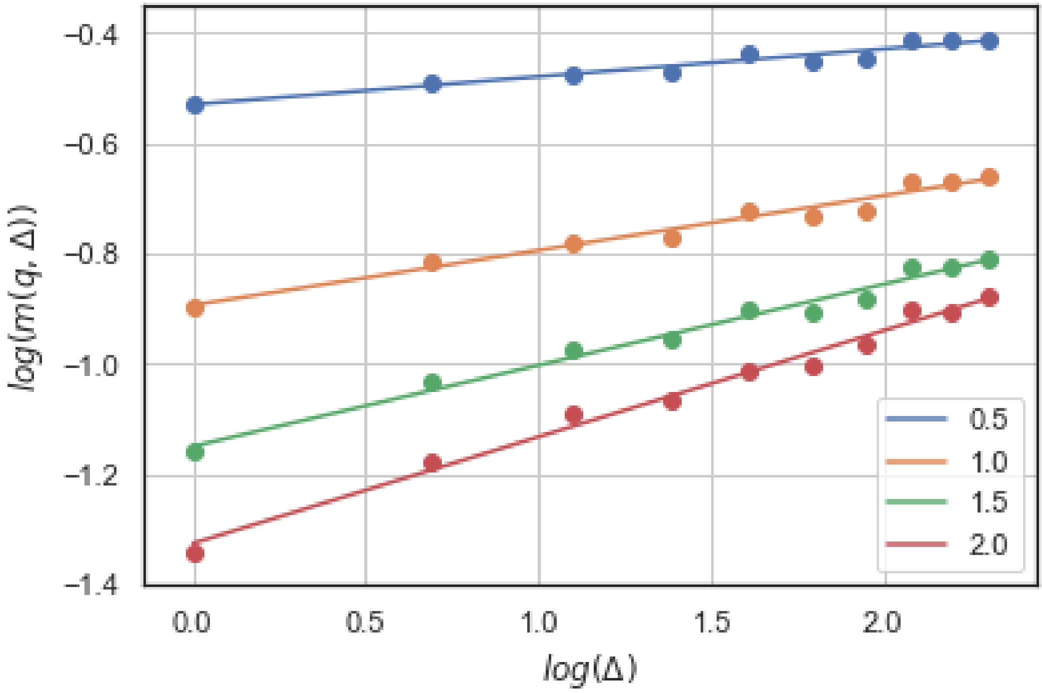

Figure 10: Evolution of q-moment of the fractional Brownien motion

And of ζ_q is:

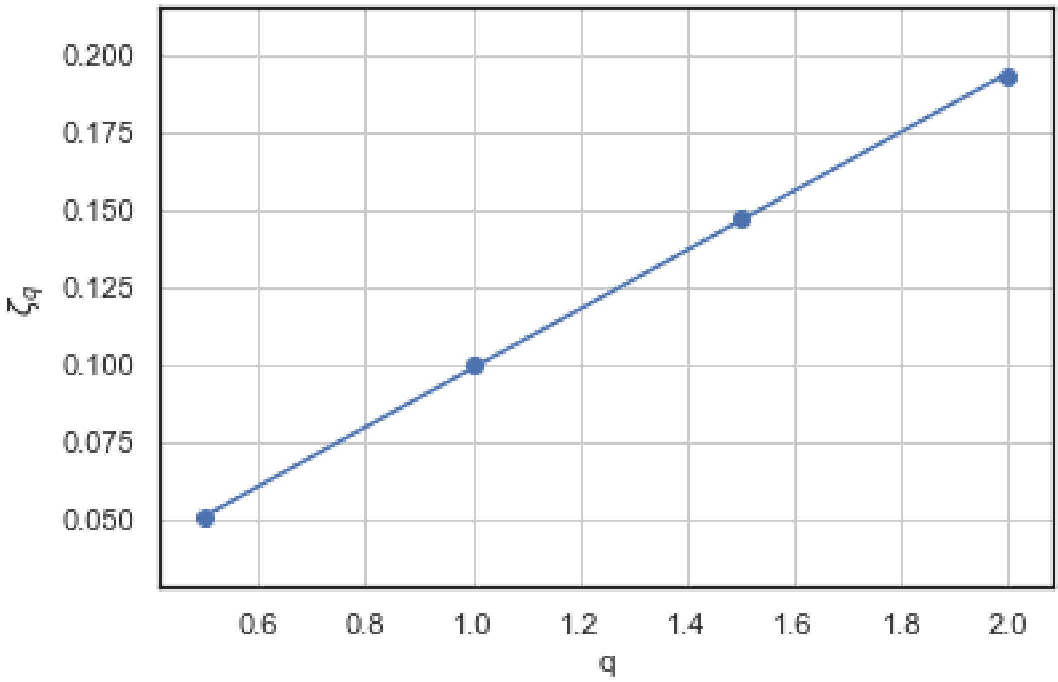

Figure 11: Evolution of ζ_q as a function of q

The value $H = 0.0951765$ obtained by the calculating the moment, close to the theoretical value $H = 0.1$ used to define the fractional Brownien option.

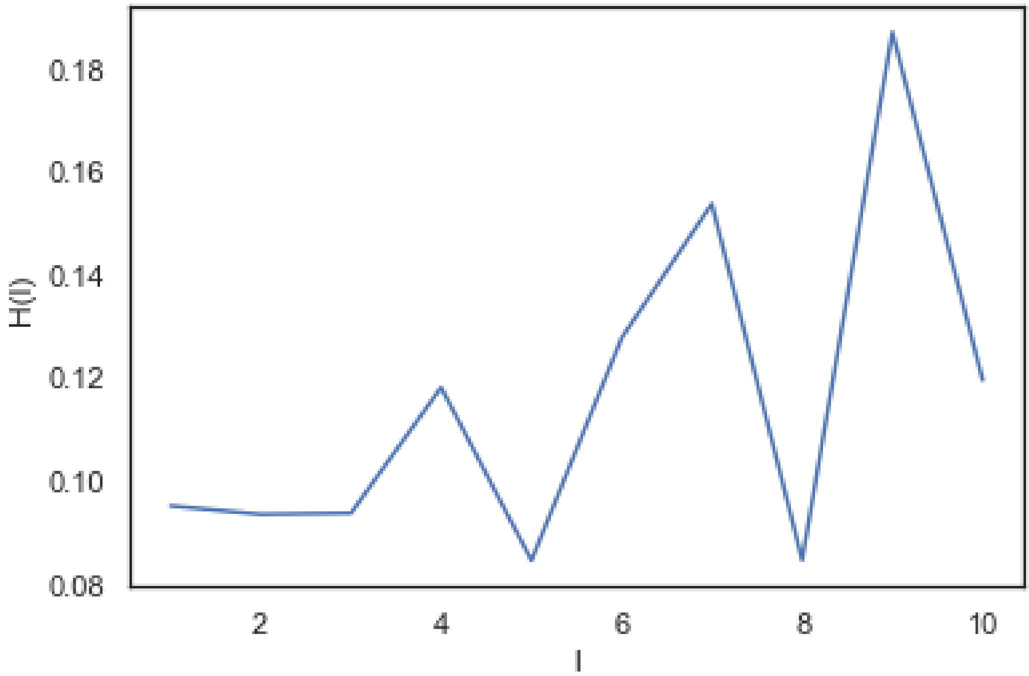

Figure 12: Evolution of H as a function of l

The value of H varies around the theoretical value, with a greater variations as l increases.

2 Implied volatility in the Lifted Heston model

2.1

The call's price in risk neutral approach is given by:

$$\begin{aligned} C_0 &= e^{-r_{int}T}\mathbb{E}^{\mathbb{Q}}[(S_T - K)_+] \\ &= e^{-r_{int}T}\int_{\mathbb{R}} f(x)q(x)dx \\ &= e^{-r_{int}T}\int_{\mathbb{R}} e^{-\omega x}f(x)e^{\omega x}q(x)dx \end{aligned}$$

Where $q(x)$ is the density distribution function of log price and $f(x) = (e^x - K)_+$.

In addition, the parameter ω has been introduced to ensure convergence integrability's conditions.

Using Plancherel-Parseval identity, we obtain:

$$C_0 = e^{-r_{int}T}\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(u + i\omega)\bar{\hat{q}}(u + i\omega)du$$

Where $\hat{f}$ and $\hat{q}$ designates respectively the fourrier transform of the function f and q.

q being the fourrier transform of the probability density of the log price, this is indeed its characteristic function:

$$\hat{q}(u) = \phi_T(u) = \mathbb{E}[e^{iu\, log(S_T)}]$$

Using the schales relation:

$$C_0 = e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty}\hat{f}(u+i\omega)\bar{\hat{q}}(u+i\omega)du + e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty}\hat{f}(-u+i\omega)\bar{\hat{q}}(-u+i\omega)du$$

- Let's compute $\bar{\hat{q}}(u+i\omega)$ and $\bar{\hat{q}}(-u+i\omega)$ as a function of ϕ_T.

We have:

$$\bar{\hat{q}}(-u+i\omega) = \mathbb{E}[e^{-i(-u+i\omega)\, log(S_T)}] = \mathbb{E}[e^{(iu+\omega)\, log(S_T)}] = \mathbb{E}[e^{i(u-i\omega)\, log(S_T)}] = \phi_T(u-i\omega)$$

And,

$$\bar{\hat{q}}(u+i\omega) = \mathbb{E}[e^{-i(u+i\omega)\, log(S_T)}] = \mathbb{E}[e^{(-iu+\omega)\, log(S_T)}] = \mathbb{E}[e^{i(-u-i\omega)\, log(S_T)}] = \phi_T(-u-i\omega)$$

Moreover, the conjugate of $\phi_T(-u-i\omega) = \mathbb{E}[e^{(-iu+\omega)\, log(S_T)}]$ is given by:

$$\mathbb{E}[e^{(iu+\omega)\, log(S_T)}] = \mathbb{E}[e^{i(u-i\omega)\, log(S_T)}] = \phi_T(u-i\omega)$$

- Let's compute now $\hat{f}(u+i\omega)$ and $\hat{f}(-u+i\omega)$.

We have:

$$\begin{aligned}
\hat{f}(u+i\omega) &= \int_{\mathbb{R}} e^{i(u+i\omega)x} f(x)dx = \int_{\mathbb{R}} e^{(iu-\omega)x}(e^x-K)_+dx \\
&= \int_{log(K)}^{+\infty} e^{(iu-\omega)x}(e^x-K)dx = \int_{log(K)}^{+\infty} e^{(iu-\omega+1)x}dx - \int_{log(K)}^{+\infty} Ke^{(iu-\omega)x}dx \\
&= \frac{1}{iu-\omega+1}[e^{(iu-\omega+1)x}]_{log(K)}^{+\infty} - \frac{K}{iu-\omega}[e^{(iu-\omega)x}]_{log(K)}^{+\infty} \\
&= -\frac{1}{iu-\omega+1}e^{(iu-\omega+1)log(K)} + \frac{K}{iu-\omega}e^{(iu-\omega)log(K)} \\
&= Ke^{(iu-\omega)log(K)}(\frac{1}{iu-\omega} - \frac{1}{iu-\omega+1}) \\
&= \frac{Ke^{(iu-\omega)log(K)}}{(iu-\omega+1)(iu-\omega)}
\end{aligned}$$

And,

$$\begin{aligned}
\hat{f}(-u+i\omega) &= \int_{\mathbb{R}} e^{i(-u+i\omega)x} f(x)dx = \int_{\mathbb{R}} e^{-(iu+\omega)x}(e^x - K)_+ dx \\
&= \int_{log(K)}^{+\infty} e^{-(iu+\omega)x}(e^x - K)dx = \int_{log(K)}^{+\infty} e^{(-iu-\omega+1)x}dx - \int_{log(K)}^{+\infty} Ke^{-(iu+\omega)x}dx \\
&= \frac{1}{-iu-\omega+1}[e^{(-iu-\omega+1)x}]_{log(K)}^{+\infty} + \frac{K}{iu+\omega}[e^{-(iu+\omega)x}]_{log(K)}^{+\infty} \\
&= \frac{1}{iu+\omega-1}e^{(-iu-\omega+1)log(K)} - \frac{K}{iu+\omega}e^{-(iu+\omega)log(K)} \\
&= Ke^{-(iu+\omega)log(K)}(\frac{1}{iu+\omega-1} - \frac{1}{iu+\omega}) \\
&= \frac{Ke^{-(iu+\omega)log(K)}}{(iu+\omega-1)(iu+\omega)}
\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}
C_0 = e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{Ke^{(iu-\omega)log(K)}}{(iu-\omega+1)(iu-\omega)}\phi_T(-u-i\omega)du \\
+e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{Ke^{(-iu-\omega)log(K)}}{(iu+\omega-1)(iu+\omega)}\phi_T(u-i\omega)du
\end{aligned}$$

If we take $\omega = \alpha_2 + 1$, we obtain:

$$\begin{aligned}
C_0 = e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{Ke^{(iu-(\alpha_2+1))log(K)}}{(iu-\alpha_2)(iu-(\alpha_2+1))}\phi_T(-u-i(\alpha_2+1))du \\
+e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{Ke^{(-iu-(\alpha_2+1))log(K)}}{(iu+\alpha_2)(iu+\alpha_2+1)}\phi_T(u-i(\alpha_2+1))du \\
C_0 = e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{e^{(iu+\alpha_2)log(K)}}{(-iu+\alpha_2)(-iu+\alpha_2+1)}\phi_T(-u-i(\alpha_2+1))du \\
+e^{-r_{int}T}\frac{1}{2\pi}\int_0^{+\infty} \frac{e^{(-iu-\alpha_2)log(K)}}{(iu+\alpha_2)(iu+\alpha_2+1)}\phi_T(u-i(\alpha_2+1))du
\end{aligned}$$

The two terms inside the two integrals are complementary, so we obtain:

$$\begin{aligned}
C_0 &= e^{-r_{int}T}\frac{1}{2\pi}2\mathrm{Re}(\int_0^{+\infty} \frac{e^{(-iu-\alpha_2)log(K)}}{(iu+\alpha_2)(iu+\alpha_2+1)}\phi_T(u-i(\alpha_2+1)))du \\
&= \frac{e^{-r_{int}T-\alpha_2 log(K)}}{\pi}\int_0^{+\infty} \mathrm{Re}[\frac{e^{-iulog(K)}}{(iu+\alpha_2)(iu+\alpha_2+1)}\phi_T(u-i(\alpha_2+1))]du
\end{aligned}$$

2.2

Ch_Lifted_Heston is a function implemented in Pyhton to approximate the characteristic function in the Lifted Heston model (See the jupyter notebook).

```
def Ch_Lifted_Heston(u,S0,T,rho,lamb,theta,nu,V0,n,rn,alpha,M):
    """Function that approximates the characteristic function in the Lifted Heston model"""
    i = complex(0,1)      # The complex number i
    h = T/M     # Time discretization step
    Psi = np.zeros(n)     # Initialise the Psi functions at 0
    cj = np.array([c(alpha,rn,j+1,n) for j in range(n)])     # Compute the cj coefficients
    xj = np.array([x(alpha,rn,j+1,n) for j in range(n)])     # Compute the xj coefficients
    SumcjPsi = np.sum(cj*Psi)     # Initialise the sum of Psi function at 0
    Intg = F(i*u,SumcjPsi,rho,nu,lamb)*g0(T,V0,lamb,theta,n,rn,alpha)*h      # Initialise the integral's value
    for k in range(M):
        Psi = 1/(1+xj*h)*(Psi + F(i*u,SumcjPsi,rho,nu,lamb)*h)      # Update of Psi
        SumcjPsi = np.sum(cj*Psi)      # Update of the sum of Psi
        Intg += F(i*u,SumcjPsi,rho,nu,lamb)*g0(T-(k+1)*h,V0,lamb,theta,n,rn,alpha)*h      # Add the new term to Intg
    Phiu = np.exp(i*u*np.log(S0) + Intg)      # The final value of the fourrier transform
    return Phiu
```

Figure 13: Ch_Lifted_Heston: The characteristic function in the Lifted Heston model

2.3

Call_Price_Lifted_Heston is a function implemented in Python that approximates the call option price in the Lifted Heston model (See the jupyter notebook).

```
def Call_Price_Lifted_Heston(S0,K,T,rho,lamb,theta,nu,V0,n,rn,alpha,M,alpha2,L):
    """Function that approximates the call option price in the Lifted Heston model"""
    i = complex(0,1)      # The complex number i
    Real_Ch_Lifted_Heston = lambda u: (Ch_Lifted_Heston(u-(alpha2+1)*i,S0,T,rho,lamb,theta,nu,V0,n,rn,alpha,M)*np.exp(-i*np.log(K
    C0 = np.exp(-alpha2*np.log(K))/(np.pi) * intg.quad(Real_Ch_Lifted_Heston,0,L)[0]      # The call option price
    return C0
```

Figure 14: Call_Price_Lifted_Heston: The call option price in the Lifted Heston model

2.4

Now we consider $r_n = 1 + 10n^{-0.9}$ in order to parametrize the weights $(c_i)_{i=1}^n$ and the means reversions $(x_i)_{i=1}^n$.

Firstly, we fix a maturity T of 1 year and a truncation level L = 100. We consider then 20 equidistant log strikes between -1.2 and 0.2, and we plot the implied volatility smiles under lifted Heston model for factors n = 5, 10, 20, 50. The following figure shows the result obtained:

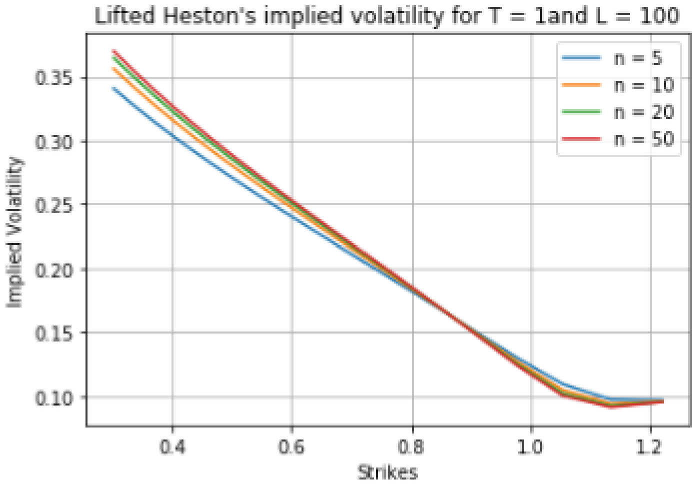

Figure 15: Implied volatility smiles for T=1 and L=100

For maturity T=1, we notice that the shape of the reproduced volatility smile under the lifted Heston model is similar for all the values of the factors considered.

In a second time, for $T = \frac{1}{26}$ and a truncation level $L = 1000$, we consider 20 equidistant log strikes between -0.15 and 0.05 and we plot the implied volatility smiles under lifted Heston model for factors n = 5, 10, 20, 50. The figure 16 shows the shape of smiles obtained:

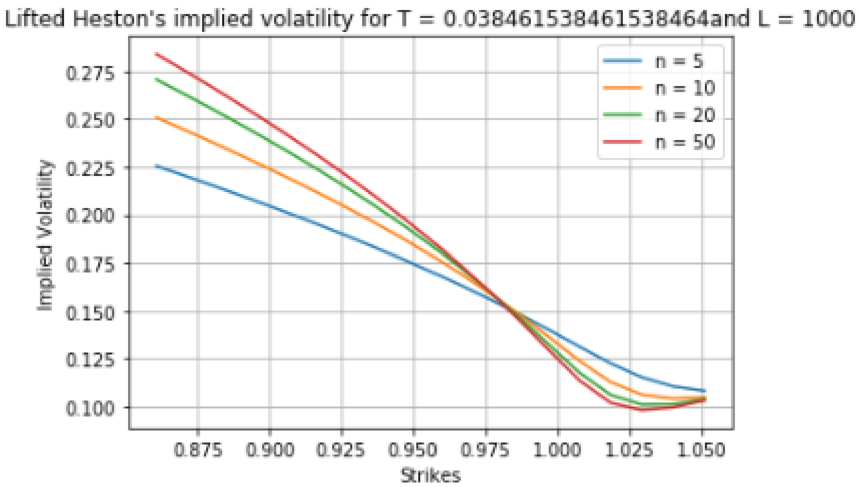

Figure 16: Implied volatility smiles for T=1/26 and L=1000

We observe that the shape of the reproduced volatility smiles differs in this case according to the value of n.
Decreasing the number of factors n in the lifted Heston model will steepen the implied volatility slice for short-maturities.
We deduce that for long maturities the fit of volatility smile is perfectly good otherwise for short maturities, it depends on the number of factors considered.