COL352 Problem Sheet 1

January 9, 2025

Problem 1. *Create a DFA that represents the language* $\{\textit{binary representation of } n \in \mathbb{N} \mid n \mod 8 \textit{ is either } 4 \textit{ or } 1\}$

Problem 2. *Create a regular expression over the alphabet* $\{0, 1\}$ *that represents the language mentioned in Problem 1.*

Problem 3. *Construct a DFA which accept the language* $L = \{w \mid w \in \{a, b\}^* \textit{ and } Na(w) \mod 3 = Nb(w) \mod 3\}$, *where* $Na(w)$ *and* $Nb(w)$ *return the number of occurrences of* a *and* b *in* w *respectively.*

Problem 4. *Construct a DFA that recognizes the following language over the alphabet* $\{0, 1\}$.

$$\{x \mid \textit{ 01 and 10 have equal number of occurrences as substrings in } x\}$$

Problem 5. *Let* $C_n = \{x \mid x \textit{ is a binary number that is a multiple of } n\}$. *Show that for each* $n \geq 1$, *the language* C_n *is regular. Provide a general construction for* C_i *for* $i \geq 0$.

Problem 6. *You already know from the lectures that regular languages are closed under complementation. Given DFAs* D_1 *and* D_2 *that recognize languages* L_1 *(over* Σ_1*) and* L_2 *(over* Σ_2*) respectively, construct an automaton* D *recognizing the following languages, if you believe that the class of regular languages is closed under the following operations. Provide a counterexample otherwise.*

1. ***Difference:*** $L_1 \setminus L_2 := \{x \mid x \in L_1 \textit{ and } x \notin L_2\}$
2. ***Star:*** $L_1^* := \{w_1 w_2 \dots w_n \mid w_i \in L_1 \textit{ for } n \geq 0, \textit{ and every } 1 \leq i \leq n\}$

Problem 7. *Design an efficient algorithm that takes input the description of a DFA* D *and determines if the resulting language* $L(D)$ *is*

1. *Empty*
2. *Infinite*
3. Σ^*

Problem 8. *Given two DFAs,* D_1 *and* D_2, *design an efficient algorithm to determine if* $L(D_1) = L(D_2)$.

Problem 9. $L_1 = (0|1)^*0(0|1)^*1(0|1), L_2 = (0|1)^*01(0|1)^*$. *Show that the two languages are equal.*

Problem 10. *Let* L_1 *be a regular language and* L_2 *be any language (not necessarily regular) over the same alphabet* Σ. *Prove that the language* $L = \{x \in \Sigma^* \mid x \cdot y \in L_1 \textit{ for some } y \in L_2\}$ *is regular by defining a DFA for* L *starting from a DFA for* L_1 *and the language* L_2.

Problem 11. $\phi : \Sigma^* \to \Gamma^*$ *is called a homomorphism over strings if for all* $x, y \in \Sigma^*$, $\phi(xy) = \phi(x)\phi(y)$. *Show that if* L *is a regular language, then* $\phi(L) := \{y \in \Gamma^* \mid y = \phi(x), x \in L\}$ *where* ϕ *is a homomorphism as defined above is also regular.*

Problem 12. *Prove that the class of regular languages is closed under inverse homomorphisms. That is, prove that if* $L \subseteq \Gamma^*$ *is a regular language and* $\phi : \Sigma^* \to \Gamma^*$ *is a string homomorphism, then* $\phi^{-1}(L) = \{x \in \Sigma^* \mid f(x) \in L\}$ *is regular.*

Problem 13. *Prove that the class of* ***non-regular*** *languages is* ***not*** *closed under the Union operation.* ***Hint:*** *use the closure properties mentioned in the lectures.*

Problem 14. *Let L be an arbitrary regular language. Prove that the following languages are regular.*

1. $\{x \mid x \cdot \mathit{reverse}(x) \in L\}$
2. $\{x \mid x \cdot \mathit{reverse}(x) \cdot x \in L\}$.
3. $\{x \mid xxx \in L\}$

Problem 15. *Let L be a regular language with DFA D. We define $\mathit{Pre}(L) = \{x \mid x \text{ is a prefix of some } y \in L\}$. Show that Pre(L) is regular by constructing a DFA.*

Problem 16. *Let A be any language. Define DROP-OUT(A) to be the language containing all strings that can be obtained by removing one symbol from a string in A. Thus, DROP-OUT(A) $= \{xz \mid xyz \in A \text{ where } x, z \in \Sigma^*, y \in \Sigma\}$. Show that the class of regular languages is closed under the DROP-OUT operation.*