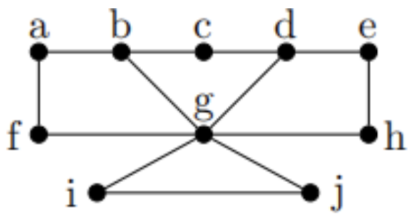
Birla Institute of Technology & Science, Pilani

Department of Mathematics

Second Semester 2021-2022

MATH F243 **ProblemSheet-3**

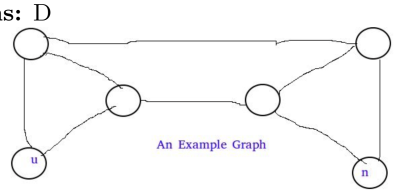


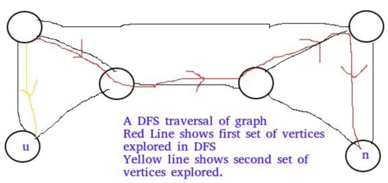
Q.1

a)Is the above graph contains an euler path or an euler cicruit?
b)Is the above graph Hamiltonian?

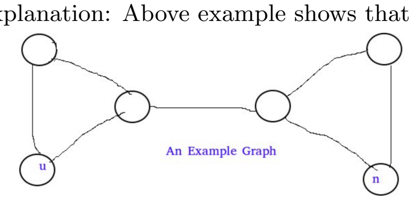
Ans:
It does contain an euler path but not euler circuit. It's not an Hamiltonian graph because it's vertex connectivity is one. (Another reason: It is not a Hamiltonian graph because Hamilton Circuit cannot have a sub-circuit which does not contain all vertices)

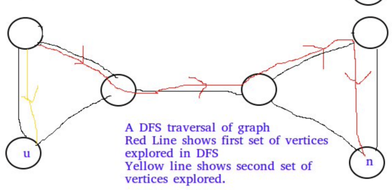
Q.2 Let T be a depth first search tree in an undirected graph G. Vertices u and n are leaves of this tree T. The degrees of both u and n in G are at least 2. which one of the following statements is true?
(A) There must exist a vertex w adjacent to both u and n in G
(B) There must exist a vertex w whose removal disconnects u and n in G
(C) There must exist a cycle in G containing u and n
(D) There must exist a cycle in G containing u and all its neighbours in G.
Ans: D





Explanation: Above example shows that A and B are FALSE:



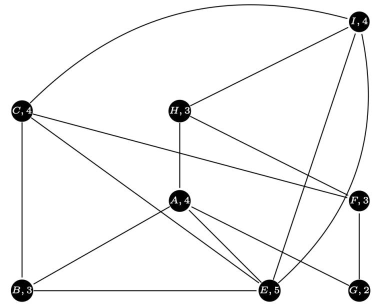


Above example shows C is false:

Q.3 Let G be a simple undirected graph. Let T_D be a depth first search tree of G. Let T_B be a breadth first search tree of G. Consider the following statements.(I) No edge of G is a cross edge with respect to T_D. (A cross edge in G is between two nodes neither of which is an ancestor of the other in T_D.) (II) For every edge (u,v) of G, if u is at depth i and v is at depth j in T_B, then $|i{-}j| = 1$.
Which of the statements above must necessarily be true?
Ans: Only statement I is correct
Explanation: There are four types of edges can yield in DFS. These are tree, forward, back, and cross edges. In undirected connected graph, forward and back egdes are the same thing. A cross edge in a graph is an edge that goes from a vertex v to another vertex u such that u is neither an ancestor nor descendant of v. Therefore, cross edge is not possible in undirected graph. So, statement (I) is correct.

For statement (II) take counterexample of complete graph of three vertices, i.e., K3 with XYZ, where X is source and Y and Z are in same level. Also,there is an edge between vertices Y and Z, i.e., $|i{-}j| = 0 \neq 1$ in BFS. So, statement became false.

Option (A) is correct.



Q.4 (a) Label the degree of each vertex in Graph 1.
(b) Determine if Graph 1 has an Euler path or an Euler cycle. If it does, then find one. If not, then explain why not.
Ans:
Graph 1 is connected and vertices F, B, H, and E have odd degree. Euler's theorem implies that Graph 1 cannot have an Euler path or cycle since we have more than 2 odd- degree vertices.
(c) Determine if Graph 1 has an Euler path or an Euler cycle. If it does, then find one. If not, then explain why not.
Ans: All the indegrees and outdegrees are equal except for vertices I and H. Thus, by Euler's Theorem, there is an Euler path from H to I given by (16,15,3,14,12,19,1,8,7,6,17,2,21,10,5,9,18,20,4,11,13).

Q.5

	Urbana	Gotham	Metropolis	Bedrock	Quahog
Urbana		1	2	3	3
Gotham	1		4	2	5
Metropolis	2	4		7	4
Bedrock	3	2	7		13
Quahog	3	5	4	13	

The distances between various cities are given in the table below.
(a) Draw the corresponding graph to the table.
(b) How many Hamiltonian cycles exist in the graph?
Use Kruskal's algorithm to find the minimal spanning tree of the graph from part (a). Draw the minimal spanning tree below.
Ans: The number of Hamiltonian cycles in a complete graph with n vertices is given by (n –1)!/ 2. Thus we have, 4!/2 = 12 Hamiltonian cycles in the above graph.

Q.6 (Ex 6.12) Let G be a 3-regular graph of order 12 and H a 4-regular graph of order 11.
a)Is G + H Eulerian?
b)Is G + H Hamiltonian?

Ans: For v in V(G), d(v) = 3 + 11 = 14. For v in V(H), d(v) = 4 + 12 = 16.

a)Is G + H Eulerian? So every vertex has even degree and hence is Eulerian.

b)Is G + H Hamiltonian? Note that there are 23 vertices in total. For any pair of vertices u and v, either: both are from G and so d(u) + d(v) = 14 + 14 = 28. both are from H and so d(u) + d(v) = 16 + 16 = 32. One from each and so d(u) + d(v) = 14 + 16 = 30. In every case d(u) + d(v) > 23 and so the graph is Hamiltonian.

Q.7 Prove that The $4 \times n$ chessboard has no knight's tour. A knight can move from one square to another that differs by 1 in one coordinate and by 2 in other coordinate.
Ans:
Let G be the graph having a vertex for each square and an edge for each pair of squares whose positions differ by a knight's move. Every neighbor of a square in the top or bottom row is in the middle two rows, so the top and bottom squares form an independent set. Deleting the 2n squares in the middle rows leaves 2n components remaining; that is not enough to prohibit the tour. Instead, note that every neighbor of a white square in the top and bottom rows is a black square in the middle two rows. Therefore, if we delete the n black squares in the middle two rows, the white squares in the top and bottom rows become n isolated vertices, and there remain 2n other vertices in the graph, which must form at least one more component. Hence we have found a set of n vertices whose deletion leaves at least n+1 components, which means that G cannot be Hamiltonian.

Q.8 The edges of a connected graph with 2k odd vertices can be partitioned into k trails if k > 0. (Assume graph is connected, because the conclusion is not true for G=H1+H2 when H1 has some odd vertices and H2 is Eulerian).
Ans: (induction on e(G)). If e(G) = 1, then G=K2, and we have one trail. If G has an even vertex x adjacent to an odd vertex y, then G0 = $G–xy$ has the same number of odd vertices as G. The trail decomposition of G0 guaranteed by the induction hypothesis has one trail ending at x and no trail ending at y. Add xy to the trail ending at x to obtain the desired decomposition of G. If G has no even vertex adjacent to an odd vertex, then G is Eulerian or every vertex of G is odd. In this case, deleting an edge xy reduces k, and we can add xy as a trail of length one to the decomposition of $G–xy$ guaranteed by the induction hypothesis.

Q.9 Let G=(V_n,E_n) such that: G's vertices are words over $\sigma = \{a, b, c, d\}$ with length n, such that there aren't two adjacent equal chars. An edge is defined to be between two vertices that are different by only one char.
Does the graph contain an Euler cycle?
Ans: Claim: For n¿3, the graph G=(V_n,E_n)) has no Euler cycle.

When n is even, we consider the string w=$a_1a_2...a_n$ with $a_{2i\,-1}$=a for i=1,2,....,k, a2=b, and a_{2i}=c for i=2,...,k.

Then we see that w has an odd degree. There's only one word that differ from w at a_3. This is because a2=b and a4=c. There's two words that differ from w at a_1,a_n. There's two words that differ from w at a_i where $i \neq 3$. This is because for $a_{i\,-1}$=a_{i+1}. Hence, there are 2(2k1)+1 neighbors of w. The graph has no Euler cycle.

When n is odd, consider the string $q=b_1b_2...b_n$ with $b_{2i-1}=a$ for $i=1,2,...,k+1$. $b_2=b$, and $b_{2i}=c$ for $i=2,3,...,k$. By the same argument as above, w has odd degree. The graph contains no Euler cycle.

For n=3, the word bac has degree 5.