PARIS SACLAY/PARIS EVRY UNIVERSITY

NUMERICAL FINANCE

Practical work: Dynamic Hedging Strategy in the Black and Scholes framework

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1 Intoduction

This practical work sets out steps on how to calculate the fair price from a theoretical standpoint, for an option at maturity that depends only on the value of the asset at maturity, specifically for a pay-off with polynomial growth, The example given is that of a call option that has a simple formula for fair price based on Black and Scholes, followed by how to replicate the pay-off of a portfolio composed of risky and non-risky assets with an investment strategy in a discrete way, as is the case in real world applications.

2 Question 1

We denote T as strictly positive real value.

The risky asset follows a dynamic given by:

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

The solution of this equation is wildly known and can be expressed as follows:

$$S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}$$

The value of $\mathbf{E}^{\mathbb{P}}(|S_T|^p)$ is:

$$\mathbf{E}^{\mathbb{P}}(|S_T|^p) = S_0^p exp(p(\mu-\sigma^2/2)T)\mathbf{E}^{\mathbb{P}}(exp(p\sigma W_T))$$

As $exp(\lambda W_t - \lambda^2 t/2)$ is a martingale for λ a real value.

The value of $\mathbf{E}^{\mathbb{P}}(|S_T|^p)$ is then:

$$\mathbf{E}^{\mathbb{P}}(|S_T|^p) = exp(p(\mu-\sigma^2/2)T)\ exp(p^2(\sigma^2/2)T) < +\infty$$

It is clear that $\mathbf{E}^{\mathbb{P}}(\Phi(S_T)) < +\infty$ as $0 \leq \Phi(x) \leq C(1+|x|^p)$ wich equates to $\Phi(S_T) \in \mathbf{L}^1(\mathbb{P})$

3 Question 2

Under the risk free probability $\mathbb{Q}$ the actualised price of the option is a martingale. In other terms, $e^{-rt}V_t$ is a martingale, $\mathbf{E}^{\mathbb{Q}}(e^{-rT}V_T/\mathcal{F}_t) = e^{-rt}V_t$

Then by replacing V_T by its expression:

$$V_t = exp(-r(T-t))\mathbf{E}^{\mathbb{Q}}(\Phi(S_T)/\mathcal{F}_t)$$

Under the risk neutral probability S_t is written as $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t^{\mathbb{Q}}}$

S_T/S_t is a function of $T-t$ and $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}$, as a consequence S_T/S_t is independent of $\mathcal{F}_t$

$\Phi(S_T) = \Phi((S_T/S_t)S_t)$ with S_t is $\mathcal{F}_t$ measurable. Which justify the following equality:

$$V_t = exp(-r(T-t))\mathbf{E}^{\mathbb{Q}}(\Phi(S_T)/\mathcal{F}_t) = exp(-r(T-t))\mathbf{E}^{\mathbb{Q}}(\Phi(S_T)/S_t)$$

4 Question 3

From question 2, $\Phi(S_T) = \Phi((S_T/S_t)S_t)$ with (S_T/S_t) independent of S_t. It follows that $V_t = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}(\Phi(S_T)/\mathcal{S}_t) = v(t, S_t)$ with $v(t,x) = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}(\Phi(xS_T/S_t))$

$e^{-rt}V_t$ is a martingale, which ensures that the expectation of its dynamic is null.

Using the Ito formula, the variation can be written as:

$$d(e^{-rt}V_t) = -re^{-rt}V_t dt + e^{-rt}dV_t$$

Using the Ito formula on V_t as $V_t = v(t, S_t)$ and $v \in C^{1,2}([0,T[\times\mathbb{R}_+^*, \mathbb{R}) \cap C^0([0,T] \times \mathbb{R}_+^*, \mathbb{R})$ (Smoothness conditions)

$$dV_t = \partial_t v(t, S_t)dt + \partial_x v(t, S_t)dS_t + \frac{1}{2}\partial_x^2 v(t, S_t)d < S, S >_t$$

By replacing dS_t with its dynamic and calculating the bracket. The expressing obtained is:

$$dV_t = \partial_t v(t, S_t) + (rS_t\partial_x v(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 v(t, S_t))dt + \sigma S_t \partial_x v(t, S_t)dW_t^{\mathbb{Q}}$$

As $\mathbf{E}^{\mathbb{Q}}(e^{-rt}\sigma S_t\partial_x v(t, S_t)dW_t^{\mathbb{Q}}) = 0$, The calculation of the expectation of the dynamic gives.

$$\mathbf{E}^{\mathbb{Q}}(d(e^{-rt}V_t)) = e^{-rt}(-rv(t, S_t) + \partial_t v(t, S_t) + rS_t\partial_x v(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 v(t, S_t))dt = 0$$

This results in the equation:

$$\partial_t v(t, x) + rx\partial_x v(t, x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t, x) - rv(t, x) = 0$$

with $V_T = v(T, S_T) = \Phi(S_T)$, which gives $v(T, x) = \Phi(x)$

It is clear that the equation verified by v can be written as:

$$\begin{cases} \partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x) - rv(t,x) = 0, \text{with } (t,x) \in [0,T[\times\mathbb{R}_+^* \\ v(T,x) = \Phi(x) \ x \in \mathbb{R}_+^* \end{cases}$$

The replacement of S_t by a positive real value is justified as S_t can take any value in $\mathbb{R}_+^*$, as S_t is a log-normal random variable.

5 Bonus question

In the case of a continuous pay-off function, the smoothness conditions are verified. To arrive to such conclusion, first for a fixed time $t \in [0, T[$, the value of the fair time under risk free probability can be expressed as follows;

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}(\Phi(\frac{S_T}{S_t}S_t)/\mathcal{F}_t) \\
&= v(t, S_t)
\end{aligned}
$$

$\frac{S_T}{S_t}$ can be expressed as $e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}})}$ which is equivalent to $e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}N}$ with N is a normal random variable $\mathcal{N}(0,1)$. $v(t,x)$ is then expressed as:

$$
v(t,x) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}y})e^{-y^2/2}dy
$$

Justifying the differentiablity at x

As x varies in $\mathbb{R}_+^*$, we consider the function $h(t,x) = e^{r(T-t)}\sqrt{2\pi}v(t,e^x)$ with $x \in \mathbb{R}$. If h is derivable then the same can be said about v.

To simplify the expression, new variables are defined $\alpha = \sigma\sqrt{T-t}$ and $\beta = (r - \frac{1}{2}\sigma^2)(T-t)$

The expression of the integral is simplified greatly for h:

$$
\begin{aligned}
h(t,x) &= \int_{\mathbb{R}}\Phi(e^{\beta+x+\alpha y})e^{-y^2/2}dy \\
&= \frac{1}{\alpha}\int_{\mathbb{R}}\Phi(e^{\beta+z})e^{-\frac{(z-x)^2}{2\alpha^2}}dz \\
&= \frac{1}{\alpha}\int_{\mathbb{R}}r(x,z)dz
\end{aligned}
$$

With the variable change $z = x + \alpha y$

The function r is differentiable at x for $x \in \mathbb{R}$, and the derivative is given by

$$
\partial_x r(x,z) = -\frac{x}{\alpha^2}\Phi(e^{\beta+z})e^{-\frac{(z-x)^2}{2\alpha^2}}
$$

The second derivative and is given by

$$
\partial_x^2 r(x,z) = -\frac{1}{\alpha^2}\Phi(e^{\beta+z})e^{-\frac{(z-x)^2}{2\alpha^2}} + \frac{x^2}{\alpha^4}\Phi(e^{\beta+z})e^{-\frac{(z-x)^2}{2\alpha^2}}
$$

Consider $R > 0$ and $|x| \leq R$ as Φ has polynomial growth.

$$
\Phi(e^{\beta+z})e^{-\frac{(z-x)^2}{2\alpha^2}} \leq C(1+e^{p(\beta+z)})e^{-\frac{z^2}{2\alpha^2}}e^{\frac{R|z|}{\alpha^2}}
$$

As $(z-x)^2 = z^2 + x^2 - 2xz \geq z^2 - 2R|z|$. Then

$$
|\partial_x r(x,z)| \leq \frac{R}{\alpha^2}C(1+e^{p(\beta+z)})e^{-\frac{z^2}{2\alpha^2}}e^{\frac{R|z|}{\alpha^2}} = o(e^{-\frac{z^2}{4\alpha^2}}) \text{ Independent of } x \text{ and integrable}
$$

And

$|\partial_x^2 r(x,z)| \leq C(\frac{R^2}{\alpha^4}+\frac{1}{\alpha^2})(1+e^{p(\beta+z)})e^{-\frac{z^2}{2\alpha^2}}e^{\frac{R|z|}{\alpha^2}} = o(e^{-\frac{z^2}{4\alpha^2}})$ Independent of x and integrable

It is clear that the function $h(t,.) \in C^2(\mathbb{R},\mathbb{R})$ which equates to $v(t,.) \in C^2(\mathbb{R}_+^*,\mathbb{R})$ because $\mathbb{R} = \bigcup_{R>0}[-R,R]$.

Justifying the differentiablity at t

for $x \in \mathbb{R}_+^*$

For simplification the variable $T-t$ is considered as t the function v is denoted as $u(t,x) = \sqrt{2\pi}v(T-t,x)$, the variable $\alpha = r - \frac{1}{2}\sigma^2$

$$u(t,x) = e^{-rt}\int_{\mathbb{R}} \Phi(xe^{\alpha t+\sigma\sqrt{t}y})e^{-y^2/2}dy$$

With variable change $z = \alpha t + \sigma\sqrt{t}y$

$$\begin{aligned}
u(t,x) &= \frac{e^{-rt}}{\sigma\sqrt{t}}\int_{\mathbb{R}} \Phi(xe^{z})e^{-\frac{(z-\alpha t)^2}{2\sigma^2 t}}dz \\
&= \frac{e^{-rt}}{\sigma\sqrt{t}}e^{\frac{\alpha}{\sigma^2}}\int_{\mathbb{R}} \Phi(xe^{z})e^{-\frac{z^2}{2\sigma^2 t}}e^{-\frac{\alpha^2 t}{2\sigma^2}}dz \\
&= \frac{e^{-rt}}{\sigma\sqrt{t}}e^{\frac{\alpha}{\sigma^2}}\int_{\mathbb{R}} g(t,z)dz
\end{aligned}$$

$t \longrightarrow \frac{e^{-rt}}{\sigma\sqrt{t}}e^{\frac{\alpha}{\sigma^2}}$ is differentiable

$$\partial_t g(t,z) = \Phi(xe^z)\frac{z^2}{2\sigma^2 t^2}e^{-\frac{z^2}{2\sigma^2 t}}e^{-\frac{\alpha^2 t}{2\sigma^2}} - \frac{\alpha^2}{2\sigma^2}e^{-\frac{z^2}{2\sigma^2 t}}e^{-\frac{\alpha^2 t}{2\sigma^2}}$$

For $\gamma > 0$, t is taken in $[\gamma, T]$, Then

$|\partial_t g(t,z)| \leq C(\frac{z^2}{2\sigma^2\gamma^2}+\frac{\alpha^2}{2\sigma^2})(1+x^p e^{pz})e^{-\frac{z^2}{2\sigma^2 T}} = o(e^{-\frac{z^2}{4\sigma^2 T}})$ Independent of t and integrable

$v(t,.) \in C^1([0,T[,\mathbb{R})$

Justifying the continuity at T

The only smoothness condition that should be addressed is the continuity at T. It is obvious that $\Phi(xe^{\alpha t+\sigma\sqrt{t}y}) \underset{t\to 0^+}{\longrightarrow} \Phi(x)$ and also:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\Phi(xe^{\alpha t+\sigma\sqrt{t}y})e^{-y^2/2} &\leq \frac{C}{\sqrt{2\pi}}(1+xe^{p(|\alpha|t+\sigma\sqrt{t}y)})e^{-y^2/2} \\
&\leq \frac{C}{\sqrt{2\pi}}(1+xe^{p(|\alpha|T+\sigma\sqrt{T}|y|)})e^{-y^2/2} = o(e^{-y^2/4}) \text{ Independent of } t
\end{aligned}$$

With dominated convergence theorem, it is clear that

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi(xe^{\alpha t+\sigma\sqrt{t}y})e^{-y^2/2}dt \underset{t\to 0^+}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi(x)e^{-y^2/2}dt = \Phi(x)$$

Which is equivalent to:

$$\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi(xe^{\alpha t+\sigma\sqrt{t}y})e^{-y^2/2}dt \underset{t\to 0^+}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi(x)e^{-y^2/2}dt = \Phi(x)$$

It yields that $v \in C^{1,2}([0,T[\times\mathbb{R}_+^*,\mathbb{R})\cap C^0([0,T]\times\mathbb{R}_+^*,\mathbb{R})$

6 Question 5

In case of $\Phi(x) = (x-K)^+$ the value of the fair price at t (for $t < T$) using the Black and Scholes formula is equal to:

$$V_t = v(t,S_t) = C(S_t,K,r,T-t,\sigma) = S_t\mathcal{N}(d_1(S_t,K,r,T-t,\sigma)) - Ke^{-r(T-t)}\mathcal{N}(d_2(S_t,K,r,T-t,\sigma))$$

With $d_1(S_t,K,r,T-t,\sigma) = \frac{1}{\sigma\sqrt{T-t}}(ln(\frac{S_t}{K})+(r+\frac{1}{2}\sigma^2)(T-t))$ and $d2(S_t,K,r,T-t,\sigma) = d1(S_t,K,r,T-t,\sigma)-\sigma\sqrt{T-t}$

The value of δ_t in the replication strategy is given by $\delta_t = \partial_x v(t,x)$

$$\begin{aligned}\partial_x v(t,x) = \mathcal{N}(d_1(x,K,r,T-t,\sigma)) + \frac{1}{\sigma\sqrt{T-t}}\phi(d_1(x,K,r,T-t,\sigma))\\ - Ke^{-r(T-t)}\frac{1}{x\sigma\sqrt{T-t}}\phi(d_2(x,K,r,T-t,\sigma))\end{aligned}$$

With $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

We have:

$$\begin{aligned}d_2^2 &= d_1^2 - 2\sigma d_1\sqrt{T-t} + \sigma^2(T-t)\\ &= d_1^2 - 2\ ln(\frac{x}{K}) - 2r(T-t)\end{aligned}$$

Then $e^{-d_2^2/2} = \frac{x}{K}e^{r(T-t)}e^{-d_1^2/2}$, it follows that:

$$\delta_t = \partial_x v(t,x) = \mathcal{N}(d1)$$

Based on the value of δ_t, we create a matrix of M simulated asset values for N steps such as $h = T/N$. It is straightforward to obtain the values of the portfolio at each step based on the formula:

$$\begin{aligned}V^h_{t_{i+1}} - V^h_{t_i} &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta^0_{t_i}(S^0_{t_{i+1}} - S^0_{t_i})\\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta^0_{t_i}S^0_{t_i}(exp(rh)-1)\\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V^h_{t_i} - \delta_{t_i}S_{t_i})(exp(rh)-1)\end{aligned}$$

To compare the precision of the approximation for the different values of h, the value of the discriticized portfolio is compared to the real value of V_t according to the Black & Scholes formula. It is clear from the graphs below that when the value of h approaches 0, which is equivalent to a higher value of N, the portfolio constructed with the investment strategy updated at each time t_i gets closer to the real value of the fair price.

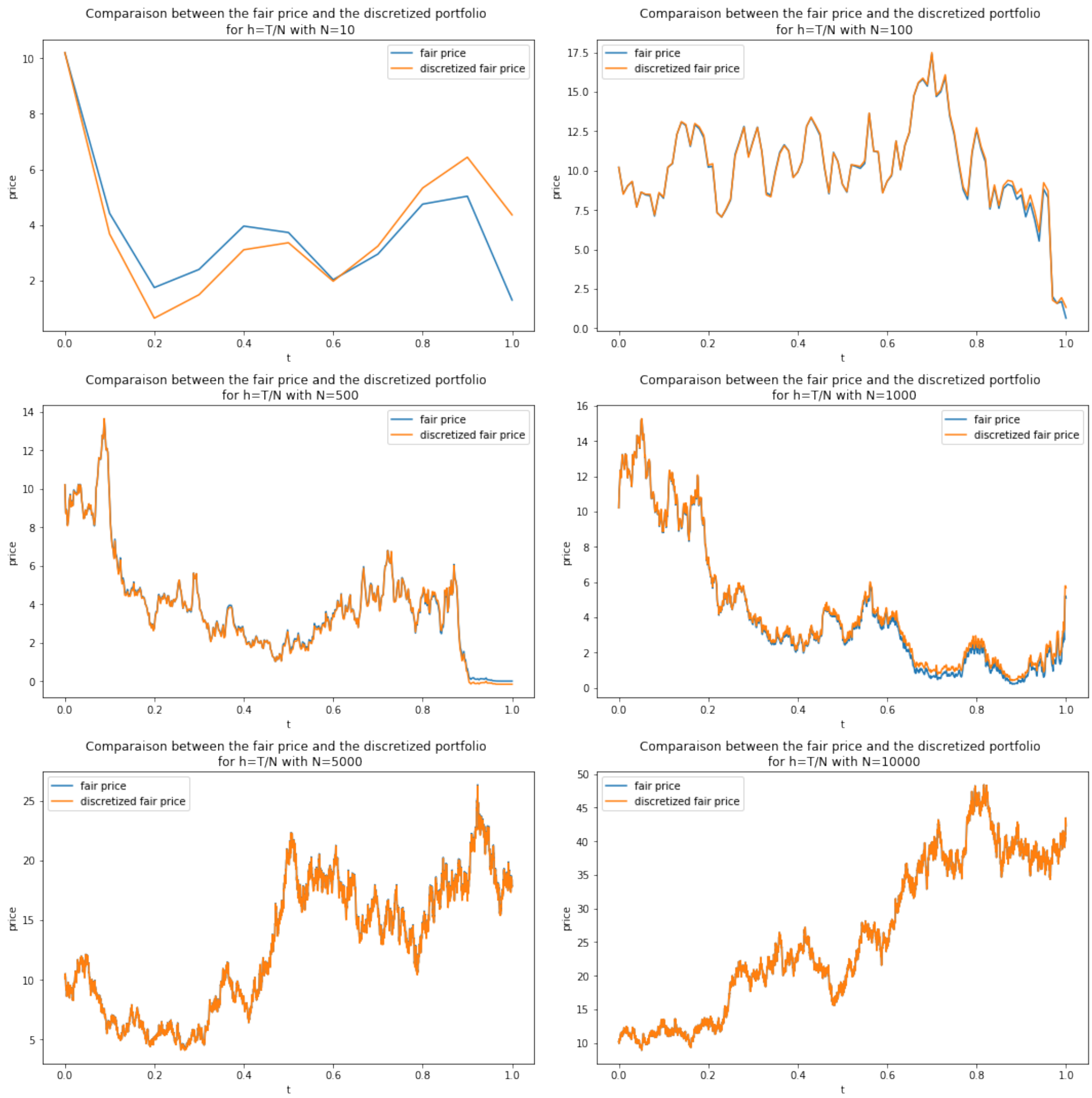

Figure 1: Comparison between the approximation portfolio to the fair price for different values of h

The values of the parameters used are $S_0 = 100$, $\sigma = 0.02$; $mu = 0.03$, $r = 0.015$, $K = 97$, $T = 1$ as stated in the question.

The difference between the actual portfolio and the constructed portfolio is that of an inherent variance associated with the approximation as presented in the figures below :

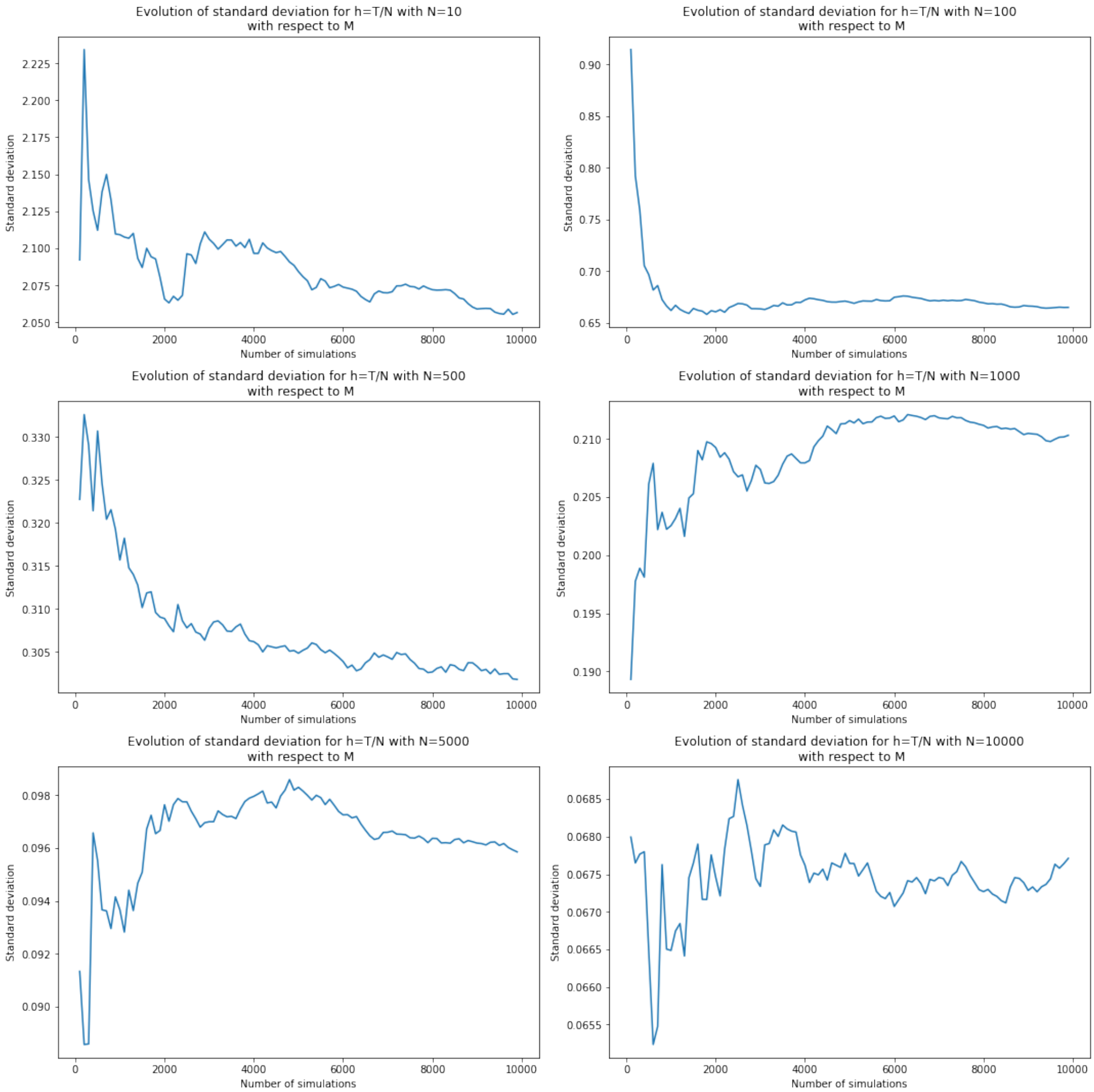

Figure 2: Comparison of the approximation to the payoff with the investement strategy for different values of h

The figure shows the standard deviation of the difference between the payoff and the approximate payoff. The standard deviation $std(V_T^h - V_T)$ is calculated based on the Central Limit Theorem, since the M represents the number of the simulated realization of $(S_t)_{t\in[0,T]}$. For the chosen number of simulations, it is clear that the standard deviation converges to a constant that depends on h. As the step h decreases, the standard deviation approaches 0 and the approximate payoff converges to the real payoff.

From the graphs above, we can deduce that a number of simulations equal to $4,000$ for the central limit theorem produces values close to the real standard deviation. The above remark about the evolution of the standard deviation with respect to h is confirmed by the plot below.

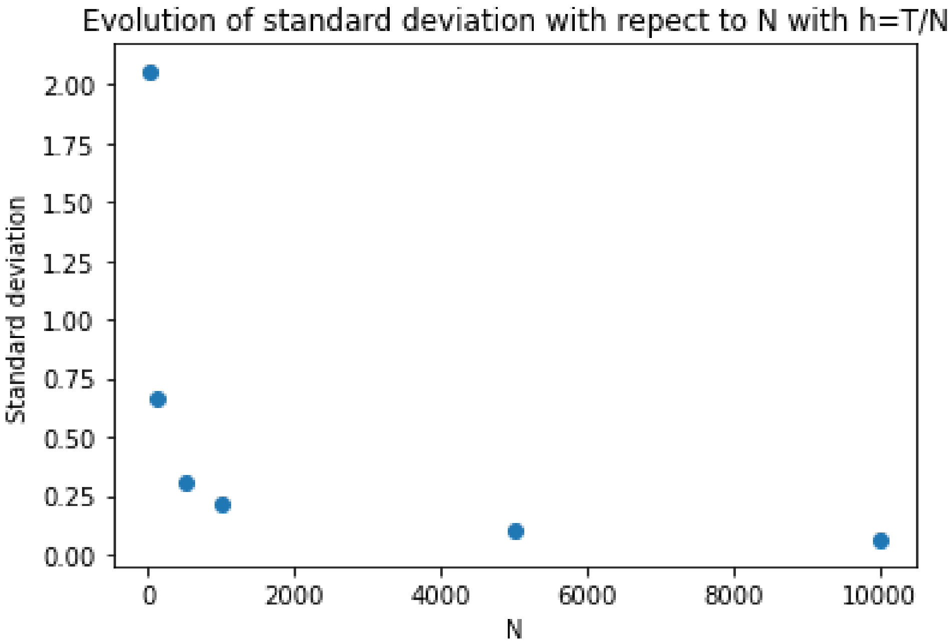

Figure 3: Comparison of the approximation to the payoff with the investment strategy for different values of h

7 Conclusion

The Black and Scholes formula marks a pivotal stage in the history of finance. An elegant and easy-to-use expression has influenced many developments. The expression of the call requires the use of such formula. The investment and hedging strategy with delta for this type of options is simple and straightforward. This course is a foretaste of what will follow and the model and methods that will be introduced in the future.