Paris Saclay/Paris Evry University

Numerical finance

Practical work: Homework 2

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1 Introduction

This second assignment provides an overview of some of the main topics in numerical finance, introducing methods such as the resolution of stochastic differential equations based on approximations such as the Euler and Milstein schemas, the calculation of payoff expectation and the method of calculating prices and Greeks mainly Delta and Gamma based on various methodologies, it also addresses the topic of variance reduction.

2 Weak strong error

The equation to be solved is written as follows:

$$X_t = x + \int_0^t bX_s ds + \int_0^t \sigma X_s dW_s$$

With $(\sigma, b) \in \mathbb{R}^2$

The dynamics of this equation is expressed as :

$$dX_t = bX_t dt + \sigma X_t dW_t$$

The dynamics of this equation match the underlying dynamics of Black&Scholes. The explicit solution of this equation is :

$$X_t = xe^{(b-\frac{\sigma^2}{2})t+\sigma W_t}$$

The following equation shows how to simulate each step for N the number of steps, and T the time period:

$$\begin{cases} X_0 = x \\ X_{t_{i+1}} = X_{t_i} e^{(b-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z_i} \quad \forall i \in \{0, 1, ..., N-1\} \end{cases}$$

With $t_i = \frac{iT}{N}$ and $h = \frac{T}{N}$ and $Z_i \sim \mathcal{N}(0, 1)$

With Euler's scheme, the functions within the integral are approximated by their values in t_i.

The values of the stochastic process are updated as follows :

$$\begin{cases} X_0 = x \\ X_{t_{i+1}} = (bh + 1 + \sigma\sqrt{h}Z_i)X_{t_i} \quad \forall i \in \{0, 1, ..., N-1\} \end{cases}$$

The formula is arrived at by this calculation :

$$\begin{aligned} dX_t &= \int_{t_i}^{t_{i+1}} bX_s ds + \int_{t_i}^{t_{i+1}} \sigma X_s dW_s \\ &\approx \int_{t_i}^{t_{i+1}} bX_{t_i} ds + \int_{t_i}^{t_{i+1}} \sigma X_{t_i} dW_s \\ &\approx bX_{t_i}h + \sigma X_{t_i}(W_{t_{i+1}} - W_{t_i}) \end{aligned}$$

The Euler method yields a precise representation of the stochastic process as shown in the figure, the values of the constants being given by : $b = 0.02$, $\sigma = 0.01$, $x = 100$ and $h \in (2^{-4}, 2^{-8}, 2^{-10})$.

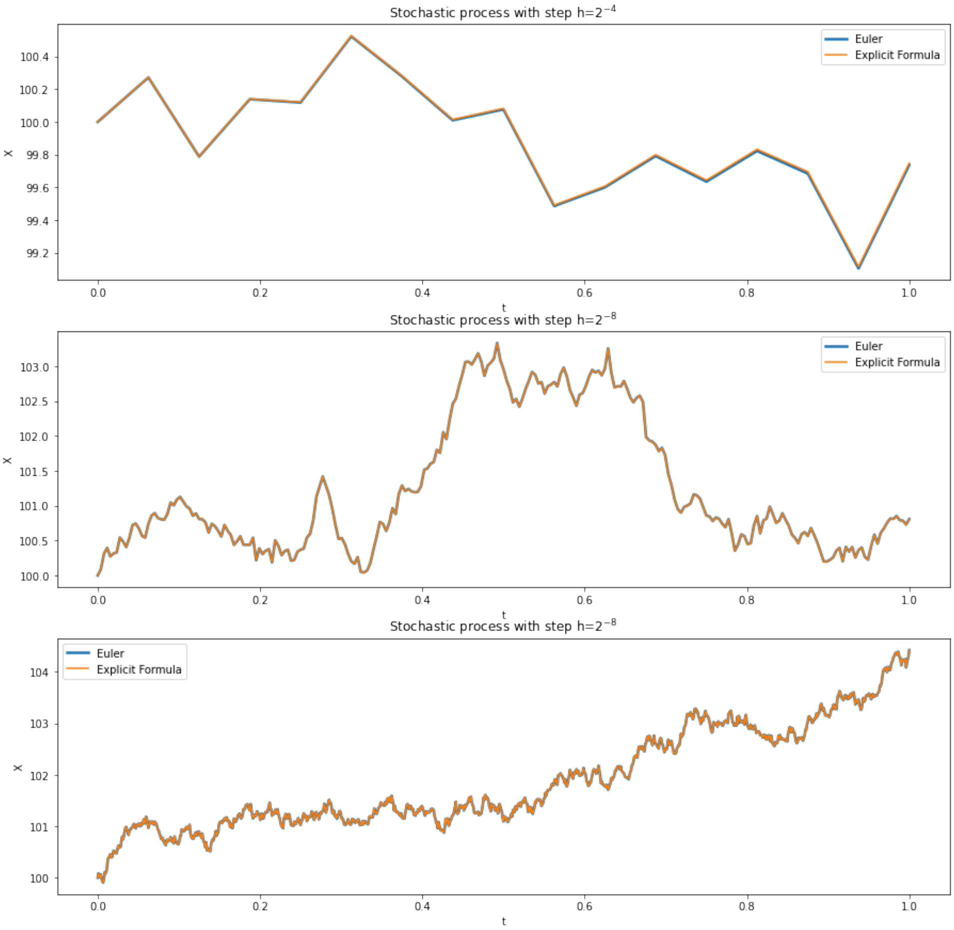

Figure 1: Comparison between the explicit formula and the approximated solution with Euler's schema for different values of h

However, when a higher value is chosen for σ, which is equivalent to a more volatile plot, the error is larger, especially in the case of $h = 2^{-4}$, as can be seen below for $sigma = 0.5$.

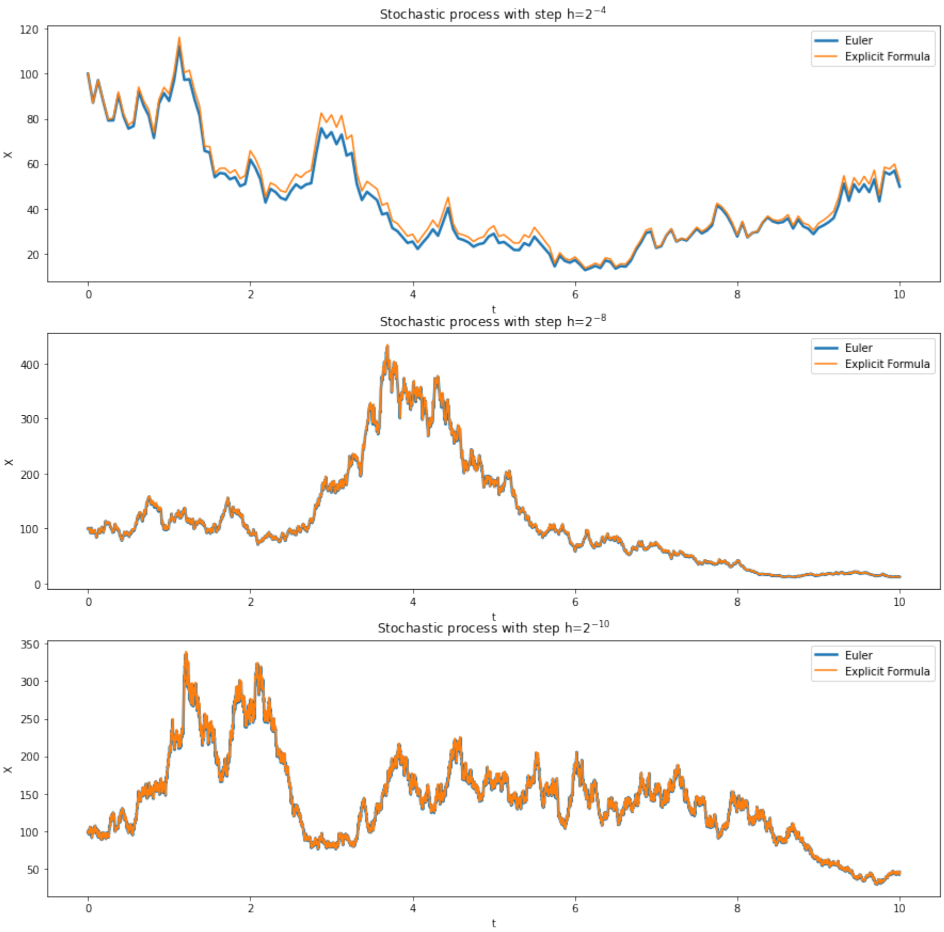

Figure 2: Comparison between the explicit formula and the approximated solution with Euler's schema for different values of h

In this section, Milstein's scheme will be discussed but first, an approximation is necessary to simulate the integral that appears in the formula.

$$\int_{t_i}^{t_{i+1}} (W_s - W_{t_i}) dW_s$$

At first, the dynamic of the stochastic of W_t^2 is calculated on the basis of Ito Lemma :

$$\begin{aligned} dW_t^2 &= 2W_t dW_t + d < W_., W_. >_t \\ &= 2W_t dW_t + dt \end{aligned}$$

From the above dynamic

$$\int_{t_i}^{t_{i+1}} W_s dW_s = \frac{1}{2}(W_{t_{i+1}}^2 - W_{t_i}^2 - h)$$

Then

$$\int_{t_i}^{t_{i+1}} (W_s - W_{t_i}) dW_s = \frac{1}{2}((W_{t_{i+1}} - W_{t_i})^2 - h)$$

As the approximation is obtained, based on the Milstein's schema, the following integral is approximated:

$$\int_{t_i}^{t_{i+1}} \sigma(X_s)dW_s \approx \int_{t_i}^{t_{i+1}} \sigma(X_{t_i} + b(X_{t_i})(s - t_{t_i}) + \sigma(X_{t_i})(W_s - W_{t_i}))dWs$$
$$\approx \sigma(X_{t_i})(W_{i+1} - W_{t_i}) + (\sigma'\sigma)(X_{t_i}) \int_{t_i}^{t_{i+1}} (W_s - W_{t_i})dW_s$$

The Milstein schema is then given by:

$$X_{t_{i+1}} = X_{t_i} + b(X_{t_i})h + \sigma(X_{t_i})(W_{t_{i+1}} - W_{t_i}) + \frac{1}{2}(\sigma'\sigma)(X_{t_i})((W_{t_{i+1}} - W_{t_i})^2 - h)$$

In the case of the first equation as $b : x \longrightarrow bx$ and $\sigma : x \longrightarrow \sigma x$ are clearly Lipschitz and continuous. The approximation is valid and is given below:

$$X_{t_{i+1}} = X_{t_i}(1 + bh + \sigma\sqrt{h}Z_i + \frac{1}{2}\sigma^2 h(Z_i^2 - 1))$$

The equation with the Mistein's schema is then :

$$\begin{cases} X_0 = x \\ X_{t_{i+1}} = X_{t_i}(1 + bh + \sigma\sqrt{h}Z_i + \frac{1}{2}\sigma^2 h(Z_i^2 - 1)) \quad \forall i \in \{0, 1, ..., N-1\} \end{cases}$$

To emphasize the difference between the Euler and Milstein schemas in terms of strong error, the following expectations should be calculated :

$$\mathbb{E}(\underset{s \le t}{sup} \, |X_s - X_s^{N,M}|) \text{ and } \mathbb{E}(\underset{s \le t}{sup} \, |X_s - X_s^{N,E}|)$$

Which can be approximated using the Monte Carlo method. The parameters used are $X_0 = 100$, $b = 0.01$, $\sigma = 0.02$ and $T = 10$ The number of simulations employed is 1000 and the number of steps varies from 50 to 1000 with a step of 50.

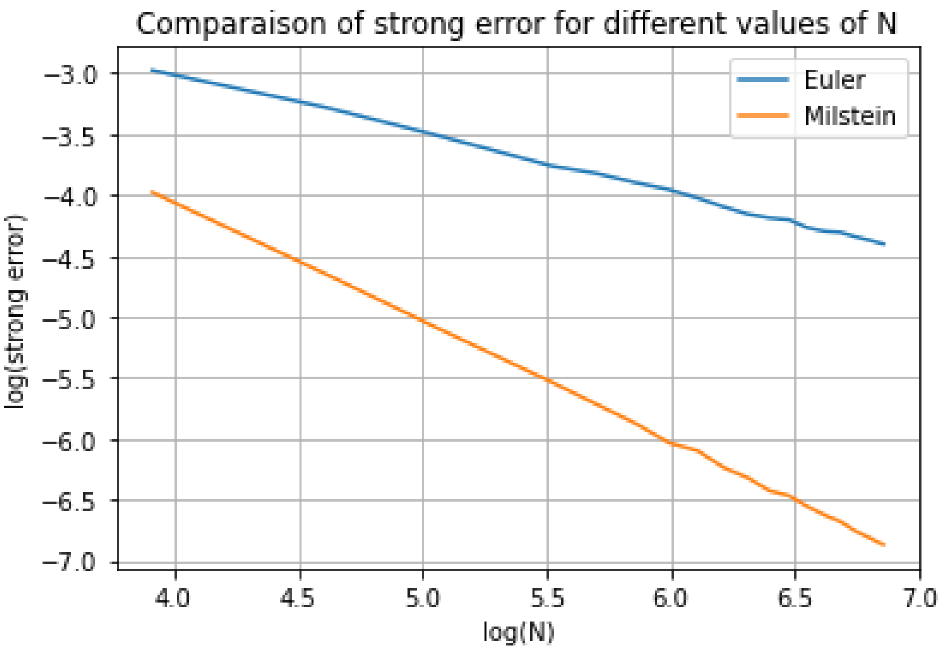

Figure 3: Comparison Euler and Milstein in terms of strong error

Using a regression model, the Milstein slope is given by -0.98 and the slope of the Euler scheme is -0.49 which coincides exactly with the theoretical measurements, since the Milstein scheme is of the order of $\frac{1}{N}$ and the Euler scheme is of the order of $\frac{1}{\sqrt{N}}$.

The expectation of the solution of the differential equation is equal to $e^{ln(X_0)+(b-\frac{\sigma^2}{2})T+\frac{1}{2}\sigma^2 T} = X_0 e^{bT}$ since the solution of the stochastic differential equation has a closed formula which follows a log-normal distribution $\mathcal{LN}(ln(X_0) + (b - \frac{\sigma^2}{2})T, \sigma^2 T)$.

$E_{n,N}$ is defined as $\frac{1}{n}\sum_{i=1}^{n} \hat{X}_1^{j,N}$ where $\hat{X}_1^{j,N}$ represents a realization of time 1 of a Euler scheme with a step equal to $\frac{1}{N}$, $N \in \mathbb{N}^*$ and $\hat{X}_1^{j,N}$ are independent for $j \in \{1, ..., n\}$. The error $\frac{1}{m}\sum_{i=1}^{m} |E^i_{n,N} - \mathbb{E}(X_1)|$ is calculated, with $E^i_{n,N}$ a realization of the approximate expectation with N the number of steps, and the step h varying from 10^{-3} to 10^{-1} with 20 points equally spaced in logarithmic space, $m = 100$, and $n = 10^i$ with $i \in \{2, ..., 5\}$. The constants are $b = 0.01$, $\sigma = 0.02$, $X_0 = 100$ and $T = 1$.

$n = 10^6$ is omitted because it was not feasible in a reasonable time frame even with a Cloud machine with 32GB of Ram and 8 high-performance cores and custom multiprocessor code. However, this will not change the conclusions and comments.

The following plot is drawn:

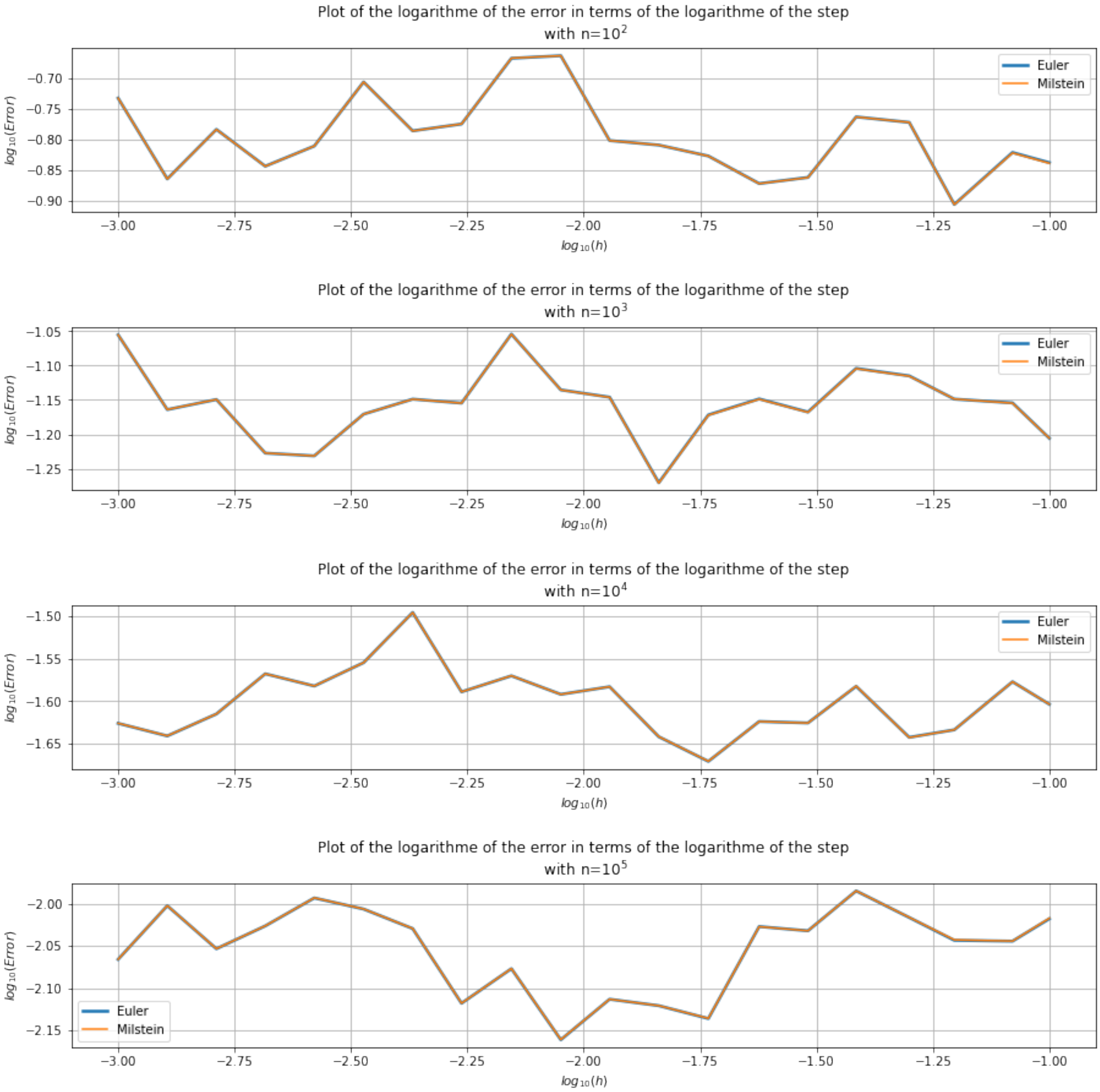

Figure 4: The logarithm of the mean absolute error of expectation computed with Monte Carlo for 100 observations

Euler and Milstein yield the same error, the number n of simulations is the most influential when calculating the error, it varies around a constant value independently of the choice of N, This is a stark difference with the previous section where the large error differs between the Mistein and Euler. The mean absolute error of the expectation depends also on the number of realizations for the calculation of the expectation of the Monte Carlo approximation, compared to the strong error which depends only the number of steps.

$$\frac{1}{n}\sum_{i=1}^{n}\hat{X}_1^{j,N} - E(X_1) = \frac{1}{n}\sum_{i=1}^{n}\hat{X}_1^{j,N} - E(\hat{X}_1^N) + E(\hat{X}_1^N) - E(X_1)$$

and $E(\hat{X}_1^N) = (1+bh)^N$

In the case of the Milstein and Euler schema $X_1^N = X_0 \prod_{i=1}^n (1 + bh + \sigma h Z_i)$ for Euler and $X_1^N = X_0 \prod_{i=1}^n (1 + bh + \sigma h Z_i + \frac{h}{2}(Z_i^2 - 1))$ for Milstein as $\mathbb{E}(Z^2) = 1$, therefore, the error does not include the term added to Milstein's schema relative to Euler's schema, and because b is chosen small, as it is the case in this calculation, and the error depends on the Monte Carlo approximation $\frac{1}{n}\sum_{i=1}^n \hat{X}_1^{j,N} - E(X_1) = \frac{1}{n}\sum_{i=1}^n \hat{X}_1^{j,N} - E(\hat{X}_1^N)$ which is dependent in n and of order $1/\sqrt{n}$ for both schemas, and also on the error introduced of order h which comes from the approximation of $X_0 e^{bh}$ with $X_0(1 + bh)^N$.

To show the effect of h, select $b = 0.5$ and $\sigma = 0.6$. The new plot is given by :

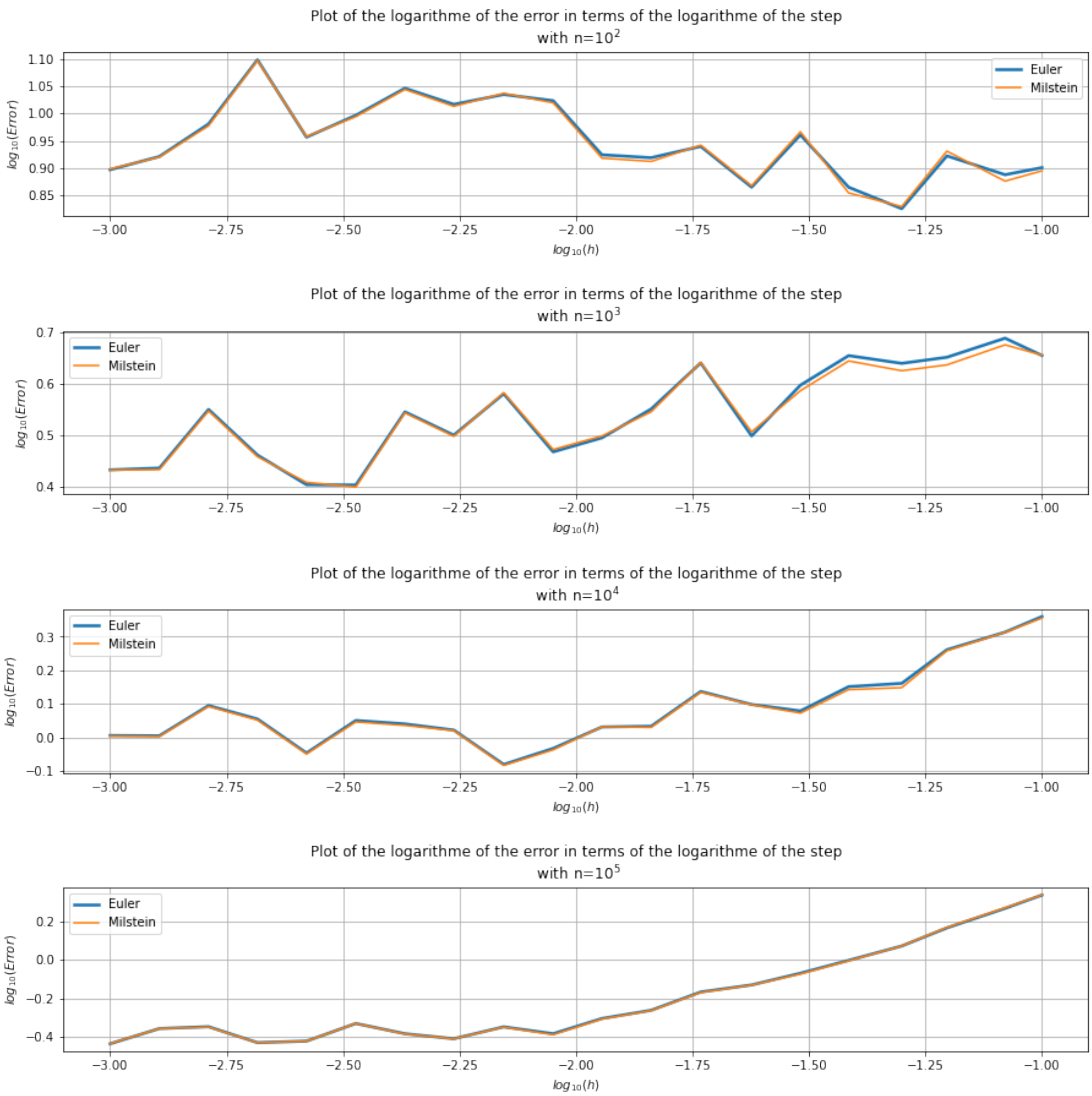

Figure 5: The logarithm of the mean absolute error of expectation computed with Monte Carlo for 100 observations

For n large enough, the effect of h is more pronounced, and as we saw earlier, in both cases the error is almost the same for both schemes, and it is an increasing function of h if the effect of n is negligible.

3 Sensitivity of Option prices by the Monte-Carlo method

3.1 Finite Difference Estimation

Function v is written as $v(t,s) = \mathbb{E}_s[f(S_{T-t})]$, suppose sufficient regularity of the function, then by Taylor expansion of $v(T,s)$, we obtain that:

$$v(T, s+\epsilon) = v(T,s) + \frac{\partial v}{\partial s}(T,s)\epsilon + \frac{1}{2}\frac{\partial^2 v}{\partial^2 s}(T,\epsilon)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

And

$$v(T, s-\epsilon) = v(T,s) - \frac{\partial v}{\partial s}(T,s)\epsilon + \frac{1}{2}\frac{\partial^2 v}{\partial^2 s}(T,\epsilon)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

We find that:

$$\begin{aligned}\Delta_\epsilon &= (2\epsilon)^{-1}(v(T,s+\epsilon) - v(T,s-\epsilon)) \\ &= \frac{\partial v}{\partial s}(T,s) + \mathcal{O}(\epsilon^2) \\ &= \Delta + \mathcal{O}(\epsilon^2)\end{aligned}$$

Which equates to that $\Delta_\epsilon \underset{\epsilon\longrightarrow 0}{\longrightarrow} \Delta$

For Γ, the Taylor series expansion is given by:

$$v(T, s+\epsilon) = v(T,s) + \frac{\partial v}{\partial s}(T,s)\epsilon + \frac{1}{2}\frac{\partial^2 v}{\partial^2 s}(T,\epsilon)\epsilon^2 + +\frac{1}{6}\frac{\partial^3 v}{\partial^3 s}(T,\epsilon)\epsilon^3 + \mathcal{O}(\epsilon^4)$$

And

$$v(T, s-\epsilon) = v(T,s) - \frac{\partial v}{\partial s}(T,s)\epsilon + \frac{1}{2}\frac{\partial^2 v}{\partial^2 s}(T,\epsilon)\epsilon^2 - \frac{1}{6}\frac{\partial^3 v}{\partial^3 s}(T,\epsilon)\epsilon^3 + \mathcal{O}(\epsilon^4)$$

By summing the two expressions:

$$v(T,s+\epsilon) + v(T,s-\epsilon) - 2v(T,s) = \frac{\partial^2 v}{\partial^2 s}(T,\epsilon)\epsilon^2 + \mathcal{O}(\epsilon^4)$$

Then

$$\Gamma_\epsilon = \Gamma + \mathcal{O}(\epsilon^2)$$

Which equates to that $\Gamma_\epsilon \underset{\epsilon\longrightarrow 0}{\longrightarrow} \Gamma$

The order of convergence of the two functions is in the order of ϵ^2.

A function is implemented to simulate Δ_ϵ and Γ_ϵ, to obtain an approximation, the Monte Carlo approach is used with:

$$\Delta_\epsilon \approx \frac{1}{2\epsilon N}\sum_{i=1}^{N}(f(S_T^{i,S0+\epsilon}) - f(S_T^{i,S0-\epsilon}))$$

With $S_T^{i,x} = xe^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z_i}$ and $Z_i \sim \mathcal{N}(0,1)$.

For Γ The approximation is calculated as such.

$$\Delta_\epsilon \approx \frac{1}{\epsilon^2 N}\sum_{i=1}^{N}(f(S_T^{i,S0+\epsilon}) + f(S_T^{i,S0-\epsilon}) - 2f(S_T^{i,S0}))$$

With the values of Z_i are taken independently for each term for *Delta* and Γ in the case of independent simulations. Expectations are computed independently of each other, which is likely to introduce more variability and instability that we will discuss later.

In what follows, the call option is utilized to further validate the calculation and implementation of the method since the Δ and Γ can be easily obtained by deriving the Black&Scholes price formula.

The discounted payoff of the call option is $e^{-rT}(S_T - K)^+$. which corresponds to the f function. The delta in Black&Scholes is given by :

$$\Delta = \mathcal{N}(d_1(S_0, K, r, \sigma, T))$$

With $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-x^2}{2}}dx$ and $d_1(S_0, K, r, \sigma, T) = \frac{\ln(\frac{S_0}{K}+(r+\frac{\sigma^2}{2})T)}{\sigma\sqrt{T}}$
Γ under the Black&Scholes Framework can be expressed as such:

$$\Gamma = \frac{\mathcal{N}'(d_1(S_0, K, r, \sigma, T))}{S_0\sigma\sqrt{T}}$$

The parameters chosen are $S_0 = K = 100$, $r = 0.02$, $\sigma = 0.35$, $T = 1$, $\epsilon = 10$ and $N = 10000$.

By testing different values, we can observe that there is a balance between the choice of ϵ and N as ϵ is taken smaller; the simulations N necessary for a proper convergence is much larger.

The actual values of Γ and Δ for the selected parameters are given by :

$$\Delta = 0.591786$$
$$\Gamma = 0.01109532$$

With the approximation method with independent realisations the values of the *Delta* and *Gamma* are obtained by:

$$\Delta_\epsilon^{ind} = 0.581059$$
$$\Gamma_\epsilon^{ind} = 0.0149522$$

With the dependency between the simulations the obtained result is given by:

$$\Delta_\epsilon^{dep} = 0.59519$$
$$\Gamma_\epsilon^{dep} = 0.011104$$

The result obtained with the second method is better and fluctuates less when the simulations are redone.

The variance of the estimator is calculated to evaluate the quality of the method. The Monte Carlo estimator for *Dellta* is:

$$\Delta_\epsilon \approx \frac{1}{2\epsilon N}\sum_{i=1}^{N}(f(S_T^{i,S_0+\epsilon}) - f(S_T^{i,S_0-\epsilon}))$$

The variance of this quantity is given by:

$$Var(\Delta_\epsilon) = \frac{1}{4\epsilon^2 N^2}\sum_{i=1}^{N}(Var(f(S_T^{i,S_0+\epsilon}) - f(S_T^{i,S_0-\epsilon})))$$

As each simulation follow the same law:

$$\begin{aligned}Var(\Delta_\epsilon) &= \frac{1}{4\epsilon^2 N}(Var(f(S_T^{S_0+\epsilon}) - f(S_T^{S_0-\epsilon})))\\ &= \frac{1}{4\epsilon^2 N}(Var(f(S_T^{S_0+\epsilon}) - f(S_T^{S_0-\epsilon})))\end{aligned}$$

We use an unbiased estimator of the variance:

$$Var(X) \approx \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{x})^2$$

for a set of realisations $(x_i)_i$ of the random variable X and $\hat{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

For Γ, the theoretical value of the variance is given by:

$$Var(\Gamma_\epsilon) = \frac{1}{\epsilon^4 N}(Var(f(S_T^{S_0+\epsilon}) + f(S_T^{S_0-\epsilon}) - 2f(S_T^{S_0})))$$

For independent simulations for each variance estimation for each term in Δ and Γ, the variance obtained is:

$$\begin{aligned}Var(\Delta_\epsilon^{ind}) &= 0.00034\\ Var(\Gamma_\epsilon^{ind}) &= 3.9425.10^{-5}\end{aligned}$$

For the same simulation for each term, the variance obtained is:

$$\begin{aligned}Var(\Delta_\epsilon^{dep}) &= 4.2185.10^{-5}\\ Var(\Gamma_\epsilon^{dep}) &= 5.9566.10^{-8}\end{aligned}$$

It is clear as it is to be expected that the variance $\sim \frac{1}{N}$. In addition, it is confirmed that Δ with dependent simulation yields the best results.

The error comes from two approximations. The first is the derivative approximation and the second approximation based on the Monte Carlo method. The latter have an approximate confidence interval of the order of $\frac{1}{\sqrt{N}}$ as:

$$\Delta_\epsilon^N - \Delta_\epsilon \sim \frac{\sigma_\epsilon}{\sqrt{N}}\mathcal{N}(0,1)$$

σ_ϵ is close to $\sqrt{Var(\frac{\partial v}{\partial s}(T,S_T))}$ which is independent of ϵ, which leads to the conclusion that the above difference is proportional to $\frac{1}{\sqrt{N}}$. The same order is given for the gamma approximation.

Because the error of the approximation of Δ as Δ_ϵ is $\mathcal{O}(\epsilon^2)$. We choose $N \sim 1/\epsilon^4$ (proportional to), to have an error proportional to ϵ^2 a fixed order. The can be said for Γ.

f is assumed to have polynomial growth, $v(T,.)$ have to be $\mathcal{C}^4(\mathbb{R},\mathbb{R})$. Let x be in $[-R,R]$. The parameter μ is defined as $\mu = (r-\sigma^2/2)T$

$$\begin{aligned} g(x) &= \mathbb{E}(f(e^x e^{(r-\sigma^2/2)T+\sigma W_T})) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\mathbb{R}} f(e^{x+y+\mu})e^{-y^2/(2\sigma^2 T)}dy \end{aligned}$$

By Variable change:

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\mathbb{R}} f(e^{y+\mu})e^{-(y-x)^2/(2\sigma^2 T)}dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\mathbb{R}} h(x,y)dy \end{aligned}$$

based on the fact that $(x-y)^2 \geq x^2 - 2|xy| + y^2 \geq y^2 - 2R|y|$. h and because f has a polynomial growth, and the derivatives of order m pf h are proportional to x to a linear combination of $h(x,y)(y-x)^k$ with $k \in \{0,...,m\}$, and $|y-x|^k \leq (|y|+R)^k \leq \mathcal{O}(e^{-y})$. It is clear that $|\frac{\partial^m h}{\partial x^m}(x,y)| \leq \mathcal{O}(e^{-y^2/(4\sigma^2 T)})$ independent of x. This proves that the function $v(T,.)$ is $\mathcal{C}^\infty$, which justifies the fineness of the function.It can also proved that $v(t,.)$ is $\mathcal{C}^\infty(]0,T])$.

3.2 Estimation by flow technique

The flow technique is another way to calculate the Delta value without using an approximation of the derivative. The derivative of the discounted payoff of a call option in the distribution sense is given by :

$$\begin{cases} f'(S_T) = e^{-rT} \text{ if } S_T \geq K \\ f'(S_T) = 0 \text{ otherwise} \end{cases}$$

The value to be computed to obtain Δ is:

$$\Delta = \mathbb{E}(f'(S_T)\frac{S_T}{S_0})$$

from the flow method and keeping the same parameters as for the above part. The value obtained for Δ is :

$$\Delta = 0.59512$$

Since this method uses an estimator as specified in the section above, the variance of the approximation is :

$$\begin{aligned} Var(\Delta^N_{Flow}) &= \frac{1}{N} Var(f'(S_T)S_T/S_0) \\ &= 4.64209.10^{-5} \end{aligned}$$

The variance is comparable to the approximation of the derivative using the same realizations for the normal random variable.

3.3 Estimation by Malliavin calculus type approach

With Millavin method, Δ can be expressed as:

$$\mathbb{E}(f(S_T)\frac{W_T}{S_0\sigma T})$$

And Γ:

$$\mathbb{E}(\frac{f(S_T)}{S_0^2\sigma T}[\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}])$$

In the case where f admits a derivative in the sense of the distribution and the conditions specified in the Milliavin technique are verified, Γ becomes :

$$\Gamma = \mathbb{E}(f'(S_T)\frac{S_T W_T}{S_0^2\sigma T} - f(S_T)\frac{W_T}{S_0^2\sigma T})$$

Applied to the call option, the values of Γ and Δ and the variance in the case of Δ and Γ are based on the second expression :

$$\begin{aligned} \Delta &= 0.596933 \\ \Gamma &= 0.011185 \\ Var(\Delta^N_{Milliavin}) &= 0.0002312 \\ Var(\Gamma^N_{Milliavin}) &= 2.59570.10^{-8} \end{aligned}$$

This method yields a better approximation of Γ based on the comparison between the variance value of the Milliavin method and the finite difference method. However, the finite difference method provides a better variance in terms of delta than the Milliavin method, as does the flow method. The finite-difference method requires choosing an epsilon estimate compared to Milliavin and the flow method, which are both straightforward. The Milliavin method is preferable in the sense that the flow method requires the derivative of the f function which, in some cases, cannot be derived directly from its expression. In the case of a call option, the expressions is easily obtained.

3.4 Variance Reduction

In the case of variance reduction, it is feasible to reduce the variance estimate by introducing a known quantity with a zero expectation. This is the control variable method. In the case of a Vanilla option, the variable to choose is $\beta(S_T - \mathbb{E}(S_T))$ which has zero expectation, such as the variance of the quantity $g(S_T) + \beta(S_T - \mathbb{E}(S_T))$ is minimal. The minimum is reached for $\beta = -(Var(S_T))^{-1} Cov(g(S_T), S_T)$. This also works for the call option, the quantity is calculated numerically. It can easily be applied to the call option, especially for the flow method where $g : x \longrightarrow f'(x)x/S_0$. The resulting variance is $8.59777.10^{-6}$ for the flow method and the variance is reduced.

For importance sampling, a change of probability. The stochastic process $H_T^\lambda = e^{\lambda W_T - \lambda^2 T/2}$ is a martingale. Under this probability $\mathbb{P}^\lambda$, $W_T^h = W_T - \lambda T$ is Brownian motion under $\mathbb{P}^\lambda$, and the expectation based on the Girsanov Theorem:

$$\mathbb{E}(g(S_T)) = \mathbb{E}^\lambda((H_T^\lambda)^{-1} g(S_T))$$

S_T is simulated under $\mathbb{P}^\lambda$ with $\lambda > 0$.

Another method is to use the following approximation of the discounted payoff function:

$$\Psi_R(S_T) = \begin{cases} e^{-rT}(S_T - K) \text{ if } S_T > K + R \\ e^{-rT}\frac{(S_T-(K-R))^2}{4R} \text{ if } S_T \in]K-R, K+R] \\ 0 \text{ Otherwise} \end{cases}$$

the derivative of this function is given by:

$$\Psi'_R(S_T) = \begin{cases} e^{-rT} \text{ if } S_T > K + R \\ e^{-rT}\frac{S_T-(K-R)}{2R} \text{ if } S_T \in]K-R, K+R] \\ 0 \text{ Otherwise} \end{cases}$$

The call discounted payoff function is denoted Φ. The Delta is approximated with the given formula:

$$\Delta = \mathbb{E}[\Psi'_R(S_T)\frac{S_T}{S_0}] + \mathbb{E}[(\Phi - \Psi_R)(S_T)\frac{W_T}{S_0 \sigma T}]$$

Using the Monte Carlo approximation with the same realizations, the evolution of the log base 10 of Δ in terms of the log base 10 of R. It is clear that for R<10, Δ is roughly constant, then it decreases to a minimum and then the function increases. The increase is almost exponential between R= 100 and ∞.

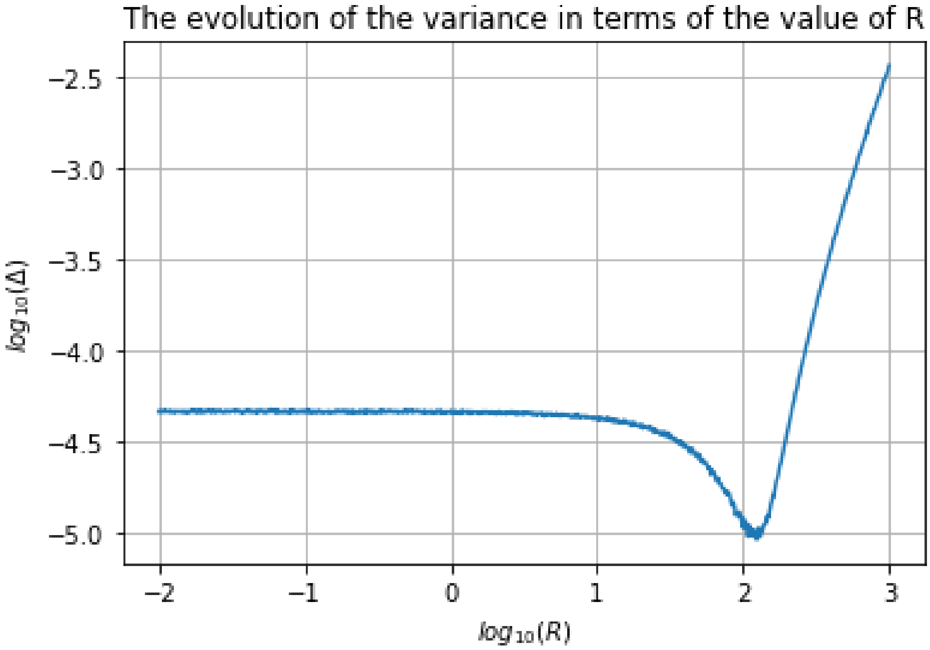

Figure 6: The evolution of the variance in terms of the value R

The minimum corresponds to R equal to 124.19 and the minimum is $9.14.10^{-6}$ which is better than the result obtained with the ordinary flow and the Milliavin method.

By analogy, Γ can be calculated as follows:

$$\Gamma = \mathbb{E}(\Psi'_R(S_T)\frac{S_T W_T}{S_0^2 \sigma T} - \Psi_R(S_T)\frac{W_T}{S_0^2 \sigma T}) + \mathbb{E}(\frac{\Phi(S_T) - \Psi_R(S_T)}{S_0^2 \sigma T}[\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}])$$

The minimum is $2.29.10^{-8}$ and corresponds to 76.53 which is lower than the Milliavin method, however, it is comparable. For Γ, the evolution of the error is given by the following graph:

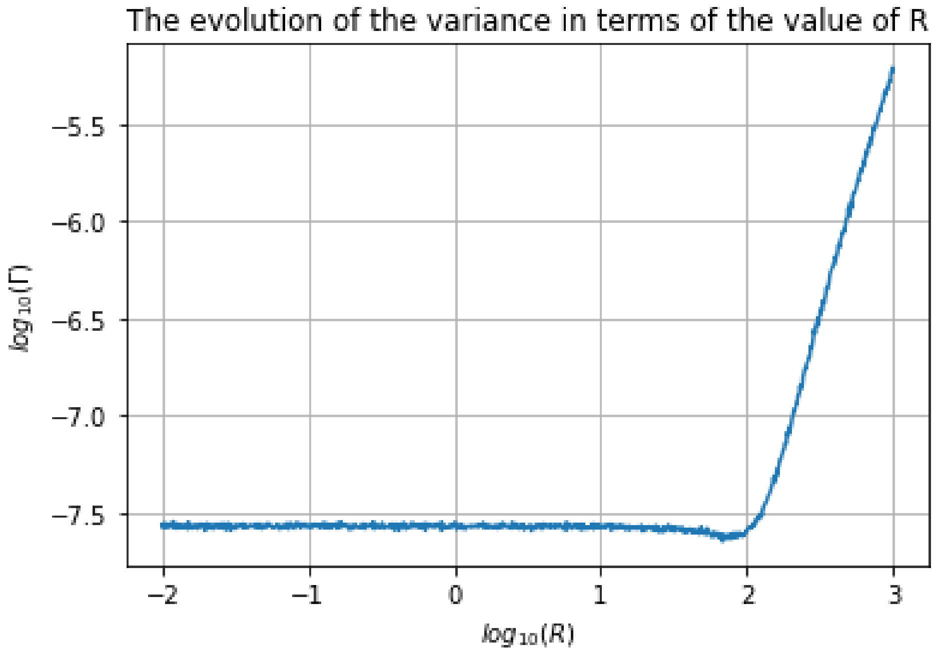

Figure 7: The evolution of the variance in terms of the value R

This practical assignment introduced theoretical methods with their practical applications. It provides a guide to the main method used in the daily work of a quantitative analyst, from pricing to approximations of differential equations.

4 Conclusion

5 Written proofs

Démonstration Milstein

Quelques notations

On pose $\varphi_t = \max\{t_i \mid t_i \leq t\}$

et $\tau_t = \min\{t_i \mid t_i > t\}$

On prend $\overline{X}_t$ la solution par approximation de Milstein

X_t la solution exacte

L'approximation de Milstein donne

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})\, h + \sigma(\overline{X}_{t_i})(W_{t_{i+1}} - W_{t_i}) + \sigma(\overline{X}_{t_i})\,\sigma'(\overline{X}_{t_i}) \int_{t_i}^{t_{i+1}} (W_s - W_{t_i})\, dW_s$$

Le schéma peut être étendu naturellement en une fonction continue

$$\overline{X}_t = \overline{X}_{\varphi_t} + b(\overline{X}_{\varphi_t})(t - \varphi_t) + \sigma(\overline{X}_{\varphi_t})(W_t - W_{\varphi_t}) + \sigma(\overline{X}_{\varphi_t})\,\sigma'(\overline{X}_{\varphi_t}) \int_{\varphi_t}^{t} (W_s - W_{\varphi_s})\, dW_s$$

En particulier

$$\overline{X}_t = \overline{X}_{\varphi_t} + \int_{\varphi_t}^{t} b(\overline{X}_{\varphi_s})\, ds + \int_{\varphi_t}^{t} \sigma(\overline{X}_{\varphi_s})\, dW_s + \int_{\varphi_t}^{t} \sigma(\overline{X}_{\varphi_s})\,\sigma'(\overline{X}_{\varphi_s})(W_s - W_{\varphi_s})\, dW_s$$

Par la suite

$$\overline{X}_t = x + \int_0^t b(\overline{X}_{\varphi_s})\, ds + \int_0^t \sigma(\overline{X}_{\varphi_s})\, dW_s + \int_0^t \sigma(\overline{X}_{\varphi_s})\,\sigma'(\overline{X}_{\varphi_s})(W_s - W_{\varphi_s})\, dW_s$$

On note en particulier que la solution exacte vérifie

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s$$

$X_{u} - \bar{X}_{u}$

$= \int_0^u (b(X_s) - b(\bar{X}_{\varphi_s}))\, ds$

$+ \int_0^u (\sigma(X_s) - \sigma(\bar{X}_{\varphi_s}) - \sigma(\bar{X}_{\varphi_s})\sigma'(\bar{X}_{\varphi_s})(W_s - W_{\varphi_s})\, dW_s$

On note particulièrement que pour $p \geq 1$, On sait que

$$E\left(\sup_{s \leq T} |X_s|^p \right) < +\infty$$

On peut aussi montrer par récurrence simple que :

$$E\left(\sup_{s \leq T} |\bar{X}_s|^p \right) < +\infty$$

par l'inégalité de convexité

$$|(X_u - \bar{X}_u)^p| \leq 3^{p-1} \left| \left(\int_0^u (b(X_s) - b(X_{\varphi_s})) \right)^p \right|$$

$$+ 3^{p-1} \left| \int_0^u b(X_{\varphi_s}) - b(\bar{X}_{\varphi_s})\, ds \right|^p$$

$$+ 3^{p-1} \left| \int_0^u \sigma(X_s) - \sigma(\bar{X}_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})(W_s - W_{\varphi_s})\, dW_s \right|^p$$

En passant au sup et à l'espérance

on a

$$E\left(\sup_{s \leq T} \left| \int_0^u b(X_{\varphi_s}) - b(\bar{X}_{\varphi_s})\, ds \right|^p \right)$$

$$\leq T^{p-1} \int_0^T E\left(|b(X_{\varphi_s}) - b(\bar{X}_{\varphi_s})|^p \right) . ds$$

Inégalité de Hölder

On a par Itô, en prenant $b \in \mathcal{C}^2_b$

$$d\,b(X_t) = b'(X_t)\,dX_t + \frac{1}{2} b''(X_t)\,\sigma^2(X_t)\,dt$$
$$= \left[b b' + \frac{\sigma^2 b''}{2} \right](X_t)\,dt + \sigma b'(X_t)\,dW_t$$

On a

$$d(\gamma_t - t)\left(b(X_t) - b(X_{\varphi_t})\right)$$
$$= -\left(b(X_t) - b(X_{\varphi_t})\right)dt + (\gamma_t - t)\,d\,b(X_t)$$

on obtient

$$(\gamma_t - \gamma_t)\left(b(X_{\gamma_t}) - \cancel{b}(X_{\varphi_t})\right) - (\gamma_t - \cancel{\varphi_t})\left(b(X_{\varphi_t}) - b(X_{\varphi_t})\right)$$
$$= -\int_{\varphi_t}^{\gamma_t} \left(b(X_s) - b(X_{\varphi_s})\right) ds$$
$$+ \int_{\varphi_t}^{\gamma_t} (\gamma_s - s) \left[\sigma b'(X_s)\,dW_s + \left[b b' + \frac{\sigma^2 b''}{2}\right][X_s]\,ds\right]$$

On obtient simplement que

$$\int_0^{\varphi_{2l}} \left(b(X_s) - b(X_{\varphi_s})\right) ds$$
$$= \int_0^{\varphi_{2l}} (\gamma_s - s) \left[\sigma b'(X_s)\,dW_s + \left[b b' + \frac{\sigma^2 b''}{2}\right](X_s)\,ds\right]$$

Donc

$$\left|\int_0^{\varphi_{2l}} \left(b(X_s) - b(X_{\varphi_s})\right) ds\right|^p$$
$$\leq 2^{p-1} \left|\int_0^{\varphi_{2l}} (\gamma_s - s)\left[\sigma b'(X_s)\,dW_s\right]\right|^p$$
$$+ 2^{p-1} \left|\int_0^{\varphi_{2l}} (\gamma_s - s)\left[b b' + \frac{\sigma^2 b''}{2}\right](X_s)\,ds\right|^p$$

En se basant sur l'inégalité de BDG $\exists C_{p,T}$

$$E\left(\sup_{s \leq T} \left| \int_0^{\varphi_s} (\gamma_r - r) \left[\sigma b'(X_r) \, dW_r \right] \right|^p \right)$$

$$\leq C_{p,T} \; E\left(\left(\int_0^{\varphi_T} (\gamma_r - r)^2 (\sigma b')^2 (X_r) \, dr \right)^{p/2} \right)$$

car $\int_0^{\varphi_s} (\gamma_r - r) \, \sigma b'(X_r) \, dW_r$ est

une martingale

car $(\gamma_r - r) \, \sigma b'(X_r) \in \mathcal{L}^2(W)$

$$\leq C_{p,T} \; T^{p/2 - 1} \int_0^{\varphi_T} E\left(\left| (\gamma_r - r)^p \, (\sigma b')^p (X_r) \right| \right) dr$$

σ est Lipshitzienne

Donc $\exists K$ tel que $|\sigma(x)| \leq K(1 + |x|)$

En particulier

$\exists K_p$ tel que $|\sigma(x)|^p \leq K_p(1 + |x|^p)$

simple à démontrer

vue que $\lim_{x \to +\infty} \frac{(1 + |x|)^p}{1 + |x|^p} = 1$

Donc puisque b' est bornée

Donc $E\left(|\sigma^p(X_r)| \, |b'(X_r)^p| \right)$

$$\leq \|b'\|_\infty^p \, K_p \left[1 + E\left(\sup_{s \leq T} |X_r|^p \right) \right)$$

Donc clairement il existe une constante

C_1 tel que

$$E\left(\sup_{s \leq T} \left| \int_0^{\varphi_s} (\gamma_r - r) \left[\sigma b'(X_r) \right] dW_r \right|^p \right)$$

$\leq C_1 \times h^p$ vue que

$(\gamma_r - r)^p \leq h^p$

Par un raisonnement similaire.

On trouve que $\exists\, C_2$ une constante tel que

$$E\Big(\sup_{s\leq T}\Big|\int_0^{\varphi_u}(\gamma_s - s)\left[bb' + \frac{\sigma^2 b''}{2}\right](X_s)\,ds\Big|^p\Big)$$

$$\leq C_2 \times h^p$$

Il suffit de s'appuyer sur l'inégalité de Hölder et sur le fait que b' et b'' sont deux fonctions bornée et sur le fait que b et σ sont Lipschitziennes

$$\sup_{u\in[0,T]}\Big|\Big(\int_{\tau_u}^{u}|b(X_s) - b(X_{\varphi_s})|\,ds\Big)^p\Big|$$

$$\leq \frac{T^{p-1}}{N^{p-1}}\sup_{u\in[0,T]}\int_{\tau_u}^{u}|b(X_s) - b(X_{\varphi_s})|^p\,ds$$

$$\leq \frac{T^{p-1}}{N^{p-1}}\max_{0\leq k\leq N-1}\int_{t_k}^{t_{k+1}}|b(X_s) - b(X_{\varphi_s})|^p\,ds$$

$$\leq h^{p-1}\int_0^T|b(X_s) - b(X_{\varphi_s})|^p\,ds$$

b est Lipschitzienne

on pose K_b la constante

$$\leq h^{p-1}K_b\int_0^T|X_s - X_{\varphi_s}|^p\,ds$$

$$X_s - X_{\varphi_s} = \int_{\varphi_s}^{s} b(X_r)\,dr + \int_{\varphi_s}^{s}\sigma(X_r)\,dW_r$$

avec des argument similaire à ce qui a été fait avant, on peut montre simplement que $E\Big(\int_0^T|X_s - X_{\varphi_s}|^p\,ds\Big)\leq C_3\times h$

avec C_3 une constante

par inégalité de convexité

et en utilisant le fait que

$$|b(x)|^p \leq C_4\,(1+|x|^p)$$

et $|\sigma(x)|^p \leq C_4\,(1+|x|^p)$

on trouve finalement que

$$E\Big(\sup_{u\in[0,T]} \Big| \int_0^u (b(X_s) - b(X_{\varphi_s}))ds \Big|^p \Big)$$

est inférieur à une quantité proportionnelle à h^p

On applique Itô sur

$\sigma(X_t)$

on suppose que $\sigma \in C_b^{2}$

$$d\sigma(X_t) = \sigma'(X_t)\,dX_t + \frac{1}{2}\sigma''(X_t)\,\sigma^2(X_t)\,dt$$

$$= \Big[b\sigma' + \frac{\sigma^2\sigma''}{2} \Big](X_t)\,dt$$

$$+ (\sigma'\sigma)(X_t)\,dW_t$$

On obtient que

$$\sigma(X_s) - \sigma(X_{\varphi_s}) - \sigma\sigma'(X_{\varphi_s})(W_s - W_{\varphi_s})$$

$$= \int_{\varphi_s}^{s} (\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_s}))\,dW_r$$

$$+ \int_{\varphi_s}^{s} \Big[b\sigma' + \frac{\sigma^2\sigma''}{2} \Big](X_r)\,dr$$

on a $\quad \sigma(X_s) - \sigma(\bar{X}_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})(W_s - W_{\varphi_s})$

$$= \sigma(X_s) - \sigma(X_{\varphi_s}) - \sigma\sigma'(X_{\varphi_s})(W_s - W_{\varphi_s})$$

$$+ \sigma(X_s) - \sigma(\bar{X}_{\varphi_s})$$

$$+ \big(\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})\big)(W_s - W_{\varphi_s})$$

$E\left(|\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r})|^p \right)$

$\leq 2^{p-1} \|\sigma'\|_\infty^{2p} E\left(|X_r - X_{\varphi_r}|^p\right)$
$+ 2^{p-1} \|\sigma''\|_\infty^{p} E\left(|\sigma(X_{\varphi_r})|^p \, |X_r - X_{\varphi_r}|^p\right)$

$X_r - X_{\varphi_r}$ est indépendante de $\mathcal{F}_{\varphi_r}$

et $\sigma(X_{\varphi_r})$ est $\mathcal{F}_{\varphi_r}$ mesurable

Donc $E\left(|\sigma(X_{\varphi_r})|^p \, |X_r - X_{\varphi_r}|^p\right)$
$= E\left(|\sigma(X_{\varphi_r})|^p\right) E\left(|X_r - X_{\varphi_r}|^p\right)$

Il suffit de montrer

$E\left(|X_r - X_{\varphi_r}|^p\right)$ inférieur à une quantité proportionnelle à $h^{p/2}$

Ceci est simple vue que

$$X_r - X_{\varphi_r} = \int_{\varphi_r}^{r} b(X_u)\, du + \int_{\varphi_r}^{r} \sigma(X_u)\, dW_u$$

en se basant sur l'inégalité de la convexité, sur l'inégalité de Hölder et BDG on trouve rapidement le résultat attendu

~~Donc~~ en se basant aussi sur le fait que σ et b sont lipschitzienne et sur le fait que

$E\left(\sup_{s \leq T} |X_s|^p \right) < +\infty$

on s'appuie sur ces deux éléments pour trouver une borne supérieure à

$E\left(|\sigma(X_{\varphi_r})|^p\right)$

D'où le résultat que

$E\left(|\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r})|^p \right)$ est inférieur à une quantité proportionnelle à $h^{p/2}$

On a

$$E\left(\left|\sigma(X_s) - \sigma(X_{\varphi_s}) - (\sigma\sigma')(X_{\varphi_s})(W_s - W_{\varphi_s})\right|^p\right)$$

$$\leq 2^{p-1} E\left(\left|\int_{\varphi_s}^{s} (\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r}))\, dW_r\right|^p\right)$$

$$+ 2^{p-1} h^{p-1} \int_{\varphi_s}^{s} E\left(\left| b\sigma' + \frac{\sigma^2\sigma''}{2}\right|^p (X_r)\right) dr$$

En se basant sur des inégalités simples

on trouve que

$$\int_{\varphi_s}^{s} E\left(\left| b\sigma' + \frac{\sigma^2\sigma''}{2}\right|^p (X_r)\right) dr$$

est inférieur à une quantité proportionnelle à h

En se basant sur l'inégalité BDG

$\exists\ K_{p,T}$ tel que

$$E\left(\left|\int_{\varphi_s}^{s} (\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r}))\, dW_r\right|^p\right)$$

$$\leq K_{p,r}\, E\left(\left(\int_{\varphi_s}^{s} (\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r}))^2 dr\right)^{p/2}\right)$$

$$\leq K_{p,r}\, h^{\frac{p}{2}-1} \int_{\varphi_s}^{s} E\left(\left|\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r})\right|^p\right) dr$$

On a $\quad \sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r})$

$$= \sigma'(X_r)\left(\sigma(X_r) - \sigma(X_{\varphi_r})\right) + \sigma(X_{\varphi_r})\left(\sigma'(X_r) - \sigma'(X_{\varphi_r})\right)$$

$$\left|\sigma\sigma'(X_r) - \sigma\sigma'(X_{\varphi_r})\right|^p$$

$$\leq 2^{p-1}\left|\sigma'(X_r)\left(\cancel{X}\sigma(X_r) - \sigma(X_{\varphi_r})\right)\right|^p$$

$$+ 2^{p-1}\left|\sigma(X_{\varphi_r})\right|^p\left|\sigma'(X_r) - \sigma'(X_{\varphi_r})\right|^p$$

$$\leq 2^{p-1}\|\sigma'\|_\infty^p \left|\sigma(X_r) - \sigma(X_{\varphi_r})\right|^p$$

$$+ 2^{p-1}\|\sigma''\|_\infty^p \left|\sigma(X_{\varphi_r})\right|^p \left|X_r - X_{\varphi_r}\right|^p$$

et donc par la suite

$$E\left(\left| \int_{\varphi_s}^{s} (\sigma\sigma'(X_{\theta}) - \sigma\sigma'(X_{\varphi_\theta})) \, dW_\theta \right|^p \right)$$

est inférieur à une quantité $\sim h^p$

et donc

$$E\left(|\sigma(X_s) - \sigma(X_{\varphi_s}) - (\sigma\sigma')(X_{\varphi_s})(W_s - W_{\varphi_s})|^p \right)$$

est inférieur à $\sim h^p$

On se focalise sur la quantité.

$$|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p \, |W_s - W_{\varphi_s}|^p$$

on a

$$\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})$$

$$= \sigma(X_{\varphi_s}) \left[\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s}) \right]$$

$$+ \sigma'(\bar{X}_{\varphi_s}) \left[\sigma(X_{\varphi_s}) - \sigma(\bar{X}_{\varphi_s}) \right]$$

on obtient

$$E\Big(\underbrace{|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p}_{\mathcal{F}_{\varphi_s} \text{ mesurable}} \; \underbrace{|W_s - W_{\varphi_s}|^p}_{\text{indépendante de } \mathcal{F}_{\varphi_s}} \Big)$$

$$= E(|W_s - W_{\varphi_s}|^p) \; E(|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p)$$

on sait que $E(|W_s - W_{\varphi_s}|^p)$ est inférieur à $\sim h^{p/2}$

et $E(|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p)$

$$\leq 2^{p-1} \|\sigma'\|_\infty^p \, E(|\sigma(X_{\varphi_s}) - \sigma(\bar{X}_{\varphi_s})|^p)$$

$$+ 2^{2p-1} \underbrace{\left(E(\sigma^{2p}(X_{\varphi_s})) \right)^{1/2} \left(E(|\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s})|^p) \right)^{1/2}}_{\text{Cauchy Shwartz}} \times \|\sigma'\|_\infty^{p/2}$$

car $|\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s})| \leq (2\|\sigma'\|_\infty)^{1/2} |\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s})|^{1/2}$

Donc

$$E\left(|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p\right]$$

$$\leq 2^{p-1}\|\sigma'\|_\infty^p \|\sigma'\|_\infty^p \; E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

$$+\|\sigma'\|_\infty^{\frac{p}{2}} 2^{2p-1} \left|E(\sigma^{2p}(X_{\varphi_s}))\right|^{1/2} \left|E\left(|\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s})|\right)^p\right)^{1/2}$$

Donc

$$E\left(|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p\right) E\left(|W_s - W_{\varphi_s}|^p\right)$$

$$\leq C_5\Big(2^{p-1} h^{p/2} \|\sigma'\|_\infty^{2p} \; E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

$$+\|\sigma'\|_\infty^{\frac{p}{2}} C_5 2^{2p-1} h^{p/2} \left|E(\sigma^{2p}(X_{\varphi_s}))\right|^{1/2} \left|E\left((\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s}))^p\right)\right|^{1/2}$$

avec C_5 une constante venant de $E\left(|W_s - W_{\varphi_s}|^p\right)$

on a $\quad ab \leq \frac{a^2+b^2}{2}$

Donc

$$2^{2p-1} h^{p/2} \left|E(\sigma^{2p}(X_{\varphi_s})\right|^{1/2} \left|E\left((\sigma'(X_{\varphi_s}) - \sigma'(\bar{X}_{\varphi_s}))^p\right)\right|^{1/2}$$

$$\leq \frac{2^{(2p-1)2} h^p \left|E(\sigma^{2p}(X_{\varphi_s})\right| + \|\sigma'\|_\infty^p \; E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)}{2}$$

par la suite la quantité

$$E\left(|\sigma\sigma'(X_{\varphi_s}) - \sigma\sigma'(\bar{X}_{\varphi_s})|^p \, |W_s - W_{\varphi_s}|^p\right)$$

$$\leq \tilde{a} \sim \frac{T^p}{N^p} + E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

$$\sim h^p + E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

La quantité

$$E\left(|b(X_{\varphi_s}) - b(\bar{X}_{\varphi_s})|^p\right)$$

$$\leq \|b'\|_\infty \; E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

La quantité

$$E\left(|\sigma(X_{\varphi_s}) - \sigma(\bar{X}_{\varphi_s})|^p\right)$$

$$\leq \|\sigma'\|_\infty \; E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

En regroupant les éléments on trouve finalement qu'il existe une constante γ tel que

$$E\left(\sup_{0\leq u\leq T} |X_u - \bar{X}_{\varphi_u}|^p\right)$$

$$\leq \gamma h^p + \gamma \int_0^t E\left(|X_{\varphi_s} - \bar{X}_{\varphi_s}|^p\right)$$

$$\leq \gamma h^p + \gamma \int_0^t E\left(\sup_{0\leq u\leq s} |X_u - \bar{X}_u|^p\right)$$

En posant $f(s) = E\left(\sup_{0\leq u\leq s} |X_u - \bar{X}_u|^p\right)$

$$f(T) \leq \gamma h^p + \gamma \int_0^T f(s)\, ds \quad \text{pour } T > 0$$

par le lemme de Gronwell.

$$f(t) \leq \gamma h^p e^{\gamma T}$$

D'où

$$E\left(\sup_{0\leq u\leq T} |X_u - \bar{X}_u|^p\right) \leq C h^p$$

avec C une constante

Méthode de Flow

f est une fonction Lipschitz continue

et f est differentiable presque partout

Donc f' est bornée presque partout

on a $$E(f(S_T)) = E\left(f\left(\frac{S_T}{S_0} S_0\right)\right)$$

S_0 est indépendante de $\frac{S_T}{S_0}$

on pose $\vartheta(x) = E\left(f\left(\frac{S_T}{S_0} x\right)\right)$ bien défini f Lipschitz

$$\vartheta(x) = E\left(f\left(e^{(r - \frac{\sigma^2}{2})T + \sigma W_T} x\right)\right)$$

on pose g la densité de $\frac{S_T}{S_0}$ qui suit une loi log normale

$$\vartheta(x) = \int_{\mathbb{R}} f(yx)\, g(y)\, dy$$

on pose $h(x) = f(yx)\, g(y)$

En dérivant dans les endroit où f est différentiable

$$h'(x) = y f'(yx)\, g(y)$$

$$|h'(x)| \leq C y\, g(y)$$

$y \to y\, g(y)$ est $\mathcal{L}^1$ car $E(|S_T|) < +\infty$

avec C la valeur supérieure de f'

Donc ϑ est dérivable

$$\vartheta'(x) = \int_{\mathbb{R}} y f'(yx)\, g(y)\, dy$$

$$= E\left(\frac{S_T}{S_0} \times f'\left(\frac{S_T}{S_0} x\right)\right)$$

Donc $\Delta = \vartheta'(S_0) = E\left(f'(S_T) \frac{S_T}{S_0}\right)$

Estimation par Malliavin

(1) On garde la même notation que dans la méthode de Elowe

$$v(x) = E\left(f\left(\frac{S_T}{S_0}x\right)\right)$$

on pose $h(x) = v(e^x)$

$$h(x) = E\left(e^{x+\sigma W_T + (r-\frac{\sigma^2}{2})T}\right)$$

on pose $\quad \gamma = (r - \frac{\sigma^2}{2})T$

v est bien définie

car $f \in \mathcal{L}^2(\mu)$

$$h(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi T}} f(e^{x+\sigma y+\gamma}) e^{-\frac{y^2}{2T}} dy$$

par changement de variable

$$h(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi T}} f(e^{\sigma y+\gamma}) e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}} dy$$

on pose $r_y(x) = \dfrac{1}{\sqrt{2\pi T}} f(e^{\sigma y+\gamma}) e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}}$

$$r_y'(x) = \frac{y-\frac{x}{\sigma}}{\sigma\sqrt{2\pi T}\,T} f(e^{\sigma y+\gamma}) e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}}$$

On prend $x \in [-R, R]$ avec $R > 0$

on a $|r_y'(x)|$

$$\leq \frac{|y| + \frac{|x|}{\sigma}}{\sqrt{2\pi T}\,T\sigma} |f(e^{\sigma y+\gamma})| e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}}$$

$$\left(y-\frac{x}{\sigma}\right)^2 = y^2 - \frac{2xy}{\sigma} + \frac{x^2}{\sigma^2} \geq y^2 - \frac{2|R||y|}{\sigma}$$

$$|h_y'(x)| \leq \frac{|y| + \frac{R}{\sigma}}{\sqrt{2\pi}\, T\sigma} \, |f(e^{\sigma y + x})| \, e^{-\frac{y^2}{2T}} \, e^{\frac{|R||y|}{\sigma T}}$$

On a

$$\int_{\mathbb{R}} \frac{|y| + \frac{R}{\sigma}}{\sqrt{2\pi T}\, T\sigma} |f(e^{\sigma y + x})| \, e^{-y^2/2T} \, e^{\frac{|R||y|}{\sigma T}} dy$$

$$\leq \left(\int_{\mathbb{R}} \left(\frac{|y| + \frac{R}{\sigma}}{\sqrt{2\pi T}\, T\sigma} \right)^2 e^{\frac{2|R||y|}{\sigma T}} e^{-y^2/2T} dy \right)^{1/2}$$

Cauchy Swartz

$$\left(\int_{\mathbb{R}} f(e^{\sigma y + x})^2 e^{-y^2/2T} dy \right)^{1/2}$$

$$< +\infty$$

D'où on peut appliquer la dérivation, vue que $E(f(S_T)^2) < +\infty$

on obtient

$$h'(x) = \int_{\mathbb{R}} \frac{y - \frac{x}{\sigma}}{\sqrt{2\pi T}\, T\sigma} f(e^{\sigma y + x}) \, e^{-\frac{(y - \frac{x}{\sigma})^2}{2T}}$$

par changement de variable

$$= \int_{\mathbb{R}} \frac{y}{\sqrt{2\pi T}\, \sigma T} f(e^{\sigma y + x}) \, e^{-\frac{y^2}{2T}}$$

$$= E\left(f\left(\frac{S_T}{S_0} e^{x}\right) \frac{W_T}{\sigma T} \right)$$

$$v(x) = h(\ln(x))$$

$$v'(x) = \frac{1}{x} h'(\ln(x))$$

$$= \frac{1}{x} E\left(f\left(\frac{S_T}{S_0} x\right) \frac{W_T}{\sigma T} \right)$$

Donc $\Delta = E\left(f(S_T) \frac{W_T}{\sigma T S_0} \right)$

h admet une dérivée seconde

$$h_y''(x) = -\frac{1}{\sigma^2\sqrt{2\pi T}\,T} f(e^{\sigma y+r})\, e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}}$$

$$+ \frac{(y-\frac{x}{\sigma})^2}{\sigma^2\sqrt{2\pi T}\,T} f(e^{\sigma y+r})\, e^{-\frac{(y-\frac{x}{\sigma})^2}{2T}}$$

$$|h_y''(x)| \leq \frac{1}{\sigma^2 T\sqrt{2\pi T}} |f(e^{\sigma y+r})|\, e^{-\frac{y^2}{2T}} e^{\frac{|R||y|}{T\sigma}}$$

$$\underbrace{+ \frac{y^2+2|y|\frac{R}{\sigma}+\frac{R^2}{\sigma^2}}{\sigma^2 T\sqrt{2\pi T}} |f(e^{\sigma y+r})|\, e^{-\frac{y^2}{2T}} e^{\frac{|R||y|}{T\sigma}}}_{u(y)}$$

on trouve

$$\int_{\mathbb{R}} u(y)\, dy$$

convexité

Cauchy

Swartz

$$\leq \left(2\int_{\mathbb{R}} \frac{1}{(\sigma^2 T\sqrt{2\pi T})^2} e^{\frac{2|R||y|}{T\sigma}} e^{-\frac{y^2}{2T}}\, dy + 2\int_{\mathbb{R}} \left(\frac{y^2+2|y|\frac{R}{\sigma}+\frac{R^2}{\sigma^2}}{\sigma^2 T\sqrt{2\pi T}}\right)^2 e^{\frac{2|R||y|}{T\sigma}}\, dy \right)^{1/2}$$

$$\left(\int_{\mathbb{R}} f(e^{\sigma y+r})^2 e^{-\frac{y^2}{2T}}\, dy \right)^{1/2}$$

$$< +\infty$$

D'où h admet une dérivée second sur $[-R, R]$ $\forall R > 0$

$$h''(x) = \int_{\mathbb{R}} h_y''(x)\, dy$$

Changement de variable

$$= \int_{\mathbb{R}} \frac{y^2}{\sigma^2\sqrt{2\pi T}\,T} f(e^{x+\sigma y+r})\, e^{-\frac{y^2}{2T}}\, dy$$

$$- \int_{\mathbb{R}} \frac{1}{\sigma^2\sqrt{2\pi T}\,T} f(e^{x+\sigma y+r})\, e^{-y^2/2T}\, dy$$

on obtient

$$h''(x) = E\left(\frac{W_T^2}{\sigma^2 T} f\left(\frac{S_T}{S_0} e^{x} \right) \right)$$

$$- \frac{1}{\sigma^2 T} E\left(f\left(\frac{S_T}{S_0} e^{x} \right) \right)$$

$$h(x) = \vartheta(e^{x})$$

$$\vartheta(x) = h(\ln(x))$$

$$\vartheta'(x) = \frac{1}{x} h'(\ln(x))$$

$$\vartheta''(x) = -\frac{1}{x^2} h'(\ln(x)) + \frac{1}{x^2} h''(\ln(x))$$

$$\vartheta''(x) = -\frac{1}{x^2} E\left(f\left(\frac{S_T}{S_0} x \right) \frac{W_T}{\sigma T} \right)$$

$$+ \frac{1}{x^2} E\left(\frac{W_T^2}{\sigma^2 T} f\left(\frac{S_T}{S_0} x \right) \right)$$

$$- \frac{1}{\sigma^2 T x^2} E\left(f\left(\frac{S_T}{S_0} x \right) \right)$$

$$\Gamma = E\left(\frac{f(S_T)}{S_0^2 \sigma T} \left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right) \right)$$

(3) si f satisfait les conditions comme dans la partie ~~Eloix~~

$$\Delta = E\left(f(S_T) \frac{W_T}{S_0 \sigma T} \right)$$

on pose

$$\Theta(x) = \frac{1}{x\sigma T} E\left(f\left(\frac{S_T}{S_0} x \right) \frac{W_T}{1} \right)$$

$$= \frac{1}{x\sigma T} \int_{\mathbb{R}} y\, f(x e^{\sigma y + r}) e^{-\frac{y^2}{2T}} dy$$

puisque f est Lipschitzienne Alors $|f(x)| \leq K(1+|x|)$

on pose

$$r_y(x) = y\, f(x e^{\sigma y + r})\, e^{-\frac{y^2}{2T}}\, dy$$

Alors $$r'_y(x) = y\, e^{\sigma y + r} f(x e^{\sigma y + r})\, e^{-y^2/2T}\, dy$$

on obtient

$$|r'_y(x)|$$

$$\leq K\, y\, e^{\sigma y + r}\left(1 + |x| e^{\sigma y + r}\right) e^{-y^2/2T}$$

pour $x \in [0, R]$

$$\leq K\, y\, e^{\sigma y + r}\left(1 + R e^{\sigma y + r}\right) e^{-y^2/2T}$$

$$= O\left(e^{-y^2/4T}\right)$$

Donc on peut appliquer la dérivation

$$v'(x) = -\frac{1}{x^2 \sigma T} \int_{\mathbb{R}} y\, f(x e^{\sigma y + r})\, e^{-y^2/2T}\, dy$$

$$+ \frac{1}{x \sigma T} \int_{\mathbb{R}} y\, e^{\sigma y + r} f(x e^{\sigma y + r})\, e^{-y^2/2T}\, dy$$

$$= E\left(f'\left(\frac{S_T}{S_0} x\right) \frac{S_T W_T}{S_0 x \sigma T}\right) - E\left[\frac{W_T f\left(\frac{S_T}{S_0} x\right)}{x^2 \sigma T}\right]$$

On obtient

$$P = E\left(f'(S_T) \frac{S_T W_T}{S_0^2 \sigma T} - f(S_T) \frac{W_T}{S_0^2 \sigma T}\right)$$