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HANDBOOK OF MATHEMATICS

LOGARITHM

LOGARITHM OF A NUMBER :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

(a) $\log_a N = x$, read as log of N to the base a $\Leftrightarrow a^x = N$
If a = 10 then we write log N or $\log_{10} N$ and if a = e we write ln N or $\log_e N$ (Natural log)

(b) Necessary conditions : $N > 0$; $a > 0$; $a \neq 1$

(c) $\log_a 1 = 0$

(d) $\log_a a = 1$

(e) $\log_{1/a} a = -1$

(f) $\log_a(x.y) = \log_a x + \log_a y$; x, y > 0

(g) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; x, y > 0

(h) $\log_a x^p = p\log_a x$; x > 0

(i) $\log_{a^q} x = \frac{1}{q}\log_a x$; x > 0

(j) $\log_a x = \frac{1}{\log_x a}$; x > 0, x ≠ 1

(k) $\log_a x = \log_b x / \log_b a$; x > 0, a, b > 0, b ≠ 1, a ≠ 1

(l) $\log_a b.\log_b c.\log_c d = \log_a d$; a, b, c, d > 0, ≠ 1

(m) $a^{\log_a x} = x$; a > 0, a ≠ 1

(n) $a^{\log_b c} = c^{\log_b a}$; a,b,c > 0; b ≠ 1

(o) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$

(p) $\log_a x = \log_a y \Rightarrow x = y$; x, y > 0 ; a > 0, a ≠ 1

(q) $e^{\ln a^x} = a^x$

(r) $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\ell n2 = 0.693$, $\ell n10 = 2.303$

(s) If a > 1 then $\log_a x < p \Rightarrow 0 < x < a^p$

(t) If a > 1 then $\log_a x > p \Rightarrow x > a^p$

(u) If 0 < a < 1 then $\log_a x < p \Rightarrow x > a^p$

(v) If 0 < a < 1 then $\log_a x > p \Rightarrow 0 < x < a^p$

TRIGONOMETRIC RATIOS & IDENTITIES

1. RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

1 Radian $= \frac{180}{\pi}$ degree $\approx$ 57°17'15" (approximately)

1 degree $= \frac{\pi}{180}$ radian $\approx$ 0.0175 radian

2. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

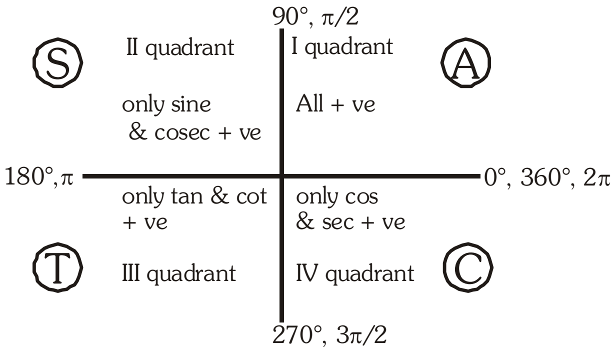
(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(c) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(d) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(e) If $\text{cosec}\theta + \cot\theta = k \Rightarrow \text{cosec}\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\,\theta = k + \frac{1}{k}$

3. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b) $\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\cos(90° - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$ $\quad$ $\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$ $\quad$ $\cos(180° + \theta) = -\cos\theta$

$\sin(270° - \theta) = -\cos\theta$ $\quad$ $\cos(270° - \theta) = -\sin\theta$

$\sin(270° + \theta) = -\cos\theta$ $\quad$ $\cos(270° + \theta) = \sin\theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(ii) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$, where $n \in I$

5. IMPORTANT TRIGONOMETRIC FORMULAE :

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

(ii) $\sin(A - B) = \sin A\cos B - \cos A\sin B$.

(iii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B\cot A + 1}{\cot B - \cot A}$

(ix) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

(x) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(xii) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

(xiii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xiv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xv) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xvi) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(xvii) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(xviii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(xix) $1 + \cos 2\theta = 2\cos^2\theta$ or $|\cos\theta| = \sqrt{\frac{1+\cos 2\theta}{2}}$

(xx) $1 - \cos 2\theta = 2\sin^2\theta$ or $|\sin\theta| = \sqrt{\frac{1-\cos 2\theta}{2}}$

(xxi) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta}$

(xxii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(xxiii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(xxiv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(xxv) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

(xxviii) sin (A + B + C)

= sinAcosBcosC + sinBcosAcosC + sinCcosAcosB − sinAsinBsinC

= ΣsinA cosB cosC − Πsin A

= cosA cosB cosC [tanA + tanB + tanC − tanA tanB tanC]

(xxix) cos (A + B + C)

= cosA cosB cosC − sinA sinB cosC − sinA cosB sinC − cosA sinB sinC

= Πcos A − Σsin A sin B cos C

= cosA cosB cosC [1 − tanA tanB − tanB tanC − tanC tanA]

(xxx) tan (A + B + C)

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6. VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° etc.

(a) $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \sin\frac{\pi}{10}$

(b) $\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \cos\frac{\pi}{5}$

(c) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \sin\frac{\pi}{12}$

(d) $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(f) $\tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(g) $\tan(22.5°) = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(h) $\tan(67.5°) = \sqrt{2}+1 = \cot(22.5°)$

7. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(a) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$, i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$

(c) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ (or $a^2\sin^2\theta + b^2\mathrm{cosec}^2\theta$) is either $2ab$ (when $|a| \geq |b|$) or $a^2 + b^2$ (when $|a| \leq |b|$).

8. IMPORTANT RESULTS :

(a) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta.\cos(60°-\theta)\cos(60°+\theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta\tan(60°-\theta)\tan(60°+\theta) = \tan 3\theta$

(d) $\cot\theta\cot(60°-\theta)\cot(60°+\theta) = \cot 3\theta$

(e) **(i)** $\sin^2\theta + \sin^2(60°+\theta) + \sin^2(60°-\theta) = \frac{3}{2}$

(ii) $\cos^2\theta + \cos^2(60°+\theta) + \cos^2(60°-\theta) = \frac{3}{2}$

(f) **(i)** If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(ii) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n+1)\frac{\pi}{2}, n \in I$

(g) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$

(h) $\cot A - \tan A = 2\cot 2A$

9. CONDITIONAL IDENTITIES :

If $A + B + C = 180°$, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(c) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(d) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(e) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(f) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(g) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

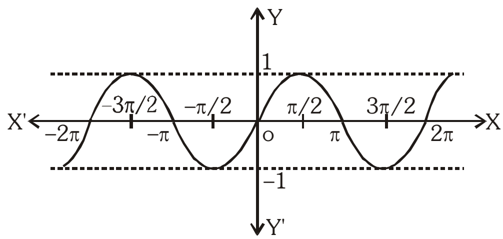
(h) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

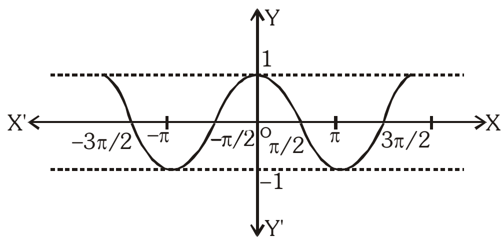
T-Ratio	Domain	Range	Period
$\sin x$	R	$[-1,1]$	2π
$\cos x$	R	$[-1,1]$	2π
$\tan x$	$R-\{(2n+1)\pi/2 ; n\in I\}$	R	π
$\cot x$	$R-[n\pi : n \in I]$	R	π
$\sec x$	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π
$\text{cosec } x$	$R-\{n\pi : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π

11. GRAPH OF TRIGONOMETRIC FUNCTIONS :

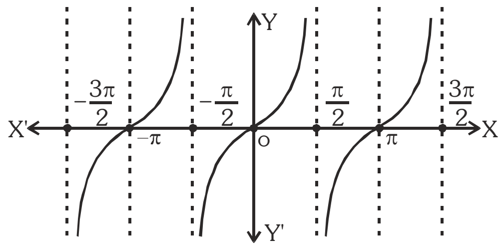
(a) y = sinx



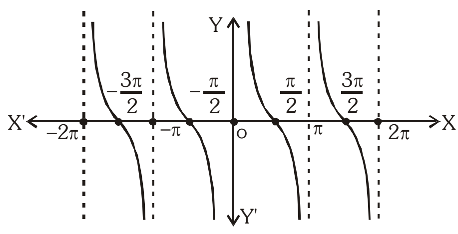
(b) y = cosx



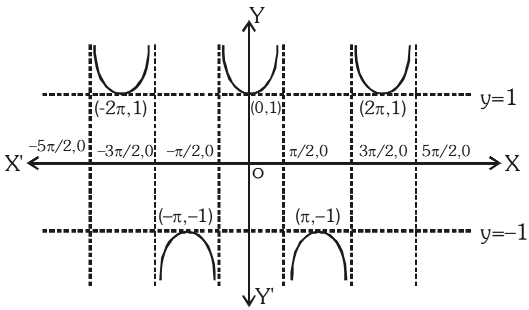
(c) y = tanx



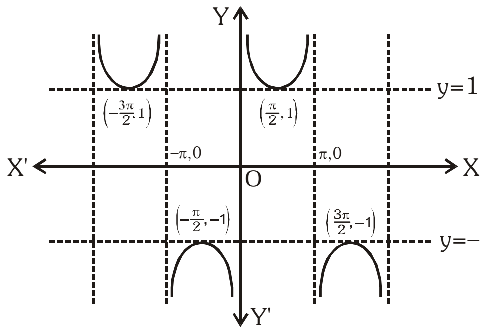
(d) y = cotx



(e) y = secx



(f) y = cosecx



12. IMPORTANT NOTE :

(a) The sum of interior angles of a polygon of n-sides

$= (n - 2) \times 180° = (n - 2)\pi$.

(b) Each interior angle of a regular polygon of n sides

$$= \frac{(n-2)}{n} \times 180° = \frac{(n-2)}{n}\pi .$$

(c) Sum of exterior angles of a polygon of any number of sides

$= 360° = 2\pi$.

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TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometric equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equations is called a solution of the trigonometric equation.

(a) **Principal solution :-** The solution of the trigonometric equation lying in the interval $[0, 2\pi]$.

(b) **General solution :-** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solutions of trigonometric equation.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$,

then $\theta = n\pi \pm \alpha$, $n \in I$

(j) $\sin(n\pi + \theta) = (-1)^n \sin\theta,\ n \in I$

$\cos(n\pi + \theta) = (-1)^n \cos\theta,\ n \in I$

4. GENERAL SOLUTION OF EQUATION a sin θ + b cos θ = c :

Consider, $a\sin\theta + b\cos\theta = c$ .... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has the solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi,\ \frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to

$$\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$$

Now this equation can be solved easily.

5. GENERAL SOLUTION OF EQUATION OF FORM :

$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}\times\cos^2 x + \ldots\ldots\ldots + a_n\cos^n x = 0$

$a_0, a_1, \ldots\ldots a_n$ are real numbers

Such an equation is solved by dividing equation both sides by $\cos^n x$.

6. IMPORTANT TIPS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(e) Check that denominator is not zero at any stage while solving equations.

(f) **(i)** If tan θ or sec θ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(ii) If cot θ or cosec θ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as tan θ and sec θ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.

QUADRATIC EQUATION

1. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $\mathbf{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then ;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $|\alpha - \beta| = \sqrt{D}\,/\,|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

(e) If α, β are roots of equation $ax^2 + bx + c = 0$, we have identity in x as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

2. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3. COMMON ROOTS OF TWO QUADRATIC EQUATIONS

(a) Atleast one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(b) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

4. ROOTS UNDER PARTICULAR CASES

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(b) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $ac < 0 \Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c

$\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a

$\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

5. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION :

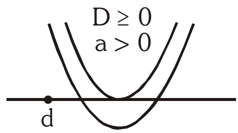
Maximum or Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

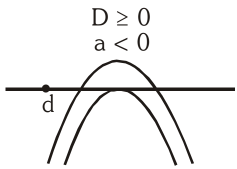
$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if } a > 0 \qquad \& \qquad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if } a < 0.$$

6. LOCATION OF ROOTS :

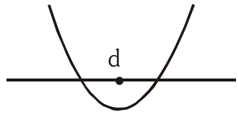
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

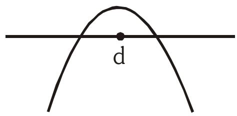
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are **$D \geq 0$ and $a.f(d) > 0$ & $(-b/2a) > d$.**



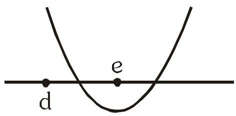


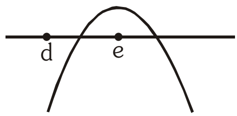
(b) Condition for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is **$a.f(d) < 0$**.





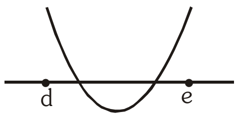
(c) Condition for exactly one root of $f(x) = 0$ to lie in the interval (d,e) i.e. $d < x < e$ is **$f(d).\ f(e) < 0$**

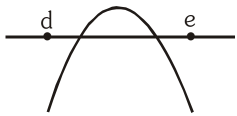




(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$ and $a.f(d) > 0$ & $af(e) > 0$ and $d < (-b/2a) < e$





7. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

8. THEORY OF EQUATIONS :

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ are the roots of the equation ;

$f(x) = a_0\,x^n + a_1\,x^{n-1} + a_2\,x^{n-2} + \ldots\ldots + a_{n-1}\,x + a_n = 0$

where a_0 , $a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3$$

$$= -\frac{a_3}{a_0}, \ldots\ldots, \alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Note :

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its constant term, when coefficient of highest degree term is (+)ve {If not then make it (+) ve}.
Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose constant term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

(iv) Rational root theorem : If a rational number $\frac{p}{q}$ $(p, q \in Z_0)$ is a root of polynomial equation with integral coefficient $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_0 = 0$, then p divides a_0 and q divides a_n.

SEQUENCE & SERIES

1. ARITHMETIC PROGRESSION (AP) :

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the **common difference**. If 'a' is the first term & 'd' is the common difference, then AP can be written as a, a + d, a + 2d, ..., a + (n – 1) d, ...

(a) n^{th} term of this AP $\boxed{T_n = a + (n - 1)d}$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]}$

where ℓ is the last term.

(c) Also n^{th} term $\boxed{T_n = S_n - S_{n-1}}$

Note :

(i) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(ii) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in n, in such case the coefficient of n is the common difference of the A.P. i.e. A

(iii) Three numbers in AP can be taken as a – d, a, a + d; four numbers in AP can be taken as a – 3d, a – d, a + d, a + 3d five numbers in AP are a – 2d, a – d, a, a + d, a + 2d & six terms in AP are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iv) If a, b, c are in A.P., then $b = \frac{a+c}{2}$

(v) If a_1, a_2, a_3....... and b_1, b_2, b_3 .... are two A.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$......... are also in A.P.

(vi) (a) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same common difference.

(b) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd & d/k respectively, where d is common difference of original A.P.

(vii) Any term of an AP (except the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2}, \ k < r$$

2. GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term $\boxed{T_n = a\, r^{n-1}}$

(b) Sum of the first n terms $\boxed{S_n = \frac{a(r^n - 1)}{r - 1}, \ \text{if } r \neq 1}$

(c) Sum of infinite GP when $|r| < 1$ $(n \to \infty, \ r^n \to 0)$

$$\boxed{S_\infty = \frac{a}{1 - r}; |r| < 1}$$

(d) If a, b, c are in GP $\Rightarrow b^2 = ac \Rightarrow$ loga, logb, logc, are in A.P.

Note :

(i) In a G.P. product of k^{th} term from beginning and k^{th} term from the last is always constant which equal to product of first term and last term.

(ii) Three numbers in **G.P.** : $\mathbf{a/r, \; a, \; ar}$

Five numbers in **G.P.** : $\mathbf{a/r^2, a/r, a, ar, ar^2}$

Four numbers in **G.P.** : $\mathbf{a/r^3, \; a/r, \; ar, \; ar^3}$

Six numbers in **G.P.** : $\mathbf{a/r^5, a/r^3, \; a/r, \; ar, \; ar^3, ar^5}$

(iii) If each term of a **G.P.** be raised to the same power, then resulting series is also a **G.P.**

(iv) If each term of a G.P. be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

(v) If a_1, a_2, a_3 ..... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ..... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ...... will also form a G.P. common ratio will be $r_1 r_2$ and $\frac{r_1}{r_2}$ respectively.

(vi) In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e. $\mathbf{T_r = \sqrt{T_{r-k}T_{r+k}}}$, $k < r$

(vii) If $\mathbf{a_1, a_2, a_3 ..... a_n}$ is a **G.P.** of **non zero, non negative terms,** then $\mathbf{\log a_1, \log a_2, ... \log a_n}$ is an **A.P.** and **vice-versa.**

3. HARMONIC PROGRESSION (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP. If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP & vice-versa. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ... \frac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero. If a, b, c are in

$$HP \Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{a}{c} = \frac{a-b}{b-c}$$

4. MEANS

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

n-arithmetic means between two numbers :

If a,b are any two given numbers & a, A_1, A_2, ..., A_n, b are in AP then A_1, A_2,...A_n are the n AM's between a & b, then

$A_1 = a + d$, $A_2 = a + 2d$,......, $A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b i.e. $\frac{a+b}{2}$

(b) Geometric mean (GM) :

If a, b, c are in GP, then b is the GM between a & c i.e. $b^2 = ac$, therefore $b = \sqrt{ac}$

n-geometric means between two numbers :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in GP then G_1, G_2, G_3 ,......G_n are n GMs between a & b.

$G_1 = ar$, $G_2 = ar^2$, ... $G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b i.e. $\sqrt{ab}$

(c) Harmonic mean (HM) :

If a, b, c are in HP, then b is HM between a & c, then $b = \frac{2ac}{a+c}$.

Important note :

(i) If A, G, H, are respectively AM, GM, HM between two positive number a & b then

(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$

(c) $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + \ldots.. + a_n}{n}$

$G = (a_1 a_2 \ldots\ldots.. a_n)^{1/n}$ and $H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots. + \frac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

5. ARITHMETICO - GEOMETRIC SERIES :

Sum of First n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a+d)r + (a+2d)r^2 + \ldots\ldots\ldots + [a + (n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]\ r^n}{1-r}$, $r \neq 1$

Sum to infinity :

If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

6. SIGMA NOTATIONS

Theorems :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$; where k is a constant.

7. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actually counting):

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

(a) Simultaneous occurrence of both events in a definite order is m x n. This can be extended to any number of events (known as multiplication principle).

(b) Happening of exactly one of the events is m + n (known as addition principle).

2. FACTORIAL :

A Useful Notation : $n! = n(n-1)(n-2) \ldots\ldots 3.\ 2.\ 1$;

$n! = n.\ (n-1)!$ where $n \in W$

$0! = 1! = 1$

$(2n)! = 2^n.\ n!\ [1.\ 3.\ 5.\ 7 \ldots\ldots (2n-1)]$

Note that :

(i) Factorial of negative integers is not defined.

(ii) Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots$$

3. PERMUTATION :

(a) ${}^nP_r$ denotes the number of permutations (arrangements) of n different things, taken r at a time ($n \in N$, $r \in W$, $n \geq r$)

$${}^nP_r = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}$$

(b) The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n-(p+q+r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

(c) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r!\ .\ {}^{n-1}C_{r-1}$

(d) The number of permutation of n different objects taken r at a time, when repetition be allowed any number of times is $n \times n \times n$ .... r times $= n^r$.

(e) **(i)** The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \dfrac{{}^nP_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\dfrac{(n-1)!}{2}$.

(ii) The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\dfrac{{}^nP_r}{r}$

4. COMBINATION :

(a) ${}^nC_r$ denotes the number of combinations (selections) of n different things taken r at a time, and ${}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{{}^nP_r}{r!}$ where $r \le n$; $n \in N$ and $r \in W$. ${}^nC_r$ is also denoted by $\dbinom{n}{r}$ or A_r^n or $C(n, r)$.

(b) The number of combination of n different things taking r at a time.

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

(c) Given n different objects , the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots.. + {}^nC_n = 2^n - 1$. This can also be stated as the total number of non-empty combinations of n distinct things.

(d) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +……things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\ldots\ldots\ldots -1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p + 1)(q + 1)(r + 1)\, 2^n - 1$

5. DIVISORS :

Let $N = p^a . q^b . r^c$ …….where p, q, r…….. are distinct primes & a, b, c……. are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is

$= (a + 1)(b + 1)(c + 1)\ldots\ldots.$

(b) The sum of these divisors is $(p^0 + p^1 + p^2 + \ldots.+ p^a)$

$(q^0 + q^1 + q^2 + \ldots.+ q^b)(r^0 + r^1 + r^2 + \ldots.+ r^c)\ldots$

(c) Number of ways in which N can be resolved as a product of two factors is

$\frac{1}{2}(a+1)(b+1)(c+1)\ldots\ldots$ if N is not a perfect square

$\frac{1}{2}\left[(a+1)(b+1)(c+1)\ldots\ldots+1\right]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other, is equal to 2^{n-1} where n is the number of different prime factors in N.

6. DIVISION INTO GROUPS AND DISTRIBUTION :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups containing m & n things respectively is : $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, then number of ways in which 2n distinct objects can be divided into two equal groups is $\frac{(2n)!}{n!\ n!\ 2!}$; as in any one way it is possible to inter change the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways = $\frac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of such grouping $= \frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each ,m groups containing q objects each is equal to $\frac{n!}{(p!)^{\ell}(q!)^{m}\ell!m!}$

Here $\ell p + mq = n$

(d) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is p^n

(e) Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.

7. DERANGEMENT :

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-........+(-1)^n\frac{1}{n!}\right]$$

8. IMPORTANT RESULT :

(a) Number of rectangles of any size in a square of size $n \times n$ is $\sum_{r=1}^{n} r^3$ & number of squares of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangles of any size in a rectangle of size $n \times p$ $(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ & number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$

(c) If there are n points in a plane of which m(<n) are collinear :

(i) Total number of lines obtained by joining these points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangle ${}^nC_3 - {}^mC_3$

(d) Maximum number of point of intersection of n circles is ${}^nP_2$ & n lines is ${}^nC_2$.

BINOMIAL THEOREM

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n$

$= \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$, where $n \in N$.

1. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}.\, y^r$

(b) Middle term :

The middle term (s) is the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\, x^{n/2}.\, y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term independent of x :

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

2. SOME RESULTS ON BINOMIAL COEFFICIENTS :

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + C_1 + C_2 + ... = C_n = 2^n$, $C_r = {}^nC_r$

(d) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$, $C_r = {}^nC_r$

(e) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$, $C_r = {}^nC_r$

3. Greatest coefficient & greatest term in expansion of $(x + a)^n$:

(a) If n is even, greatest binomial coefficient is ${}^nC_{n/2}$

If n is odd, greatest binomial coefficient is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \&\ T_{p+1} \ \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is an integer equal to p} \\ T_{q+1} \quad \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is non integer and} \in (q, q+1), q \in I \end{cases}$$

4. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in \mathbb{R}$, then $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$ ∞ provided $|x| < 1$.

Note :

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots \infty$

5. EXPONENTIAL SERIES :

(a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots\infty$; where x may be any real or complex number & $e = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$

(b) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + ...\infty$, where $a > 0$

6. LOGARITHMIC SERIES :

(a) $\ell n(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 < x \le 1$

(b) $\ell n(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....\infty$, where $-1 \le x < 1$

(c) $\ell n\frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right)$, $|x| < 1$

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COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form a + ib where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called real part of z (a = Re z) and 'b' is called imaginary part of z (b = Im z).

Every Complex Number Can Be Regarded As

- Purely real if b = 0
- Imaginary if $b \neq 0$
  - Purely imaginary if a = 0

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non-negative.

2. CONJUGATE COMPLEX :

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$ which is real

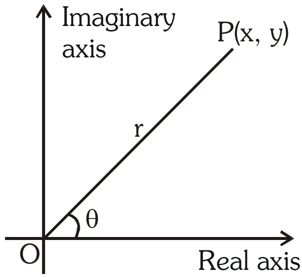
(iv) If z is purely real then $z - \bar{z} = 0$

(v) If z is purely imaginary then $z + \bar{z} = 0$

3. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x , y).



Length OP is called **modulus** of the complex number denoted by $|z|$ & θ is called the **principal argument or amplitude**, $(\theta \in (-\pi, \pi])$.

e.g. $|z| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis), $x > 0$

Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

(b) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as $\text{CiS}\,\theta$.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(c) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $\overline{(\overline{z})} = z$ **(b)** $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

(c) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$ **(d)** $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$

(e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$; $z_2 \neq 0$

(f) If f is a polynomial with real coefficient such that $f(\alpha + i\beta) = x + iy$, then $f(\alpha - i\beta) = x - iy$.

5. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$ **(b)** $|z| \geq \text{Re}(z)$ **(c)** $|z| \geq \text{Im}(z)$

(d) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$ **(e)** $z\overline{z} = |z|^2$ **(f)** $|z_1 z_2| = |z_1|.|z_2|$

(g) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$ **(h)** $|z^n| = |z|^n$

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}\left(z_1\overline{z}_2\right)$

or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(j) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(k) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangular Inequality]**

(l) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangular Inequality]**

(m) If $\left|z + \frac{1}{z}\right| = a$ $(a > 0)$, then $\max |z| = \frac{a + \sqrt{a^2 + 4}}{2}$

& $\min |z| = \frac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$

6. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) (i) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi$; $k \in I$

(ii) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$; $k \in I$

(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$,
where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(b) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}(z)$

7. DE'MOIVER'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of roots of a complex quantity should be determined using theory of equation.

8. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1 , $\omega = \dfrac{-1 + i\sqrt{3}}{2} = e^{i2\pi/3}$

& $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2} = e^{i4\pi/3}$

(b) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1 + \omega^r + \omega^{2r} = \begin{cases} 0, & r \text{ is not integral multiple of } 3 \\ 3, & r \text{ is multiple of } 3 \end{cases}$$

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

9. SQUARE ROOT OF COMPLEX NUMBER :

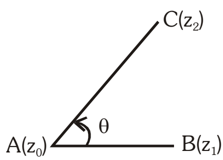
$$\sqrt{a + ib} = \pm \left\{ \sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}} \right\} \text{ for } b > 0$$

& $\pm \left\{ \sqrt{\dfrac{|z| + a}{2}} - i\sqrt{\dfrac{|z| - a}{2}} \right\}$ for $b < 0$ where $|z| = \sqrt{a^2 + b^2}$.

10. ROTATION :

$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|} e^{i\theta}$$

Take θ in anticlockwise direction



11. GEOMETRY IN COMPLEX NUMBER :

(a) Distance formula : $|z_1 - z_2|$ = distance between the points z_1 & z_2 on the Argand plane.

(b) Section formula : If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m+n}$ divides the join of z_1 & z_2 in the ratio m : n.

(c) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$
- Orthocentre of the Δ ABC

$$= \frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

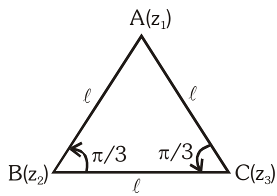
or $\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$

- Incentre of the $\Delta ABC = \frac{(az_1 + bz_2 + cz_3)}{(a+b+c)}$
- Circumcentre of the ΔABC

$$= \frac{(z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$$

11. RESULT RELATED WITH TRIANGLE :

(a) Equilateral triangle :



$$\frac{z_1 - z_2}{\ell} = \frac{z_3 - z_2}{\ell} e^{i\pi/3} \quad \text{......(i)}$$

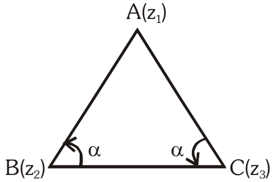
Also $\frac{z_2 - z_3}{\ell} = \frac{z_1 - z_3}{\ell} . e^{i\pi/3}$ ......(ii) from (i) & (ii)

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\text{or} \quad \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

(b) Isosceles triangle :

If



then $4\cos^2\alpha\,(z_1 - z_2)(z_3 - z_1) = (z_3 - z_2)^2$

(c) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

12. EQUATION OF LINE THROUGH POINTS z_1 & z_2 :

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\bar{a}z + a\bar{z} + b = 0}$ where $a \in C$ & $b \in R$.

Note :

(i) Complex slope of line joining points z_1 & z_2 is $\dfrac{(z_2 - z_1)}{\overline{(z_2 - z_1)}}$. Also note that slope of a line in Cartesian plane is different from complex slope of a line in Argand plane.

(ii) Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\dfrac{a}{\bar{a}}$, $b \in R$

(iii) Two lines with complex slope μ_1 & μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$.

(iv) Length of perpendicular from point $A(\alpha)$ to line $\bar{a}z + a\bar{z} + b = 0$ is $\dfrac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

13. EQUATION OF CIRCLE :

(a) Circle whose centre is z_0 & radius = r

$|z - z_0| = r$

(b) General equation of circle is

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$

centre '$-a$' & radius = $\sqrt{|a|^2 - b}$

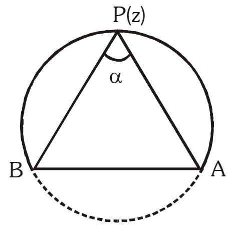
(c) Diameter form $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

or $\quad \arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\dfrac{\pi}{2}$

(d) Equation $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.

(e) Equation $|z - z_1|^2 + |z - z_2|^2 = k$

represent circle if $k \geq \dfrac{1}{2}|z_1 - z_2|^2$



(f) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha \qquad 0 < \alpha < \pi,\ \alpha \neq \dfrac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

14. STANDARD LOCI :

(a) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) if $2k > |z_1 - z_2|$ $\Rightarrow$ An ellipse

(ii) If $2k = |z_1 - z_2|$ $\Rightarrow$ A line segment

(iii) If $2k < |z_1 - z_2|$ $\Rightarrow$ No solution

(b) Equation $||z - z_1| - |z - z_2|| = 2k$ (a constant) represent

(i) If $2k < |z_1 - z_2|$ $\Rightarrow$ A hyperbola

(ii) If $2k = |z_1 - z_2|$ $\Rightarrow$ A line ray

(iii) $2k > |z_1 - z_2|$ $\Rightarrow$ No solution

DETERMINANT

1. MINORS :

The minor of a given element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have " 9 minors".

2. COFACTORS :

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element : $C_{ij} = (-1)^{i+j}. M_{ij}$;

Important Note :

Consider $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_2 be cofactor of b_2 and so on, then,

(i) $a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \ldots\ldots\ldots\ldots\ldots = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = \ldots\ldots\ldots\ldots = 0$

3. PROPERTIES OF DETERMINANTS:

(a) The value of a determinants remains unaltered, if the rows & corresponding columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

(c) If a determinant has any two rows (or columns) identical or in same proportion, then its value is zero.

(d) If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

(e) $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) e.g.

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D' = D.$$

Note : While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when x = a, then x – a is a factor of Δ.

Again, if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r; r = 1, 2, 3 are three differentiable functions.

then $$\frac{d}{dx}D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$$

4. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D' is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D' = D^{n-1}$

5. SPECIAL DETERMINANTS :

(a) Symmetric Determinant :

Elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(b) Skew Symmetric Determinant :

If $a_{ij} = -a_{ji}$ then the determinant is said to be a skew symmetric determinant. Here all the principal diagonal elements are zero. The value of a skew symmetric determinant of odd order is zero and of even order is perfect square.

e.g. $$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

(c) Other Important Determinants :

(i) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

6. SYSTEM OF EQUATION :

(a) System of equation involving two variable :

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Consistent (System of equation has solution)

- **unique solution**: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or $a_1b_2 - a_2b_1 \neq 0$ (Equations represents intersecting lines)
- **Infinite solution**: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Equations represents coincident lines)

Inconsistent (System of equation has no solution)

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Equations represents parallel disjoint lines)

If $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$

(b) System of equations involving three variables :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

To solve this system we first define following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is used to solve the system.

Check value of Δ

- $\Delta \neq 0$ → Consistent system and has unique solution $x = \dfrac{\Delta_1}{\Delta},\ y = \dfrac{\Delta_2}{\Delta},\ z = \dfrac{\Delta_3}{\Delta}$
- $\Delta = 0$ → Check the values of Δ_1, Δ_2 and Δ_3
  - Atleast one of Δ_1, Δ_2 and Δ_3 is not zero → Inconsistent system
  - $\Delta_1 = \Delta_2 = \Delta_3 = 0$ → Put z = t and solve any two equations to get the values of x & y in terms of t
    - If these values of x, y and z in terms of t satisfy third equation → Consistent system (System has infinite solutions)
    - The values of x, y, z doesn't satisfy third equation → Inconsistent system

Note :

(i) **Trivial solution :** In the solution set of system of equation if all the variables assumes zero, then such a solution set is called Trivial solution otherwise the solution is called non-trivial solution.

(ii) If $d_1 = d_2 = d_3 = 0$ then system of linear equation is known as system of Homogeneous linear equation which always posses atleast one solution (0, 0, 0).

(iii) If system of homogeneous linear equation posses non-zero/non-trivial solution then $\Delta = 0$.

In such case given system has infinite solutions.

MATRICES

1. INTRODUCTION :

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m ×n matrix.
In compact form, the matrix is represented by $A=[a_{ij}]_{m\times n}$.

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \ldots\ldots\ldots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : $(A = O_{m\times n})$, An m × n matrix whose all entries are zero.

(d) Horizontal Matrix : A matrix of order m × n is a horizontal matrix if n > m.

(e) Vertical Matrix : A matrix of order m × n is a vertical matrix if m > n.

(f) Square Matrix : (Order n) If number of rows = number of column, then matrix is a square matrix.

Note :

(i) The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

(ii) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn}$ are called Diagonal Elements. The line along which the diagonal elements lie is called "Principal or Leading diagonal. "The quantity $\sum a_{ii}$ = trace of the matrix written as, t_r (A)

3. SQUARE MATRICES :

SQUARE MATRICES

↓

Triangular Matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{pmatrix}$$

Upper Triangular $a_{ij} = 0 \ \forall \ i > j$

Lower Triangular $a_{ij} = 0 \ \forall \ i < j$

Diagonal Matrix denoted as
$A = \text{diag}(a_{11}, a_{22} \ldots\ldots, a_{nn})$
where $a_{ij} = 0$ for $i \neq j$

Scalar Matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \longrightarrow$$

If $a_{11} = a_{22} = a_{33} = a$

Unit or Identity Matrix

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If $a_{11} = a_{22} = a_{33} = 1$

Note :

(i) Minimum number of zeros in triangular matrix of order n
$= n(n - 1)/2$.

(ii) Minimum number of zeros in a diagonal matrix of order n
$= n(n - 1)$.

(iii) Null square matrix is also a diagonal matrix.

4. EQUALITY OF MATRICES :

Matrices $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

(a) both have the same order.

(b) $a_{ij} = b_{ij}$ for each pair of i & j .

5. ALGEBRA OF MATRICES :

(I) Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) Addition of matrices is commutative : $A + B = B + A$

(b) Matrix addition is associative : $(A + B) + C = A + (B + C)$

(c) $A + O = O + A$ (Additive identity)

(d) $A + (-A) = (-A) + A = O$ (Additive inverse)

(II) Multiplication of A Matrix By A Scalar :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

(III) Multiplication of matrices (Row by Column) :

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is $m \times p$

& $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

(IV) Properties of Matrix Multiplication :

(a) $AB = O \not\Rightarrow A = O$ or $B = O$ (in general)

Note :

If A and B are two non-zero matrices such that $AB = O$, then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = -BA$ then A and B are said to anticommute

(b) Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then $(AB)C = A(BC)$

(c) Distributivity :

$$\left.\begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned}\right] \text{Provided A,B \& C are conformable}$$

for respective products

(V) Positive Integral Powers of A square matrix :

(a) $A^m A^n = A^{m+n}$

(b) $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^m = I$, $m,n \in N$

6. CHARACTERISTIC EQUATION :

Let A be a square matrix. Then the polynomial in x, $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called characteristic equation of A.

7. CAYLEY - HAMILTON THEOREM :

Every square matrix A satisfy its characteristic equation i.e. $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$ is the characteristic equation of matrix A, then $a_0A^n + a_1A^{n-1} + \ldots\ldots + a_{n-1}A + a_nI = 0$

8. TRANSPOSE OF A MATRIX : (Changing rows & columns)

Let A be any matrix of order $m \times n$. Then transpose of A is A^T or A' of order $n \times m$ and $(A^T)_{ij} = (A)_{ji}$.

Properties of transpose :

If A^T & B^T denote the transpose of A and B

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A\,B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, where k is a scalar.

General : $(A_1 . A_2, \ldots\ldots A_n)^T = A_n^T . \ldots\ldots . A_2^T . A_1^T$
(reversal law for transpose)

9. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $A A^T = I$

Note :

(i) The determinant value of orthogonal matrix is either 1 or −1. Hence orthogonal matrix is always invertible

(ii) $AA^T = I = A^T A$ Hence $A^{-1} = A^T$.

10. SOME SPECIAL SQUARE MATRICES :

(a) Idempotent Matrix : A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following :

(i) $A^n = A \ \forall \ n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1

(iii) If idempotent matrix is invertible then it will be an identity matrix i.e. I.

(b) Periodic Matrix : A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) Nilpotent Matrix : A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$.

Note that a nilpotent matrix will not be invertible.

(d) Involutary Matrix : If $A^2 = I$, the matrix is said to be an involutary matrix.

Note that $A = A^{-1}$ for an involutary matrix.

(e) If A and B are square matrices of same order and AB = BA then

$$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \ldots\ldots\ldots + {}^nC_nB^n$$

11. SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

For symmetric matrix **$A = A^T$** i.e. $a_{ij} = a_{ji}\ \forall$ i,j.

Note : Maximum number of distinct entries in any symmetric matrix of order n is $\dfrac{n(n+1)}{2}$.

(b) Skew symmetric matrix :

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $A^T = -A$ i.e. $a_{ij} = -a_{ji}\ \forall$ i & j. Hence if A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0\ \forall$ i.

Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

(c) Properties of symmetric & skew symmetric matrix :

(i) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(ii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iii) If A & B are symmetric matrices then,

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(iv) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and $$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

12. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then $(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

= Transpose of cofactor matrix.

Note :

If A be a square matrix of order n, then

(i) $A(\text{adj } A) = |A|\, I_n = (\text{adj } A) . A$

(ii) $|\text{adj } A| = |A|^{n-1}$, $n \geq 2$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, $|A| \neq 0$.

(iv) $\text{adj } (AB) = (\text{adj } B) (\text{adj } A)$

(v) $\text{adj } (KA) = K^{n-1} (\text{adj } A)$, where K is a scalar

13. INVERSE OF A MATRIX (Reciprocal Matrix) :

A square matrix A said to be invertible (non singular) if there exists a matrix B such that, $AB = I$ (Note that $AB = I \Leftrightarrow BA = I$)
B is called the inverse (reciprocal) of A and is denoted by A^{-1} . Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A|\, I_n$

$\Rightarrow A^{-1}. A(\text{adj } A) = A^{-1} I_n |A|$

$\Rightarrow I_n(\text{adj } A) = A^{-1} |A| I_n$

$\therefore A^{-1} = \dfrac{(\text{adj } A)}{|A|}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$

Theorem : If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Note :

(i) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible ,

(a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k};\ k \in N$

(iii) $|A^{-1}| = \dfrac{1}{|A|}$.

14. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

Example :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow AX = B \quad \Rightarrow A^{-1}AX = A^{-1}B$, if $|A| \neq 0$.

$$\Rightarrow X = A^{-1}B = \frac{\text{Adj } A}{|A|}.B$$

Note :

(i) If $|A| \neq 0$, system is consistent having unique solution

(ii) If $|A| \neq 0$ & (adj A) . B $\neq$ O(Null matrix), system is consistent having unique non-trivial solution.

(iii) If $|A| \neq 0$ & (adj A) . B = O (Null matrix), system is consistent having trivial solution.

(iv) If $|A| = 0$, then **matrix method fails**

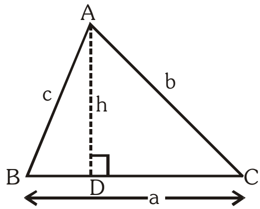
If (adj A) . B = O (null matrix) $\downarrow$ Infinite solutions or no solution

If (adj A) . B $\neq$ O $\downarrow$ Inconsistent (no solution)

PROPERTIES AND SOLUTIONS OF TRIANGLE

1. SINE FORMULAE :

In any triangle ABC



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

2. COSINE FORMULAE :

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

(b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. PROJECTION FORMULAE :

(a) $b \cos C + c \cos B = a$

(b) $c \cos A + a \cos C = b$

(c) $a \cos B + b \cos A = c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(b) $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(c) $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

5. HALF ANGLE FORMULAE :

$$s=\frac{a+b+c}{2} = \text{semi-perimeter of triangle.}$$

(a) (i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ **(ii)** $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ **(ii)** $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$

(ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\frac{\Delta}{s(s-b)}$

(iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{\Delta}{s(s-c)}$

(d) Area of Triangle

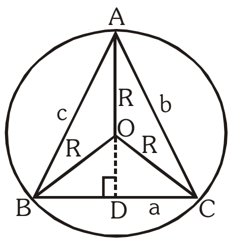
$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

$$=\frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}$$

6. RADIUS OF THE CIRCUMCIRCLE 'R' :

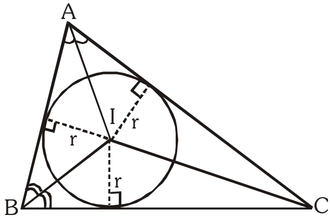
Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.



$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}.$$

7. RADIUS OF THE INCIRCLE 'r' :

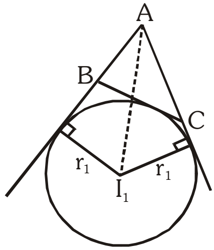
Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

8. RADII OF THE EX-CIRCLES :



Point of intersection of two external angle and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to angle A of ΔABC and so on then :

(a) $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(b) $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

(c) $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

9. LENGTH OF ANGLE BISECTOR, MEDIANS & ALTITUDE :

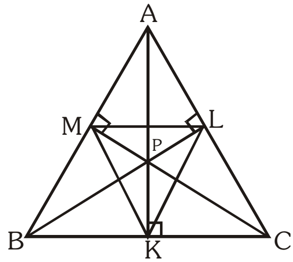
If m_a, β_a & h_a are the lengths of a median, an angle bisector & altitude from the angle A then,

$$\frac{1}{2}\sqrt{b^2+c^2+2bc\cos A} = m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$, $h_a = \frac{a}{\cot B + \cot C}$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$

10. ORTHOCENTRE AND ORTHIC TRIANGLE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the orthic triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of orthocentre from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB

(d) The sides of the orthic triangle are a cos A (= R sin 2A), bcosB(= R sin2B) and c cos C (=R sin 2C) and its angles are $\pi - 2A$, π -2B and $\pi - 2C$

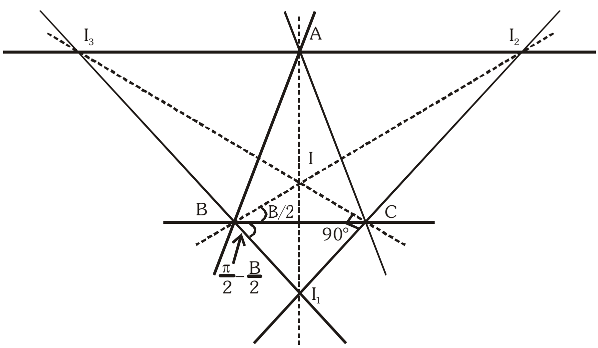
(e) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

(f) Area of orthic triangle = $2\Delta\cos A\cos B\cos C$

$$= \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C$$

(g) Circumradii of orthic triangle = R/2

11. EX-CENTRAL TRIANGLE :



(a) The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

(b) Incentre I of ΔABC is the orthocentre of the excentral $\Delta\, I_1I_2I_3$.

(c) ΔABC is the orthic triangle of the $\Delta\, I_1I_2I_3$.

(d) The sides of the excentral triangle are

$$4R\cos\frac{A}{2},\ \ 4R\cos\frac{B}{2} \text{ and } 4R\cos\frac{C}{2}$$

and its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(e) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre

$$= R\sqrt{1-8\cos A\cos B\cos C}$$

(b) The distance between circumcentre and incentre $= \sqrt{R^2-2Rr}$

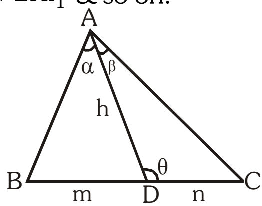
(c) The distance between incentre and orthocentre

$$= \sqrt{2r^2-4R^2\cos A\cos B\cos C}$$

(d) The distance between circumcentre & excentre is

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2+2Rr_1} \text{ \& so on.}$$

13. m-n THEOREM :



$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

$(m + n)\cot\theta = n\cot B - m\cot C.$

14. IMPORTANT POINTS :

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = bcosB, then the triangle is isosceles or right angled.

(b) In Right Angle Triangle :

(i) $a^2 + b^2 + c^2 = 8R^2$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle :

(i) $R = 2r$

(ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$

(iv) area $= \frac{\sqrt{3}a^2}{4}$

(v) $R = \frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) at mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1, except in case of equilateral triangle. In equilateral triangle all these centres coincide.

15. REGULAR POLYGON :

Consider a 'n' sided regular polygon of side length 'a'

(a) Radius of incircle of this polygon $r = \frac{a}{2}\cot\frac{\pi}{n}$

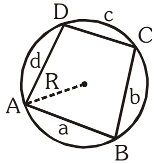
(b) Radius of circumcircle of this polygon $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

(c) Perimeter & area of regular polygon

Perimeter = na = $2nr\tan\frac{\pi}{n} = 2nR\sin\frac{\pi}{n}$

Area $= \frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{4}na^2\cot\frac{\pi}{n}$

16. CYCLIC QUADRILATERAL :



(a) Quadrilateral ABCD is cyclic if $\angle A + \angle C = \pi$ $= \angle B + \angle C$

(opposite angle are supplementary angles)

(b) Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a + b + c + d$

(c) $\cos B = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$ & similarly other angles

(d) Ptolemy's theorem : If ABCD is cyclic quadrilateral, then AC . BD = AB . CD + BC . AD

17. SOLUTION OF TRIANGLE :

Case-I : Three sides are given then to find out three angles use

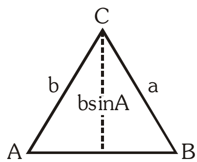
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac} \quad \& \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Case-II : Two sides & included angle are given :

Let sides a, b & angle C are given then use $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$ and find value of A – B ....(i)

& $\dfrac{A+B}{2} = 90° - \dfrac{C}{2}$ ....(ii) $\quad c = \dfrac{a\sin C}{\sin A}$ ....(iii)

Case-III :



Two sides a, b & angle A opposite to one of them is given

(a) If $a < b\sin A$ No triangle exist

(b) If $a = b\sin A$ & A is acute, then one triangle exist which is right angled.

(c) $a > b\sin A$, $a < b$ & A is acute, then two triangles exist

(d) $a > b\sin A$, $a > b$ & A is acute, then one triangle exist

(e) $a > b\sin A$ & A is obtuse, then there is one triangle if $a > b$ & no triangle if $a < b$.

Note : Case-III can be analysed algebraically using Cosine rule as $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, which is quadratic in c.

18. ANGLES OF ELEVATION AND DEPRESSION :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

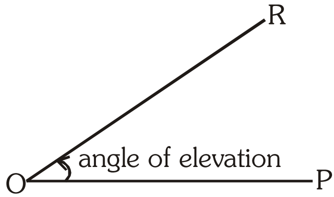


Fig. (a)

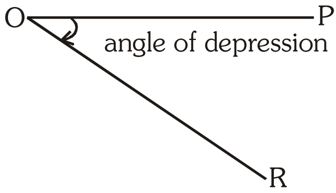


Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

STRAIGHT LINE

1. RELATION BETWEEN CARTESIAN CO-ORDINATE & POLAR CO-ORDINATE SYSTEM

If (x,y) are Cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and AC $\neq$ BD; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; AC $\neq$ BD; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

3. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m : n is given by :

(a) For internal division : $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

(b) For external division : $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

(c) Line $ax + by + c = 0$ divides line joining points $P(x_1,y_1)$ & $Q(x_2, y_2)$ in ratio $= -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

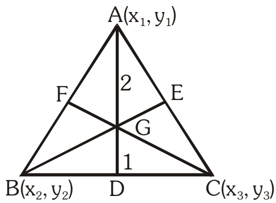
4. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

(a) Centroid :

(i) The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices).

(ii) Centroid divides the median in the ratio of 2 : 1.

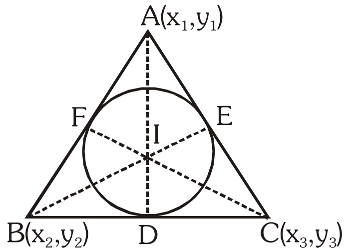


(iii) Co-ordinates of centroid $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

(iv) If P is any internal point of triangle such that area of ΔAPB, ΔAPC and ΔBPC are same then P must be centroid.

(b) Incenter :

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of a circle touching all the sides of a triangle.



Co-ordinates of incenter $I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Where a, b, c are the sides of triangle ABC.

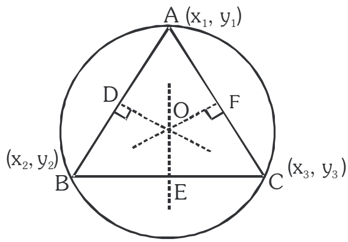
Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a, (c+a):b, (a+b):c$

(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$.



Also it is a centre of a circle touching all the vertices of a triangle.

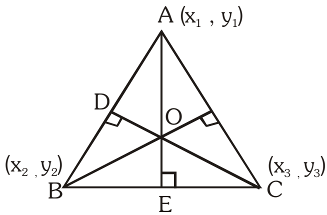
Note :

(i) If a triangle is right angle, then its circumcenter is mid point of hypotenuse.

(ii) Find perpendicular bisector of any two sides and solve them to find circumcentre.

(d) Orthocenter :

It is the point of intersection of perpendicular drawn from vertices on opposite sides of a triangle and can be obtained by solving the equation of any two altitudes.



Note :

If a triangle is right angled triangle, then orthocenter is the point where right angle is formed.

Remarks :

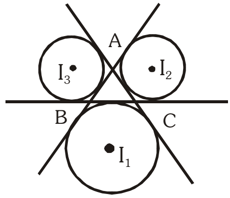
(i) If the triangle is equilateral, then centroid, incentre, orthocenter and circumcenter coincides.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio $2:1$

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

(e) Ex-centers :

The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are



$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

5. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]|$$

To remember the above formula, take the help of the following method :

$$=\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) Area of Equilateral triangle

If altitude of any equilateral triangle is P, then its area = $\frac{P^2}{\sqrt{3}}$.

If 'a' be the side of equilateral triangle, then its area = $\left(\frac{a^2\sqrt{3}}{4}\right)$

(iii) Area of quadrilateral whose consecutive vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) & (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

6. CONDITION OF COLLINEARITY FOR THREE POINTS :

Three points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if any one of the given point lies on the line passing through the remaining two points. Thus the required condition is -

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1} \quad \text{or} \quad \frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

7. EQUATION OF STRAIGHT LINE :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & b $\neq 0$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance a is $y = a$ or $y = -a$

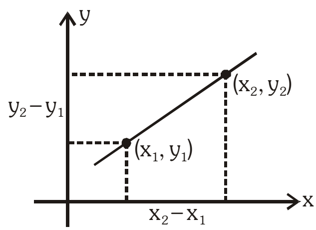
(b) Equation of x-axis is $y = 0$

(c) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

8. SLOPE OF LINE :

If a given line makes an angle θ $(0° \leq \theta < 180°,\ \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $m=\tan\theta$. Obviously the slope of the x-axis and line parallel to it is zero and y-axis and line parallel to it does not exist.



If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB $= \frac{y_2 - y_1}{x_2 - x_1}$

9. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. then the equation of this straight line is written as : $\mathbf{y = mx + c}$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $\mathbf{y - y_1 = m(x-x_1)}$

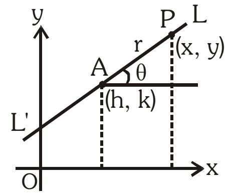
(c) Two point form : Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) Intercept form : If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\mathbf{\frac{x}{a} + \frac{y}{b} = 1}$

(e) Normal form : If p is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive), where $0 \le \alpha < 2\pi$.

(f) Parametric form : To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r then $x - h = r\cos\theta$, $y - k = r\sin\theta$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'.

Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

(g) General form : We know that a first degree equation in x and y, **ax + by + c = 0** always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept made by this line on x-axis $= -\frac{c}{a}$ and intercept made by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

10. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) Intersecting $\Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

11. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1,y_1) on the line ax+by+c=0 is $= \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

In particular the length of the perpendicular from the origin on the line ax + by + c = 0 is $P = \frac{|c|}{\sqrt{a^2+b^2}}$

12. DISTANCE BETWEEN TWO PARALLEL LINES :

(a) The distance between two parallel lines $ax + by + c_1=0$ and $ax+by+c_2=0$ is $= \dfrac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$.

13. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

14. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y=mx+c$ is written as :

$$\mathbf{y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)}$$

15. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

16. CONCURRENCY OF LINES :

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

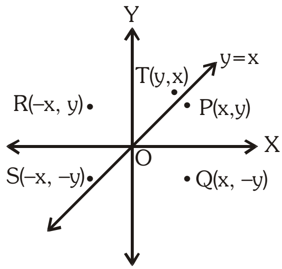
are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note :

If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}.

17. REFLECTION OF A POINT :

Let P(x,y) be any point, then its image with respect to



(a) x-axis is Q(x, –y)

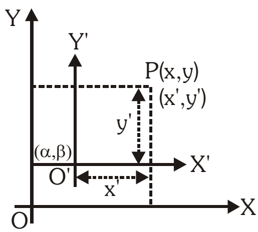
(b) y-axis is R(–x, y)

(c) origin is S(–x, –y)

(d) line y=x is T(y, x)

18. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :

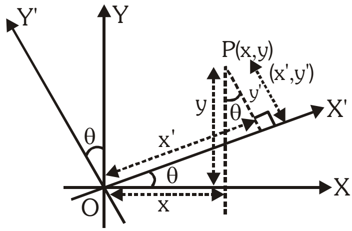


If coordinates of any point P(x, y) with respect to new origin (α, β) will be (x', y') then $x = x' + \alpha$, $\quad y = y' + \beta$

or $\quad x' = x - \alpha$, $\quad y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) Rotation of axes without shifting the origin :



Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$

then $\quad x = x' \cos\theta - y' \sin\theta$

$\qquad\quad y = x' \sin\theta + y' \cos\theta$

and $\quad x' = x\cos\theta + y\sin\theta$

$\qquad\quad y' = -x\sin\theta + y\cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

19. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad ....(1)$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (1)

(b) Equation of bisector of acute/obtuse angles :

See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by −1 to make the constant terms positive

Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

20. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x+b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

21. GENERAL EQUATION AND HOMOGENEOUS EQUATION OF SECOND DEGREE :

(a) A general equation of second degree $\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represent a pair of straight lines if $\Delta = abc+2fgh-af^2 - bg^2 - ch^2 = 0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(b) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are

(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(ii) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2^{nd} degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y=\left(\frac{-h\pm\sqrt{h^2-ab}}{b}\right)x \;\equiv y=m_1x \;\&\; y=m_2x$$

and $m_1+m_2=-\frac{2h}{b}$; $m_1m_2=\frac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angles to each other is $a+b=0$. i.e. co-efficient of x^2 + co-efficient of $y^2=0$.

(ii) Coincident is $h^2=ab$.

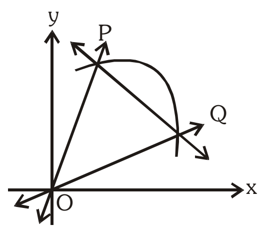
(iii) Equally inclined to the axis of x is $h=0$. i.e. coeff. of $xy=0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$, $a\neq b$, $h\neq 0$.

(e) Pair of straight lines perpendicular to the lines $ax^2+2hxy+by^2=0$ and through origin are given by $bx^2-2hxy+ay^2=0$.

(f) If lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are parallel then distance between them is $=2\sqrt{\frac{g^2-ac}{a(a+b)}}$

22. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :



Let the equation of curve be :

$ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(i)

and straight line be

$\ell x+my+n=0$ ......(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by -

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

23. STANDARD RESULTS :

(a) Area of rhombus formed by lines $a|x|+b|y|+c=0$ or $\pm ax \pm by + c = 0$ is $\frac{2c^2}{|ab|}$.

(b) Area of triangle formed by line $ax+by+c=0$ and axes is $\frac{c^2}{2|ab|}$.

(c) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax+by+c=0$ is given by $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$

(d) Image of point (x_1, y_1) w.r. to the line $ax+by+c=0$ is given by

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

CIRCLE

1. DEFINITION :

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

2. STANDARD EQUATIONS OF THE CIRCLE :

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is $\mathbf{(x-h)^2 + (y-k)^2 = r^2}$

(b) General equation of circle :

$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$, where g, f, c are constants and centre is (–g, –f)

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

Note : The general quadratic equation in x and y,

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :

(i) coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$

(ii) coefficient of $xy = 0$ or $h = 0$

(iii) $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on:

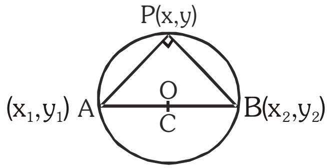
(i) x-axis $= 2\sqrt{g^2 - c}$ (ii) y-axis $= 2\sqrt{f^2 - c}$

Note :

Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.

(d) Diameter form of circle :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle then the equation of the circle is given by



$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

(e) The parametric forms of the circle :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are

$$x = r\cos\theta,\ y = r\sin\theta\ ;\ \theta \in [0, 2\pi)$$

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is

$x = h + r\cos\theta,\ y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

are $x = -g + \sqrt{g^2 + f^2 - c}\ \cos\theta,\ y = -f + \sqrt{g^2 + f^2 - c}\ \sin\theta$

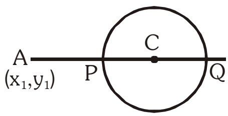
where θ is parameter.

Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

3. POSITION OF A POINT W.R.T CIRCLE :

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1,y_1) then :



Point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >$ or $=$ or < 0.

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

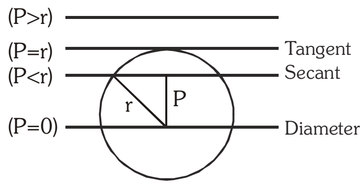
(c) The power of point is given by S_1.

4. TANGENT LINE OF CIRCLE :

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.



(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1,y_1) on the circle $x^2+ y^2 = a^2$ is $\mathbf{xx_1 + yy_1 = a^2}$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is $\mathbf{x\cos t + y\sin t = a}$

(2) The point of intersection of the tangents at the points P(α) and Q(β) is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y+y_1) + c = 0}$

(iv) If line y = mx + c is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact points are $\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is

$\mathbf{y = mx \pm a\sqrt{1+m^2}}$

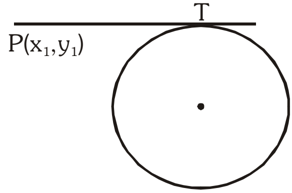
(v) The equation of tangent with slope m of the circle $(x - h)^2 + (y - k)^2 = a^2$ is

$\mathbf{(y - k) = m(x - h) \pm a\sqrt{1+m^2}}$

Note :

To get the equation of tangent at the point $(x_1\ y_1)$ on any curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\frac{x+x_1}{2}$ in place of x, $\frac{y+y_1}{2}$ in place of y, $\frac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$) :



The length of tangent drawn from point (x_1,y_1) out side the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

(d) Equation of Pair of tangents ($SS_1 = T^2$) :

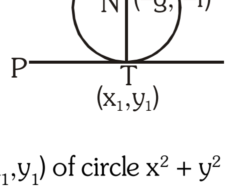
Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2 \text{ or } SS_1 = T^2$$

5. NORMAL OF CIRCLE :

Normal at a point of the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

(a) Equation of normal at point (x_1, y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

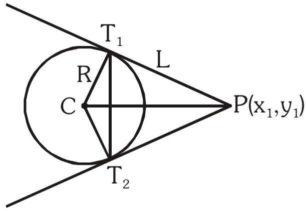


$$y - y_1 = \left(\frac{y_1+f}{x_1+g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$.

6. CHORD OF CONTACT :

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :



$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. T = 0 same as equation of tangent).

7. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$) :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $\mathbf{y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)}$.

This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

which is designated by $\mathbf{T = S_1}$.

8. DIRECTOR CIRCLE :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let the circle be $x^2 + y^2 = a^2$, then the equation of director circle is $x^2 + y^2 = 2a^2$.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

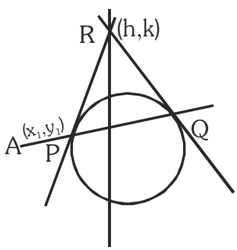
Note :

The director circle of

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

9. POLE AND POLAR :

Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle S =0 in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of



point R is called polar of the point A and point A is called the pole, with respect to the given circle.

The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1+yy_1+g(x+x_1) + f(y+y_1)+c=0$

Pole of a given line with respect to a circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

10. FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

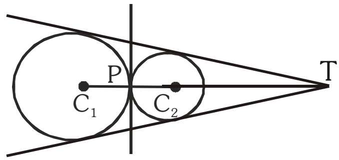
(d) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

11. DIRECT AND TRANSVERSE COMMON TANGENTS :

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

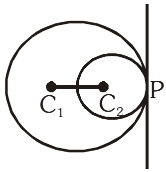
(a) Both circles will touch :

(i) Externally if $C_1C_2 = r_1 + r_2$, point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).



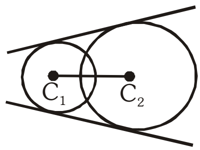
In this case there are **three common tangents.**

(ii) Internally if $C_1C_2 = |r_1 - r_2|$, point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**



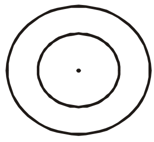
(b) The circles will intersect :

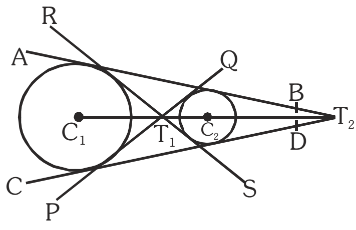
when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**



(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.





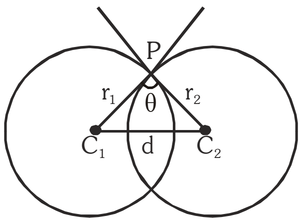
(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.

Note : Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

12. THE ANGLE OF INTERSECTION OF TWO CIRCLES :

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles.



then $\cos\theta = \dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}$ or $\boxed{\cos\theta = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)}$

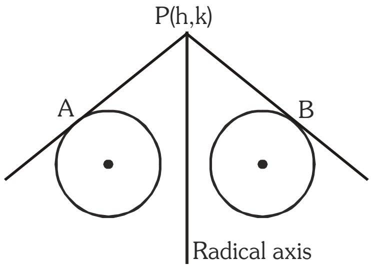
Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is

$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

13. RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$) :

Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal is called the radical axis. If two circles are -



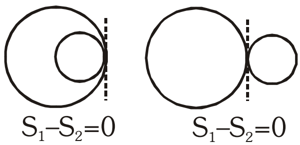
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1 y + c_1 = 0$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

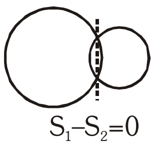
Then the equation of radical axis is given by $\mathbf{S_1 - S_2 = 0}$

Note :

(i) If two circles touches each other then common tangent is radical axis.



(ii) If two circles cuts each other then common chord is radical axis.



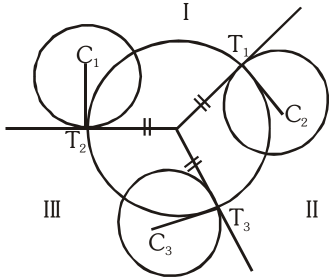
(iii) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

14. Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

Note :



(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

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PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the FOCUS.

(b) The fixed straight line is called the DIRECTRIX.

(c) The constant ratio is called the ECCENTRICITY denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the AXIS.

(e) A point of intersection of a conic with its axis is called a VERTEX.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

Case (ii) When the focus does not lie on the directrix :

The conic represents :

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; D \neq 0$ $h^2 = ab$	$0 < e < 1; D \neq 0$ $h^2 < ab$	$D \neq 0\ ; e > 1$ $h^2 > ab$	$e > 1\ ; D \neq 0$ $h^2 > ab\ ;\ a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4\ ax$. For this parabola :

(i) Vertex is (0, 0) **(ii)** Focus is (a, 0)

(iii) Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE with respect to axis as diameter.

(d) Latus rectum :

A focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

(i) Length of the latus rectum = 4a.

(ii) Length of the semi latus rectum = 2a.

(iii) Ends of the latus rectum are L(a, 2a) & L'(a, – 2a)

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

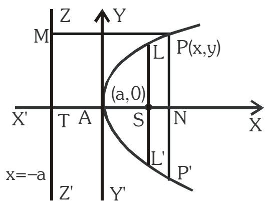
(iii) Two parabolas are said to be equal if they have latus rectum of same length.

5. PARAMETRIC REPRESENTATION :

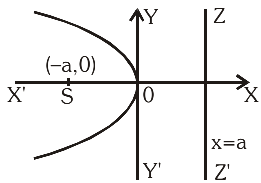
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. TYPE OF PARABOLA :

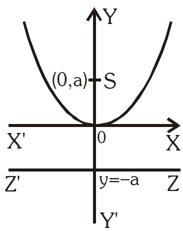
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



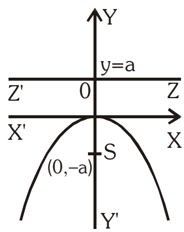
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Fous	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2,2at)$	x+a
$y^2 =-4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2,2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2=4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a–h=0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x–h+a
$(x-p)^2=4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y–q+b

7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm$

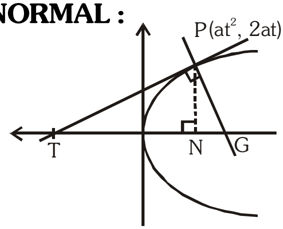
$\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : length of the focal chord making an angle α with the x-axis is $4a\,\text{cosec}^2\,\alpha$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semi latus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$yy_1 = 2a(x + x_1)$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form :

Equation of tangent to the given parabola at its point P(t), is -

$ty = x + at^2$

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points)

12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

13. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$, where :

$S \equiv y^2 - 4ax$; $S_1 \equiv y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

14. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

15. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

This reduced to $T = S_1$

where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

16. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola $y^2 = 4ax$ is $y = 2a/m$, where m = slope of parallel chords.

17. CONORMAL POINTS :

Foot of the normals of three concurrent normals are called conormals point.

(i) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(ii) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

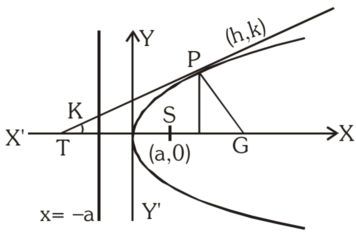
(iii) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(iv) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(v) If three normals are drawn from point (h, 0) on parabola $y^2 = 4ax$, then $h > 2a$ and one of the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the



parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord

i.e. $2a = \dfrac{2bc}{b+c}$ or $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a}$.

(f) Image of the focus lies on directrix with respect to any tangent of parabola $y^2 = 4ax$.

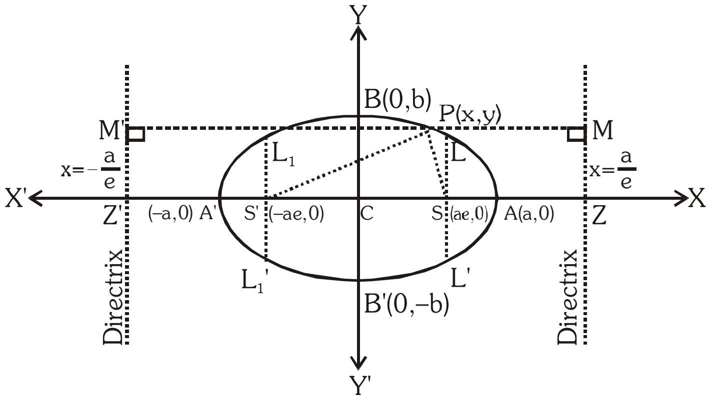
ELLIPSE

1. STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axis along the co-ordinate axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where a > b & $b^2 = a^2(1-e^2)$

$\Rightarrow a^2 - b^2 = a^2 e^2$.

where e = eccentricity (0 < e < 1).



FOCI : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(a) Equation of directrices :

$$x=\frac{a}{e} \;\;\&\;\; x=-\frac{a}{e}.$$

(b) Vertices :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (Z)** $\left(\pm\frac{a}{e}, 0\right)$.

(d) Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0,-b)$ & $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axis : The major & minor axis together are called **Principal Axis** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate** with respect to major axis as diameter.

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$$

(ii) Equation of latus rectum : $x = \pm\, ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$,

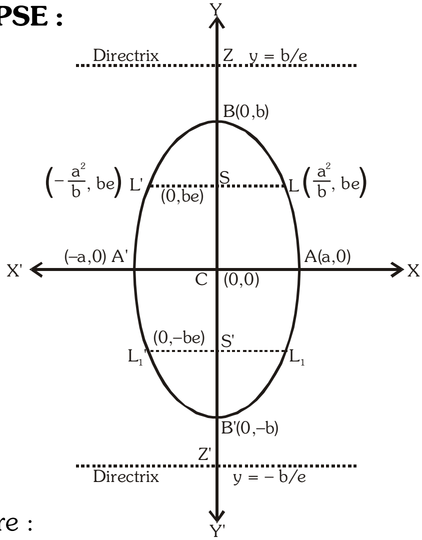
$L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) Focal radii : $SP = a - ex$ and $S'P = a + ex$
$\Rightarrow$ $SP + S'P = 2a$ = Major axis.

(l) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

2. ANOTHER FORM OF ELLIPSE :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \ (a<b)$$



(a) AA' = Minor axis = 2a

(b) BB' = Major axis = 2b

(c) $a^2 = b^2(1-e^2)$

(d) Latus rectum

$$LL' = L_1L_1' = \frac{2a^2}{b}.$$

equation $y = \pm be$

(e) Ends of the latus rectum are :

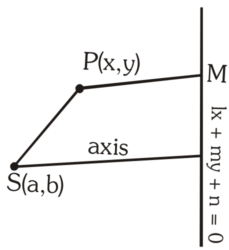
$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) Equation of directrix $y = \pm \dfrac{b}{e}$.

(g) Eccentricity : $e = \sqrt{1-\dfrac{a^2}{b^2}}$

3. GENERAL EQUATION OF AN ELLIPSE :

Let (a,b) be the focus S, and $lx + my + n = 0$ is the equation of directrix. Let P(x,y) be any point on the ellipse. Then by definition.



$\Rightarrow$ SP = e PM (e is the eccentricity)

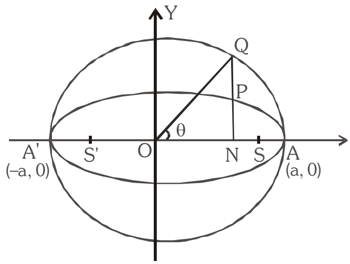
$$\Rightarrow (x-a)^2 + (y-b)^2 = e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$$

$$\Rightarrow (l^2+m^2)\{(x-a)^2+(y-b)^2\} = e^2\{lx+my+n\}^2$$

4. POSITION OF A POINT W.R.T. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 > <$ or $= 0$.

5. AUXILIARY CIRCLE/ECCENTRIC ANGLE :



A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse ($0 \le \theta < 2\pi$).

Note that $\frac{l(PN)}{l(QN)}=\frac{b}{a}=\frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6. PARAMATRIC REPRESENTATION :

The equations $x = a \cos\theta$ & $y = b \sin\theta$ together represent the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a \cos\theta, b \sin\theta)$ is on the ellipse then ;

$Q(\theta) \equiv (a \cos\theta, a \sin\theta)$ is on the auxiliary circle.

7. LINE AND AN ELLIPSE :

The line y = mx + c meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two real points, coincident or imaginary according as c^2 is $<$ = or $>$ $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

8. TANGENT TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) Point form :

Equation of tangent to the given ellipse at its point (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

(b) Slope form :

Equation of tangent to the given ellipse whose slope is 'm', is

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

Point of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(c) Parametric form :

Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

9. NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) Point form : Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(b) Slope form : Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

(c) Parametric form : Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\text{cosec}\,\theta = (a^2 - b^2)$.

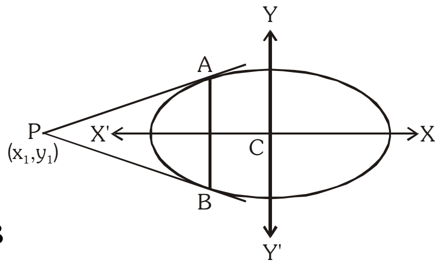
10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

then the equation of the chord of contact AB is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1,y_1)

11. PAIR OR TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

i.e. $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

12. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

13. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

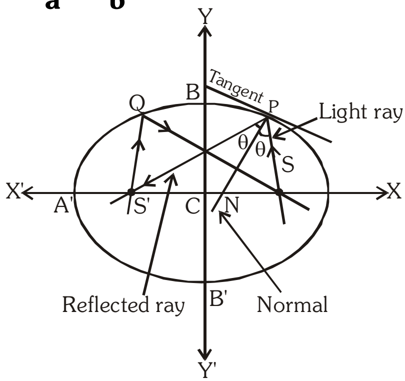
whose mid-point be (x_1,y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

14. IMPORTANT HIGHLIGHTS for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(I) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa.



(II) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(III) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha+\beta+\gamma+\delta=(2n+1)\pi$.

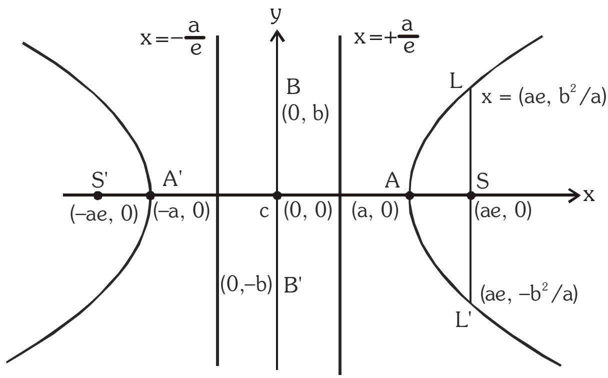
(IV) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are four concyclic points then sum of their eccentric angles is even multiple of π.
i.e. $\alpha+\beta+\gamma+\delta=2n\pi$

(V) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle.

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. ($e > 1$).

1. STANDARD EQUATION & DEFINITION(S) :



Standard equation of the hyperbola is $\mathbf{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$,

where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \left(\dfrac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \dfrac{a}{e}$ & $x = -\dfrac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \dfrac{2b^2}{a} = \dfrac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a\,(e^2 - 1)$

$= 2e$(distance from focus to directrix)

(iii) Ends : $\left(ae, \dfrac{b^2}{a}\right), \left(ae, \dfrac{-b^2}{a}\right)$; $\left(-ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axis of the hyperbola.

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left|\,|PS| - |PS'|\,\right| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axis of one hyperbola are respectively the conjugate & the transverse axis of the other are called **Conjugate Hyperbolas** of each other. eg. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ &

$-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

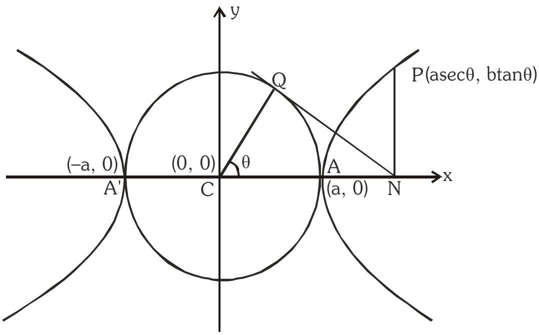
(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**.

Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of it's latus rectum is equal to it's transverse or conjugate axis.

4. AUXILIARY CIRCLE :



A circle drawn with centre C & transverse axis as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive , zero or negative according as the point (x_1, y_1) lies within , upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is $\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1 y_1)$ to the hyperbola.

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = mx \pm\sqrt{a^2m^2 - b^2}$. Point of contact are

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2-b^2}}\right)$$

Note that there are two parallel tangents having the same slope m.

(c) Parametric form : Equation of the tangent to the given hyperbola at the point (a sec θ , b tan θ) is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1.$$

Note : Point of intersection of the tangents at θ_1 & θ_2 is

$$x=a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\quad y=b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$$

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(a) Point form : Equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2=a^2e^2$.

(b) Slope form : The equation of normal of slope m to the given hyperbola is $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are

$$\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}},\mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$$

(c) Parametric form : The equation of the normal at the point P (a sec θ , b tan θ) to the given hyperbola is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2.$$

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2+y^2=a^2-b^2$.

If $b^2<a^2$ this circle is real ; if $b^2=a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T = 0$ at (x_1,y_1)

11. PAIR OR TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and a pair of tangents PA, PB can be drawn to it from P. Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

i.e. $$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

12. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

whose mid-point be (x_1,y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $$\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$$

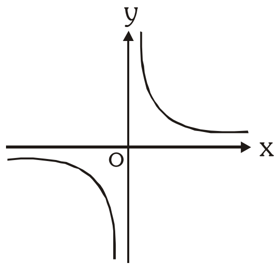
13. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

Combined equation of asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

14. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1+t_2)$ with slope $m = \frac{-1}{t_1t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$

& at $P(t)$ is $\frac{x}{t}+ty = 2c$.

(d) Equation of normal is $y-\frac{c}{t}=t^2(x-ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

15. IMPORTANT HIGHLIGHTS :

(i) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii.

(ii) Reflection property of the hyperbola : An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

FUNCTION

1. DEFINITION :

If to every value (considered as real unless other-wise stated) of a variable x, which belongs to a set A, there corresponds one and only one finite value of the quantity y which belong to set B, then y is said to be a function of x and written as $f : A \rightarrow B$, $y = f(x)$, x is called argument or independent variable and y is called dependent variable.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x) = y}$

y is called the image of x and x is the pre-image of y, under mapping f.

Every function $f : A \rightarrow B$ satisfies the following conditions.

(i) $f \subset A \times B$ **(ii)** $\forall a \in A \quad \exists \quad b \in B$ such that $(a, b) \in f$ and

(iii) If $(a, b) \in f$ & $(a, c) \in f \quad \Rightarrow \quad b = c$

2. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, then the set A is known as the domain of 'f' & the set B is known as co-domain of 'f'. The set of all f images of elements of A is known as the range of 'f'. Thus

Domain of $f = \{x \mid x \in A, (x, f(x)) \in f\}$

Range of $f = \{f(x) \mid x \in A, f(x) \in B\}$

Range is a subset of co-domain.

3. IMPORTANT TYPES OF FUNCTION :

(a) Polynomial function :

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots. a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note :

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax,\ a \neq 0$

(ii) There are four polynomial functions, satisfying the relation ; $f(x).\ f(1/x) = f(x) + f(1/x)$. They are :

(a) $f(x) = x^n + 1,\ n \in \mathbb{N}$

(b) $f(x) = 1 - x^n,\ n \in \mathbb{N}$

(c) $f(x) = 0$

(d) $f(x) = 2$

(iii) Domain of a polynomial function is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebraic function :

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

(c) Rational function :

A rational function is a function of the form $y = f(x) = \frac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$,

Domain : $R - \{x \mid h(x) = 0\}$

Any rational function is automatically an algebraic function.

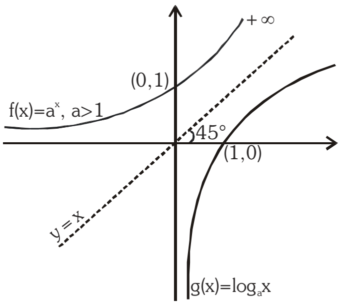
(d) Exponential and Logarithmic Function :

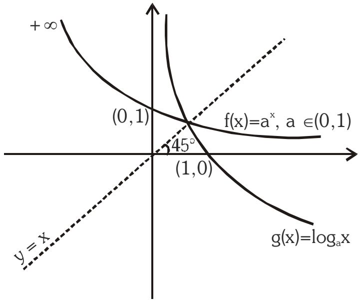
A function $f(x) = a^x (a > 0),\ a \neq 1,\ x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (Functions are mirror image of each other about the line $y = x$)

Domain of a^x is R **Range** R^+

Domain of $\log_a x$ is R^+ **Range** R

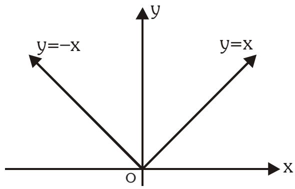




(e) Absolute value function :

It is defined as : $y = |x|$

$$= \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



Also defined as max{x, –x}

Domain : R **Range :** $[0, \infty)$

Note : $f(x)=\dfrac{1}{|x|}$ **Domain :** R–{0} **Range :** R^+

Properties of modulus function :

For any x, y, a ∈ R.

(i) $|x| \geq 0$

(ii) $|x| = |-x|$

(iii) $|xy| = |x|\,|y|$

(iv) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$; $y \neq 0$

(v) $|x| = a \Rightarrow x = \pm a,\ a > 0$

(vi) $\sqrt{x^2} = |x|$

(vii) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$. where a is positive.

(viii) $|x| \leq a \Rightarrow x \in [-a, a]$. where a is positive

(ix) $|x| > |y| \Rightarrow x^2 > y^2$

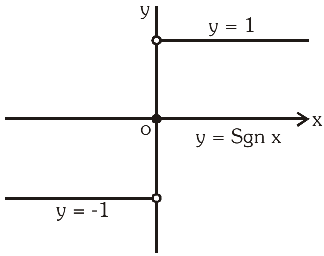
(x) $\big||x|-|y|\big| \leq |x+y| \leq |x|+|y|$

Note that (a) $|x|+|y|=|x+y| \Rightarrow xy \geq 0$

(b) $|x|+|y|=|x-y| \Rightarrow xy \leq 0$

(f) Signum function :

Signum function y = sgn (x) is defined as follows



$$y = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

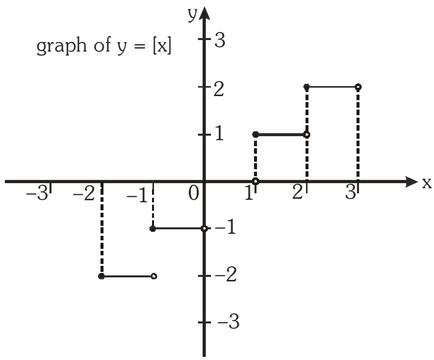
Domain : R

Range: {–1, 0, 1}

(g) Greatest integer or step up function :

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1



Domain : R

Range : I

Properties of greatest integer function :

(i) $x - 1 < [x] \le x < [x] + 1,\ 0 \le x - [x] < 1$

(ii) $[x + y] = \begin{cases} [x] + [y] & , \{x\} + \{y\} \in [0, 1) \\ [x] + [y] + 1 & , \{x\} + \{y\} \in [1, 2) \end{cases}$

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

(iii) $\{x\} + \{-x\} = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$

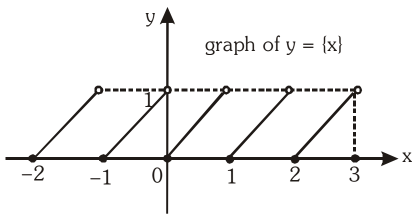
Note : $f(x) = \dfrac{1}{[x]}$

Domain : R − [0, 1) **Range :** $\{x \mid x = \frac{1}{n},\ n \in I - \{0\}\}$

(h) Fractional part function :

It is defined as : g(x) = {x} = x − [x] e.g.

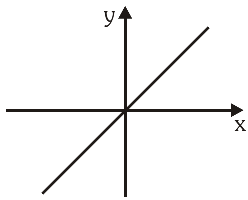
x	{x}
[−2,−1)	x+2
[−1,0)	x+1
[0,1)	x
[1,2)	x−1



Domain : R **Range :** [0,1) **Period :** 1

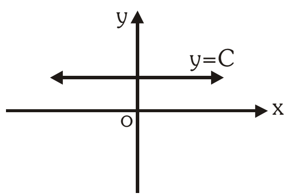
Note : $f(x)=\dfrac{1}{\{x\}}$ **Domain :** R − I **Range :** (1, ∞)

(i) Identity function :



The function $f : A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity function on A and is denoted by I_A.

(j) Constant function :



$f : A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c,\ \forall\ x \in A$, $c \in B$ is constant function.

Domain : R **Range :** {c}

(k) Trigonometric functions :

(i) Sine function : $f(x) = \sin x$

Domain : R **Range :** $[-1, 1]$, period 2π

(ii) Cosine function : $f(x) = \cos x$

Domain : R **Range :** $[-1, 1]$, period 2π

(iii) Tangent function : $f(x) = \tan x$

Domain : $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R, period π

(iv) Cosecant function : $f(x) = \operatorname{cosec} x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : $R - (-1, 1)$, period 2π

(v) Secant function : $f(x) = \sec x$

Domain : $R - \{x \mid x = (2n + 1)\pi/2 : n \in I\}$

Range : $R - (-1, 1)$, period 2π

(vi) Cotangent function : $f(x) = \cot x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : R, period π

(l) Inverse Trigonometric function :

(i) $f(x) = \sin^{-1} x$ **Domain :** $[-1, 1]$ **Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $f(x) = \cos^{-1} x$ **Domain :** $[-1, 1]$ **Range :** $[0, \pi]$

(iii) $f(x) = \tan^{-1} x$ **Domain :** R **Range :** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $f(x) = \cot^{-1} x$ **Domain :** R **Range :** $(0, \pi)$

(v) $f(x) = \operatorname{cosec}^{-1} x$ **Domain :** $R - (-1, 1)$ **Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $f(x) = \sec^{-1} x$ **Domain :** $R - (-1, 1)$ **Range :** $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

4. EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

5. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, f + g, f − g, (f . g) & (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$, domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$, domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, domain $A \cap B - \{x \mid g(x) = 0\}$

6. CLASSIFICATION OF FUNCTIONS :

(a) One-One function (Injective mapping) :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note:

(i) Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

(iii) Non-monotonic function can also be injective.

(b) Many-one function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one if $\exists\ x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Note : If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

Total number of functions
= number of one-one functions + number of many-one function

(c) Onto function (Surjective) :

If range = co-domain, then f(x) is onto.

(d) Into function :

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Note :

(i) If 'f' is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) If $f : R \to R$ is a polynomial
- (a) Of even degree, then it will neither be injective nor surjective.
- (b) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

7. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION :

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function gof : $A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'. $x \longrightarrow \boxed{f} \longrightarrow \boxed{g} \longrightarrow g(f(x))$

Properties of composite functions:

(a) In general composite of functions is not commutative i.e. $gof \neq fog$.

(b) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

(d) If gof is one-one function then f is one-one but g may not be one-one.

8. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$, then $f(x, y)$ is homogeneous function of degree n.

9. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

10. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equation $x^3 + y^3 = 1$ defines y as an implicit function of x. If y has been expressed in terms of x alone then it is called an **Explicit function.**

11. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall\, x \in A$ & $y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) \mid (x, f(x)) \in f\}$

Properties of inverse function :

(a) The inverse of a bijection is unique.

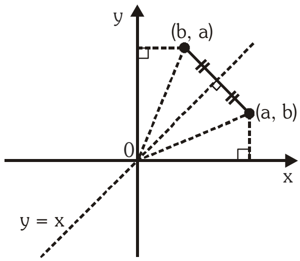
(b) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.

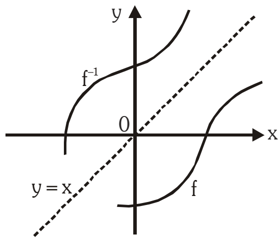
(c) The inverse of a bijection is also a bijection.

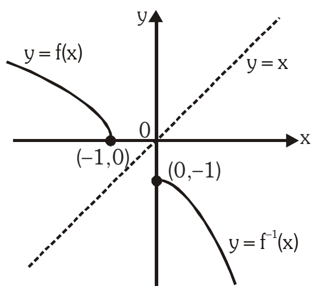
(d) If f & g are two bijections $f : A \to B$, $g : B \to C$ & gof exist, then the inverse of gof also exists and $(gof)^{-1} = f^{-1}og^{-1}$.

(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.

In general $f(x) = x \Rightarrow f(x) = f^{-1}(x)$.







The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

12. ODD & EVEN FUNCTIONS :

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

$f(-x) = f(x)$, then it is an even function

and if $f(-x) = -f(x)$, then it is an odd function

Note :

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many-one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function.

e.g. $f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$

(v) The only function which is defined on the entire number line & even and odd at the same time is f(x) = 0

(vi) If f(x) and g(x) both are even or both are odd then the function f(x) . g(x). will be even but if any one of them is odd & other is even, then f.g will be odd.

13. PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a positive number T(T >0) called the period of the function such that f(x + T) = f(x) = f(x – T), for all values of x within the domain of f(x) and least positive T if exist called fundamental period.

Note :

(i) Inverse of a periodic function does not exist.

(ii) Every constant function is periodic, with no fundamental period.

(iii) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$ (here period means fundamental period).

(iv) If f(x) has period p and g(x) has period q, then one of the period of f(x) + g(x) will be LCM of p & q. However it may not be fundamental period.

(v) If f(x) has period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ (provided each one is defined over some non empty set) also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period T/a (a > 0).

(vii) $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x|$ & $|\operatorname{cosec} x|$ are periodic function with period π.

(viii) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\operatorname{cosec}^n x$, are periodic function with period 2π when 'n' is odd or π when n is even.

(ix) $\tan^n x$, $\cos^n x$ are periodic function with period π.

14. GENERAL :

If x, y are independent variables and f is continuous function, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\ x$

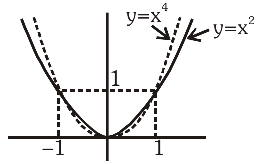
(b) $f(xy) = f(x)\ .\ f(y) \Rightarrow f(x) = x^n,\ n \in R$ or $f(x) = 0$

(c) $f(x + y) = f(x)\ .\ f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

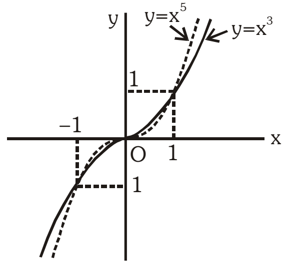
(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

15. SOME BASIC FUNCTION & THEIR GRAPH :

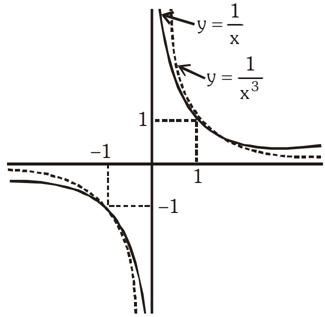
(a) $y = x^{2n}$, where $n \in N$



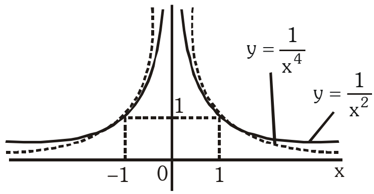
(b) $y = x^{2n+1}$, where $n \in N$



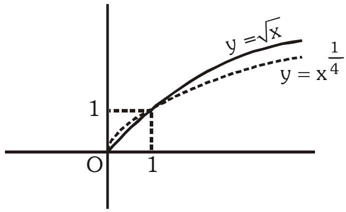
(c) $y = \dfrac{1}{x^{2n-1}}$, where $n \in N$



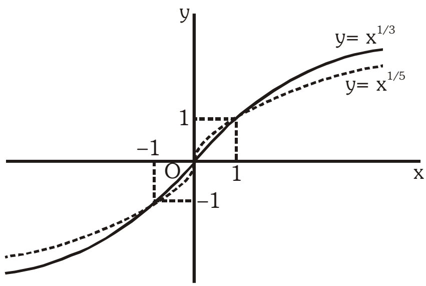
(d) $y = \dfrac{1}{x^{2n}}$, where $n \in N$



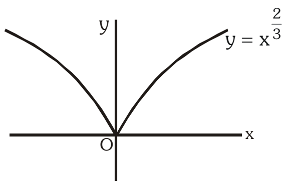
(e) $y = x^{\frac{1}{2n}}$, where $n \in N$



(f) $y = x^{\frac{1}{2n+1}}$, where $n \in N$

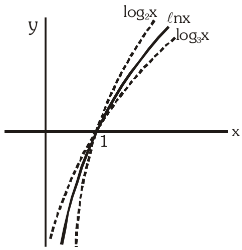


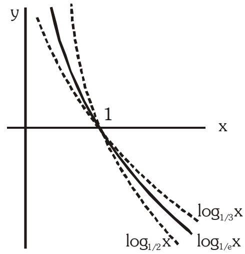
Note : $y = x^{2/3}$



(g) $y = \log_a x$

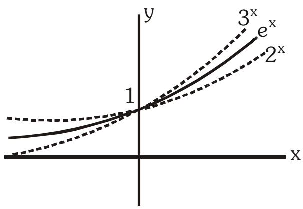
when $a > 1$ when $0 < a < 1$





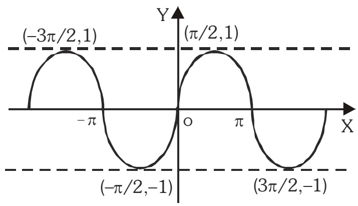
(h) $y = a^x$

$a > 1$ $0 < a < 1$

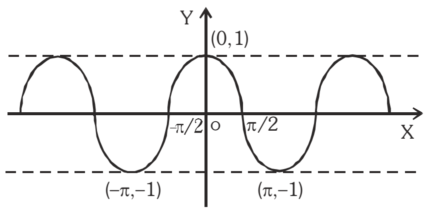


(i) Trigonometric functions :

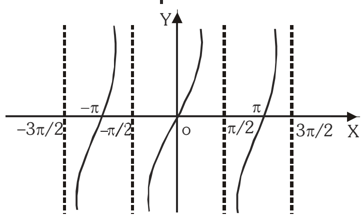
$y = \sin x$



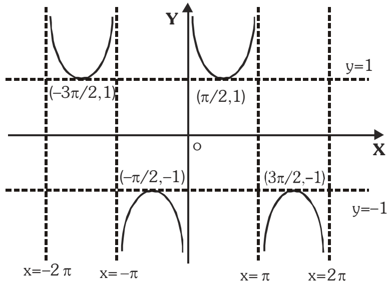
$y = \cos x$



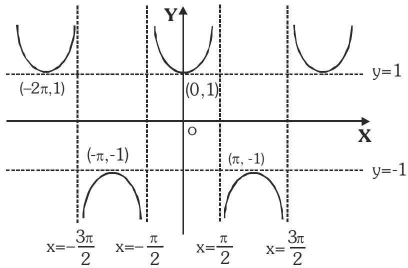
$y = \tan x$



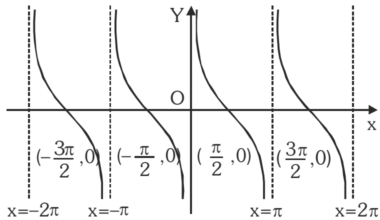
$y = \text{cosec}x$



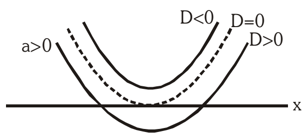
$y = \sec x$



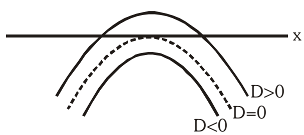
$y = \cot x$



(j) $y = ax^2 + bx + c$



vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$



where $D = b^2 - 4ac$

16. TRANSFORMATION OF GRAPH :

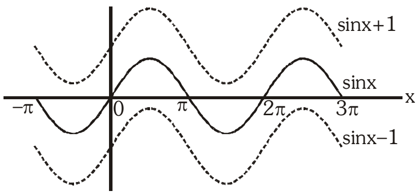
(a) when f (x) transforms to f (x) + k

if k > 0 then shift graph of f (x) upward through k

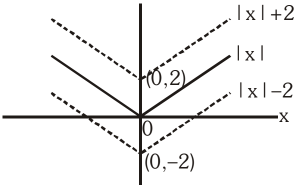
if k < 0 then shift graph of f (x) downward through k

Examples :

1.



2.



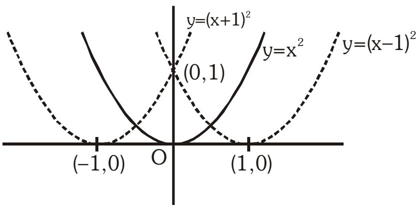
(b) f (x) transforms to f (x + k) :

if k > 0 then shift graph of f (x) through k towards left.

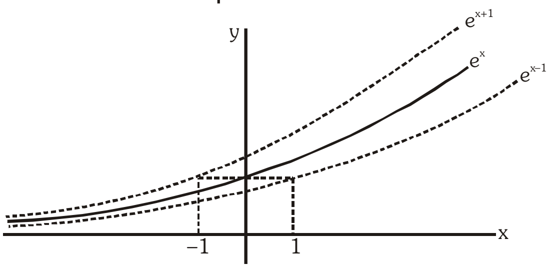
if k < 0 then shift graph of f (x) through k towards right.

Examples :

1.



2.

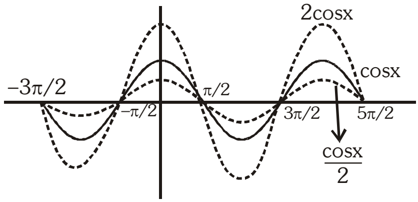


(c) f (x) transforms to kf (x) :

if k > 1 then stretch graph of f (x) k times along y-axis

if 0 < k < 1 then shrink graph of f (x), k times along y-axis

Examples :

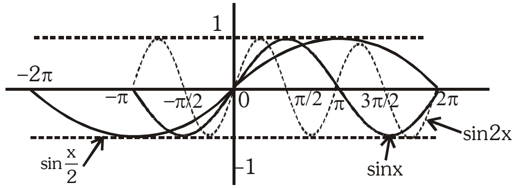


(d) f (x) transforms to f (kx) :

if k > 1 then shrink graph of f (x), 'k' times along x-axis

if 0 < k < 1 then stretch graph of f (x), 'k' times along x-axis

Examples :

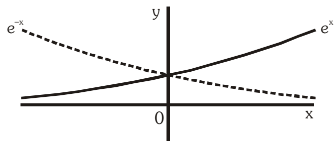


(e) f (x) transforms to f (–x) :

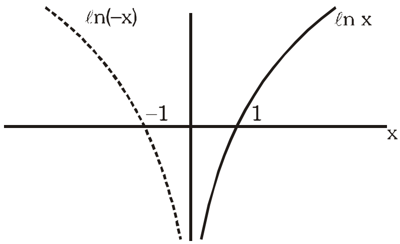
Take mirror image of the curve y = f (x) in y-axis as plane mirror

Example :

1.



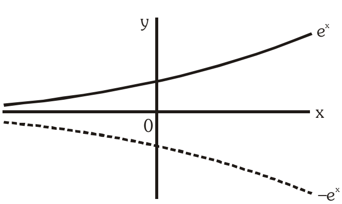
2.

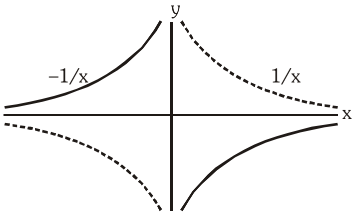


(f) f (x) transforms to $-f$ (x) :

Take image of y = f (x) in the x axis as plane mirror

Examples :

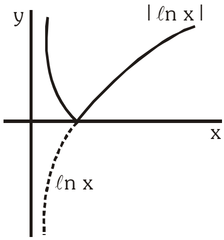


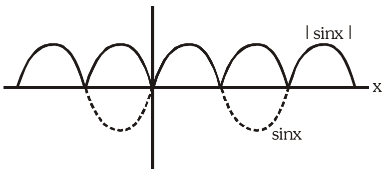


(g) f(x) transforms to $|f(x)|$:

Take mirror image (in a axis) of the portion of the graph of f(x) which lies below x-axis.

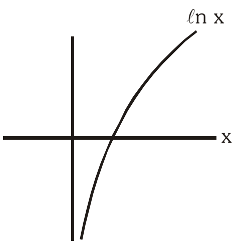
Examples :

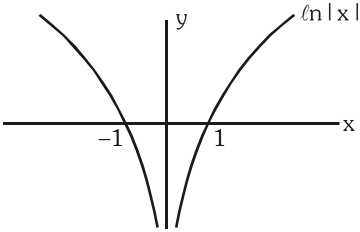


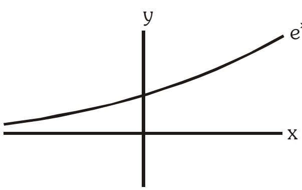


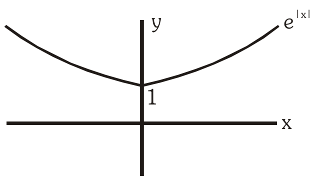
(h) f(x) transforms to $f(|x|)$:

Neglect the curve for x < 0 and take the image of curve for x ≥ 0 about y-axis.





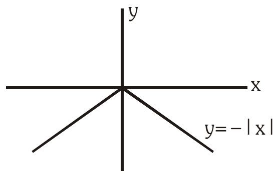


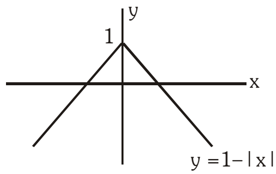


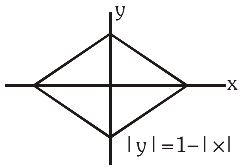
(i) $y = f(x)$ transforms to $|y| = f(x)$:

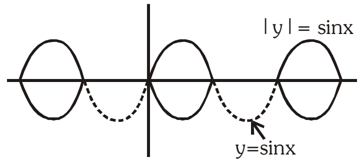
Remove the portion of graph which lies below x-axis & then take mirror image {in x axis} of remaining portion of graph

Examples :







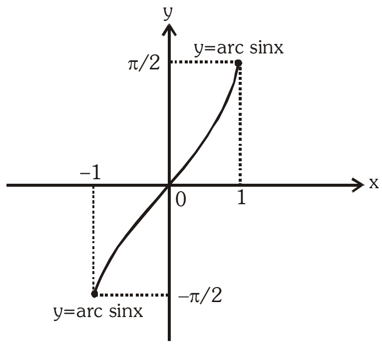


INVERSE TRIGONOMETRIC FUNCTION

1. DOMAIN, RANGE & GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f^{-1} : [-1,\ 1] \rightarrow [-\pi/2,\ \pi/2]$,

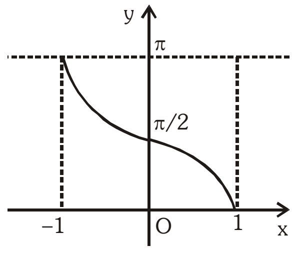
$f^{-1}(x) = \sin^{-1}(x)$



($y = \sin^{-1}x$)

(b) $f^{-1} : [-1,\ 1] \rightarrow [0,\ \pi]$,

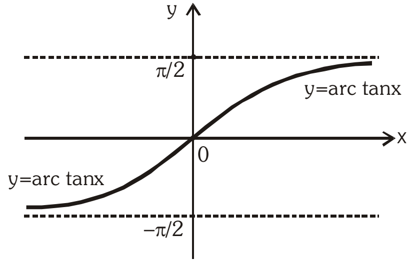
$f^{-1}(x) = \cos^{-1} x$



($y = \cos^{-1}x$)

(c) $f^{-1} : R \rightarrow (-\pi/2,\ \pi/2)$,

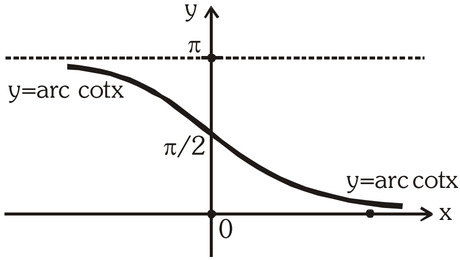
$f^{-1}(x) = \tan^{-1} x$



($y = \tan^{-1}x$)

(d) $f^{-1} : R \to (0, \pi)$,

$f^{-1}(x) = \cot^{-1} x$

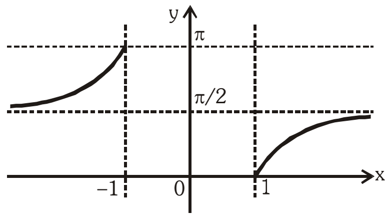


$(y = \cot^{-1}x)$

(e) $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [0, \pi/2) \cup (\pi/2, \pi]$,

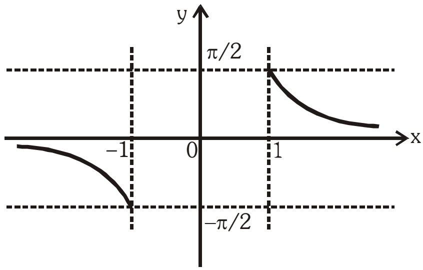
$f^{-1}(x) = \sec^{-1} x$



(f) $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [-\pi/2, 0) \cup (0, \pi/2]$,

$f^{-1}(x) = \operatorname{cosec}^{-1} x$



2. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P-1 :

(i) $y = \sin(\sin^{-1}x) = x$, $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

(ii) $y = \cos(\cos^{-1} x) = x$, $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

(iii) $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

(iv) $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

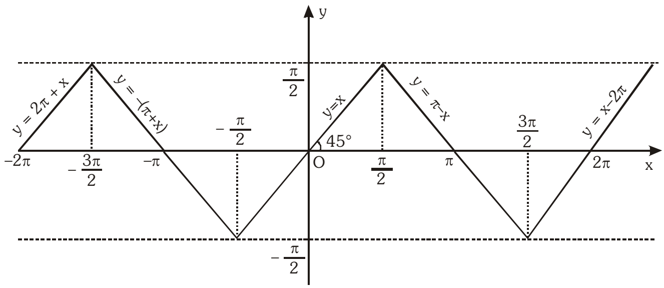
(v) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, $|x| \geq 1$, $|y| \geq 1$, y is aperiodic

(vi) $y = \sec(\sec^{-1} x) = x$, $|x| \geq 1$; $|y| \geq 1$, y is aperiodic

P-2 :

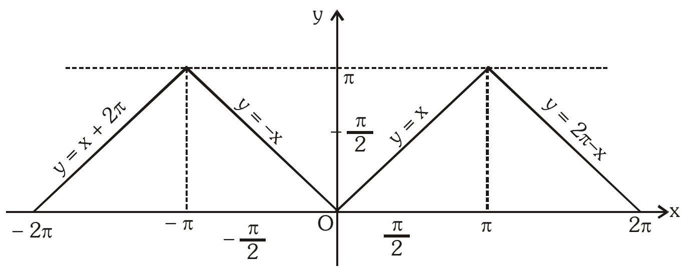
(i) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π.

$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x & , & -\dfrac{3\pi}{2}\le x\le -\dfrac{\pi}{2}\\ x & , & -\dfrac{\pi}{2}\le x\le \dfrac{\pi}{2}\\ \pi-x & , & \dfrac{\pi}{2}\le x\le \dfrac{3\pi}{2}\\ x-2\pi & , & \dfrac{3\pi}{2}\le x\le \dfrac{5\pi}{2}\\ 3\pi-x & , & \dfrac{5\pi}{2}\le x\le \dfrac{7\pi}{2}\\ x-4\pi & , & \dfrac{7\pi}{2}\le x\le \dfrac{9\pi}{2}\end{cases}$$



(ii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0,\pi]$, periodic with period 2π

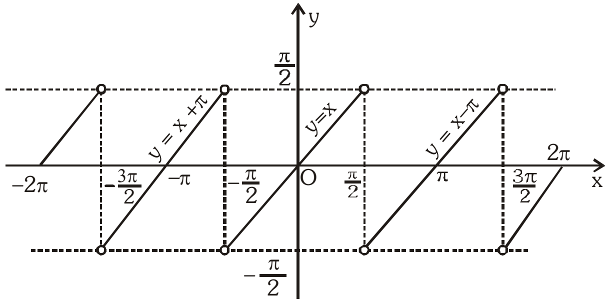
$$\cos^{-1}(\cos x)=\begin{cases}-x & , & -\pi\le x\le 0\\ x & , & 0\le x\le \pi\\ 2\pi-x & , & \pi\le x\le 2\pi\\ x-2\pi & , & 2\pi\le x\le 3\pi\\ 4\pi-x & , & 3\pi\le x\le 4\pi\end{cases}$$



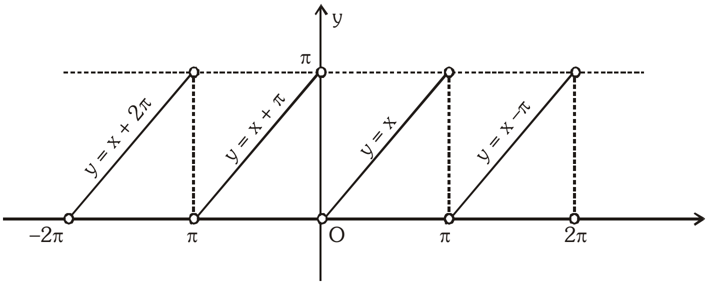
(iii) $y = \tan^{-1}(\tan x)$

$$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}; \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$, periodic with period π

$$\tan^{-1}(\tan x) = \begin{cases} x+\pi & , -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x & , -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x-\pi & , \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x-2\pi & , \frac{3\pi}{2} < x < \frac{5\pi}{2} \\ x-3\pi & , \frac{5\pi}{2} < x < \frac{7\pi}{2} \end{cases}$$

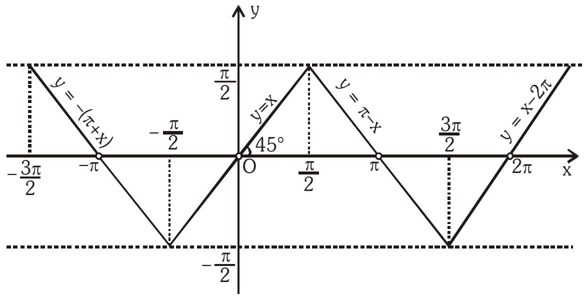


(iv) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with period π



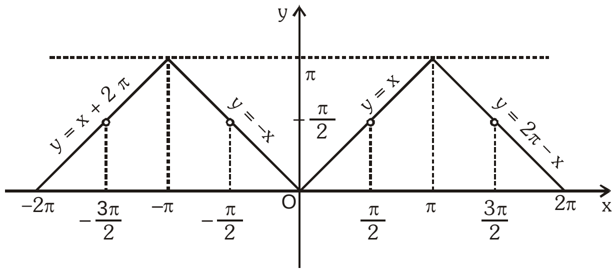
(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$,

y is periodic with period 2π.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

$$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



P-3 :

(i) $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $x \leq -1$ or $x \geq 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $x \leq -1$ or $x \geq 1$

(iii) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$; $x < 0$

P-4 :

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, $-1 \leq x \leq 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $-1 \leq x \leq 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $x \in R$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $x \leq -1$ or $x \geq 1$

(vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, $x \leq -1$ or $x \geq 1$

P-5 :

(i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $-1 \leq x \leq 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, $x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$, $|x| \geq 1$

P-6 :

(i) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, \text{ where } x > 0, y > 0 \ \& \ xy < 1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy}, \text{where } x > 0, y > 0 \ \& \ xy > 1 \\ \dfrac{\pi}{2}, \text{ where } x > 0, y > 0 \ \& \ xy = 1 \end{cases}$$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$, where $x > 0$, $y > 0$

(iii) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0$, $y > 0$ & $(x^2 + y^2) < 1$

Note that : $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \dfrac{\pi}{2}$

(iv) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0$, $y > 0$ & $x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$

(v) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(vi) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, where $x > 0, y > 0$

(vii) $$\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x < y, \quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x > y, \quad x, y > 0 \end{cases}$$

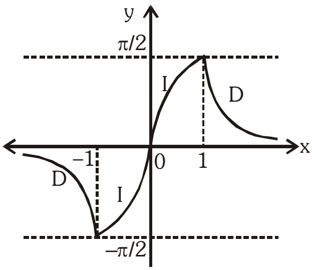
(viii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

Note : In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

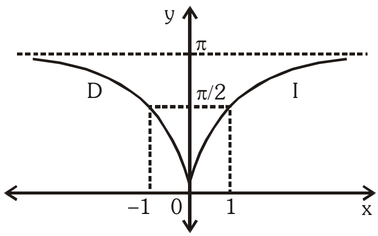
3. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$



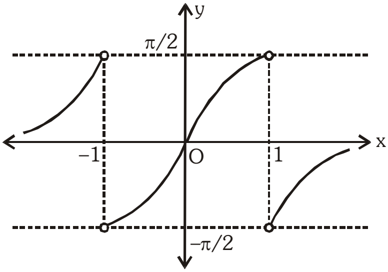
$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$

(b) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$



$$= \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

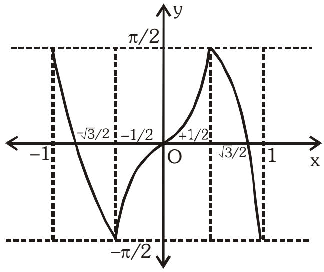
(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$



$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$

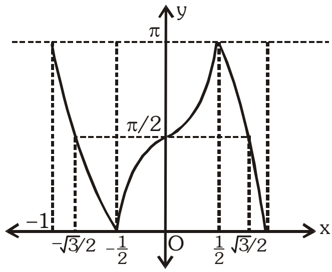
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1}x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$



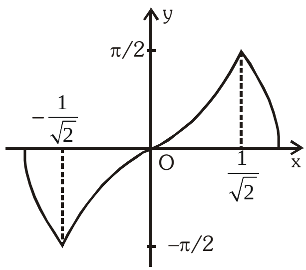
(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1}x - 2\pi & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$



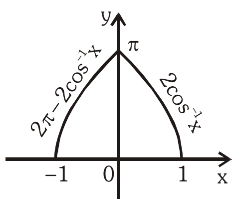
(f) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$= \begin{cases} -\left(\pi + 2\sin^{-1}x\right) & -1 \le x \le -\frac{1}{\sqrt{2}} \\ 2\sin^{-1}x & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$



(g) $\cos^{-1}(2x^2 - 1)$

$$= \begin{cases} 2\cos^{-1}x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & -1 \le x \le 0 \end{cases}$$



LIMIT

1. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

2. LEFT HAND LIMIT & RIGHT HAND LIMIT OF A FUNCTION :

Left hand limit (LHL) = $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a - h), h > 0.$

Right hand limit (RHL) = $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a + h), h > 0.$

Limit of a function f(x) is said to exist as $x \to a$ when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ = Finite and fixed quantity.

Important note :

In $\lim\limits_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

3. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\lim\limits_{x\to a}\left[f(x)+g(x)\right] = l + m$

(b) Difference rule : $\lim\limits_{x\to a}\left[f(x)-g(x)\right] = l - m$

(c) Product rule : $\lim\limits_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)} = l^m$, provided $l > 0$

(g) $\lim\limits_{x\to a} f\left[g(x)\right] = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim\limits_{x\to a} \ell n(f(x)) = \ell n[\lim\limits_{x\to a} f(x)]$; provided ℓnx is defined at x= $\lim\limits_{t\to a} f(t)$.

4. INDETERMINATE FORMS :

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty-\infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$.

Note :

We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementary algebra,

5. GENERAL METHODS TO BE USED TO EVALUATE LIMITS:

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \ldots\ldots\ldots\ldots + a^{n-1})$, $n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + \ldots\ldots\ldots\ldots + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(b) Rationalization or double rationalization :

In this method we rationalise the factor containing the square root and simplify.

(c) Limit when $x \to \infty$:

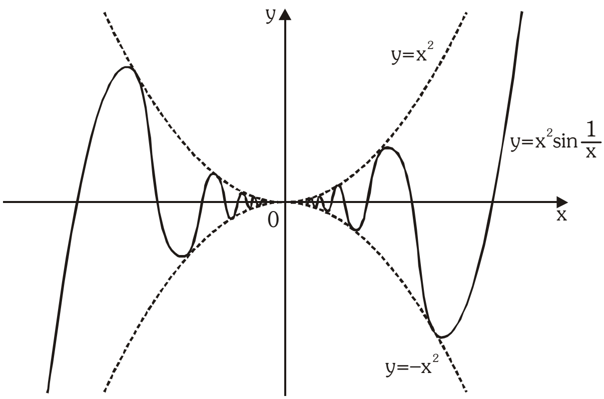
(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

(d) Squeeze play theorem (Sandwich theorem) :

If $f(x) \le g(x) \le h(x)$; $\forall x$ & $\lim\limits_{x \to a} f(x) = \ell = \lim\limits_{x \to a} h(x)$ then

$\lim\limits_{x \to a} g(x) = \ell$,



for example : $\lim\limits_{x \to 0} x^2 \sin\dfrac{1}{x} = 0$, as illustrated by the graph given.

(e) Using substitution $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ i.e.

by substituting x by a – h or a + h

6. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\tan^{-1} x}{x} = \lim_{x\to 0}\frac{\sin^{-1} x}{x}$$

[where x is measured in radians]

Further if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}\frac{\sin f(x)}{f(x)} = 1$.

7. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim\limits_{x\to 0}\frac{a^x - 1}{x} = \ell n\, a \,(a > 0)$ In particular $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}\frac{a^{f(x)} - 1}{f(x)} = \ell na,\ a > 0$

(b) $\lim\limits_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

(c) $\lim\limits_{x\to 0} (1+x)^{1/x} = e = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

(**Note :** The base and exponent depends on the same variable.)

In general, if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)} = e$

(d) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} \phi(x) = \infty$,

then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = e^k$ where $k = \lim\limits_{x\to a} \phi(x)\ [f(x) - 1]$

(e) If $\lim\limits_{x\to a} f(x) = A > 0$ & $\lim\limits_{x\to a} \phi(x) = B$ (a finite quantity),

then $\lim\limits_{x\to a}[f(x)]^{\phi(x)} = e^{B\, \ell n A} = A^B$

8. LIMIT USING SERIES EXPANSION :

Binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x\,\ell na}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + \ldots,\ a > 0$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, x \in R$

(c) $\ell n\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,\ \text{for} -1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, x \in R$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, x \in R$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots, |x| < \frac{\pi}{2}$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, x \in R$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots, x \in [-1, 1]$

(i) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots,\ n \in R,\ x \in (-1,\ 1).$

CONTINUITY

1. CONTINUOUS FUNCTIONS :

A function $f(x)$ is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a)$. Symbolically $f(x)$ is continuous at x = a

if $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) = f(a) =$ finite and fixed quantity $(h > 0)$

i.e. $\text{LHL}|_{x=a} = \text{RHL}|_{x=a} =$ value of $f(x)|_{x=a}$ = finite and fixed quantity.

At isolated points functions are considered to be continuous.

2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

- f is continuous in the open interval (a,b)
- f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a) =$ a finite quantity
- f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b) =$ a finite quantity

Note :

(i) All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

(ii) If f & g are two functions that are continuous at x = c then the function defined by : $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number, $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g(c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(iii) If f and g are continuous then fog and gof are also continuous.

(iv) If f and g are discontinuous at x = c, then f + g, f – g, f.g may still be continuous.

(v) Sum or difference of a continuous and a discontinuous function is always discontinuous.

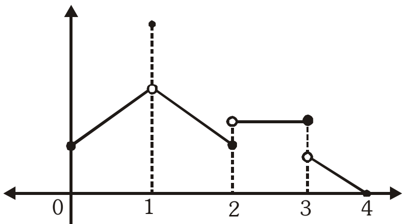
3. REASONS OF DISCONTINUITY :

(a) Limit does not exist

i.e. $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

(b) $\lim\limits_{x\to a} f(x) \neq f(a)$

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1, 2 and 3.



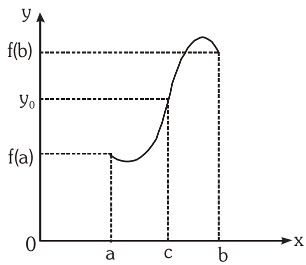
$\lim\limits_{x\to 1}$ **f(x) ≠ f(1)**

$\lim\limits_{x\to 2}$ **f(x) does not exist**

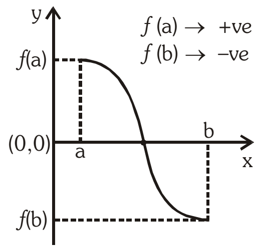
$\lim\limits_{x\to 3}$ **f(x) does not exist**

4. THE INTERMEDIATE VALUE THEOREM :

Suppose f(x) is continuous on an interval I and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), their exists a number c between a and b such that $f(c) = y_0$



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)



$f(x) = 0$ in (a, b)

Note that a function f which is continuous in [a,b] possesses the following property :

If f(a) & f(b) posses opposite signs, then there exists atleast one solution of the equation f(x) = 0 in the open interval (a,b).

DIFFERENTIABILITY

1. INTRODUCTION :

The derivative of a function 'f' is function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x)\ \text{or}\ \frac{df(x)}{dx}$$

The derivative evaluated at a point a, can be written as :

$$f'(a),\left[\frac{df(x)}{dx}\right]_{x=a},f'(x)_{x=a},\ \text{etc.}$$

2. RIGHT HAND & LEFT HAND DERIVATIVES :

(a) Right hand derivative :

The right hand derivative of f(x) at x = a denoted by $f'_{+}(a)$ is defined as :

$$f'_{+}(a)=\lim_{h\to 0^{+}}\frac{f(a+h)-f(a)}{h}$$, provided the limit exists & is finite.

(b) Left hand derivative :

The left hand derivative of f(x) at x = a denoted by $f'_{-}(a)$ is defined

as : $f'_{-}(a)=\lim\limits_{h\to 0^{+}}\dfrac{f(a-h)-f(a)}{-h}$, provided the limit exists & is finite.

(c) Derivability of function at a point :

If $f'_{+}(a)=f'_{-}(a)$ = finite quantity, then f(x) is said to be **derivable or differentiable at x = a**. In such case $f'_{+}(a)=f'_{-}(a)=f'(a)$ & it is called derivative or differential coefficient of f(x) at x = a.

Note :

(i) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

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(ii) If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be derivable at x =a & if g(a) $\neq 0$ then the function f(x)/g(x) will also be derivable at x = a.

3. IMPORTANT NOTE :

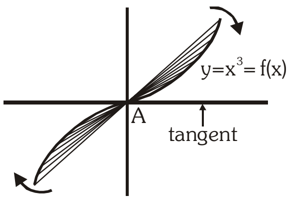
(a) Let $f'_{+}(a) = p$ & $f'_{-}(a) = q$

- **When p & q are finite :**

If p and q are finite (whether equal or not), then f is continuous at x = a but converse is NOT necessarily true.

(i) $p = q \Rightarrow$ f is differentiable at x = a $\Rightarrow$ f is continuous at x = a

e.g.



$f'_{-}(0) = 0$ and $f'_{+}(0) = 0 \Rightarrow \quad f'(0) = 0$,

here x axis is tangent to the curve at x = 0.

(ii) $p \neq q \Rightarrow$ f is not differentiable at x = a, but f is still continuous at x = a. In this case we have a sudden change in the direction of the graph of the function at x = a. This point is called a **corner point** of the function. At this point there is no tangent to the curve.

- **When p or q may not be finite :**

In this case f is not differentiable at x = a and nothing can be concluded about continuity of the function at x = a.

Note :

(a) **Corner :** If f is continuous at x = a with RHD and LHD at x = a both are finite but not equal or exactly one of them is infinite, then the point x = a is called a corner point and at this point function is not differentiable but continuous.

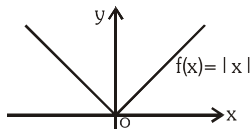
(b) **Cusp :** If f is continuous at x = a and one of RHD, LHD at x = a, approaches to ∞ and other one approaches to $-\infty$, then the point x = a is called a cusp point. At cusp point we have a vertical tangent and at this point function is not differentiable but continuous. We can observe that cusp is sharper than corner point.

(b) Geometrical interpretation of differentiability :

(i) If the function y = f(x) is differentiable at x = a, then a unique tangent can be drawn to the curve y = f(x) at P(a, f(a)) & f'(a) represent the slope of the tangent at point P.

(ii) If LHD and RHD are finite but unequal then it geometrically implies a sharp corner at x = a.

e.g. $f(x) = |x|$ is continuous but not differentiable at x = 0. A sharp corner is seen at x = 0 in the graph of $f(x) = |x|$.



(c) Vertical tangent : If y = f(x) is continuous at x = a and $\lim_{x\to a}|f'(x)|$ approaches to ∞, then y = f(x) has a vertical tangent at x = a. If a function has vertical tangent at x = a then it is non differentiable at x = a.

4. DERIVABILITY OVER AN INTERVAL :

(a) f(x) is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the open interval (a, b).

(b) f(x) is said to be derivable over the closed interval [a, b] if :

(i) f(x) is derivable in (a, b) &

(ii) for the points a and b, $f'_+(a)$ & $f'_-(b)$ exist.

Note :

(i) If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x)=f(x).g(x)$ can still be differentiable at $x = a$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x)=f(x).g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ & $g(x)$ both are non-derivable at $x=a$ then the sum function $F(x)=f(x)+g(x)$ may be a differentiable function.

(iv) If $f(x)$ is derivable at $x = a \not\Rightarrow f'(x)$ is continuous at $x = a$.

(v) Sum or difference of a differentiable and a non-differentiable function is always is non-differentiable.

METHODS OF DIFFERENTIATION

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$$ is called calculating derivative using first principle or ab initio or delta method.

2. FUNDAMENTAL THEOREMS :

If f and g are derivable function of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$, known as **SUM RULE**

(b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$, known as **PRODUCT RULE**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$,

where $g \neq 0$ known as **QUOTIENT RULE**

(e) If $y = f(u)$ & $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, known as **CHAIN RULE**

Note : In general if $y = f(u)$, then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

3. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)
(i)	x^n	nx^{n-1}
(ii)	e^x	e^x
(iii)	a^x	$a^x \ell na$, $a > 0$
(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x)\log_a e$, $a > 0$, $a \neq 1$
(vi)	sinx	cosx
(vii)	cosx	– sinx
(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx
(x)	cosecx	– cosecx . cotx
(xi)	cotx	$-\text{cosec}^2 x$
(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, \ -1 < x < 1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2}, \ x \in R$
(xvi)	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert\sqrt{x^2-1}}, \lvert x \rvert > 1$
(xvii)	$\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert\sqrt{x^2-1}}, \ \lvert x \rvert > 1$
(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in R$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(a) A function which is the product or quotient of a number of function or

(b) A function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it is convenient to take the logarithm of the function first & then differentiate.

5. DIFFERENTIATION OF IMPLICIT FUNCTION :

(a) Let function is $\phi(x, y) = 0$ then to find dy /dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy / dx together on one side to finally find dy / dx

OR $\dfrac{dy}{dx} = \dfrac{-\partial\phi/\partial x}{\partial\phi/\partial y}$ where $\dfrac{\partial\phi}{\partial x}$ & $\dfrac{\partial\phi}{\partial y}$ are partial differential coefficient of $\phi(x, y)$ w.r.to x & y respectively.

(b) In expression of dy/dx in the case of implicit functions, both x & y are present.

6. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

7. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y= f(x)$; $z = g(x)$, then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$

8. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If inverse of $y = f(x)$ is denoted as $g(x) = f^{-1}(x)$, then $g(f(x)) = x$

$\Rightarrow$ $g'(f(x))f'(x)=1$

9. HIGHER ORDER DERIVATIVE :

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy / dx) or y'] & it is called the first derivative of y w. r. t. x. If it happens that the first derivative has a derivative on (a , b) then this derivative is called second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y /dx^2) or y''. Similarly, the 3rd order derivative of y w.r.t x, if it exists, is defined by $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y''' & so on.

10. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Similarly one can also proceed columnwise.

11. L' HÔPITAL'S RULE :

(a) Applicable while calculating limits of indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point a, except, may be, at the point a itself, and $g'(x) \neq 0$, and if

$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$,

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper $+\infty$ or $-\infty$.

(b) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

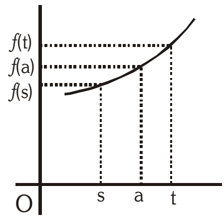
(c) Indeterminate forms of the type 1^∞, ∞^0 or 0^0 are reduced to forms of the type $0.\infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell n f(x)}$.

MONOTONICITY

1. INCREASING / DECREASING / STRICTLY INCREASING / STRICTLY DECREASING NATURE OF A FUNCTION AT A POINT :

I. Increasing at x = a :

If $f(s) \le f(a) \le f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be increasing at $x = a$.



(1) When 'a' be left end of the interval

$f(a) \le f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

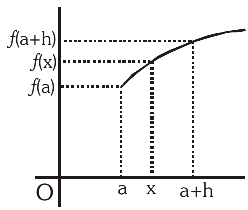
(2) When 'a' be right end of the interval

$f(x) \le f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

II. Strictly increasing at x = a :

If $f(s) < f(a) < f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be strictly increasing at $x = a$.



(1) When 'a' be left end of the interval

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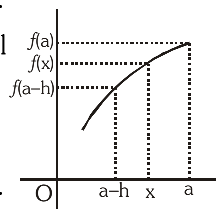


III. Decreasing at x = a :

If $f(s) \geq f(a) \geq f(t)$ when ever $s < a < t$,

where $s, t \in (a - h, a + h) \cap D_f$

for some $h > 0$, then f is said to be decreasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) \geq f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) \geq f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

IV. Strictly decreasing at x = a :

If $f(s) > f(a) > f(t)$ when ever $s < a < t$,

where $s, t \in (a - h, a + h) \cap D_f$

for some $h > 0$, then f is said to be strictly decreasing at $x = a$.

(1) When 'a' be left end of the interval

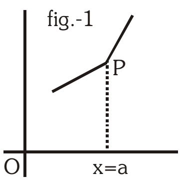
$f(a) > f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly decreasing at $x = a$.

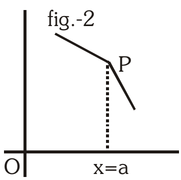
(2) When 'a' be right end of the interval

$f(x) > f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

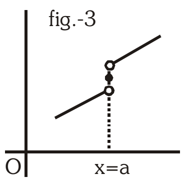
$\Rightarrow$ f is strictly decreasing at $x = a$.



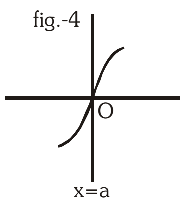
Strictly increasing at x = a



Strictly decreasing at x = a



Strictly increasing at x = a



Strictly increasing at x = a

2. INCREASING & DECREASING NATURE OF A FUNCTION OVER AN INTERVAL :

Consider an interval $I \subseteq D_f$

I. Increasing Over an Interval I :

If $\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, then f is increasing over the interval I.

II. Decreasing Over an Interval I :

If $\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, then f is decreasing over the interval I.

III. Strictly increasing over an Interval I :

If $\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$, then f is strictly increasing over the interval I.

IV. Strictly decreasing over an Interval I :

If $\forall\ x_1, x_2 \in I,\ x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)$, then f is strictly decreasing over the interval I.

Monotonic function :

If a function is either increasing or decreasing over an interval then it is said to be monotonic function over the interval.

If a function is either strictly increasing or strictly decreasing over an interval then it is said to be strictly monotonic function over the interval.

Some Important points :

(i) Increasing or monotonic increasing or non decreasing has same meaning. Similarly decreasing or monotonic decreasing or non increasing has same meaning.

(ii) If a function is strictly increasing, then it is also said to be increasing function, but converse is not necessarily true.

(iii) Functions which are increasing over some interval and decreasing over another interval are known as non-monotonic functions over the union of the intervals.

(iv) A function may be monotonic in a subset but may not be monotonic in a superset.

(v) Constant function is increasing as well as decreasing over any interval. So it is called monotonic function.

For differentiable functions :

Consider an interval I ($\subseteq D_f$) that can be [a, b] or (a, b) or [a, b) or (a, b].

(1) $f'(x) > 0\ \forall\ x \in I \Rightarrow f$ is strictly increasing function over the interval I.

(2) $f'(x) \geq 0\ \forall\ x \in I \Rightarrow f$ is increasing function over the interval I.

(3) $f'(x) \geq 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points)

$\Rightarrow f$ is strictly increasing function over the interval I.

(4) $f'(x) < 0\ \forall\ x \in I \Rightarrow f$ is strictly decreasing function over the interval I.

(5) $f'(x) \leq 0\ \forall\ x \in I \Rightarrow f$ is decreasing function over the interval I.

(6) $f'(x) \leq 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points)

$\Rightarrow f$ is strictly decreasing function over the interval I.

3. ROLLE'S THEOREM :

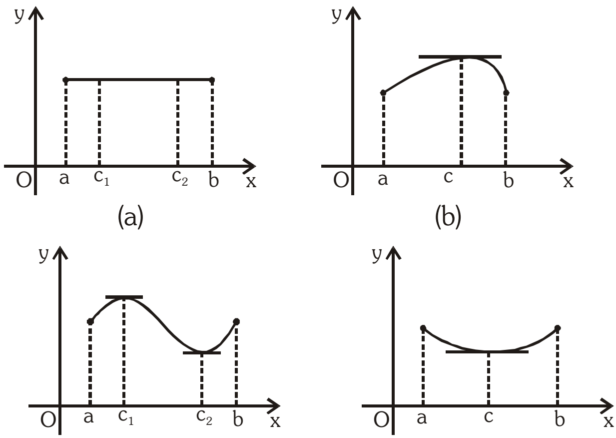
Let f be a function that satisfies the following three hypotheses :

(a) f is continuous in the closed interval [a, b].

(b) f is differentiable in the open interval (a, b)

(c) f(a) = f(b)

Then there is a number c in (a, b) such that f'(c) = 0.

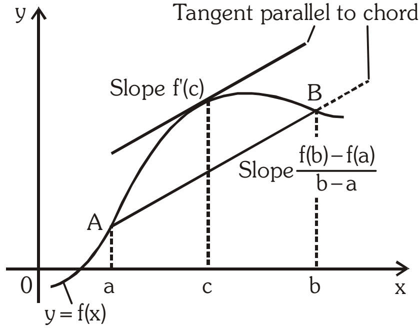


(a) (b) (c) (d)

Conclusion : If f is a differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

4. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :

Let f be a function that satisfies the following hypotheses:



(i) f is continuous in a closed interval [a, b]

(ii) f is differentiable in the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

(a) Geometrical Interpretation :

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations :

If we think of the number (f(b) – f(a))/(b – a) as the average change in f over [a, b] and f'(c) as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

5. SPECIAL NOTE :

One can make use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that

(a) f(x) & its first derivative f'(x) are continuous for $a \le x \le b$.

(b) f(a) and f(b) have opposite signs.

(c) f'(x) is different from zero for all values of x between a & b.

Then there is one & only one root of the equation f(x) = 0 in (a, b).

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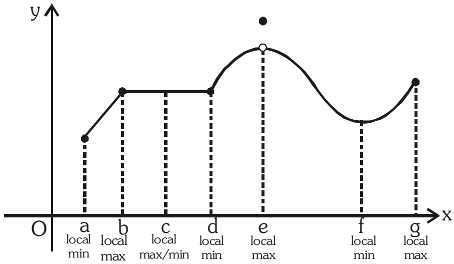
MAXIMA-MINIMA

1. INTRODUCTION :

MAXIMA & MINIMA :



(a) Local Maxima /Relative maxima :

A function $f(x)$ is said to have a local maxima at x = a if $f(a) \geq f(x)\ \forall\ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.

(b) Local Minima/Relative minima :

A function $f(x)$ is said to have a local minima at x = a if $f(a) \leq f(x)\ \forall\ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.

(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

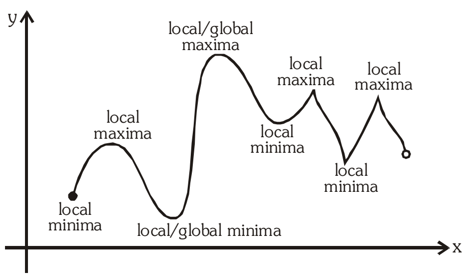
(d) Absolute minima (Global minima) :

A function f has an absolute minima at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D.

Note :

(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.



(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

(iv) It is not necessary that $f(x)$ always has local maxima/minima at end points of the given interval when they are included.

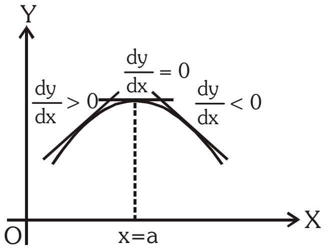
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

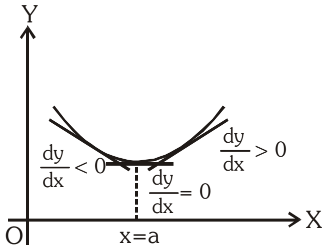
(a) First derivative test :

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from positive to negative in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local minima.**





Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $x=a$ is not a point of extremum of f.

(b) Second derivative test :

If $f(x)$ is continuous and differentiable at $x = a$ where $f'(a) = 0$ (stationary points) and $f''(a)$ also exists then for ascertaining maxima/minima at $x = a$, 2^{nd} derivative test can be used -

(i) If $f''(a) > 0 \Rightarrow x = a$ is a point of local minima

(ii) If $f''(a) < 0 \Rightarrow x = a$ is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

(c) n^{th} derivative test :

Let f(x) be a function such that $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) If n is even &

$$\begin{cases} f^n(a) > 0 \Rightarrow \text{Minima} \\ f^n(a) < 0 \Rightarrow \text{Maxima} \end{cases}$$

(ii) If n is odd then neither maxima nor minima at x = a.

3. USEFUL FORMULAE OF MENSURATION TO REMEMBER:

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base x height.

(d) Lateral surface area of prism = perimeter of the base x height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

(m) Area of a circular sector = $\frac{1}{2} r^2\theta$, when θ is in radians.

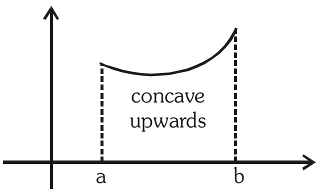
(n) Perimeter of circular sector = $2r + r\theta$.

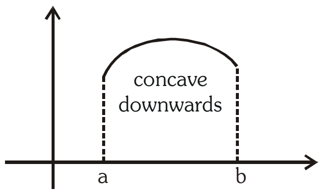
4. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE :

The sign of the 2^{nd} order derivative determines the concavity of the curve.

i.e. If $f''(x) \geq 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) \leq 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).





5. SOME SPECIAL POINTS ON A CURVE :

(a) Stationary points: The stationary points are the points of domain where $f'(x) = 0$.

(b) Critical points : There are three kinds of critical points as follows :

(i) The point at which $f'(x) = 0$

(ii) The point at which $f'(x)$ does not exists

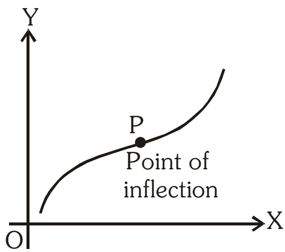
(iii) The end points of interval (if included)

These points belongs to domain of the function.

Note : Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/ local minima.

(c) Point of inflection :

A point where the graph of a function has a tangent line and where the strict concavity changes is called a point of inflection. For finding point of inflection of any function, compute the points



(x-coordinate) where $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist. Let the solution is x = a, if $\frac{d^2y}{dx^2} = 0$ at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

6. SOME STANDARD RESULTS :

(a) Rectangle of largest area inscribed in a circle is a square.

(b) The function $y = \sin^m x \cos^n x$ attains the max value at $x = \tan^{-1}\sqrt{\frac{m}{n}}$

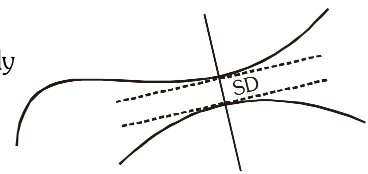
(c) If $0 < a < b$ then $|x-a| + |x-b| \geq b - a$ and equality hold when $x \in [a, b]$.

If $0 < a < b < c$ then $|x-a| + |x-b| + |x-c| \geq c - a$ and equality hold when x = b

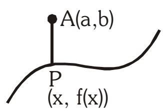
If $0 < a < b < c < d$ then $|x-a| + |x-b| + |x-c| + |x-d| \geq d - a$ and equality hold when $x \in [b,c]$.

7. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES :

Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)



Note : Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.



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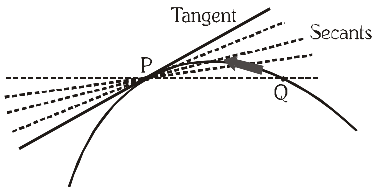
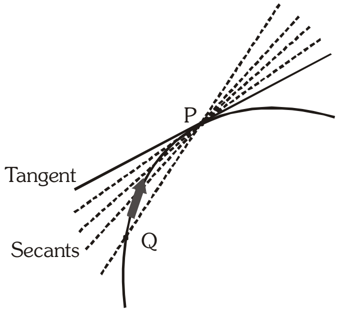
E

TANGENT & NORMAL

1. TANGENT TO THE CURVE AT A POINT :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant's slope as Q $\rightarrow$ P from either side.





2. NORMAL TO THE CURVE AT A POINT :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

3. THINGS TO REMEMBER :

(a) The value of the derivative at P (x_1,y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1,\, y_1)} = \text{Slope of tangent at } P(x_1,\ y_1) = m(\text{say}).$$

(b) Equation of tangent at $(x_1,\ y_1)$ is $y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1,\, y_1)} (x - x_1)$

(c) Equation of normal at $(x_1,\ y_1)$ is; $y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$.

Note :

(i) The point P $(x_1,\ y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx is not defined or dx/dy = 0 at that point.

(iv) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

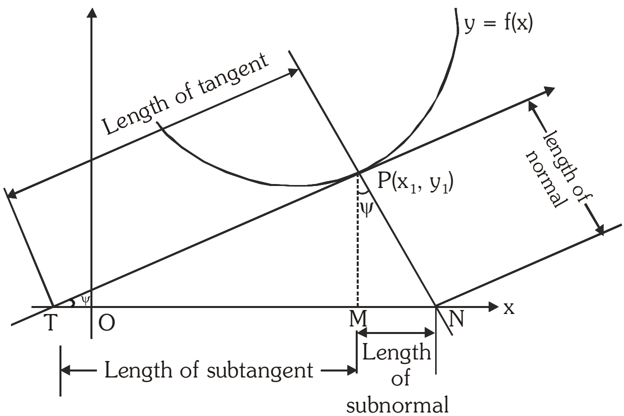
(v) If a curve passing through the origin be given by a rational/integral/algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

4. ANGLE OF INTERSECTION BETWEEN TWO CURVES :

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection. If the angle between two curves is 90° then they are called **ORTHOGONAL** curves.

Note : If the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{c}+\frac{y^2}{d}=1$, intersect each other orthogonally, then a − c = b − d.

5. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL :



(a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+\left[\ f'(x_1)\ \right]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

6. DIFFERENTIALS :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tanx then $dy = \sec^2 x\ dx$. In general $dy = f'(x)dx$ or $df(x) = f'(x)dx$

Note :

(i) d(c) = 0 where 'c' is a constant

(ii) d(u + v) = du + dv

(iii) d(uv) = udv + vdu

(iv) d(u − v) = du − dv

(v) $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$

(vi) For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in y = f(x) is $y + \Delta y = f(x + \Delta x) = f(x) + \dfrac{dy}{dx}.\Delta x$

(vii) The relation dy = f'(x) dx can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + c \Leftrightarrow \frac{d}{dx}\{F(x)+c\} = f(x)$, where c is called the **constant of integration.**

Note : If $\int f(x)dx = F(x) + c$, then $\int f(ax+b)dx = \frac{F(ax+b)}{a} + c, a \neq 0$

1. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n|\,ax+b\,| + c$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\,a}(a > 0) + c$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n\,|\sec(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \operatorname{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}.\sec(ax+b)+c$

(xiii) $\int \sec x\, dx = \ell n\left|\sec x + \tan x\right| + c$

OR $\int \sec x dx = \ell n \tan\left|\frac{\pi}{4}+\frac{x}{2}\right| + c$

(xiv) $\int \operatorname{cosec} x\, dx = \ell n\left|\operatorname{cosec} x - \cot x\right| + c$

OR $\int \operatorname{cosec} x\, dx = \ell n\left|\tan\frac{x}{2}\right| + c$ OR $-\ell n(\operatorname{cosec} x + \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right] + c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right) + c$

(xxiv) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right) + c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

2. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

Integral I = $\int f(x)\,dx$ is changed to $\int f(\phi(t))\phi'(t)\,dt$, by a suitable substitution x = ϕ(t) provided the later integral is easier to integrate.

Some standard substitution :

(1) $\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put f(x) = t & proceed.

(2) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\;dx$

Express ax^2 + bx + c in the form of perfect square & then apply the standard results.

(3) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express px + q = A (differential coefficient of quadratic term of denominator) + B.

(4) $\int e^x[f(x)+f'(x)]dx = e^x.f(x) + c$

(5) $\int [f(x)+xf'(x)]dx = x\,f(x) + c$

(6) $\int \frac{dx}{x(x^n+1)}$ $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(7) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(8) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t$.

(9) $\int \frac{dx}{a + b\sin^2 x}$ OR $\int \frac{dx}{a + b\cos^2 x}$

OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply N^r & D^r by $\sec^2 x$ & put $\tan x = t$.

(10) $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles, put $\tan \frac{x}{2} = t$

(11) $\int \frac{a.\cos x + b.\sin x + c}{p.\cos x + q.\sin x + r}dx.$

Express Numerator $(N^r) \equiv \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

(12) $\int \frac{x^2 + 1}{x^4 + Kx^2 + 1}dx$ OR $\int \frac{x^2 - 1}{x^4 + Kx^2 + 1}dx$,

where K is any constant.

Divide Nr & Dr by x^2, then put $x - \frac{1}{x} = t$ OR $x + \frac{1}{x} = t$ respectively & proceed

(13) $\int \frac{dx}{(ax + b)\sqrt{px + q}}$ & $\int \frac{dx}{(ax^2 + bx + c)\sqrt{px + q}}$; put $px + q = t^2$

(14) $\int \frac{dx}{(ax + b)\sqrt{px^2 + qx + r}}$, put $ax + b = \frac{1}{t}$;

$\int \frac{dx}{(ax^2 + bx + c)\sqrt{px^2 + qx + r}}$, put $x = \frac{1}{t}$

(15) $\int \sqrt{\frac{x - \alpha}{\beta - x}}\,dx$ OR $\int \sqrt{(x - \alpha)(\beta - x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x - \alpha}{x - \beta}}\,dx$ OR $\int \sqrt{(x - \alpha)(x - \beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

(16) To integrate $\int \sin^m x \cos^n x \, dx$.

(i) If m is odd positive integer put cos x = t.

(ii) If n is odd positive integer put sin x = t

(iii) If m + n is negative even integer then put tan x = t.

(iv) If m and n both even positive integer then use

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$

(b) Integration by parts : $\int u.v \, dx = u\int v \, dx - \int\left[\frac{du}{dx}.\int v \, dx\right] dx$

where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v \, dx\right] dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

(c) Partial fraction : Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
	where x^2+bx+c cannot be factorised further	

Note :

In competitive exams, partial fraction are generally found by inspection by noting following fact :

$$\frac{1}{(x-\alpha)(x-\beta)}=\frac{1}{(\alpha-\beta)}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right).$$

It can be applied to the case when x^2 or any other function is there in all places of x.

Example :

(1) $\frac{1}{(x^2+1)(x^2+3)}=\frac{1}{2}\left(\frac{1}{t+1}-\frac{1}{t+3}\right)$ {take $x^2=t$}

(2) $\frac{1}{x^4(x^2+1)}=\frac{1}{x^2}\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)=\frac{1}{x^4}-\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)$

(3) $\frac{1}{x^3(x^2+1)}=\frac{1}{x}\left(\frac{1}{x^2}-\frac{1}{x^2+1}\right)=\frac{1}{x^3}-\frac{1}{x(x^2+1)}$

DEFINITE INTEGRATION

1. **(a) The Fundamental Theorem of Calculus, Part 1 :**
If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_a^x f(t)dt, \qquad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

(b) The Fundamental Theorem of Calculus, Part 2 :

If f is continuous on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f, that is, a function such that $F' = f$.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation f(x) = 0 has atleast one root lying in (a,b) provided f is a continuous function in (a,b).

2. A definite integral is denoted by $\int_a^b f(x)dx$ which represents the algebraic area bounded by the curve y = f(x), the ordinates x = a, x = b and the x-axis. ex. $\int_0^{2\pi} \sin x\, dx = 0$

3. **PROPERTIES OF DEFINITE INTEGRAL :**

(a) $\int_a^b f(x)dx = \int_a^b f(t)\,dt \quad \Rightarrow \quad \int_a^b f(x)\,dx$ does not depend upon x. It is a numerical quantity.

(b) $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b]. This property to be used when f is piecewise continuous in (a, b).

(d) $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a}[f(x)+f(-x)]dx = \begin{bmatrix} 0 & ;\text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & ;\text{if } f(x) \text{ is an even function} \end{bmatrix}$

(e) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

(f) $\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \begin{bmatrix} 2\int_0^a f(x)\,dx & ; \text{ if } f(2a-x)=f(x) \\ 0 & ; \text{ if } f(2a-x)=-f(x) \end{bmatrix}$

(g) $\int_0^{nT} f(x)dx = n\int_0^T f(x)\,dx$, $(n \in I)$; where 'T' is the period of the function i.e. $f(T+x) = f(x)$

Note that : $\int_x^{T+x} f(t)dt$ will be independent of x and equal to $\int_0^T f(t)dt$

(h) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$ where f(x) is periodic with period T & $n \in I$.

(i) $\int_{ma}^{na} f(x)dx = (n-m)\int_0^a f(x)dx$, $(n, m \in I)$ if f(x) is periodic with period 'a'.

4. WALLI'S FORMULA :

(a) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3).....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)}K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) $\int_0^{\pi/2} \sin^n x . \cos^m x\,dx$

$= \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2}K$

Where $K = \begin{cases} \frac{\pi}{2} & \text{if both m and n are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$

5. DERIVATIVE OF ANTIDERIVATIVE FUNCTION (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$$

6. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots.. + f\left(a + \overline{n-1}h\right)]$$

$$= \lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh) \text{ , where } b - a = nh$$

If a = 0 & b = 1 then, $\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where nh = 1

OR $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$.

7. ESTIMATION OF DEFINITE INTEGRAL :

(a) If f(x) is continuous in [a, b] and it's range in this interval is [m, M], then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

(b) If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)\,dx$

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

(d) If $f(x) \geq 0$ on the interval [a,b], then $\int_a^b f(x)dx \geq 0$.

(e) f(x) and g(x) are two continuous function on [a, b] then

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)\,dx}$$

8. SOME STANDARD RESULTS :

(a) $\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\cos x\,dx$

(b) $\int_a^b \{x\}\,dx = \frac{b-a}{2}$; a, b $\in$ I

(c) $\int_a^b \frac{|x|}{x}dx = |b| - |a|$.

DIFFERETIAL EQUATION

1. DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a DIFFERENTIAL EQUATION.

2. SOLUTION (PRIMITIVE) OF DIFFERENTIAL EQUATION :

Finding the unknown function which satisfies given differential equation is called SOLVING OR INTEGRATING the differential equation. The solution of the differential equation is also called its PRIMITIVE, because the differential equation can be regarded as a relation derived from it.

3. ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest order differential coefficient occurring in it.

4. DEGREE OF DIFFERENTIAL EQUATION :

The degree of a differential equation which can be written as a polynomial in the derivatives, is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots\ldots\ldots = 0$ is of order m & degree p.

Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't exist.

5. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

The eliminant is the required differential equation.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

6. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE). A solution obtainable from the general solution by giving particular or initial values to the constants is called a PARTICULAR SOLUTION.

7. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Variables separable :

TYPE-1 : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable – separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$;

where c is the arbitrary constant. Consider the example $(dy/dx) = e^{x-y} + x^2.e^{-y}$.

TYPE-2 : Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials. If $x = r\cos\theta$, $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$

(ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$

(iii) $xdy - ydx = r^2\,d\theta$

Also, if $x = r\sec\theta$ & $y = r\tan\theta$ then

$x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

TYPE – 3 : $\frac{dy}{dx} = f(ax+by+c),\ b \neq 0$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x+y)^2\frac{dy}{dx} = a^2$

(b) Homogeneous equations :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ & $\phi(x, y)$ are homogeneous functions of x & y and of the same degree, is called HOMOGENEOUS. This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function. The differential equation thus, is transformed to an equation which is variables separable.

Consider the example $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$

(c) Equations reducible to the homogeneous form :

If $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$; where $a_1b_2 - a_2b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$

transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous.

Note :

(i) If $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variables separable.

(ii) If $b_1 + a_2 = 0$, then a simple cross multiplication and substituting d(xy) for xdy + ydx & integrating term by term yields the result easily.

Consider the examples $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$; $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

& $\frac{dy}{dx} = \frac{2x-y+1}{6x-5y+4}$

(iii) In an equation of the form : $yf(xy)dx + xg(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

8. LINEAR DIFFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form ;

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x).y + a_n(x) = 0,$$

where $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ are called the coefficients of the differential equation.

(a) Linear differential equations of first order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by integrating factor $e^{\int Pdx}$.

Then the solution of this equation will be $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

(b) Equations reducible to linear form :

The equation $\frac{dy}{dx} + Py = Q.y^n$ where P & Q are function, of x,

is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1}=Z$. Consider the example $(x^3y^2+xy)dx=dy$.

The equation $\frac{dy}{dx} + Py = Q\,y^n$ is called BERNOULI'S EQUATION.

9. TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

Orthogonal trajectories :

A curve making at each of its points a right angle with the curve of the family passing through that point is called an orthogonal trajectory of that family.

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x, y, \frac{-1}{y'}\right) = 0$$

The general integral of this equation $\phi(x, y, C) = 0$ gives the family of orthogonal trajectories.

Note :

Following exact differentials must be remembered :

(i) $dx + dy = d(x + y)$

(ii) $dx - dy = d(x - y)$

(iii) $xdy + y\,dx = d(xy)$

(iv) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(v) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(vi) $2(xdx + ydy) = d(x^2 + y^2)$

It should be observed that :

(i) $\dfrac{xdy + ydx}{xy} = d(\ell n xy)$

(ii) $\dfrac{dx + dy}{x + y} = d(\ell n(x + y))$

(iii) $\dfrac{xdy - ydx}{xy} = d\left(\ell n \dfrac{y}{x}\right)$

(iv) $\dfrac{ydx - xdy}{xy} = d\left(\ell n \dfrac{x}{y}\right)$

(v) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{y}{x}\right)$

(vi) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{x}{y}\right)$

(vii) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(viii) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

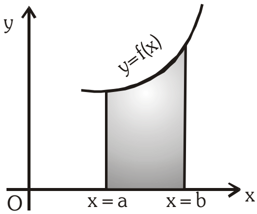
(ix) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

(x) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

AREA UNDER THE CURVE

1. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ & $x = b$ is given by,

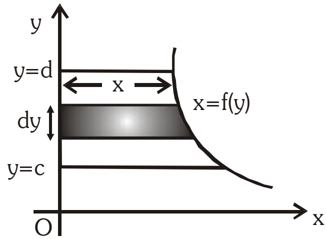
$$A = \int_a^b f(x)dx = \int_a^b y\,dx, \ f(x) \geq 0.$$



2. If the area is below the x-axis then A is negative. The convention is to consider the magnitude only i.e. $A = \left|\int_a^b y\,dx\right|$ in this case.

3. The area bounded by the curve $x = f(y)$, y-axis & abscissa $y = c$, $y = d$ is given by,

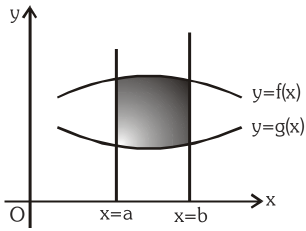
$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)dy, \ f(y) \geq 0$$



4. Area between the curves $y = f(x)$ & $y = g(x)$ between the ordinates $x = a$ & $x = b$ is given by,

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx, \ f(x) \geq g(x) \ \forall \ x \in (a, b).$$



5. Average value of a function $y = f(x)$ w.r.t. x over an interval $a \leq x \leq b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)dx$.

6. CURVE TRACING :

The following outline procedure is to be applied in Sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on replacing 'x' by '$-x$' and 'y' by '$-y$', the equation of the curve is unaltered then there is symmetry in opposite quadrants, i.e. symmetric about the origin.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x-axis & also the y-axis.

(d) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab.

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$.

(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$.

(d) Area enclosed by the parabola and its double ordinate P, Q is two-third of area of rectangle PQRS, where R,S lie on tangent at the vertex.

VECTORS

1. Physical quantities are broadly divided in two categories viz (a) Vector Quantities & (b) Scalar quantities.

(a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

(b) Scalar quantities :

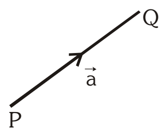
A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

2. REPRESENTATION :

Vectors are represented by directed straight line segment

magnitude of $\vec{a} = |\vec{a}|$ = length PQ

direction of $\vec{a}$ = P to Q.



3. (a) ZERO VECTOR OR NULL VECTOR :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a ZERO VECTOR . It is denoted by $\vec{O}$.

(b) UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

(c) COLLINEAR VECTORS

Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

(d) COPLANAR VECTORS

A given number of vectors are called coplanar if their supports are all parallel to the same plane.

Note that "TWO VECTORS ARE ALWAYS COPLANAR".

(e) EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(i) the same length,

(ii) the same or parallel supports and

(iii) the same sense.

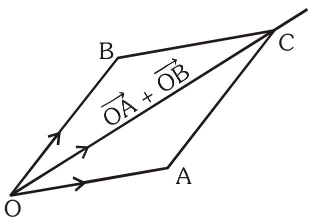
(f) Free vectors : If a vector can be translated anywhere in space without changing its magnitude & direction, then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in space keeping its magnitude & direction same.

(g) Localized vectors : For a vector of given magnitude and direction, if its initial point is fixed in space, then such a vector is called localised vector. Unless & until stated, vectors are treated as free vectors.

4. ADDITION OF VECTORS :

(a) It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(i) If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.



(ii) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)

(iii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associativity)

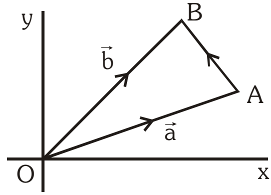
(b) Multiplication of vector by scalars :

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$ **(ii)** $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ **(iv)** $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

5. POSITION VECTOR :

Let O be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then,



$\vec{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

6. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.

7. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b}-\vec{a})$ where t is a parameter. If the line pass through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{r} = \vec{a} + t\vec{b}$, where t is a parameter.

8. TEST OF COLLINEARITY OF THREE POINTS :

(a) Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$

(b) Three points A, B, C are collinear, if any two vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ are parallel.

9. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT):

(a) $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad (0 \le \theta \le \pi)$, θ is angle between $\vec{a}$ & $\vec{b}$.

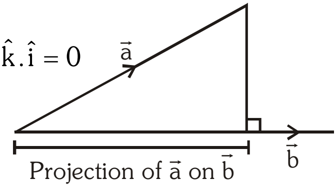
Note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

(b) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(c) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$; $(\vec{a}, \vec{b} \neq 0)$

(d) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

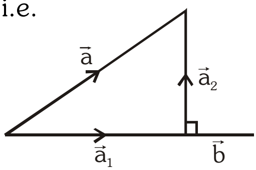
(e) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.



Note :

(i) The vector component of $\vec{a}$ along $\vec{b}$ i.e.

$\vec{a}_1 = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular



to $\vec{b}$ i.e. $\vec{a}_2 = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ $\qquad (\vec{a} = \vec{a}_1 + \vec{a}_2)$

(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by

$$\cos\phi = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \quad 0 \le \phi \le \pi$$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\vec{a}| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}}, \quad |\vec{b}| = \sqrt{b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2}}$$

(iv) $-|\vec{a}|\,|\vec{b}| \le \vec{a}.\vec{b} \le |\vec{a}|\,|\vec{b}|$

(v) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

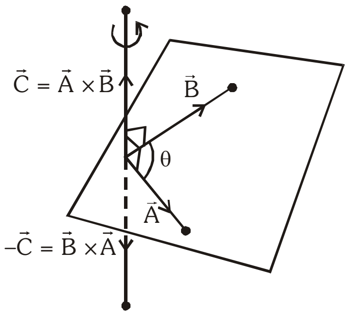
(vi) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}-\hat{b})$, $\lambda \in R^+$

(vii) $|\vec{a}\pm\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2\pm 2\vec{a}.\vec{b}$

(viii) $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

10. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system .



(b) Lagranges Identity : For any two vectors $\vec{a}$ & $\vec{b}$;

$$(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}|=\sqrt{\vec{a}^2\vec{b}^2-(\vec{a}.\vec{b})^2}$ **(ii)** $\vec{c}.\vec{a}=0$; $\vec{c}.\vec{b}=0$ and

(iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(d) $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) ($\vec{a}\neq 0$, $\vec{b}\neq 0$) i.e. $\vec{a}=K\vec{b}$, where K is a scalar

(i) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

(ii) $(m\vec{a}) \times \vec{b} = \vec{a} x (m\vec{b}) = m(\vec{a} \times \vec{b})$ where m is a scalar.

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

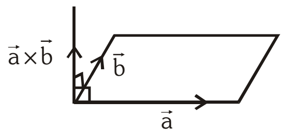
(vi) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

(e) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) (i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

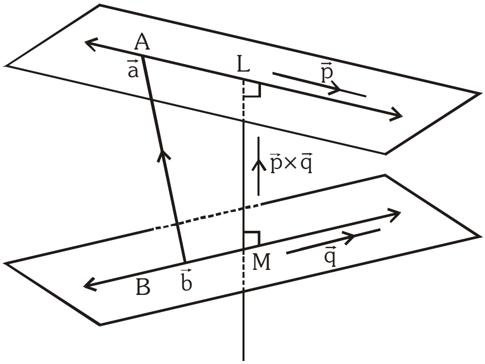
(h) Vector area :

(i) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle $ABC = \dfrac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$.

The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

(ii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$. Area of $\Delta = \dfrac{1}{2}|\vec{a} \times \vec{b}|$

11. SHORTEST DISTANCE BETWEEN TWO LINES :



Lines which do not intersect & are also not parallel are called skew lines. In other words the lines which are not coplanar are skew lines. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$

i.e. $\vec{LM}$ = | Projection of $\vec{AB}$ on $\vec{LM}$ |

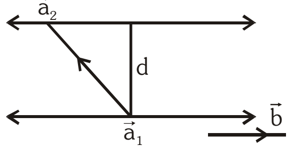
$$= |\text{Projection of } \vec{AB} \text{ on } \vec{p}\times\vec{q}\,|$$

$$= \left|\frac{\vec{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(a) The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0

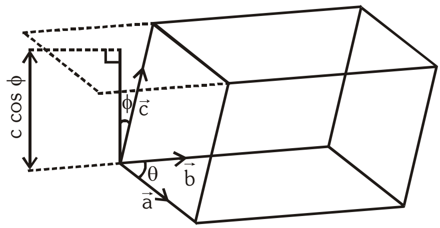
i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$ $\Rightarrow \left[\left(\vec{b}-\vec{a}\right)\ \vec{p}\ \ \vec{q}\right] = 0$

(b) If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$



12. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(a) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as: $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(b) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(c) $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(d) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(e) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(f) $[\hat{i}\,\hat{j}\,\hat{k}] = 1$; $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$; $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(g) **(i)** The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(ii) Volume of parallelopiped whose co-terminus edges are $\vec{a}, \vec{b}$ & $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$.

(h) **Remember that :**

(i) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

(ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

13. VECTOR TRIPLE PRODUCT :

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be any three vectors, then that expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

(a) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(b) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} . \vec{c})\vec{b} - (\vec{b} . \vec{c})\vec{a}$

(c) $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

14. LINEAR COMBINATIONS / LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

Linear combination of vectors :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$,........... then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots\ldots\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$,........ for any x, y, z .... $\in R$. We have the following results :

(a) If $\vec{x}_1, \vec{x}_2, \ldots.\vec{x}_n$ are n non zero vectors, & $k_1, k_2, \ldots.k_n$ are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots..k_n\vec{x}_n = 0$ $\Rightarrow k_1 = 0, k_2 = 0 \ldots\ldots k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, \ldots.\vec{x}_n$ are **linearly independent vectors**

(b) If $\vec{x}_1, \vec{x}_2, \ldots\ldots.\vec{x}_n$ are **not linearly independent** then they are said to be linear dependent vectors. i.e. if $k_1\vec{x}_1, k_2\vec{x}_2 + \ldots\ldots.k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, \ldots\vec{x}_n$ are said to be **linearly dependent.**

(c) **Fundamental theorem in plane** : let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$

(d) **Fundamental theorem in space** : let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

15. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$

where, $x+y+z+w=0$

16. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$ then the two systems are called Reciprocal System of vectors.

Note : $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

17. TETRAHEDRON :

(i) Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point of concurrecy is called the centre of the tetrahedron.

(ii) In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

(iii) The angle between any two plane faces of regular tetrahedron is $\cos^{-1}\dfrac{1}{3}$

3D-COORDINATE GEOMETRY

1. DISTANCE FORMULA :

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

2. SECTION FORMULAE :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(x, y, z)$ divide PQ in the ratio $m_1 : m_2$. Then R is

$$(x, y, z) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

Mid point of PQ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

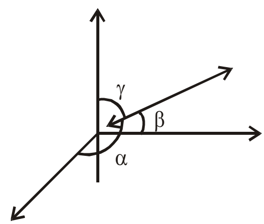
3. CENTROID OF A TRIANGLE :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

4. DIRECTION COSINES OF LINE :

If α, β, γ be the angles made by a line with x-axis, y-axis & z-axis respectively then $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are called direction cosines of a line, denoted by l, m & n respectively and the relation between ℓ, m, n is given by $\ell^2 + m^2 + n^2 = 1$



D. cosine of x-axis, y-axis & z-axis are respectively

1, 0, 0; 0, 1, 0; 0, 0, 1

5. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line.

i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

It is easy to see that there can be infinitely many sets of direction ratios for a given line.

6. RELATION BETWEEN D.C'S & D.R'S :

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2}$$

$$\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \; ; \quad m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} \; ; \quad n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

7. DIRECTION COSINE OF AXES :

Direction ratios and Direction cosines of the line joining two points :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the d.c.'s of AB are $\frac{1}{r}(x_2 - x_1)$, $\frac{1}{r}(y_2 - y_1)$, $\frac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]} = |\overrightarrow{AB}|$

8. PROJECTION OF A LINE ON ANOTHER LINE :

Let PQ be a line segment with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let L be a straight line whose d.c.'s are ℓ, m, n. Then the length of projection of PQ on the line L is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

9. ANGLE BETWEEN TWO LINES :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = l_1 l_2 + m_1m_2 + n_1n_2$. If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by

$$\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

10. PERPENDICULARITY AND PARALLELISM :

Let the two lines have their d.c.'s given by l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $l_1 l_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note:
If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

11. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) One point form : Let $A(x_1, y_1, z_1)$ be a given point on the straight line and l, m, n the d.c's of the line, then its equation is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(say)}$$

It should be noted that $P(x_1 + lr, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. AP = r. One should note that for AP = r; l, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$ but here $AP \neq r$

(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

is $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

12. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE :

Let equation of the line be

$$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \qquad \text{(say)} \qquad \ldots\ldots\ldots \text{(i)}$$

and A (α, β, γ) be the point. Any point on the line (i) is

$P(\ell r + x_1, mr + y_1, nr + z_1)$ .... (ii)

If it is the foot of the perpendicular, from A on the line, then AP is $\perp$ to the line, so $\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)m + (\gamma - z_1)n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r + x_1 - \alpha} = \frac{y-\beta}{mr + y_1 - \beta} = \frac{z-\gamma}{nr + z_1 - \gamma}$$

13. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form $ax + by + cz + d = 0$, in which a, b, c are constants, where $a^2 + b^2 + c^2 \neq 0$ (i.e. a, b, $c \neq 0$ simultaneously).

(a) Vector form of equation of plane :

If $\vec{a}$ be the position vector of a point on the plane and $\vec{n}$ be a vector normal to the plane then it's vectorial equation is given by $(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$ where $d = \vec{a}.\vec{n} =$ constant.

(b) Plane Parallel to the Coordinate Planes :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

(iv) Equation of the plane parallel to x-y plane at a distance c is $z = c$. Similarly, planes parallel to y-z plane and z-x plane are respectively $x = c$ and $y = c$.

(c) Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $by + cz + d = 0$.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

(d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(e) Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is $lx + my + nz = p$.

(f) Vectorial form of Normal equation of plane :

If $\hat{n}$ is a unit vector normal to the plane from the origin to the plane and d be the perpendicular distance of plane from origin then its vector equation is $\vec{r}.\hat{n} = d$.

(g) Equation of a Plane through three points :

The equation of the plane through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)\ (x_3, y_3, z_3)$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

14. ANGLE BETWEEN TWO PLANES :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if $aa' + bb' + cc' = 0$ and they are parallel if $a/a' = b/b' = c/c'$.

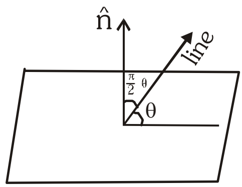
Planes parallel to a given Plane :

Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$. d' is to be found by other given condition.

15. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and $ax + by + cz + d = 0$ respectively and θ be the angle which line makes with the plane. Then $(\pi/2 - \theta)$ is the angle between the line and the normal to the plane.

So $$\sin\theta = \frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(\ell^2 + m^2 + n^2)}}$$



Line is parallel to plane if $\theta = 0$

i.e. if $al + bm + cn = 0$.

Line is $\perp$ to the plane if line is parallel to the normal of the plane

i.e. if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$.

16. CONDITION IN ORDER THAT THE LINE MAY LIE ON THE GIVEN PLANE :

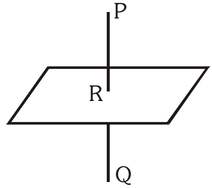
The line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ will lie on the plane $Ax + By + Cz + D = 0$

if **(a)** $A\ell + Bm + Cn = 0$ and **(b)** $Ax_1 + By_1 + Cz_1 + D = 0$

17. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs.

18. IMAGE OF A POINT IN THE PLANE :



Let the image of a point $P(x_1, y_1, z_1)$

in a plane $ax + by + cz + d = 0$ is

$Q(x_2, y_2, z_2)$ and foot of perpendicular

of point P on plane is $R(x_3, y_3, z_3)$, then

(a) $$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

(b) $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -2\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

19. CONDITION FOR COPLANARITY OF TWO LINES :

Let the two lines be

$$\frac{x - \alpha_1}{\ell_1} = \frac{y - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1} \quad \text{..........(i)}$$

and $$\frac{x - \alpha_2}{\ell_2} = \frac{y - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2} \quad \text{..........(ii)}$$

These lines will coplanar if $\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

the plane containing the two lines is $\begin{vmatrix} x - \alpha_1 & y - \beta_1 & z - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

20. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

Distance between two parallel planes $ax + by + cz + d_1 = 0$

& $ax + by + cz + d_2 = 0$ is - $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

21. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes

$u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0$, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

22. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

(a) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign give the bisector of the angle which contains the origin.

(b) Bisector of acute/obtuse angle : First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \quad \Rightarrow$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \quad \Rightarrow$ origin lies in acute angle

PROBABILITY

1. SOME BASIC TERMS AND CONCEPTS

(a) An Experiment : An action or operation resulting in two or more outcomes is called an experiment.

(b) Sample Space : The set of all possible outcomes of an experiment is called the sample space, denoted by S. An element of S is called a sample point.

(c) Event : Any subset of sample space is an event.

(d) Simple Event : An event is called a simple event if it is a singleton subset of the sample space S.

(e) Compound Events : It is the joint occurrence of two or more simple events.

(f) Equally Likely Events : A number of simple events are said to be equally likely if there is no reason for one event to occur in preference to any other event.

(g) Exhaustive Events : All the possible outcomes taken together in which an experiment can result are said to be exhaustive.

(h) Mutually Exclusive or Disjoint Events : If two events cannot occur simultaneously, then they are mutually exclusive.
If A and B are mutually exclusive, then $A \cap B = \phi$.

(i) Complement of an Event : The complement of an event A, denoted by $\bar{A}$, A' or A^C, is the set of all sample points of the space other then the sample points in A.

2. MATHEMATICAL DEFINITION OF PROBABILITY

Let the outcomes of an experiment consists of *n* exhaustive mutually exclusive and equally likely cases. Then the sample spaces S has *n* sample points. If an event A consists of m sample points, $(0 \le m \le n)$, then the probability of event A, denoted by P(A) is defined to be m/n i.e. P(A) = m/n.

Let S = $a_1, a_2, \ldots\ldots, a_n$ be the sample space

(a) $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.

(b) $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.

(c) If $A_i = \{a_i\}$, i = 1, ..., n then A_i is the event corresponding to a single sample point a_i. Then $P(A_i) = \frac{1}{n}$.

(d) $0 \le P(A) \le 1$

3. ODDS AGAINST AND ODDS IN FAVOUR OF AN EVENT :

Let there be m + n equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition, probability of occurrences of event A = $P(A) = \frac{m}{m+n}$

The probability of non-occurrence of event A = $P(A') = \frac{n}{m+n}$

$\therefore$ P(A) : P(A') = m : n

Thus the odd in favour of occurrences of the event A are defined by m : n i.e. P(A) : P(A'); and the odds against the occurrence of the event A are defined by n : m i.e. P(A') : P(A).

4. ADDITION THEOREM

(a) If A and B are any events in S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the probability of an event is a nonnegative number, it follows that

$P(A \cup B) \le P(A) + P(B)$

For three events A, B and C in S we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

General form of addition theorem

(Principle of Inclusion-Exclusion)

For n events $A_1, A_2, A_3, \ldots\ldots A_n$ in S, we have

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots\ldots \cup A_n)$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots\ldots$$

$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n)$$

(b) If A and B are mutually exclusive, then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B)$.

5. CONDITIONAL PROBABILITY :

If A and B are any events in S then the conditional probability of B relative to A, i.e. probability of occurence of B when A has occured, is given by

$$P(B/A) = \frac{P(B \cap A)}{P(A)}. \quad \text{If } P(A) \neq 0$$

6. MULTIPLICATION THEOREM

Independent event :

So if A and B are two independent events then happening of B will have no effect on A.

(a) When events are independent :

$P(A/B) = P(A)$ and $P(B/A) = P(B)$, then

$P(A \cap B) = P(A).\ P(B)$ OR $P(AB) = P(A) . P(B)$

(b) When events are not independent

The probability of simultaneous happening of two events A and B is equal to the probability of A multiplied by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect to B) i.e

$P(A \cap B) = P(A) . P(B/A)$ or $P(B) . P(A/B)$

OR

$P(AB) = P(A) . P(B/A)$ or $P(B) . P(A/B)$

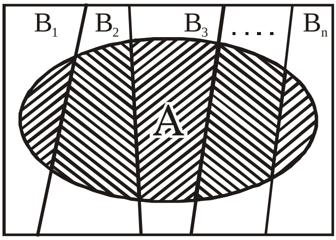
(c) Probability of at least one of the n Independent events

If p_1 , p_2 , p_3 , ... p_n are the probabilities of n independent events A_1 , A_2, A_3 ..... A_n then the probability of happening of at least one of these event is

$1 - [(1 - p_1) (1 - p_2) \ldots\ldots (1 - p_n)]$

$\Rightarrow\ P(A_1 + A_2 + A_3 + \ldots + A_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2)\, P(\bar{A}_3) \ldots P(\bar{A}_n)$

7. TOTAL PROBABILITY THEOREM :



Let an event A of an experiment occurs with its n mutually exclusive & exhaustive events $B_1, B_2, B_3, \ldots\ldots B_n$
then total probability of occurence of even A is

$$P(A) = P(AB_1) + P(AB_2) + \ldots\ldots + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$$

$$\Rightarrow\ P(A) = P(B_1)\, P(A \mid B_1) + P(B_2)\, P(A \mid B_2) + \ldots + P(B_n)\, P(A \mid B_n)$$
$$= \Sigma P(B_i)\, P(A \mid B_i)$$

8. BAYE'S THEOREM OR INVERSE PBOBABILITY :

Let A_1, A_2, ..., A_n be n mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then $P\left(\frac{A_i}{A}\right) = \frac{P(A_i)P(A/A_i)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$

9. BINOMIAL DISTRIBUTION FOR REPEATED TRIALS

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment.

Outcomes of such an experiment are known as success and failure.

Probability of success is denoted by p and probability of failure by q.

$\therefore$ p + q = 1

If binomial experiment is repeated n times, then

(a) Probability of exactly r successes in n trials = ${}^nC_r p^r q^{n-r}$

(b) Probability of at most r successes in n trails = $\sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$

(c) Probability of atleast r successes in n trails = $\sum_{\lambda=r}^{n} {}^nC_\lambda p^\lambda q^{n-\lambda}$

(d) Probability of having I^{st} success at the r^{th} trials = $p\,q^{r-1}$.

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

Note : $(p+q)^n = {}^nC_0 q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + {}^nC_r p^r q^{n-r} + \ldots\ldots + {}^nC_n p^n = 1$

10. SOME IMPORTANT RESULTS

(a) Let A and B be two events, then

(i) $P(A) + P(\bar{A}) = 1$

(ii) $P(A+B) = 1 - P(\bar{A}\bar{B})$

(iii) $P(A/B) = \dfrac{P(AB)}{P(B)}$

(iv) $P(A+B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$

(v) $A \subset B \Rightarrow P(A) \le P(B)$

(vi) $P(\bar{A}B) = P(B) - P(AB)$

(vii) $P(AB) \le P(A)\,P(B) \le P(A+B) \le P(A) + P(B)$

(viii) $P(AB) = P(A) + P(B) - P(A+B)$

(ix) P(Exactly one event) $= P(A\bar{B}) + P(\bar{A}B)$

$= P(A) + P(B) - 2P(AB) = P(A+B) - P(AB)$

(x) P(neither A nor B) $= P(\bar{A}\bar{B}) = 1 - P(A+B)$

(xi) $P(\bar{A} + \bar{B}) = 1 - P(AB)$

(b) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) $= 2^n$

(c) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) $= 6^n$

(d) Playing Cards :

(i) Total Cards : 52(26 red, 26 black)

(ii) Four suits : Heart, Diamond, Spade, Club - 13 cards each

(iii) Court Cards : 12 (4 Kings, 4 queens, 4 jacks)

(iv) Honour Cards : 16 (4 aces, 4 kings, 4 queens, 4 jacks)

(e) Probability regarding n letters and their envelopes :

If n letters are placed into n directed envelopes at random, then

(i) Probability that all letters are in right envelopes $= \frac{1}{n!}$.

(ii) Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$

(iii) Probability that no letters is in right envelopes

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$$

(iv) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$$

11. PROBABILITY DISTRIBUTION :

(a) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by :

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \quad (\text{ Since } \Sigma p_i = 1)$$

(c) Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 . p_i$

$\sigma^2 = \sum p_i x_i^2 - \mu^2$ (Note that Standard Deviation (SD) = $+\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variate 'X' is given by; $P(X = r) = {}^nC_r\, p^r q^{n-r}$ where: p = probability of success in a single trial, q = probability of failure in a single trial and p + q = 1.

(e) Mean of Binomial Probability Distribution (BPD) = np ; variance of BPD = npq.

(f) If p represents a person's chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

STATISTICS

MEASURES OF CENTRAL TENDENCY :

An average value or a central value of a distribution is the value of variable which is representative of the entire distribution, this representative value are called the measures of central tendency.

Generally there are following five measures of central tendency :

(a) Mathematical average

(i) Arithmetic mean (ii) Geometric mean

(iii) Harmonic mean

(b) Positional average

(i) Median (ii) Mode

1. ARITHMETIC MEAN/MEAN :

(i) For ungrouped dist. : If $x_1, x_2, \ldots\ldots x_n$ are n values of variate x_i then their mean $\bar{x}$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + \ldots.. + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow \Sigma x_i = n\bar{x}$

(ii) For ungrouped and grouped freq. dist. : If $x_1, x_2, \ldots. x_n$ are values of variate with corresponding frequencies $f_1, f_2, \ldots f_n$ then their mean is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots. + f_nx_n}{f_1 + f_2 + \ldots. + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

(iii) By short cut method :

Let $d_i = x_i - a$

$\therefore \bar{x} = a + \frac{\Sigma f_i d_i}{N}$, where a is assumed mean

(iv) By step deviation method :

Let $u_i = \frac{d_i}{h} = \frac{x_i - a}{h}$

$\therefore \quad \bar{x} = a + \left(\frac{\Sigma f_i u_i}{N}\right)h$

(v) Weighted mean : If $w_1, w_2, \ldots\ldots w_n$ are the weights assigned to the values $x_1, x_2, \ldots\ldots x_n$ respectively then their weighted mean is defined as

$$\text{Weighted mean} = \frac{w_1x_1 + w_2x_2 + \ldots\ldots + w_nx_n}{w_1 + \ldots\ldots + w_n} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

(vi) Combined mean : If $\bar{x}_1$ and $\bar{x}_2$ be the means of two groups having n_1 and n_2 terms respectively then the mean (combined mean) of their composite group is given by combined mean

$$= \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

If there are more than two groups then,

$$\text{combined mean} = \frac{n_1\bar{x}_1 + n_1\bar{x}_2 + n_3\bar{x}_3 + \ldots.}{n_1 + n_2 + n_3 + \ldots.}$$

(vii) Properties of Mean :

- Sum of deviations of variate about their mean is always zero i.e. $\Sigma(x_i - \bar{x}) = 0$, $\Sigma f_i(x_i - \bar{x}) = 0$
- Sum of square of deviations of variate about their mean is minimum i.e. $\Sigma(x_i - \bar{x})^2$ is minimum
- If $\bar{x}$ is the mean of variate x_i then mean of $(x_i + \lambda)$ is $\bar{x} + \lambda$

  mean of $(\lambda x_i) = \lambda\bar{x}$

  mean of $(ax_i + b)$ is $a\bar{x} + b$ (where λ, a, b are constant)
- Mean is independent of change of assumed mean i.e. it is not effected by any change in assumed mean.

2. MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divides an arranged series into two equal parts.

Formulae of median :

(i) For ungrouped distribution : Let n be the number of variate in a series then

$$\text{Median} = \begin{cases} \left(\dfrac{n+1}{2}\right)^{th} \text{ term, (when n is odd)} \\ \text{Mean of} \left(\dfrac{n}{2}\right)^{th} \text{ and } \left(\dfrac{n}{2}+1\right)^{th} \text{ terms, (when n is even)} \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and Find value of N then

$$\text{Median} = \begin{cases} \left(\dfrac{N+1}{2}\right)^{th} \text{ term, (when N is odd)} \\ \text{Mean of} \left(\dfrac{N}{2}\right)^{th} \text{ and } \left(\dfrac{N}{2}+1\right)^{th} \text{ terms, (when N is even)} \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\dfrac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class

$$\therefore \text{ Median} = \ell + \frac{\left(\dfrac{N}{2} - F\right)}{f} \times h$$

where ℓ — lower limit of median class

f — freq. of median class

F — c.f. of the class preceding median class

h — Class interval of median class

3. MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency

Method for determining mode :

(i) For ungrouped dist. : The value of that variate which is repeated maximum number of times

(ii) For ungrouped freq. dist. : The value of that variate which have maximum frequency.

(iii) For grouped freq. dist. : First we find the class which have maximum frequency, this is model calss

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where ℓ — lower limit of model class

f_0 — freq. of the model class

f_1 — freq. of the class preceding model class

f_2 — freq. of the class succeeding model class

h — class interval of model class

4. RELATION BETWEEN MEAN, MEDIAN AND MODE :

In a moderately asymmetric distribution following is the relation between mean, median and mode of a distribution. It is known as impirical formula.

Mode = 3 Median – 2 Mean

Note : (i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode coincide.

5. MEASURES OF DISPERSION :

The dispersion of a statistical distribution is the measure of deviation of its values about the their average (central) value.

Generally the following measures of dispersion are commonly used.

(i) Range

(ii) Mean deviation

(iii) Variance and standard deviation

(i) Range : The difference between the greatest and least values of variate of a distribution, are called the range of that distribution. If the distribution is grouped distribution, then its range is the difference between upper limit of the maximum class and lower limit of the minimum class.

Also, coefficient of range = $\dfrac{\text{difference of extreme values}}{\text{sum of extreme values}}$

(ii) Mean deviation (M.D.) : The mean deviation of a distribution is, the mean of absolute value of deviations of variate from their statistical average (Mean, Median, Mode).

If A is any statistical average of a distribution then mean deviation about A is defined as

Mean deviation = $\dfrac{\sum_{i=1}^{n}|x_i - A|}{n}$ (for ungrouped dist.)

Mean deviation = $\dfrac{\sum_{i=1}^{n}f_i\,|x_i - A|}{N}$ (for freq. dist.)

Note :- It is minimum when taken about the median

Coefficient of Mean deviation = $\dfrac{\text{Mean deviation}}{A}$

(where A is the central tendency about which Mean deviation is taken)

(iii) Variance and standard deviation : The variance of a distribution is, the mean of squares of deviation of variate from their mean. It is denoted by σ^2 or var(x).

The positive square root of the variance are called the standard deviation. It is denoted by σ or S.D.

Hence standard deviation = $+\sqrt{\text{variance}}$

Formulae for variance :

(i) for ungrouped dist. :

$$\sigma_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$\sigma_x^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2, \quad \text{where } d_i = x_i - a$$

(ii) For freq. dist. :

$$\sigma_x^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N}$$

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2 = \frac{\Sigma f_i x_i^2}{N} - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\sigma_u^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] \quad \text{where } u_i = \frac{d_i}{h}$$

(iii) Coefficient of S.D. $= \frac{\sigma}{\bar{x}}$

Coefficient of variation $= \frac{\sigma}{\bar{x}} \times 100$ (in percentage)

Note :- $\sigma^2 = \sigma_x^2 = \sigma_d^2 = h^2 \sigma_u^2$

6. MATHEMATICAL PROPERTIES OF VARIANCE :

- Var.$(x_i + \lambda)$ = Var.(x_i)
  Var.$(\lambda x_i) = \lambda^2$.Var$(x_i)$
  Var$(ax_i + b) = a^2$.Var(x_i)
  where λ, a, b, are constant

- If means of two series containing n_1, n_2 terms are $\bar{x}_1, \bar{x}_2$ and their variance's are σ_1^2, σ_2^2 respectively and their combined mean is $\bar{x}$ then the variance σ^2 of their combined series is given by following formula

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{(n_1 + n_2)} \text{ where } d_i = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}$$

$$\text{i.e. } \sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

MATHEMATICAL REASONING

1. STATEMENT :

A declarative sentence which is either true or false but not both, is called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F.

2. SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement.

3. COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement.

Here the simple statements which form a compound statement are known as its sub statements.

4. LOGICAL CONNECTIVES :

The words or phrases which combines simple statements to form a compound statement are called logical connectives.

S.N.	Connectives	Symbol	Use	Operation
1.	and	$\wedge$	$p \wedge q$	conjunction
2.	or	$\vee$	$p \vee q$	disjunction
3.	not	$\sim$ or '	$\sim p$ or p'	negation
4.	If .... then	$\Rightarrow$ or $\rightarrow$	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

5. TRUTH TABLE :

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$(\sim p)$
T	F
F	T

Conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

6. LOGICAL EQUIVALENCE :

Two compound statements $S_1(p, q, r\ldots)$ and $S_2(p, q, r \ldots.)$ are said to be logically equivalent or simply equivalent if they have same truth values for all logical possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

i.e. $\boxed{p \to q \equiv \sim p \vee q}$

7. TAUTOLOGY AND CONTRADICTION :

(i) Tautology : A statement is said to be a tautology if it is true for all logical possibilities

i.e. its truth value always T. it is denoted by t.

(ii) **Contradiction/Fallacy :** A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

8. DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then we obtain its dual replacing t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) the connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p,q)$ is the dual of the compound statement $S(p,q)$ then

(a) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ (b) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

9. CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT ($p \to q$):

(i) **Converse :** The converse of the conditional statement $p \to q$ is defined as $q \to p$

(ii) **Inverse :** The inverse of the conditional statement $p \to q$ is defined as $\sim p \to \sim q$

(iii) **Contrapositive :** The contrapositive of conditional statement $p \to q$ is defined as $\sim q \to \sim p$.

Note : $(p \to q) \equiv (\sim q \to \sim p) \equiv (\sim p \vee q)$.

10. NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) Negation of conjunction : $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(iii) Negation of conditional : $\sim(p \rightarrow q) \equiv p \wedge \sim q$

(iv) Negation of biconditional : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\equiv (\sim p \leftrightarrow q) \equiv (p \leftrightarrow \sim q)$

As we know that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$$\therefore \sim(p \leftrightarrow q) \equiv \sim[(p \rightarrow q) \wedge (q \rightarrow p)]$$

$$\equiv \sim(p \rightarrow q) \vee \sim(q \rightarrow p)$$

$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

11. ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some low of algebra of statements are as follow

(i) Idempotent Laws :

(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

(ii) Commutative laws :

(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

(iii) Associative laws :

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(iv) Distributive laws :

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(v) De Morgan Laws :

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(vi) Involution laws (or Double negation laws) : $\sim(\sim p) \equiv p$

(vii) Identity Laws : If p is a statement and t and c are tautology and contradiction respectively then

(a) $p \wedge t \equiv p$ (b) $p \vee t \equiv t$ (c) $p \wedge c \equiv c$ (d) $p \vee c \equiv p$

(viii) Complement Laws :

(a) $p \wedge (\sim p) \equiv c$ (b) $p \vee (\sim p) \equiv t$

(c) $(\sim t) \equiv c$ (d) $(\sim c) \equiv t$

12. QUANTIFIED STATEMENTS AND QUANTIFIERS :

The words or phrases "All", "Some", "None", "There exists a" are examples of quantifiers.

A statement containing one or more of these words (or phrases) is a quantified statement.

Note : Phrases "There exists a" and "Atleast one" and the word "some" have the same meaning.

NEGATION OF QUANTIFIED STATEMENTS :

(1) '**None**' is the negation of '**at least one**' or '**some**' or '**few**'
Similarly negation of '**some**' is '**none**'

(2) The negation of "**some A are B**" or "**There exist A which is B**" is "**No A are (is) B**" or "**There does not exist any A which is B".**

(3) Negation of "**All A are B**" is "**Some A are not B**".

SETS

1. SET :

A set is a collection of well defined objects which are distinct from each other.

Set are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by a, b, c .... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$,

2. SOME IMPORTANT NUMBER SETS :

N = Set of all natural numbers

= {1, 2, 3, 4, ...}

W = Set of all whole numbers

= {0, 1, 2, 3, ...}

Z or I set of all integers

= {.... –3, –2, –1, 0, 1, 2, 3, ...}

Z^+ = Set of all +ve integers

= {1, 2, 3, ...} = N.

Z^- = Set of all –ve integers

= (–1, –2, –3, ...}

Z_0 = The set of all non-zero integers.

= {±1, ±2, ±3, ...}

Q = The set of all rational numbers.

$$= \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$$

R = the set of all real numbers.

R–Q = The set of all irrational numbers

3. REPRESENTATION OF A SET :

(i) Roster Form : In this form a set is described by listing elements, separated by commas and enclose then by curly brackets

(ii) Set Builder Form : In this case we write down a property or rule p Which gives us all the element of the set

$$A = \{x : P(x)\}$$

4. TYPES OF SETS :

Null set or Empty set : A set having no element in it is called an Empty set or a null set or void set it is denoted by ϕ or { }

A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton : A set consisting of a single element is called a singleton set.

Finite Set : A set which has only finite number of elements is called a finite set.

Order of a finite set : The number of elements in a finite set is called the order of the set A and is denoted O(A) or n(A). It is also called cardinal number of the set.

Infinite set : A set which has an infinite number of elements is called an infinite set.

Equal sets : Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A. If sets A and B are equal. We write A = B and if A and B are not equal then $A \neq B$.

Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same

i.e. $n(A) = n(B)$

Note : Equal sets are always equivalent but equivalent sets may not be equal.

Subsets : Let A and B be two sets if every element of A is an element B, then A is called a subset of B i.e. $A \subseteq B$

Proper subset : If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

Note-1 : Every set is a subset of itself i.e. $A \subseteq A$ for all A

Note-2 : Empty set ϕ is a subset of every set

Note-3 : Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

Note-4 : The total number of subsets of a finite set containing n elements is 2^n

Universal set : A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

Some Operation on Sets :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$

(iv) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

(v) **De-Morgan Laws :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(vi) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(vii) **Distributive Laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(viii) **Commutative Laws :** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(ix) **Associative Laws :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(x) $A \cap \phi = \phi$; $A \cap U = A$

$A \cup \phi = A$; $A \cup U = U$

(xi) $A \cap B \subseteq A$; $A \cap B \subseteq B$

(xii) $A \subseteq A \cup B$; $B \subseteq A \cup B$

(xiii) $A \subseteq B \Rightarrow A \cap B = A$

(xiv) $A \subseteq B \Rightarrow A \cup B = B$

Disjoint Sets :

IF $A \cap B = \phi$, then A, B are disjoint.

Note : $A \cap A' = \phi \quad \therefore$ A, A' are disjoint.

Symmetric Difference of Sets :

$A \Delta B = (A - B) \cup (B - A)$

- $(A')' = A$
- $A \subseteq B \Leftrightarrow B' \subseteq A'$

If A and B are any two sets, then

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

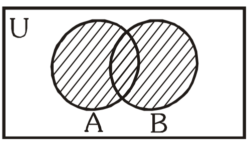
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

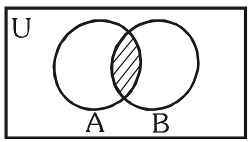
(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

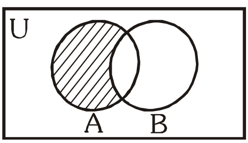
Venn Diagram :



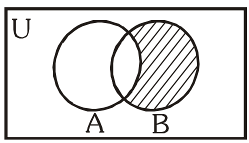
$A \cup B$



$A \cap B$

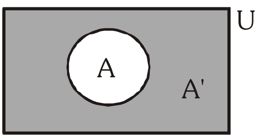


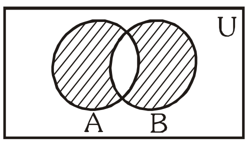
$A - B$



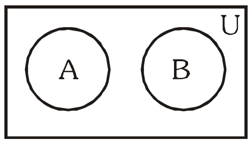
$B - A$

Clearly $(A - B) \cup (B - A) \cup (A \cup B) = A \cup B$





$(A\Delta B)=(A-B) \cup (B-A)$



Disjoint Sets

Note : $A \cap A' = \phi$, $A \cup A' = U$

5. SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint sets

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ (A – B) and (B – A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

(x) If A_1, A_2 .... A_n are finite sets, then

$$n\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - .... + (-1)^{n-1} n(A_1 \cap A_2 \cap .....A_n)$$

RELATIONS

1. INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.

thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

Total Number of Relations : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So total number of subsets of $A \times B$ is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, $\text{Domain (R)} = \{a : (a, b) \in R\}$

and, $\text{Range (R)} = \{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Inverse Relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, $\text{Domain}(R) = \text{Range}(R^{-1})$ and $\text{Range}(R) = \text{Domain}(R^{-1})$

Note : Relation on a set : If R is a relation from set A to A itself then R is called Relation on set A.

2. TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

Void Relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a , a) \notin R$.

Every Identity relation is reflexive but every reflexive relation is not identity.

Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Leftrightarrow (b, a) \in R$

i.e. $a R b \Leftrightarrow bRa$

Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

i.e. $a R b$ and $b R c \Rightarrow a R c$

Antisymmetric Relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$

Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

It is not necessary that every relation which is symmetric and transitive is also reflexive.

Important Notes