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HAJOS CONJECTURE

GRAPHS WITH TREEWIDTH 2 AND MAX DEGREE 4

CS4410

Introduction

In this paper we will discuss a new proof for Hajos Cycle Decomposition Conjecture for graphs with treewidth atmost 2 and maximum degree of any vertex in the graph as 4.

Hajos Conjecture

Hajos Conjecture states that for any Eulerian Graph G on n vertices there exist a edge decomposition of G into k cycles where $k \leq \lfloor (n-1)/2 \rfloor$.
This Conjecture has been proved for various classes of graphs including tree width 2 graphs in general as well as graphs with maximum degree 4.
Here we will show a different proof for graphs with tree width atmost 2 and maximum degree of any vertex in the graph as 4.
We will use equivalence of tree width 2 graphs with the series and parallel graphs.

Notation

All graphs considered in the paper are simple (no loops and multiple edges) and undirected.
For any graph G, $V(G)$ represents the set of vertices of the graph and $E(G)$ represents the set of edges of the graph.
We use $deg(v)$ to mean degree of vertex $v \in V(G)$.
We will use the notation $tw(G) = k$ to mean that G has tree width k.
A path $P(G)$ in G is a sequence of vertices $v_1 v_2 ....1 v_i v_{i+1} ..... v_n$ of length $n-1$ where $v_i \in V(G)$ and $v_i v_{i+1} \in E(G)$.
A cycle $C(G)$ in G is a sequence of vertices $v_1 v_2 .... v_i v_{i+1} ..... v_{n-1} v_n v_1$ of length n where $v_i \in V(G)$ and $v_i v_{i+1} \in E(G)$ and $v_n v_1 \in E(G)$.

Our main result in the paper is the following theorem.

Theorem 1

Let G be an Eulerian Graph with $deg(v) \leq 4 \; \forall v \in V(G)$ and $tw(G) = 2$ then G has a cycle decomposition into $\lfloor (n-1)/2 \rfloor$ cycles.

We will state a few basic results that we will use in the proof of Theorem 1.

Results

Definition - Series Parallel Graphs

We inductively define Series-Parallel graphs.

(a) K_2 is a Series-Parallel Graph.

(b) G is a series parallel graph if either $G = G_1 +_s G_2$ or $G_1 +_p G_2$ where G_1 and G_2 are arbitrary Series-Parallel graphs on atleast 2 vertices.

We define the operation $+_s$ and $+_p$.

(a) **Series Join Operation -** $+_s$
Consider 2 graphs G_1 and G_2, then their series join $G_1 +_s G_2$ is obtained by identifying a source vertex s_1, a sink vertex t_1 in G_1 and a sources vertex s_2, a sink vertex t_2 in G_2 and joining G_1 and G_2 by combining t_1 and s_2 into a single vertex.
The new graph is said to have s_1 as the source vertex and t_2 as the sink vertex

(b) **Parallel Join Operation -** $+_p$
Consider 2 graphs G_1 and G_2, then their parallel join $G_1 +_p G_2$ is obtained by identifying a source vertex s_1, a sink vertex t_1 in G_1 and a sources vertex s_2, a sink vertex t_2 in G_2 and joining G_1 and G_2 by combining s_1 and s_2 into a single vertex say s and t_1 and

t_2 into a single vertex say t.

The new graph is said to have s as the source vertex and t as the sink vertex.

Lemma 1

A Graph G is Series-Parallel iff $tw(G) = 2$.

Lemma 2

If G is a 2-connected graph then there exist 2 vertex disjoint paths connecting every pair of vertex in G.
Therefore every pair of vertices lie in a common cycle.

Lemma 3

G is an Eulerian Graph iff every vertex in G has even degree.

Lemma 4

If G has tree width 2 then G has no K_4 as a minor.

Proof

Our proof technique is to find the minimum counter example to the above conjecture and arrive at a contradiction.
Let G be an Eulerian Graph such that $tw(G) = 2$ and $\Delta(G) \leq 4$.
From Lemma 1 G is a Series-Parallel Graph.
Also let G be a minimum counter example to Hajos Conjecture i.e any graph with lesser number of vertices and edges satisfy Hajos Conjecture.
Now we will prove a few lemmas before proving **Theorem 1**.

Lemma 5

Let G be the graph as described above then G must be 2 connected.

Proof

Let for the sake of contradiction G be not 2 connected then it must have a cut vertex say $v \in V(G)$.
Let G_1 and G_2 be the components obtained after removing v.
It is easy to see that no cycle in $G_1 \cup v$ will have an edge in $G_2 \cup v$ and no cycle in $G_2 \cup v$ will have an edge in $G_1 \cup v$.
Therefore this means that $G_1 \cup v$ and $G_2 \cup v$ must be Eulerian. This is true as v must have even number of edges incident from both G_1 and G_2 otherwise we cannot have an Euler Tour in G.
Now let $G_1 \cup v$ have n_1 vertices and $G_2 \cup v$ have n_2 vertices.
We have -

$$n_2 + n_2 = n + 1$$

Since $n_1, n_2 < n$, therefore $G_1 \cup v$ and $G_2 \cup v$ cannot be minimum counter examples. Hence they satisfy Hajos Conjecture and we have cycle decompositions C_1 and C_2 respectively. Now C_1 must be disjoint from C_2, hence $C = C_1 \cup C_2$ is a cycle decomposition for G.

$$|C| = |C_1| + |C_2| \leq \lfloor (n_1 - 1)/2 \rfloor + \lfloor (n_2 - 1)/2 \rfloor \leq \lfloor (n_1 + n_2 - 2)/2 \rfloor \leq \lfloor (n - 1)/2 \rfloor$$

Which is a contradiction.

Lemma 6

G cannot be a series join of 2 Series Parallel Graphs.

Proof

This follows immediately from the previous lemma. If G can be expressed as series join of G_1 and G_2 then the vertex representing the combination of sink t_1 of G_1 and source s_2 of G_2 must be a cut vertex.
Hence by the previous lemma this is not possible.

Hence G is a parallel join of 2 Series Parallel graphs.
From now on we will refer to these 2 graphs as G_1 and G_2 i.e. $G = G_1 +_p G_2$.

Proof Of Theorem 1

Consider G, by lemma 6 -
$G = G_1 +_p +G_2$ where G_1 or G_2 are Series-Parallel Graphs.
Let s and t be the source and sink vertices formed after the above operation on G_1 and G_2.
We will prove the theorem by considering different cases based on the degree of vertices s and t in G.

Case 1

$deg(s) = 2$ and $deg(t) = 2$

Now both and s and t will have exactly one neighbour in G_1 and G_2.
Since G is 2-connected there exist a cycle $C(G)$ through s and t in G. The cycle will pass through edges both in G_1 and G_2.
Now consider the graph $G' = G - C(G)$.

The remaining graph has $n-2$ vertices and since G is eulerian so is $G^{'} = G - C(G)$.
Now since $G^{'}$ is smaller than G therefore it satisfies Hajos Conjecture and we get a decomposition $\mathcal{C}(G^{'})$ for $G^{'}$.
We can extend this decomposition to a decomposition for G by adding $C(G)$.
Therefore $\mathcal{C}(G) = \mathcal{C}(G^{'}) \cup C(G)$

Hence $|\mathcal{C}(G)| = |\mathcal{C}(G^{'})| + 1$. Thus,
$|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor + 1 \leq \lfloor (n-1)/2 \rfloor$.
Hence Proved.

Now we will consider cases where degree of atleast one of the vertices s and t is 4.
Here we have 3 types of situations where we will define 'relaxations' so that we can get the decompositions easily. We will consider cases with respect to s, those with respect to t are symmetric.

(a) **Type 1**

s has degree 4 and it has 2 neighbours v_1, v_2 in G_1 and it has 2 neighbours v_3, v_4 in G_2.
And v_1 is not adjacent to v_2 and v_3 is not adjacent to v_4.
In this case we will remove s from the graph and add the edges (v_1, v_2) and (v_3, v_4).
The graph will remain Eulerian as every vertex still has even degree.

(b) **Type 2**

s has degree 4 and and it has 2 neighbours v_1, v_2 in G_1 and 2 neighbours v_3, v_4 in G_2.
But one of the pairs (v_1, v_2) and (v_3, v_4) are adjacent.
Say v_1, v_2 is adjacent.
Then we will remove a cycle $C(G)$ containing s and t from G.
Note that there exist a cycle that passes through edges in both G_1 and G_2 we are considering such a cycle here.
Definitely $C(G)$ must contain one of the edges $(s, v1)$ or (s, v_2) from G_1. We can always modify the cycle so that if it passes through $(s, v1)$ we make it pass through $v_1, v2$ and then v_2, s.
Now removing this cycle removes the edge v_1, v_2 from the graph.

We can do a similar modification in case v_3, v_4 are also adjacent as it would be independent of the modification here.

(c) **Type 3**

s has degree 4 and it has 3 neighbours v_1, v_2, v_3 in G_1 and 1 neighbour v_4 in G_2.
We have considered without loss of generality that G_1 has 3 neighbours.
Now here we will use an important fact as stated by Lemma 4, that G has no K_4 minor.
Therefore the 3 neighbours of s v_1, v_2 and v_3 cannot form a clique i.e. K_3, as the subgraph induced by s, v_1, v_2, v_3 will be a K_4.
Therefore there exist a pair of neighbours of s say v_1, v_2 that are not neighbours.
Then we will remove a cycle $C(G)$ containing s and t from G.
Note that there exist a cycle that passes through edges in both G_1 and G_2, we are considering such a cycle here.
After removing the cycle $C(G)$ we will remove the vertex s and add the edge v_1, v_2.
Note that after removing the cycle G_2 already gets disconnected from s.

Now that we have considered all the types there is an important point to note that there will be no clash in the relaxation procedure for s and t even if they are of different types apart from the fact that we need to a remove a cycle through s and t in some cases and not in some.
But if s of type 2 and t is of type 1, then s requires removal of a cycle but t does not. We will handle this issue in case 6.
All the other changes apart from removing the cycle through s and t are local.
Now we are ready to handle all the other cases.

Case 2

s is of type 1 or type 2 and t has degree 2 Such a graph is not possible as it has to be wriiten as a union of 2 graph each of which have exactly one vertex of odd degree (degree 1 here).

We know that every graph has even number of vertices with odd degree.

Case 3

s is of type 3 and t has degree 2.
In this case we will relax s as explained above and this will involve the isolation of vertex t as exactly one cycle passes through it.
Since relaxation of s involves removal of s as well we now have an eulerian graph on $n-2$ vertices.
By the minimality of G, $G - \{s,t\}$ satisfies Hajos Conjecture and we have a cycle decomposition for it $\mathcal{C}(G^{'})$. Now conisder the cycle which passes through the edge $e = \{v_1, v_2\}$ where e is the edge added during relaxation of s.
Then we can add s and restore the edges s, v_1 and s, v_2 and extend the cycle to a cycle containing s in G.
Finally adding $C(G)$ the cycle removed during relaxation of s we get a decompostion for $\mathcal{C}(G)$ for G.
$|\mathcal{C}(G)| = |\mathcal{C}(G^{'})| + 1$.
Therefore $|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor + 1 \leq \lfloor (n-1)/2 \rfloor$.
Hence Proved

Case 4

s is of type 3 and t is of type 3.
In this case we relax s and t simultaneously with the removal of a cycle $C(G)$.
Then we remove s and t as described in the relaxation process.
By the minimality of G, $G^{'} = G - \{s,t\}$ satisfies Hajos Conjecture and we have a cycle decomposition for it $\mathcal{C}(G^{'})$. Now as described in the above case we can add back both s and t and remove the edges we had added to the neighbours of s and t to extend the cycles including those edges in $G^{'}$ to cycles in G.
Hence the size of the deomposition does not increase here and by adding $C(G)$ we get a decomposition $\mathcal{C}(G)$ for G.
And as we noted-

$|\mathcal{C}(G)| = |\mathcal{C}(G')| + 1$.
Therefore $|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor + 1 \leq \lfloor (n-1)/2 \rfloor$.
Hence Proved

Case 5

s is of type 1 or type 2 and t is of type 3.
Such a graph is not possible as it will be a result of parallel operation of 2 graphs G_1, G_2 each of which have one vertex of odd degree and such G_1 and G_2 cannot exist.

Case 6

s is of type 1 or type 2 and t is of type 1 or type 2.
Let the neighbours of s be u_1, u_2, u_3, u_4 and the neighbours of t be v_1, v_2, v_3, v_4.

(a) **Case a**

If both s and t is of type 2 then we will follow the standard relaxation for both of them.
That is we will remove a cycle $C(G)$ through s and t according to the process described before.
Now we will remove s.
This will cause the vertex adjacent to s in G_1 say u_1 and in G_2 say u_3 to have odd degree. We will simply add an edge between them, call it e.
Similar process will be done for t and we will add an edge f.
Now we have $G' = G - \{s,t\}$ in which e and f join the 2 components $G_1 - \{s,t\}$ and $G_2 - \{s,t\}$.
Now G' satisfies Hajos Conjecture and we have a cycle decomposition for it $\mathcal{C}(G')$. Any cycle through e will have to pass through f and we can add s and t back remove the edges e and f and extend it to a cycle in G.
And by adding $C(G)$ we get a decomposition $\mathcal{C}(G)$ for G.
Now-
$|\mathcal{C}(G)| = |\mathcal{C}(G')| + 1$.

Therefore $|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor + 1 \leq \lfloor (n-1)/2 \rfloor$.

(b) **Case b**

If both s and t is of type 1 then we will follow the standard relaxation for both of them.
Its easy to see that the relaxation itself gives us a graph $G^{'}$ with 2 components $G_1 - \{s,t\}$ and $G_2 - \{s,t\}$.
By the minimality of G, $G^{'} = G - \{s,t\}$ satisfies Hajos Conjecture and we have a cycle decomposition for it $\mathcal{C}(G^{'})$.

We can add back s and t and remove the edges we added during the relaxation. We can now extend the cycles passing through these removed edges to pass through s and t.
In this case we remove no cycle therefore after extending the cycles to that in G we get a decomposition $\mathcal{C}(G)$ for G.
$|\mathcal{C}(G)| = |\mathcal{C}(G^{'})|$.
Therefore $|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor < \lfloor (n-1)/2 \rfloor$.

(c) **Case c**

s is of type 2 and t is of type 1.
Let the neighbours of s be u_1, u_2, u_3, u_4 and the neighbours of t be v_1, v_2, v_3, v_4. In this case we change the relaxation slightly.
We relax s as normal and do not relax t, removing a cycle $C(G)$ through s and t.
Now s and t both have degree 2.
Now we remove both s and t.
This will cause the vertex adjacent to s in G_1 say u_1 and in G_2 say u_3 to have odd degree. And the vertex adjacent to t in G_1 say v_1 and in G_2 say v_3 to have odd degree.
We will add edges $e = u_1, u_3$ and $f = v_1, v_3$.
Now we have $G^{'} = G - \{s,t\}$ in which e and f join the 2 components $G_1 - \{s,t\}$ and $G_2 - \{s,t\}$.
Now $G^{'}$ satisfies Hajos Conjecture and we have a cycle decomposition for it $\mathcal{C}(G^{'})$. Any

cycle through e will have to pass through f and we can add s and t back remove the edges e and f and extend it to a cycle in G.
And by adding $C(G)$ we get a decomposition $\mathcal{C}(G)$ for G.
Now-
$|\mathcal{C}(G)| = |\mathcal{C}(G^{'})| + 1$.
Therefore $|\mathcal{C}(G)| \leq \lfloor (n-3)/2 \rfloor + 1 \leq \lfloor (n-1)/2 \rfloor$.

Hence Proved