# Birla Institute of Technology & Science, Pilani

**Department of Mathematics**

Second Semester 2021-2022

**MATH F243** **ProblemSheet-1**

**Q.1** Let T be a tree:
a) Use the fact that there is a unique path between any pair of vertices in a tree to show that if v is an internal vertex of T then T - v is not connected.
b) If v is a leaf in a tree T show that T - v is connected.
**Ans:** a) Use the fact that there is a unique path between any pair of vertices in a tree to show that if v is an internal vertex of T then T - v is not connected. By the definition of internal d(v) > 1, so v has at least two neighbors in T, u1 and u2 say. But since T is a tree there is a unique path from u1 to u2, which must be u1, v, u2 Now T - v has no u1 u2–path, and so is disconnected.

b) Show that if v is a leaf in T then T - v is connected. Let T be a tree and v a leaf of T. By the definition of leaf d(v) = 1, so v has exactly one neighbor, u say. Let x and y be any pair of vertices in T - v. Since T is a tree there is a unique xy–path in T, P say. The degree of an internal vertex of a path must be at least 2, but the degree of v is one, so v does not appear as an internal vertex of P. Thus P only contains vertices in T - v and so connects x and y in T - v.

**Q.2** How many different total number of connected undirected simple graphs are possible on the 6 labeled vertices such that degree of the vertices a1, a3, and a5 is one?
**Ans:** 108
Vertices a2, a4, a6 can be in a straight line in 3 different ways with the other 3 vertices having 3 different possible positions giving $3 \times 3^3$ And when a2, a4, a6 form a cycle, the other threes again have 3 possible positions giving $3^3$.

**Q.3** Explain why in every social gathering there are at least two persons who are friends with the same number of persons.
**Ans:** $2^M$ where $M = {}^nC_2$ (Hand-shaking theorem)

**Q.4** Let G be a simple graph where the vertices correspond to each of the squares of an $8 \times 8$ chess board and where two squares are adjacent if, and only if, a knight can go from one square to the other in one move. What is/are the possible degree(s) of a vertex in G? How many vertices have each degree? How many edges does G have?
**Ans:** G has 168 edges
A knight can move two squares vertically and one square horizontally, or two squares horizontally and one square vertically, on a chessboard. Checking all possibilities we see that, the four "corner" vertices of the graph have degree 2; the eight "edge" vertices that are next to the corners have degree 3; twenty vertices ( remaining 16 edge vertices plus four more that are next to the corners but not on the edge) have degree 4; sixteen vertices have degree 6; the remaining 16 "interior" vertices have degree 8. For the last part of the question use Hand-Shaking Theorem.

**Q.5** If a simple graph with no isolated vertices has no induced subgraph with exactly two edges, then it is a complete graph.
**Ans:**
Let G be such a graph. If G is disconnected, then edges from two components yield four vertices that induce a subgraph with two edges. If G is connected and not complete, then G has nonadjacent vertices say, x and y. Let Q be a shortest x, y-path; it has length at least 2. Any three successive vertices on Q induce $P_3$, with two edges.

**Q.6** Prove that there exist a graph G with degree sequence d = $(d_1, \ldots, d_{2k})$ $d_{2i} = {}_{2i-1} = i$ $1 < i < k$.
**Ans:**This is the degree list for the bipartite graph with vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ defined by $x_r$ $y_s$ if and only if r +s > k. Since the neighborhood of $x_r$ is $y_k$ , $y_{k+1}, \ldots, y_{kr+1}$, the degree of $x_r$ is r. Thus the graph has two vertices of each degree from 1 to k.

**Q.7** Give a counterexample to the following statement and add a hypothesis to correct it:
If e is a cut-edge in G, then at least one endpoint of e is a cut vertex.
**Ans:** $P_2$ provides a counterexample to the given statement. The corrected assertion is: Let e be a cutedge in G. If the component of G containing e has more than two vertices, then at least one endpoint of G is a cut vertex.
For the proof of the corrected assertion, let H be the component of G containing e and let u and v be the endpoints of e. Further let H1 and H2 be the two components of $H \setminus e$ with u ∈ V (H1) and v ∈ V (H2). Since n(H) > 3, (at least) one of H1 and H2 has (at least) two vertices. Because of symmetry we can assume that it is H1. Choose w ∈ $V(H1) \setminus u$. Every w, v-path uses the edge e. Therefore deleting u separates w from v.

**Q.8** Prove that if a connected graph G remains connected after removing an edge e from G iff e is part of some cycle/circuit in G.
**Ans:** ( =>): Let e be an arbitrary edge connecting vertices u and v in G. We know that G - e is connected. So there is another uv-path so that u and v are connected and paths that contained e in G will now be connected with the help of the uv-path in G-e. So, in G, the union of the uv-path and the edge will contain a circuit hence e is part of a cycle/circuit.
(<=): Edge e (connects vertices u and v) is part of a cycle say C in graph G. So, in G-e we have the uv-path C-e. Now, any path in G that contained e will now be connected by a subset of the path C-e. So every pair of vertices in G-e remains connected

**Q.9** Let G be a connected graph with at least three vertices. Form G' from G by adding an edge between every pair of vertices which are distance 2 apart in G. Show that G' formed in this way has no cut-vertices.
**Ans:**
Let G be a connected graph with at least three vertices. Let G' be formed from G by adding an edge between every pair of vertices which are distance 2 apart in G. Clearly any cut vertex of G' is also a cut vertex of G since we have only added edges. Suppose u is a cut vertex of G and let x and y be in different components of G - u. Since G is connected there must be an paths from x to u and v to w where v and w are neighbors of u. But since v and w are both neighbors of u they are distance 2 apart in G and so have an edge between them in G'. Thus there is an xy-path in G - u.

**Q.10** Compute the number of non-isomorphic (labelled) simple graphs of order 5 and size 4 such that in each graph there exists a vertex v with deg v $> 2$.

**Ans:** 125

60(degree 3, connected) + 5(degree 4, connected) + 60(degree 3, disconnected)

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