An Analysis of the Worst-Case Profit Loss of Merging Firms Due to Merger Paradox *

Sarthak Motwani
Supervisor: Prof. Thirumulanathan D

1 Introduction

The Merger Paradox is a well-known and counter-intuitive result in the field of Industrial organization. It says that under certain assumptions, the total profits of the merging firms before the merger happens (i.e., the pre-merger state) is typically more than the profit of the merged firm in the post-merger state. We are assuming that the Market does not become a Monopoly for the merged firm after the merger happens. This observation is interestingly counter intuitive in the sense that it is normal to believe that when firms merge, the Market moves closer to being a Monopoly for the merged firm and the Market control of the merged firm should increase. However, that is typically not the case. Even though the total number of firms in the market decrease, the new market leads to a new strategic interaction among the firms in which this merged firm acts as a single unit and thus gives rise to a new equilibrium. In this new equilibrium, it can be observed under certain assumptions that the Merger Paradox result typically holds.

This write-up is in continuation to paper [1]. In [1], the authors consider the Cournot setting of the market. Under this setting, they analyse the worst-case fraction of loss in profits of the merged firm as compared to the cumulative profits of the merging firms in the pre merger state. They show that in a market with a concave demand function and $n \geq 3$ firms and each firm having a convex cost function, when k out of n merge, they may lose in equilibrium at most a fraction of $1 - 1/k$ of their total pre-merger equilibrium profits and this bound is asymptotically tight. The main result in [1] is the following:

Theorem. *Consider a market with an affine demand function and three firms with convex cost functions. When two of the firms merge, they may lose at most a fraction of* $1/9$ *of their total pre-merger profits in equilibrium. Moreover, there exists such a market, in which the two merging firms lose exactly a fraction of 1/9 of their total profits by merging.*

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They also show that whenever all the n firms have linear and same cost functions, then a merger of k out of n firms causes the merging firms to lose a fraction of exactly $max(0, 1 - \frac{(n+1)^2}{k(n-k+2)^2})$. Thus, when $n = 3$ and $k = 2$, this ratio equals 1/9, which is the same as the worst fraction of profit loss as described in the theorem. Considering this observation, the proof of the above theorem in [1] also proceeds in a way that transforms the convex costs of the merging firms into linear costs in the pre merger and post merger state. Then, using algebraic manipulation, they show that the ratio of post merger and pre merger profits in the original market is atleast the ratio of the same in the transformed market, which is atleast 8/9.

In our analysis in this write-up, we fulfil the following objectives:

- We try to prove a generalization of this result when k out n firms merge in a Cournot market under the same assumptions as in [1], along the same lines as the proof given in [1]. However, we show, using counter examples, that in order to find the worst-case fraction of profit loss, the proof cannot proceed in the way completely analogous to the one used in [1].
- Including all the assumptions used in the main result in [1], we consider a market with 4 firms out of which 2 firms merge. Considering the merging firms to have quadratic cost functions and the coefficients of the linear terms as the same, we prove that the worst case fraction of profit loss in this market is indeed $1 - 25/32 = 7/32$.

2 Preliminaries

All the following notations, assumptions and preliminary results are taken directly from [1]:

2.1 Notations

- We consider a Cournot game with n firms, denoted $F_1, F_2, \ldots, F_n$. In the pre-merger state, each firm F_i chooses a quantity $x_i \in [0, \infty)$ to be supplied by it. The inverse demand function is denoted by $P(x)$ and the cost function of F_i is denoted $C_i(x)$. The profit of F_i is denoted by $\pi_i(x_1, x_2, \ldots, x_n)$.
- In the post-merge state, we assume that k out of n firms merge and denote the merged firm by $F_{1,2,\ldots,k}$. A strategy $\tilde{x}_{1,2,\ldots,k}$ of the merged firm can be represented by a vector of quantities $(\tilde{x_1}, ..., \tilde{x_k})$, such that $\tilde{x_i}$ is the quantity produced by factory i. The cost function of the merged firm is denoted by $\tilde{C}_{1,2,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$ and its profit is denoted by $\tilde{\pi}_{1,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$.

2.2 Assumptions

- The function $P()$ is affine of the form $P(x) = a - bx$ where $a, b \geq 0$.

- The functions $C_i()$ are all continuously differentiable, non-decreasing and convex. Moreover, they all maintain that $C_i(0) = 0$.

2.3 Known Properties of Cournot Markets

Lemma 1 to 6 are taken directly from [1]. Lemma 7 (a result already known) is a generalization of lemma 6 in [1].

Lemma 1. *The following are necessary and sufficient conditions for a vector $(x_1, x_2, \dots, x_n)$ to be a Cournot equilibrium, which exists and is unique in markets with a concave demand and convex costs. For each $i \in [n]$,*

$$\text{If } x_i > 0 : C_i'(x_i) = P(x_1 + x_2 + \dots x_n) + x_i P'(x_1 + x_2 + \dots + x_n)$$

$$\text{If } x_i = 0 : C_i'(x_i) \geq P(x_1 + x_2 + \dots + x_n) + x_i P'(x_1 + x_2 + \dots + x_n)$$

Lemma 2. *If firms $F_1, F_2, \dots, F_k$ merge in a market with a concave demand and convex costs, for some $k \leq n$, the function $\tilde{C}_{1,2,\dots,k}(\tilde{x}_{1,2,\dots,k})$ maintains all the assumptions of cost functions (as defined above) in the post-merger market. Moreover, if each factory F_i produces a quantity of x_i units after merging, then for each $i \in [k]$:*

$$\text{If } \tilde{x}_i > 0 : {C_i}'(\tilde{x}_i) = \tilde{C}'_{1,2,\dots,k}(\tilde{x}_{1,2,\dots,k})$$

$$\text{If } \tilde{x}_i = 0 : {C_i}'(\tilde{x}_i) \geq \tilde{C}'_{1,2,\dots,k}(\tilde{x}_{1,2,\dots,k})$$

Lemma 3. *If some firms merge, each of the other firms produces in equilibrium, in markets with a concave demand and convex costs, at least the amount it produced before that merging. That is, if $x = (x_1, x_2, \dots, x_n)$ is the Cournot equilibrium before $F_1, F_2, \dots, F_k$ merged, for some $k \leq n$, and $(\tilde{x}_{1,2,\dots,k}, \tilde{x}_{k+1}, \dots, \tilde{x}_n)$ is the equilibrium afterwards, then $\tilde{x_i} \geq x_i$ for each $i \geq k+1$.*

Lemma 4. *If some firm F_i improves its cost function $C_i()$ to $\hat{C}_i()$, in the sense that for each $x \geq 0$ it holds that $C_i'(x) \geq \hat{C}_i'(x)$, then both F_i's production level and its profit in equilibrium can only increase in markets with a concave demand and convex costs. The production level and the profit of each of the other firms can only decrease in that case.*

Lemma 5. *If some firm F_i leaves the market, then the profit of each of the other firms in equilibrium is at least as it was before F_i left, in markets with a concave demand and convex costs. On the other hand, if some firm joins the market, the profit of the others can only decrease in equilibrium.*

Lemma 6. *Consider a market with a concave demand function and $n \geq 3$ firms, each having a convex cost function. When $F_1, F_2, \dots, F_k$ merge, for some $k \leq n$, the profit of the merged firm in equilibrium is at least the profit of F_i in equilibrium before merging, for any $i \in [k]$.*

Lemma 7. *Consider that k out of n firms merge in the Cournot oligopoly market. Let the inverse demand function $P(s) = a - bs$, and the cost functions of the firms $C_j^M(\cdot)$ be convex. Assume that the merging firms lose a fraction of η of their total pre-merger profits. Then, there exists a linear function $C_j^{LIN}(\cdot)$, such that replacing the cost functions of the non-merging firm by it, yields a market in which the fraction of loss η^{LIN} is at least the fraction of loss before replacement.*

3 Project Work

3.1 Proof of Theorem in [1]

In [1], the proof for the result proceeds in the following way. Say there's a market M_1 having three firms, F_1, F_2, F_3 in which F_1 and F_2 merge. The demand function is affine. The cost functions of the merging firms are convex and because of lemma 7, the cost function of the non-merging firm is linear. They construct a market M_2 which is obtained from M_1 by replacing the cost functions of F_i by the same function $C^{M_2}(x) = c \cdot x$, where $c = \tilde{C}'^{M_1}_{1,2}(\tilde{x}^{M_1}_{1,2})$ for all $i \in \{1, 2\}$. They construct another market M_3, obtained from M_1 by replacing the cost function $C_i^{M_1}(x)$ by the function $C_i^{M_3}(x) = c_i^{M_3} \cdot x$ where $c_i^{M_3} = C_i'^{M_1}(x_i^{M_1})$ for $i = \{1, 2\}$. Using the known results of a Cournot Market and some algebraic manipulation, they get to the following inequality towards the end:

$$\frac{\tilde{\pi}^{M_1}_{1,2}}{\sum_{i=1}^{2} \pi_i^{M_1}(x^{M_1})} \geq \frac{\tilde{\pi}^{M_2}_{1,2}}{\sum_{i=1}^{2} \pi_i^{M_2}(x^{M_1})} \geq \frac{(n+1)^2}{k(n-k+2)^2} \tag{1}$$

3.2 Problems in Generalizing the above Inequality

The first inequality is easier to generalize from $n = 3$ and $k = 2$ to a general n and k. However, the generalization of the second inequality is not straightforward. We need to prove that the following inequality holds under the same assumptions.

$$\frac{\tilde{\pi}^{M_2}_{1,2,\ldots,k}}{\sum_{i=1}^{k} \pi_i^{M_2}(x^{M_1})} \geq \frac{(n+1)^2}{k(n-k+2)^2} \tag{2}$$

This is a known result that the above inequality is equivalent to the following inequality:

$$0 \geq -z \cdot (\sum_{j=1}^{k} y_j) \cdot (n - 2k + 1) - (n - k + 1) \cdot (\sum_{j=1}^{k} y_j)^2 \tag{3}$$

Here, the notations $z = b - (n - k + 2) \cdot c + \sum_{j=k}^{n} c_j^{M_2}$ and $y_i = c_i^{M_3} - k \cdot c$. This inequality holds under the assumption that in market M_1, the average of the marginal cost functions of the merging firms is always greater than the marginal cost function of the merged firm, that is $\sum_{i=1}^{k} y_i \geq 0$. However, we show using a counter-example that the marginal cost function of the merged firm can exceed the average of the marginal cost functions of the merging firms. We also show using another example that because $\sum_{i=1}^{k} y_i$ can be negative, the inequality (3) need need not hold.

3.3 Counter Examples

The first example is the counter example for the assumption that the average of the marginal cost functions of the merging firms is always greater than the marginal cost function of the merged firm.

Example 1: Consider a market with the demand function $P(S) = a - bS$ where $a > 0, b > 0$ and three firms with the following cost functions (Assume $k > 0$):

$$C_1(x_1) = x_1 \qquad C_2(x_2) = \frac{k{x_2}^2}{2} \qquad C_3(x_3) = 0$$

Soving the Cournot equilibrium for this market, we get:

$$\pi_1(x) = (a - b(x_1 + x_2 + x_3))x_1 - x_1$$

$$\frac{\partial \pi_1(x)}{\partial x_1} = (a - b(x_1 + x_2 + x_3)) - bx_1 - 1 = 0$$

$$\implies 2bx_1 + bx_2 + bx_3 = a - 1 \tag{4}$$

$$\pi_2(x) = (a - b(x_1 + x_2 + x_3))x_2 - \frac{k{x_2}^2}{2}$$

$$\frac{\partial \pi_2(x)}{\partial x_2} = (a - b(x_1 + x_2 + x_3)) - bx_2 - kx_2 = 0$$

$$\implies bx_1 + (2b + k)x_2 + bx_3 = a \tag{5}$$

$$\pi_3(x) = (a - b(x_1 + x_2 + x_3))x_3$$

$$\frac{\partial \pi_3(x)}{\partial x_3} = (a - b(x_1 + x_2 + x_3)) - bx_3 = 0$$

$$\implies bx_1 + bx_2 + 2bx_3 = a \tag{6}$$

On simultaneously solving these three equations we get:

$$x_1 = \frac{b(a-3) + k(a-2)}{(4b+3k)b} \qquad x_2 = \frac{a+1}{4b+3k} \qquad x_3 = \frac{(a+1)(b+k)}{(4b+3k)b}$$

Note that if $a \geq 3$, then $x_1 = \frac{b(a-3)+k(a-2)}{(4b+3k)b} > 0$ and if $a \leq 2$, then $x_1 = 0$. For $2 < a < 3$, x_1 may be 0 or positive depending upon b and k. Let us assume that a, b and k are such that $x_1 > 0$. Now, the marginal costs of each firm at the equilibrium are:

$$C_1'(x_1) = 1 \qquad C_2'(x_2) = \frac{k(a+1)}{4b+3k} \qquad C_3'(x_3) = 0$$

$$\text{Average marginal cost: } \frac{C_1'(x_1)+C_2'(x_2)}{2} = \frac{1+\frac{k(a+1)}{4b+3k}}{2} = \frac{4b+4k+ak}{2(4b+3k)} \tag{7}$$

Now, let us assume that firms 1 and 2 merge and produce quantities $\tilde{x_1}+\tilde{x_2} = \tilde{x}_{1,2}$. From lemma 2 in [1],
If $\tilde{x_1} > 0$ and $\tilde{x_2} > 0$, then $C_1'(\tilde{x_1}) = C_2'(\tilde{x_2}) \implies 1 = k\tilde{x_2}$, that is, $\tilde{x_2} = \frac{1}{k}$.
Note that $C_2'(\tilde{x_2}) \leq 1$ if $\tilde{x_2} \leq \frac{1}{k}$. Therefore,
If $\tilde{x}_{1,2} \leq \frac{1}{k}$, then $\tilde{x_1} = 0$ and $\tilde{x_2} = \tilde{x}_{1,2}$ and $\tilde{C'}_{1,2}(\tilde{x}_{1,2}) = k\tilde{x_2} = k\tilde{x}_{1,2}$,
And if $\tilde{x}_{1,2} > \frac{1}{k}$, then $\tilde{x_2} = \frac{1}{k}$ and $\tilde{x_1} = \tilde{x}_{1,2} - \frac{1}{k}$ and $\tilde{C'}_{1,2}(\tilde{x}_{1,2}) = 1$.
Case 1: If the post-merger equilibrium is such that the merged firm produces a quantity $\tilde{x}_{1,2} > \frac{1}{k}$. In this case, $\tilde{C'}_{1,2}(\tilde{x}_{1,2}) = 1$. Solving the Cournot equilibrium in this case,

$$\begin{aligned} \tilde{\pi}_{1,2}(\tilde{x}) &= (a - b(\tilde{x}_{1,2}+\tilde{x_3}))\tilde{x}_{1,2} - \tilde{C}_{1,2}(\tilde{x}_{1,2}) \\ \frac{\partial \tilde{\pi}_{1,2}(\tilde{x})}{\partial \tilde{x}_{1,2}} &= (a - b(\tilde{x}_{1,2}+\tilde{x_3})) - b\tilde{x}_{1,2} - 1 = 0 \\ &\implies 2b\tilde{x}_{1,2} + b\tilde{x_3} = a-1 \end{aligned} \tag{8}$$

$$\begin{aligned} \tilde{\pi}_3(\tilde{x}) &= (a - b(\tilde{x}_{1,2}+\tilde{x_3}))\tilde{x_3} - C_3(\tilde{x_3}) \\ \frac{\partial \tilde{\pi}_3(\tilde{x})}{\partial \tilde{x_3}} &= (a - b(\tilde{x}_{1,2}+\tilde{x_3})) - b\tilde{x_3} = 0 \\ &\implies b\tilde{x}_{1,2} + 2b\tilde{x_3} = a \end{aligned} \tag{9}$$

Solving these two equations simultaneously, we get:

$$\tilde{x}_{1,2} = \frac{a-2}{3b} \qquad\qquad \tilde{x_3} = \frac{a+1}{3b}$$

Clearly, $\tilde{x}_{1,2} > 0$ because $a > 2$. Now, for this case to be valid for this solution, $\tilde{x}_{1,2} > \frac{1}{k} \implies k > \frac{3b}{a-2}$. Note that $k > \frac{3b}{a-2} \implies k(a-2) > 3b \implies b(a-3) + k(a-2) > ab > 0$. So, $x_1 > 0$ is already ensured in this case.
Consider the inequality, $\frac{C_1'(x_1)+C_2'(x_2)}{2} \geq \tilde{C'}_{1,2}(\tilde{x}_{1,2})$, which in this case is $\frac{4b+4k+ak}{2(4b+3k)} \geq 1 \iff 4b+4k+ak \geq 8b+6k \iff k \geq \frac{4b}{a-2}$. Thus this inequality holds iff $k \geq \frac{4b}{a-2}$. Clearly, $k > \frac{3b}{a-2}$ does not necessarily imply $k \geq \frac{4b}{a-2}$ and hence, for $k \in (\frac{3b}{a-2}, \frac{4b}{a-2})$ along with $a > 2$, we get a contradiction to our assumption.

Case 2: If the post-merger equilibrium is such that the merged firm produces a quantity $\tilde{x}_{1,2} \leq \frac{1}{k}$. In this case, $\tilde{C'}_{1,2}(\tilde{x}_{1,2}) = k\tilde{x}_{1,2}$. Solving the Cournot equilibrium in this case,

$$\begin{aligned} \tilde{\pi}_{1,2}(\tilde{x}) &= (a - b(\tilde{x}_{1,2}+\tilde{x_3}))\tilde{x}_{1,2} - \tilde{C}_{1,2}(\tilde{x}_{1,2}) \\ \frac{\partial \tilde{\pi}_{1,2}(\tilde{x})}{\partial \tilde{x}_{1,2}} &= (a - b(\tilde{x}_{1,2}+\tilde{x_3})) - b\tilde{x}_{1,2} - k\tilde{x}_{1,2} = 0 \\ &\implies (2b+k)\tilde{x}_{1,2} + b\tilde{x_3} = a \end{aligned} \tag{10}$$

$$
\begin{aligned}
\tilde{\pi}_3(\tilde{x}) &= (a - b(\tilde{x}_{1,2} + \tilde{x_3}))\tilde{x}_3 - C_3(\tilde{x_3}) \\
\frac{\partial \tilde{\pi}_3(\tilde{x})}{\partial \tilde{x}_3} &= (a - b(\tilde{x}_{1,2} + \tilde{x_3})) - b\tilde{x}_3 = 0 \\
&\implies b\tilde{x}_{1,2} + 2b\tilde{x_3} = a
\end{aligned} \tag{11}
$$

Solving these two equations simultaneously, we get:

$$
\tilde{x}_{1,2} = \frac{a}{3b + 2k} \qquad\qquad \tilde{x_3} = \frac{a(b+k)}{b(3b+2k)}
$$

For this case to be valid for this solution, $\tilde{x}_{1,2} \leq \frac{1}{k} \implies k \leq \frac{3b}{a-2}$. Note that $k \leq \frac{3b}{a-2} \implies k(a-2) \leq 3b \implies b(a-3) + k(a-2) \leq ab$. The maximum value of the numerator of x_1 in this case is $ab > 0$, implying that this case does not completely exclude the possibility of $x_1 > 0$. Thus, it is possible to choose values of a, b and k such that this case will be realised in the post-merger state. In this case $\tilde{C'}_{1,2}(\tilde{x}_{1,2}) = \frac{ak}{3b+2k}$. Consider the inequality, $\frac{C_1'(x_1)+C_2'(x_2)}{2} \geq \tilde{C'}_{1,2}(\tilde{x}_{1,2})$, which in this case is $\frac{4b+4k+ak}{2(4b+4k)} \geq \frac{ak}{3b+2k}$. Since this is a more complex inequality, we work with some numerical values of a and b to give a counter example for this case.

Numerical Example: Take $a = 4$ and $b = 1$. For the case 2 to be valid, $k \leq \frac{3}{2}$. Now, consider the last inequality of case 2 after substituting $a = 4, b = 1$, $\frac{4+8k}{2(4+4k)} \geq \frac{4k}{3+2k} \iff (1+2k)(3+2k) \geq 8k(1+k) \iff 4k^2 \leq 3 \implies k \leq \frac{\sqrt{3}}{2}$. Therefore, this assumption is violated in this case when $k > \frac{\sqrt{3}}{2}$. Hence, for $k \in (\frac{\sqrt{3}}{2}, \frac{3}{2}]$, our assumption is violated.
Checking this numerical example for case 1 as well, we observe that for $k \in (3/2, 2)$, the assumption is violated. Taking the union of the two intervals obtained in case 1 and case 2 that for this numerical example, we observe that for $k \in (\frac{\sqrt{3}}{2}, 2)$, our assumption is violated.

Example 2:- The second example is a counter example for the ~~assumpt~~ fact that the following inequality is true.

$$\frac{\tilde{\pi}^{M2}_{1,2,\cdots K}}{\sum_{i=1}^{K} \pi_i^{M2}(x^{M1})} \geq \frac{(n+1)^2}{K(n-K+2)^2} \quad \text{———} (*)$$

Consider 4 firms with the following cost functions :-

$C_1(x_1) = x_1$, $\quad C_2(x_2) = \frac{K}{2}(x_2)^2$, $\quad C_3(x_3) = \frac{1}{4}x_3$, $\quad C_4(x_4) = \frac{1}{4}\cdot x_4$

$K > 0$

And the inverse demand function $P(s) = 5 - 5s$

For the first firm,

$$\pi_1(X) = (5 - 5(x_1+x_2+x_3+x_4))\cdot x_1 - x_1$$

$$\frac{\partial \pi_1}{\partial x_1} = 5 - 5(x_1+x_2+x_3+x_4) - 5x_1 - 1 = 0$$

$$\boxed{4 = 10x_1 + 5x_2 + 5x_3 + 5x_4} \quad \text{———} (1)$$

$$\pi_2(X) = (5 - 5(x_1+x_2+x_3+x_4))\cdot x_2 - \frac{K}{2}(x_2)^2$$

$$\frac{\partial \pi_2}{\partial x_2} = 5 - 5(x_1+x_2+x_3+x_4) - 5x_2 - Kx_2 = 0$$

$$\boxed{5 = 5x_1 + (10+K)x_2 + 5x_3 + 5x_4} \quad \text{———} (2)$$

$$\pi_3(X) = (5 - 5(x_1+x_2+x_3+x_4))\cdot x_3 - \frac{1}{4}x_3$$

$$\frac{\partial \pi_3}{\partial x_3} = 5 - 5(x_1+x_2+x_3+x_4) - 5x_3 - \frac{1}{4} = 0$$

$$\boxed{\frac{19}{4} = 5x_1 + 5x_2 + 10x_3 + 5x_4} \quad \text{———} (3)$$

$$\pi_4(X) = (5 - 5(x_1+x_2+x_3+x_4))\, x_4 - \frac{1}{4}x_4$$

$$\frac{\partial \pi_4}{\partial x_4} = 5 - 5(x_1+x_2+x_3+x_4) - 5x_4 - \frac{1}{4} = 0$$

$$\boxed{\frac{19}{4} = 5x_1 + 5x_2 + 5x_3 + 10x_4} \quad \text{———} (4)$$

Solving these 4 equations, we get

$$x_1 = \frac{k+3}{2(4k+25)}, \quad x_2 = \frac{13}{2(4k+25)}, \quad x_3 = x_4 = \frac{22k+105}{20(4k+25)}$$

$$C_1'(x_1) = 1, \quad C_2'(x_2) = kx_2 = \frac{13k}{2(4k+25)}$$

$$\frac{C_1'(x_1) + C_2'(x_2)}{2} = \frac{1 + \frac{13k}{2(4k+25)}}{2} = \frac{13k + 8k + 50}{4(4k+25)}$$

$$\Rightarrow \boxed{\frac{C_1'(x_1) + C_2'(x_2)}{2} = \frac{21k+50}{4(4k+25)}} \quad \text{--- (5)}$$

Assume that firms 1 and 2 merge,

- If $\tilde{x}_1 > 0$ and $\tilde{x}_2 > 0 \Rightarrow 1 = k\tilde{x}_2 \Rightarrow \tilde{x}_2 = \frac{1}{k}$

  Thus, this case is true only if
  $\tilde{x}_{1,2} > \frac{1}{k}$

  In this case, $\tilde{x}_2 = \frac{1}{k}$, $\tilde{x}_1 = \tilde{x}_{1,2} - \frac{1}{k}$ and $\tilde{C}_{1,2}'(\tilde{x}_{1,2}) = 1$

- If $\tilde{x}_{1,2} \leq \frac{1}{k}$, $\tilde{x}_1 = 0$, $\tilde{x}_2 = \tilde{x}_{1,2}$ and $\tilde{C}_{1,2}'(\tilde{x}_{1,2}) = k\tilde{x}_{1,2}$

$\boxed{\text{Case-I}}$ Assume that the post-merger equilibrium quantity is such that $\tilde{x}_{1,2} > \frac{1}{k} \Rightarrow \tilde{C}_{1,2}'(\tilde{x}_{1,2}) = 1$

$$\tilde{\pi}_{1,2}(\tilde{x}) = (5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4)) \cdot \tilde{x}_{1,2} - \tilde{C}_{1,2}(\tilde{x}_{1,2})$$

$$\frac{\partial \tilde{\pi}_{1,2}}{\partial \tilde{x}_{1,2}} = 5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4) - 5(\tilde{x}_{1,2}) - 1 = 0$$

$$\boxed{4 = 10\tilde{x}_{1,2} + 5\tilde{x}_3 + 5\tilde{x}_4} \quad \text{--- (6)}$$

$$\tilde{\pi}_3(\tilde{x}) = (5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4)) \cdot \tilde{x}_3 - C_3(\tilde{x}_3)$$

$$\frac{\partial \tilde{x}_3}{\partial \tilde{x}_3} = 5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4) - 5\tilde{x}_3 - \frac{1}{4} = 0$$

$$\boxed{\frac{19}{4} = 5\tilde{x}_{1,2} + 10\tilde{x}_3 + 5\tilde{x}_4} \quad \text{—— (7)}$$

$$\tilde{x}_4(\tilde{x}) = (5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4))\tilde{x}_4 - c_4(\tilde{x}_4)$$

$$\frac{\partial \tilde{x}_4}{\partial \tilde{x}_4} = 5 - 5(\tilde{x}_{1,2} + \tilde{x}_3 + \tilde{x}_4) - 5\tilde{x}_4 - \frac{1}{4} = 0$$

$$\boxed{\frac{19}{4} = 5\tilde{x}_{1,2} + 5\tilde{x}_3 + 10\tilde{x}_4} \quad \text{—— (8)}$$

Solving (6), (7), (8),

$\tilde{x}_{1,2} = \frac{1}{8}$, $\tilde{x}_2 = \tilde{x}_3 = \frac{11}{40}$

This case is valid only if $\frac{1}{8} > \frac{1}{K} \Rightarrow \boxed{K > 8}$

$$\tilde{c}_{1,2}'(\tilde{x}_{1,2}) = 1$$

$$\Sigma Y_i^\circ \geq 0 \Rightarrow \frac{c_1'(x_1) + c_2'(x_2)}{2} \geq \tilde{c}_{1,2}'(\tilde{x}_{1,2})$$

$$\Rightarrow \frac{21K + 50}{4(4K + 25)} \geq 1 \Rightarrow 21K + 50 \geq 16K + 100$$

$$\Rightarrow 5K \geq 50 \Rightarrow \boxed{K \geq 10}$$

Thus, this assumption (that $\Sigma Y_i^\circ \geq 0$) is ~~the~~ violated for this case when $K \in (8, 10)$

(i) Take $K = 9$, $\Sigma Y_i^\circ = 1 + \frac{13 \times 9}{2(36 + 25)} - 2 = \frac{117}{122} - 1$

$$= \frac{-5}{117} < 0$$

(ii) Take $K = 11$, $\Sigma Y_i^\circ = 1 + \frac{13 \times 11}{2(44 + 25)} - 2 = -1 + \frac{143}{138}$

$$= \frac{5}{138} > 0$$

Thus, we've found a K for which $\Sigma Y_i > 0$ and a K for which $\Sigma Y_i < 0$. Consider the ~~last~~ inequality $(*)$, which as shown above is equivalent to :-

$$0 \geq -Z\left(\sum_{j=1}^{k} y_j\right)(n-2k+1) - (n-k+1)\left(\sum_{j=1}^{k} y_j\right)^2$$

Denote $P = \sum_{j=1}^{k} y_j$

$$Q(P) = -Z.P.(n-2k+1) - (n-k+1)P^2$$

We know that $Q(P) > 0$ if

$$P \in \left(\frac{-Z(n-2k+1)}{n-k+1}, 0\right)$$

In this particular example,

$$Z = b - (n-k+2).c + \sum_{j=k}^{n} c_j^{M_2} = 5 - 4\times1 + 1 + \frac{1}{4} + \frac{1}{4}$$

$$= 1 + 1 + \frac{1}{2} = \frac{5}{2}$$

$$\Rightarrow \frac{-Z(n-2k+1)}{n-k+1} = \frac{-\frac{5}{2}\times 1}{3} = \frac{-5}{6}$$

Thus, if $\Sigma Y_i \in \left(\frac{-5}{6}, 0\right)$, ~~th~~ the inequality is violated

When $K=9$, $\Sigma Y_i = \frac{-5}{117}$ (as calculated above)

We can see that,

$$-Z\left(\sum_{j=1}^{k} Y_j\right)(n-2k+1) - (n-k+1)\left(\sum_{j=1}^{k} Y_j\right)^2$$

$$= -\frac{5}{2} \cdot \frac{-5}{117} \times 1 - 3\left(\frac{5}{117}\right)^2 = 0 \cdot 1013 > 0$$

Therefore, $\boxed{\text{inequality } (*) \text{ is violated}}$.

3.4 Inference from Counter Examples

We observe that the second inequality in the following inequalities need not hold.

$$\frac{\tilde{\pi}_{1,2}^{M_1}}{\sum_{i=1}^{k} \pi_i^{M_1}(x^{M_1})} \geq \frac{\tilde{\pi}_{1,2}^{M_2}}{\sum_{i=1}^{k} \pi_i^{M_2}(x^{M_1})} \geq \frac{(n+1)^2}{k(n-k+2)^2}$$

However, that does not necessarily mean that the worst case fraction of loss in profits can be more than $1 - \frac{(n+1)^2}{k(n-k+2)^2}$. This is because it is still possible that $\frac{\tilde{\pi}_{1,2}^{M_2}}{\sum_{i=1}^{k} \pi_i^{M_2}(x^{M_1})} \leq \frac{(n+1)^2}{k(n-k+2)^2}$, which we saw in the second counter example, but $\frac{\tilde{\pi}_{1,2}^{M_1}}{\sum_{i=1}^{k} \pi_i^{M_1}(x^{M_1})} \geq \frac{(n+1)^2}{k(n-k+2)^2}$.

In fact, we worked out a large number of examples and observed that the worst-case would never cross $1 - \frac{(n+1)^2}{k(n-k+2)^2}$. Thus, we try to come up with some other methodology to attempt to prove the general result. In order to get an in depth understanding of why all the examples are giving this number as the worst-case fraction of profit loss, we work out the proof for the worst-case fraction of profit loss in a market in which the merging firms have quadratic costs.

3.5 Merger of 2 out of 4 firms with Quadratic Costs

Theorem. *Consider a market with affine demand function $P(S) = b - aS$ and four firms. Let the cost functions for firms be be $C_1(x_1) = c_1x_1 + \tilde{c}_1{x_1}^2, C_2(x_2) = c_2x_2 + \tilde{c}_2{x_2}^2, C_3(x_3) = c_3x_3$ and $C_4(x_4) = c_4x_4$ and $c_1, \tilde{c}_1, c_2, \tilde{c}_2, c_3, c_4 \geq 0$. Assume that firms 1 and 2 merge. Given $c_1 = c_2$, the worst-case fraction of loss in profits of the merging firms is $1 - 25/32 = 7/32$.*

Proof:- Given, $C_i(x_i) = c_i x_i + \tilde{c}_i x_i^2$, $c_i, \tilde{c}_i \geq 0$,

$i = 1, 2$, $C_3(x_3) = c_3 x_3$, $C_4(x_4) = c_4 x_4$

$P(s) = b - as$

$$\pi_1(X) = (b - a(x_1 + x_2 + x_3 + x_4)) \cdot x_1 - c_1 x_1 - \tilde{c}_1 x_1^2$$

$$\frac{\partial \pi_1}{\partial x_1} = b - a(x_1 + x_2 + x_3 + x_4) - a x_1 - c_1 - 2\tilde{c}_1 x_1 = 0$$

$$\boxed{2(a + \tilde{c}_1) x_1 + a(x_2 + x_3 + x_4) = b - c_1} \quad \text{—— (1)}$$

$$\pi_2(X) = (b - a(x_1 + x_2 + x_3 + x_4)) \cdot x_2 - c_2 x_2 - \tilde{c}_2 x_2^2$$

$$\frac{\partial \pi_2}{\partial x_2} = b - a(x_1 + x_2 + x_3 + x_4) - a x_2 - c_2 - 2\tilde{c}_2 x_2$$

$$\boxed{2(a + \tilde{c}_2) x_2 + a(x_1 + x_3 + x_4) = b - c_2} \quad \text{—— (2)}$$

$$\pi_3(X) = (b - a(x_1 + x_2 + x_3 + x_4)) \cdot x_3 - c_3 x_3$$

$$\frac{\partial \pi_3}{\partial x_3} = (b - a(x_1 + x_2 + x_3 + x_4)) - a x_3 - c_3 = 0$$

$$\boxed{2a x_3 + a(x_1 + x_2 + x_4) = b - c_3} \quad \text{—— (3)}$$

$$\pi_4(X) = (b - a(x_1 + x_2 + x_3 + x_4)) \cdot x_4 - c_4 x_4$$

$$\frac{\partial \pi_4}{\partial x_4} = (b - a(x_1 + x_2 + x_3 + x_4)) - a x_4 - c_4 = 0$$

$$\boxed{2a x_4 + a(x_1 + x_2 + x_3) = b - c_4} \quad \text{—— (4)}$$

Solving (1), (2), (3), (4), we get :-

$$\boxed{x_1 = \frac{b + c_3 + c_4 + \dfrac{a(c_2 - c_1)}{a + 2\tilde{c}_2} - 3c_1}{4a + 6\tilde{c}_1 + a\left(\dfrac{a + 2\tilde{c}_1}{a + 2\tilde{c}_2}\right)}}$$

$$\boxed{x_2 = \dfrac{b + c_3 + c_4 + \left(\dfrac{a(c_1 - c_2)}{a + 2\tilde{c}_1}\right) - 3c_2}{4a + 6\tilde{c}_2 + a\left(\dfrac{a + 2\tilde{c}_2}{a + 2\tilde{c}_1}\right)}}$$

$$\boxed{x_3 = \dfrac{b + c_4 - 2c_3 + \dfrac{a(c_1 - c_3)}{a + 2\tilde{c}_1} + \dfrac{a(c_2 - c_3)}{a + 2\tilde{c}_2}}{3a + \dfrac{a^2}{a + 2\tilde{c}_1} + \dfrac{a^2}{a + 2\tilde{c}_2}}}$$

$$\boxed{x_4 = \dfrac{b + c_3 - 2c_4 + \dfrac{a(c_1 - c_4)}{a + 2\tilde{c}_1} + \dfrac{a(c_2 - c_4)}{a + 2\tilde{c}_2}}{3a + \dfrac{a^2}{a + 2\tilde{c}_1} + \dfrac{a^2}{a + 2\tilde{c}_2}}}$$

<u>Profits</u> :- $\pi_i(X) = a(x_i)^2 + (C_i'(x_i))\cdot x_i - C_i(x_i)$

$$\pi_1(X) = a x_1^2 + x_1(c_1 + 2\tilde{c}_1 x_1) - c_1 x_1 - \tilde{c}_1 x_1^2$$

$$= a(x_1)^2 + \tilde{c}_1 x_1^2 = (a + \tilde{c}_1) x_1^2$$

$$\pi_2(X) = (a + \tilde{c}_2) x_2^2$$

$$\pi_3(X) = a x_3^2$$

$$\pi_4(X) = a x_4^2$$

Firms 1 and 2 merge :- Let the merged firm produced the quantity $\bar{x}_{1,2}$. from Lemma 2 in [1], assuming $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$.

$$\bar{x}_1 + \bar{x}_2 = \bar{x}, \quad c_1'(\bar{x}_1) = c_2'(\bar{x}_2)$$

$$c_1 + 2\tilde{c}_1 \bar{x}_1 = c_2 + 2\tilde{c}_2(\bar{x}_{1,2} - \bar{x}_1)$$

This gives, $\boxed{\bar{x}_1 = \dfrac{c_2 + 2\tilde{c}_2\bar{x}_{1,2} - c_1}{2(\tilde{c}_1 + \tilde{c}_2)}}$ — (5) and $\boxed{\bar{x}_2 = \dfrac{c_1 + 2\tilde{c}_1\bar{x}_{1,2} - c_2}{2(\tilde{c}_1 + \tilde{c}_2)}}$ — (6)

Cost function of the merged firm,

$$\tilde{C}_{1,2}(\tilde{x}_{1,2}) = \tilde{C}_1(\tilde{x}_1) + \tilde{C}_2(\tilde{x}_2)$$

Putting $\tilde{x}_1$ and $\tilde{x}_2$ from ⑤ and ⑥, we get:

$$\tilde{C}_{1,2}(\tilde{x}_{1,2}) = \frac{-1(c_1-c_2)^2}{4(\tilde{c}_1+\tilde{c}_2)} + \frac{(c_1\tilde{c}_2+\tilde{c}_1 c_2)\cdot\tilde{x}}{\tilde{c}_1+\tilde{c}_2}$$

$$+ \frac{\tilde{c}_1\tilde{c}_2}{\tilde{c}_1+\tilde{c}_2}(\tilde{x})^2$$

$$\boxed{\tilde{C}'_{1,2}(\tilde{x}_{1,2}) = \frac{c_1\tilde{c}_2+\tilde{c}_1 c_2+2\tilde{c}_1\tilde{c}_2\tilde{x}_{1,2}}{\tilde{c}_1+\tilde{c}_2}}$$

Solving the Post-Merger Cournot equilibrium.

$$\tilde{\pi}_{1,2}(\tilde{X}) = (b-a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4))\cdot\tilde{x}_{1,2} - \tilde{C}_{1,2}(\tilde{x}_{1,2})$$

$$\frac{\partial\tilde{\pi}_{1,2}}{\partial\tilde{x}_{1,2}} = b - a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4) - a\tilde{x}_{1,2} - \frac{c_1\tilde{c}_2+\tilde{c}_1c_2+2\tilde{c}_1\tilde{c}_2\tilde{x}_{1,2}}{\tilde{c}_1+\tilde{c}_2}$$

$$\boxed{\left(2a+\frac{2\tilde{c}_1\tilde{c}_2}{\tilde{c}_1+\tilde{c}_2}\right)\cdot\tilde{x}_{1,2} + a\tilde{x}_3 + a\tilde{x}_4 = b - \frac{c_1\tilde{c}_2+\tilde{c}_1c_2}{\tilde{c}_1+\tilde{c}_2}} \quad \text{— ⑦}$$

$$\tilde{\pi}_3(\tilde{X}) = (b-a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4))\cdot\tilde{x}_3 - C_3(\tilde{x}_3)$$

$$\frac{\partial\tilde{\pi}_3}{\partial\tilde{x}_3} = b - a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4) - a\tilde{x}_3 - c_3 = 0$$

$$\boxed{a\tilde{x}_{1,2} + 2a\tilde{x}_3 + a\tilde{x}_4 = b - c_3} \quad \text{— ⑧}$$

$$\tilde{\pi}_4(\tilde{X}) = (b-a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4))\cdot\tilde{x}_4 - C_4(\tilde{x}_4)$$

$$\frac{\partial\tilde{\pi}_4}{\partial\tilde{x}_4} = b - a(\tilde{x}_{1,2}+\tilde{x}_3+\tilde{x}_4) - a\tilde{x}_4 - c_4 = 0$$

$$\boxed{a\tilde{x}_{1,2} + a\tilde{x}_3 + 2a\tilde{x}_4 = 0} \quad \text{— ⑨}$$

Solving ⑦, ⑧ and ⑨, we get :-

$$\boxed{\tilde{x}_{1,2} = \frac{(b+c_3+c_4)(\tilde{c}_1+\tilde{c}_2) - 3c_1\tilde{c}_2 - 3\tilde{c}_1 c_2}{2(3\tilde{c}_1\tilde{c}_2 + 2a(\tilde{c}_1+\tilde{c}_2))}}$$

$$\boxed{\tilde{x}_3 = \frac{(b-2c_3+c_4)(2\tilde{c}_1\tilde{c}_2 + a\tilde{c}_1 + a\tilde{c}_2) + a(\tilde{c}_1(c_2-3) + \tilde{c}_2(c_1-3))}{2a(3\tilde{c}_1\tilde{c}_2 + 2a\tilde{c}_1 + 2a\tilde{c}_2)}}$$

$$\boxed{\tilde{x}_4 = \frac{(b-2c_4+c_3)(2\tilde{c}_1\tilde{c}_2 + a\tilde{c}_1 + a\tilde{c}_2) + a(\tilde{c}_1(c_2-3) + \tilde{c}_2(c_1-3))}{2a(3\tilde{c}_1\tilde{c}_2 + 2a\tilde{c}_1 + 2a\tilde{c}_2)}}$$

Profit of the Merged firm :-

$$\tilde{\pi}_{1,2}(\tilde{x}_{1,2}) = a(\tilde{x}_{1,2})^2 + (\tilde{c}'_{1,2}(\tilde{x}_{1,2}))\cdot\tilde{x}_{1,2} - \tilde{c}_{1,2}(\tilde{x}_{1,2})$$

$$\tilde{\pi}_{1,2}(\tilde{x}_{1,2}) = a(\tilde{x}_{1,2})^2 + \frac{\tilde{c}_1\tilde{c}_2(\tilde{x}_{1,2})^2 + \frac{1}{4}(c_1-c_2)^2}{(\tilde{c}_1+\tilde{c}_2)}$$

$$\boxed{\tilde{\pi}_{1,2}(\tilde{x}_{1,2}) = \frac{(\tilde{c}_1\tilde{c}_2 + a\tilde{c}_1 + a\tilde{c}_2)\tilde{x}^2 + \frac{1}{4}(c_1-c_2)^2}{(\tilde{c}_1+\tilde{c}_2)}}$$

Pre-merger profit $\Rightarrow \pi_1(x_1) = (a+\tilde{c}_1)x_1^2$, $\pi_2(x_2) = (a+\tilde{c}_2)\cdot x_2^2$

$$\pi_1(x_1)+\pi_2(x_2) = (a+\tilde{c}_1)\left[\frac{(b+c_3+c_4-3c_1)(a+2\tilde{c}_2) + a(c_2-c_1)}{(4a+6\tilde{c}_1)(a+2\tilde{c}_2) + a(a+2\tilde{c}_1)}\right]^2 + (a+\tilde{c}_2)\left[\frac{(b+c_3+c_4-3c_2)(a+2\tilde{c}_1) + a(c_1-c_2)}{(4a+6\tilde{c}_2)(a+2\tilde{c}_1) + a(a+2\tilde{c}_2)}\right]^2$$

Take $c_1 = c_2 = c$,

Pre-merger profit =

$$(a+\tilde{c}_1)\left[\frac{(b+c_3+c_4-3c)(a+2\tilde{c}_2)}{(4a+6\tilde{c}_1)(a+2\tilde{c}_2)+a(a+2\tilde{c}_1)}\right]^2 + (a+\tilde{c}_2)\left[\frac{(b+c_3+c_4-3c)(a+2\tilde{c}_1)}{(4a+6\tilde{c}_2)(a+2\tilde{c}_1)+a(a+2\tilde{c}_2)}\right]^2$$

$$= (b+c_3+c_4-3c)^2\left(\left[\frac{(a+\tilde{c}_1)(a+2\tilde{c}_2)^2}{((4a+6\tilde{c}_1)(a+2\tilde{c}_2)+a(a+2\tilde{c}_1))^2}\right] + \left[\frac{(a+\tilde{c}_2)(a+2\tilde{c}_1)^2}{(4a+6\tilde{c}_2)(a+2\tilde{c}_1)+a(a+\tilde{c}_2)}\right]\right) \quad (10)$$

$$\text{Let } P = \frac{(a+\tilde{c}_1)(a+2\tilde{c}_2)^2}{((4a+6\tilde{c}_1)(a+2\tilde{c}_2)+a(a+2\tilde{c}_1))^2} + \frac{(a+\tilde{c}_2)(a+2\tilde{c}_1)^2}{((4a+6\tilde{c}_2)(a+2\tilde{c}_1)+a(a+2\tilde{c}_2))^2}$$

$$\text{Post-Merger Profit} = \frac{(\tilde{c}_1\tilde{c}_2 + a\tilde{c}_1 + a\tilde{c}_2)(\tilde{x}_{12})^2 + \frac{1}{4}(c_1-c_2)^2}{\tilde{c}_1+\tilde{c}_2}$$

Putting the value of $\tilde{x}_{1,2}$, we get

At $c_1 = c_2$

$$\text{Post-merger Profit} = \left(\frac{\tilde{c}_1\tilde{c}_2 + a(\tilde{c}_1+\tilde{c}_2)}{\tilde{c}_1+\tilde{c}_2}\right)\left[\frac{(b+c_3+c_4-3c)(\tilde{c}_1+\tilde{c}_2)}{2(3\tilde{c}_1\tilde{c}_2+2a(\tilde{c}_1+\tilde{c}_2))}\right]^2 \quad (11)$$

The Ratio of the Post-merger and Pre-merger profit.

$$\text{Ratio} = \frac{\text{Post-Merger Profit}}{\text{Pre-merger Profit}}$$

$$\text{Ratio} = \frac{(\tilde{c}_1\tilde{c}_2 + a(\tilde{c}_1+\tilde{c}_2))\cancel{(b+c_3+c_4-3c)^2}(\tilde{c}_1+\tilde{c}_2)^{\cancel{2}} \cdot P}{\cancel{(\tilde{c}_1+\tilde{c}_2)} \cdot 4(3\tilde{c}_1\tilde{c}_2+2a(\tilde{c}_1+\tilde{c}_2)) \cdot \cancel{(b+c_3+c_4-3c)^2}}$$

$$\boxed{\text{Ratio} = \frac{(\tilde{c}_1\tilde{c}_2 + a(\tilde{c}_1+\tilde{c}_2))(\tilde{c}_1+\tilde{c}_2)}{4(3\tilde{c}_1\tilde{c}_2+2a(\tilde{c}_1+\tilde{c}_2)) \cdot P}} \quad (12)$$

• Now, we prove the following:-

(i) Fixing $\tilde{\tau}_1 = \tilde{\tau}_2 = \varepsilon$, we show that the minimum value of this ratio occurs when $\varepsilon \to 0$ and the minimum value equals $\frac{25}{32}$.

(ii) Let $\tilde{\tau}_2 = k \cdot \tilde{\tau}_1$. We prove that for any $k > 0$, minimum value of this ratio occurs when $\tau_1 \to 0$.

Proof (i) Let $\tilde{\tau}_1 = \tau_2 = \varepsilon$, then

$$\text{Ratio} = \frac{(\varepsilon^2 + 2a\varepsilon) \cdot 2\varepsilon\,(5a^2 + 16a\varepsilon + 12\varepsilon^2]^2}{4(3\varepsilon^2 + 4a\varepsilon)^2 (2a^3 + 10a^2\varepsilon + 8\varepsilon^3 + 16a\varepsilon^2]}.$$

$$= \frac{\cancel{2\varepsilon^2}(\varepsilon + 2a)(5a^2 + 16a\varepsilon + 12\varepsilon^2]^2}{2\,4\cancel{\varepsilon^2}(3\varepsilon + 4a)^2 (2a^3 + 10a^2\varepsilon + 8\varepsilon^3 + 16a\varepsilon^2]}$$

$$= \frac{(\varepsilon + 2a)\cancel{(2\varepsilon + a)^2}(6\varepsilon + 5a)^2}{4(3\varepsilon + 4a)^2 \cancel{(2\varepsilon + a)^2}(\varepsilon + a)} = \frac{(\varepsilon + 2a)(6\varepsilon + 5a)^2}{4(3\varepsilon + 4a)^2(\varepsilon + a)}$$

Observe that $\lim\limits_{\varepsilon \to 0} \frac{(\varepsilon + 2a)(6\varepsilon + 5a)^2}{4(3\varepsilon + 4a)^2(\varepsilon + a)} = \frac{2a \times 25a^2}{4 \times 16a^2 \times a} = \frac{25}{32}$

Consider, $f(\varepsilon) = \frac{(\varepsilon + 2a)(6\varepsilon + 5a)^2}{(\varepsilon + a)(6\varepsilon + 8a)^2}$

$$\frac{df}{d\varepsilon} = \frac{\left[\left((6\varepsilon + 5a)^2 + (\varepsilon + 2a) \cdot 12 \cdot (6\varepsilon + 5a)\right)(\varepsilon + a)(6\varepsilon + 8a)^2 - (\varepsilon + 2a)(6\varepsilon + 5a)^2\left[(6\varepsilon + 8a)^2 + (6\varepsilon + 8a) \cdot 12 \cdot (\varepsilon + a)\right]\right]}{\left((\varepsilon + a)(6\varepsilon + 8a)^2\right)^2}$$

$$= \frac{(6\varepsilon + 5a)(6\varepsilon + 8a)\left[(18\varepsilon + 29a)(\varepsilon + a)(6\varepsilon + 8a) - (\varepsilon + 2a)(6\varepsilon + 5a)(18\varepsilon + 20a)\right)}{\left((\varepsilon + a)(6\varepsilon + 8a)^2\right)^2}.$$

$$\frac{df}{d\varepsilon} = \frac{(6\varepsilon+5a)(6\varepsilon+8a)\left[30a^2\varepsilon+32a^3\right]}{\left((\varepsilon+a)(6\varepsilon+8a)^2\right)^2}$$

Therefore, $\dfrac{df}{d\varepsilon} > 0 \quad \forall\ \varepsilon > 0$

Hence, the minimum value of this ratio occurs when $\varepsilon \to 0$ and the minimum value is equal to $\dfrac{25}{32}$.

<u>Proof (ii)</u> :- let $\tilde{c}_1 = x$ and $\tilde{c}_2 = k\tilde{c}_1 = kx$, $k > 0$.

$$\text{Ratio} = \frac{(kx^2 + a(k+1)x)(k+1)x\left[5a^2+8a(k+1)x+12kx^2\right]^2}{4\left[3kx^2+2a(k+1)x\right]^2\left[2a^3+(5a^2+4kx^2)(k+1)x+4a(k+1)^2x^2\right]}$$

$$= \frac{(4k(k+1)x^3+4a(k+1)^2x^2]\left[5a^2+8a(k+1)x+12kx^2\right]^2}{(8a(k+1)x+12kx^2]^2\left[2a^3+5a^2(k+1)x+4k(k+1)x^3+4a(k+1)^2x^2\right]}$$

let $P(x) = 4k(k+1)x^3 + 4a(k+1)^2x^2$

and $Q(x) = 8a(k+1)x + 12kx^2$

$$\therefore\ \text{Ratio} = \frac{(P(x))(5a^2+Q(x))^2}{(Q(x))^2(2a^3+5a^2(k+1)x+P(x))}$$

Differentiating the ratio wrt x, we get :-

$$\frac{d\,\text{Ratio}}{dx} = \frac{\begin{aligned}&\left[P'(5a^2+Q)^2+2P(5a^2+Q)Q'\right]Q^2\left[2a^3+5a^2(k+1)x+P\right]\\ &\quad - P(5a^2+Q]^2\left[2QQ'(2a^3+5a^2(k+1)x+P]+Q^2[P'+5a^2(k+1)]\right]\end{aligned}}{Q^4(2a^3+5a^2(k+1)x+P)^2}$$

$$P'(x) = \underline{12(K)(K+1)x^2} + 8a(K+1)^2 x^2$$

$$Q'(x) = 8a(K+1) + 24Kx$$

On substituting $P(x)$, $Q(x)$, $P'(x)$, $Q'(x)$ and expanding, we get :-

$$\frac{d\,\text{Ratio}}{dx} = \frac{\begin{aligned}&\big(8960a^9K^5x^4 + 19200a^9K^4x^4 + 12800a^9K^2x^4 \\ &+ 19200a^9Kx^4 + 12800a^9K^3x^4 + 8960a^9x^4 + 14336a^9x^4 \\ &+ 14336a^8K^6x^5 + 98816a^8K^5x^5 + 208640a^8K^4x^5 + \\ &248320a^8K^3x^5 + 208640a^8K^2x^5 + 98816a^8Kx^5 + 14336a^8x^5 \\ &+ 107520a^7K^6x^6 + 624960a^7K^5x^6 + 1308480a^7K^4x^6 + \\ &1308480a^7K^3x^6 + 624960a^7K^3x^6 + 624960a^7K^2x^6 \cancel{+107520} \\ &+ 107520a^7Kx^6 + 578560a^6K^6x^7 + 2538880a^6K^5x^7 + \\ &3920640a^6K^4x^7 + 578560a^6K^2x^7 + 102400a^5K^7x^8 + \\ &1917440a^5K^6x^8 + 5378560a^5K^5x^8 + 5378560a^5K^4x^8 \\ &+ 1917440a^5K^3x^8 + 102400a^5K^2x^8 + 430080a^4K^7x^9 + \\ &3148800a^4K^6x^9 + 5437440a^4K^5x^9 + 3148800a^4K^4x^9 + \\ &430080a^4K^3x^9 + 599040a^3K^7x^{10} + 2294784a^3K^6x^{10} \\ &+ 2294784a^3K^5x^{10} + 599040a^3K^4x^{10} + 276480a^2K^7x^{11} \\ &+ 552960a^2K^6x^{11} + 276480a^2K^5x^{11}\big)\end{aligned}}{Q^4\left(2a^3 + 5a^2(K+1)x + P\right)^2}$$

Therefore, every single term in $\frac{d\,\text{Ratio}}{dx}$ is greater than zero $\forall x > 0$.

Ratio attains its minimum value when $x \to 0$.

Hence, we've checked for all the lines $\tilde{c}_2 = K\tilde{c}_1$, except

Now, for $\tilde{c}_2 = 0$, substituting $\tilde{c}_2 = 0$ in the expression, we get :-

$$\text{Ratio} = \frac{(a\cdot\tilde{c}_1^{\,2})(5a^2 + 8a\tilde{c}_1)^2}{16a^2(\tilde{c}_1)^2(2a^3 + 5a^2\tilde{c}_1 + 4a\tilde{c}_1^{\,2})} = \frac{(5a^2 + 8a\tilde{c}_1)^2}{16a(2a^3 + 5a^2\tilde{c}_1 + 4a\tilde{c}_1^{\,2})}$$

$$= \frac{(5a + 8\tilde{c}_1)^2}{16(2a^2 + 5a\tilde{c}_1 + 4a\tilde{c}_1^{\,2})}$$

$$16\,\frac{d\,\text{Ratio}}{d\tilde{c}_1} = \frac{2(5a+8\tilde{c}_1)\cdot 8\,(2a^2+5a\tilde{c}_1+4\tilde{c}_1^2) - (5a+8\tilde{c}_1)^2\,[5a+8\tilde{c}_1]}{(2a^2+5a\tilde{c}_1+4a\tilde{c}_1^2)^2}$$

$$= \frac{(5a+8\tilde{c}_1)\,[32a^2+\cancel{80a\tilde{c}_1}+\cancel{64\tilde{c}_1^2}-25a^2-\cancel{64\tilde{c}_1^2}-\cancel{80a\tilde{c}_1}]}{(2a^2+5a\tilde{c}_1+4a\tilde{c}_1^2)^2}$$

$$= \frac{(5a+8\tilde{c}_1)(7a^2)}{(2a^3+5a\tilde{c}_1+4a\tilde{c}_1^2]^2}$$

$\therefore$ for $\tilde{c}_2 = 0$, $\frac{d\,\text{Ratio}}{d\tilde{c}_1} > 0$. Thus, the minimum value of this ratio occurs when $\tilde{c}_1 \to 0$.

Hence, the proof of (i) and (ii) show that the worst-case loss in fraction of profits, given $c_1 = c_2$ occur when $\tilde{c}_1 \to 0$ and $\tilde{c}_2 \to 0$ and this ratio equals $1 - \frac{25}{32} = \frac{7}{32}$.

Corner Cases: After proving the above theorem for an internal Cournot equilibria both in the pre-merger and the post-merger state, the following are the proofs for the corner cases. The proofs for the Corner cases 2, 3 and 4 are directly taken from [1].

Corner Case 1: If x^M is a corner solution for any F_i, $i \in [k]$. Consider a market $\bar{M}$ in which F_i is omitted. Note that in both the markets M and $\bar{M}$ in the pre-merger state, the equilibrium quantity produced by each firm F_j such that $j \neq i$ is the same because the necessary and sufficient conditions given in lemma 1 in [1] are identically satisfied for each F_j in M and $\bar{M}$. Denote the Cournot equilibrium of $\bar{M}$ by $x^{\bar{M}}$. Therefore,

$$\sum_{j=1}^{k} \pi_j^M(x^M) = \sum_{\substack{j=1 \\ j\neq i}}^{k} \pi_j^{\bar{M}}(x^{\bar{M}})$$

Consider the post-merger cost function of the merged firm in market M at some quantity $\tilde{x}_{1,2,\ldots,k} \geq 0$. For each firm F_j such that $j \in [k]$, from lemma 2 in [1], ${C'}_j^M(\tilde{x}_j^M) \geq \tilde{C}'^M_{1,2,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$, where the equality holds if $\tilde{x}_j^M > 0$.

Now, assume that in $\bar{M}$ the firms $F_1, F_2, \ldots, F_{i-1}, F_{i+1}, \ldots, F_k$ merge. Consider the post-merger cost function of the merged firm in market $\bar{M}$ at the same quantity $\tilde{x}_{1,2,\ldots,k} \geq 0$. For each firm F_j such that $j \in [k]\backslash\{i\}$, from lemma 2 in [1], ${C'}_j^{\bar{M}}(\tilde{x}_j^{\bar{M}}) \geq \tilde{C}'^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$, where the equality holds if $\tilde{x}_j^{\bar{M}} > 0$. In markets M and $\bar{M}$, $\tilde{x}_{1,2,\ldots,k} = \sum_{j=1}^{k} \tilde{x}_j^M = \sum_{\substack{j=1 \\ j\neq i}}^{k} \tilde{x}_j^{\bar{M}}$. Obviously, $0 = \tilde{x}_i^{\bar{M}} \leq \tilde{x}_i^M$ because F_i is omitted in $\bar{M}$. If $\tilde{x}_i^M = 0$, then $\tilde{C}'^M_{1,2,\ldots,k}(\tilde{x}_{1,2,\ldots,k}) = \tilde{C}'^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$. If $\tilde{x}_i^M > 0$, then there exists atleast one $j \in [k]\backslash\{i\}$ such that $\tilde{x}_j^{\bar{M}} > \tilde{x}_j^M \geq 0$. Hence, $\tilde{C}'^M_{1,2,\ldots,k}(\tilde{x}_{1,2,\ldots,k}) \leq {C'}_j^M(\tilde{x}_j^M) \leq {C'}_j^{\bar{M}}(\tilde{x}_j^{\bar{M}}) = \tilde{C}'^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$. The first inequality and the third equality hold because of lemma 2 in [1]. The second inequality holds because the cost functions are convex. Therefore, in either case, $\tilde{C}'^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$ is atleast $\tilde{C}'^M_{1,2,\ldots,k}(\tilde{x}_{1,2,\ldots,k})$. Since, this is true for any arbitrary quantity $\tilde{x}_{1,2,\ldots,k} \geq 0$, lemma 4 in [1] guarantees that the profit of the merged firm in M is atleast the profit of the merged firm in $\bar{M}$. That is,

$$\tilde{\pi}^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}^{\bar{M}}) \leq \tilde{\pi}^M_{1,2,\ldots,k}(\tilde{x}^M)$$

where $\tilde{x}^M$ and $\tilde{x}^{\bar{M}}$ denote the post-merger Cournot equilibria in markets M and $\bar{M}$ respectively. Hence, we get,

$$\frac{\tilde{\pi}^M_{1,2,\ldots,k}(\tilde{x}^M)}{\sum_{j=1}^{k} \pi_j^M(x^M)} \geq \frac{\tilde{\pi}^{\bar{M}}_{1,\ldots,i-1,i+1,\ldots,k}(\tilde{x}^{\bar{M}})}{\sum_{\substack{j=1 \\ j\neq i}}^{k} \pi_j^{\bar{M}}(x^{\bar{M}})} \geq \frac{n^2}{(k-1)(n-k+2)^2}$$

$$\geq \frac{(n+1)^2}{k(n-k+2)^2}$$

The last inequality follows because,

$$\frac{n^2}{(k-1)} \geq \frac{(n+1)^2}{k} \iff \frac{n^2}{(n+1)^2} \geq \frac{k-1}{k}$$

Now, $k \leq \frac{n+1}{2} \implies \frac{k-1}{k} \leq \frac{n-1}{n+1}$ and the above inequality is true if $\frac{n^2}{(n+1)^2}$ is greater than or equal to the maximum value of $\frac{k-1}{k}$. That is,

$$\frac{n^2}{(n+1)^2} \geq \frac{n-1}{n+1} \iff n^2 \geq (n-1)\cdot(n+1) \iff n^2 \geq n^2-1 \iff 0 \geq -1$$

, which is indeed true.

Corner Case 2: Similarly, if $\tilde{x}^M$ is a corner solution for $F_{1,2,\ldots,k}$, it has zero profits, and so [1, Lem. 6] implies that $0 \geq \sum_{i=1}^{k} \pi_i^M$, and the theorem trivially holds.

Corner Case 3: If x^M is a corner solution for any F_j, $j \in [n]\backslash[k]$, then there exists $c \leq c_j$, such that replacing F_j's cost function, $C_j^M(x) = c_j \cdot x$, by $\hat{C}_j^M(x) = c \cdot x$ yields a market with a unique equilibrium, which is a non-corner solution for F_j, although it produces zero quantity. In order to obtain such $c \leq c_j$, note that by [1, Lem. 1] and by the definition of a corner solution:

$$c_j = \tilde{C}_j(x_j^M) > P(x_j^M + \ldots + x_j^M) + x_j^M \cdot P'(x_1^M + \ldots + x_n^M)$$

Decrease c_j until equality is reached, and pick the $c \leq c_j$ preserving that equality. In the obtained market, x^M is a unique Cournot equilibrium, as the conditions of [1, Lem, 1] are met. Moreover, it is non-corner for F_j, but F_j indeed produces a zero quantity in the market. That replacement has no effect on the profits of F_1, F_2, $\ldots$, F_k in M before merging. In addition, by [1, Lem. 4], since $c \leq c_j$, that replacement can only decrease the profits of the merged firm $F_{1,2,\ldots,k}$. Concluding, in the obtained market, F_1, F_2, $\ldots$, F_k lose a (weakly) higher fraction of profits by merging compared to M.

Corner Case 4: If $\tilde{x}^M$ is a corner solution for any F_j, $j \in [n]\backslash[k]$, then omit it from M. This has no effect on the production levels of the merged firm in M's equilibrium, as the conditions in [1, Lem. 1] are met. However, due to [1, Lem. 5], the profits of F_1, F_2 $\ldots$, F_k before merging can only increase. So, F_1, F_2, $\ldots$, F_k lose a (weakly) higher fraction of merging in that market compared to M.

4 Future Work

The above Theorem gives good insights as to how a formal proof of the general case might proceed. Having tried out many numerical examples, we're convinced that the worst-case fraction of profit loss for the merging firms in this setup is $max(0, 1 - \frac{(n+1)^2}{k(n-k+2)^2})$. But, a major difficulty that arises when trying to attempt a formal proof of this result is the general characterization of post-merger quantities and marginal costs in terms of the pre-merger parameters. A key observation when we attempted to prove the above theorem with quadratic costs for the merging firms was that the ratio of post-merger by pre-merger profits would become $\frac{(n+1)^2}{k(n-k+2)^2}$ if we made the costs of each firm closer and closer to linear. In the above proof, given $c1 = c2$, $\tilde{c1}$ and $\tilde{c2}$ would cause the worst-case profit loss when they both tend to zero, essentially making the cost functions of the merging firms linear. Thus, an attempt to prove the general result might involve analysing only the linear terms in the cost functions of the merging firms. Henceforth, the future work involves the attempt to give a proof for the worst-case loss of fraction of profits of the merging firms for a general n and k, taking into account these observations.

References

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