Birla Institute of Technology & Science, Pilani

Department of Mathematics

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MATH F243 **ProblemSheet-4**

Q.1 Show that a connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.

Ans:

We show that a set F of edges is a disconnecting set in a graph G if and only if it disconnects some block. If deleting F leaves each block of G connected, then the full graph remains connected. If deleting F disconnects some block B, then the remainder of G cannot contain a path between distinct components of B−F, because then B would not be a maximal subgraph having no cut vertex. With this claim, the edge-connectivity of G is the minimum of the edge connectivities of its blocks, which yields the desired statement.

Q.2 For a connected graph G with at least three vertices, the following are equivalent.

A) G is 2-edge-connected.

B) Every edge of G appears in a cycle.

C) G has a closed trail containing any specified pair of edges.

D) G has a closed trail containing any specified pair of vertices.

Ans:

A $\rightarrow$ B. A connected graph is 2-edge-connected if and only if it has no cut-edges. Cut-edges are precisely the edges belonging to no cycles.

A $\rightarrow$ D. By Menger's Theorem, a 2-edge-connected graph G has two edge-disjoint x, y-paths, where x, y $\in$ V(G). Following one path and returning on the other yields a closed trail containing x and y.

D $\rightarrow$ B. Let xy be an edge. D yields a closed trail containing x and y. This breaks into two trails with endpoints x and y. At least one of them,T does not contain the edge xy. Since T is an x, y-walk, it contains an x, y-path. Since T does not contain xy, this path completes a cycle with xy.

B $\rightarrow$ C. Choose e, f $\in$ E(G); we want a closed trail through e and f. Subdivide e and f to obtain a new graph G_0, with x, y being the new vertices. Subdividing an edge does not destroy paths or cycles, although it may lengthen them. Thus G_0 is connected and has every edge on a cycle, because G has these properties. Because we have already proved the equivalence of B and D, we know that G_0 has a closed trail containing x and y. Replacing the edges incident to x and y on this trail with e and f yields a closed trail in G containing e and f .

C $\rightarrow$ D. Given a pair of vertices, choose edges incident to them. A closed trail containing these edges is a closed trail containing the original vertices.

Q.3 If G is a 2-connected graph and v $\in$ V(G), then v has a neighbor u such that G – u – v is connected.

Ans: Because G is 2-connected, G – v is connected. If G – v is 2-connected, then we may let u be any neighbor of v. If G – v is not 2-connected, let B be a block of G – v containing exactly one cut vertex of G – v, and call that cut vertex x. Now v must have a neighbor in B – x, else G – x is disconnected, with B – x as a component. Let u be a neighbor of v in B – x. Since B – u is connected, G – v – u is connected.

Q.4 Let G be a 5-connected graph, show that between any 3 distinct vertices u, v and w there are 2 cycles which have C and C' which have only the points u and v in common and do not go through w.

Ans: Let u, v and w be distinct vertices of a 5-connected graph, G. Since G is 5-connected, there are at least 5 internally disjoint uv-paths. At most one of these paths go through w. The remaining 4 paths form 2 cycles which have only the points u and v in common and do not go through w.

Q.5 G is a graph on n vertices and $2n-2$ edges. The edges of G can be partitioned into two edge-disjoint spanning trees. Which of the following is NOT true for G?

A. For every subset of k vertices, the induced subgraph has at most $2k-2$ edges.

B. The minimum cut in G has at least 2 edges.

C. There are at least 2 edge-disjoint paths between every pair of vertices.

D. There are at least 2 vertex-disjoint paths between every pair of vertices.

Ans:

There are 2 spanning trees (a spanning tree connects all n vertices) for G which are edge disjoint. A spanning tree for n nodes require n – 1 edges and so 2 edge-disjoint spanning trees requires 2n2 edges. As G has only 2n – 2 edges, it is clear that it doesn't have any edge outside that of the two spanning trees. Now lets see the cases:

Lets take any subgraph of G with k vertices. The remaining subgraph will have n – k vertices. Between these two subgraphs (provided both has at least one vertex) there should be at least 2 edges, as we have 2 spanning trees in G. So, (B) is TRUE. Also, in the subgraph with k vertices, we cannot have more than 2k – 2 edges as this would mean that in the other subgraph with n – k vertices, we would have less than 2n – 2k edges, making 2 spanning trees impossible in it. So, (A) is also TRUE.

A spanning tree covers all the vertices. So, 2 edge-disjoint spanning trees in G means, between every pair of vertices in G we have two edge-disjoint paths (length of paths may vary). So, (C) is also TRUE.

So, that leaves option (D) as answer. It is not quite hard to give a counter example for (D) due to presence of a cut vertex. Any cyclic graph makes (D) false or Take two copies of K4(complete graph on 4 vertices), G1 and G2. Let V(G1)={1,2,3,4} and V(G2)={5,6,7,8}. Construct a new graph G3 by using these two graphs G1 and G2 by merging at a vertex, say merge (4,5). The resultant graph is two edge connected, and of minimum degree 2 but there exist a cut vertex, the merged vertex.

Q.6 Consider the following problem. You are given a flow network with unit-capacity edges: it consists of a directed graph G = (V, E), a source s and a sink t. You are also given a parameter k. The goal is delete k edges so as to reduce the maximum s – t flow in G as much as possible. In other words, you should find a set of edges F so that $|F|$ = k and the maximum s – t flow in the graph G0 = (V, E \F) is as small as possible.

Ans: First observe that by removing any k edges in a graph, we reduce the capacity of any cut by at most k, and so, the min-cut will reduce by at most k. Therefore, the max-flow will reduce by at most k. Now we show that one can in fact reduce the max-flow by k. To achieve this, we take a min-cut X and remove k edges going out of it. The capacity of this cut will now become f – k, where f is the value of the max-flow. Therefore, the min-cut becomes f – k, and so, the max-flow becomes f – k.

Q.7 Let G be a k-connected graph. Show using the definitions that if G_0 is obtained from G by adding a new vertex V adjacent to at least k vertices of G, then G_0 is k-connected.

Ans: Let S be such that G_0 – S is disconnected. Let us show that $|S| \geq k$. Assume the contrary that $|S| \leq$ k – 1 . If V ∈ S, then G – (S \V) is disconnected as well. Since G is k-connected then $|S| > |S - V| > k$. This is a contradiction. If V ∉ S then G – S is connected (by k-connectivity of G) and, since the degree of V is at least k, then V is adjacent for at least one vertex of G – X. Hence, G_0 – S is connected. This is a contradiction.

Q.8 Let G be a connected graph with all degrees even. Show that G is 2-edge-connected.

Ans: As G is connected with all degrees even, it has an Euler tour. Deleting any edge from an Euler tour results in an Euler trail. So G–e has an Euler trail and all its vertices have positive degree, so it is connected. As this is true for any edge e, G is a 2-edge-connected graph.

Q.9 Prove that G is 2-connected if and only if for any three vertices x, y, z there is a path in G from x to z containing y.

Ans: We want to show that given x, y, z in G, there exists a path from x to z containing y. The idea is to construct a graph G_0 out of G and then apply Menger's theorem to G_0. To construct G_0, we add an extra vertex s to G, and connect s to the vertices x and z. By exercise 7, G_0 is 2-connected. By Menger's theorem, there are two internally vertex-disjoint s-y paths in G_0. By construction, one of them contains x and another contains z. Therefore, there is a path in G from x to z containing y. Conversely, Let x be any vertex of G. We want to show that G – x is still connected by showing that any two vertices in G – x are connected. Let y, z be any two vertices of G – x. By assumption, there is a path x . . . y . . . z in G. Then there is a path y . . . z in G – x, so these two vertices are connected in G – x.

Q.10 Prove that a graph G on at least k + 1 vertices is k-connected if and only if G – X is connected for every vertex set X of size k – 1.

Ans: By the definition of k-connectivity, if G is k-connected then G – X is connected for every set X of size k – 1.
Conversely, Assume the contrary that G = (V, E) is not k-connected. Then there is a set of vertices Y such that $|Y| \leq$ k – 1 and the graph G – Y is disconnected. Hence, there are two vertices x and y, which lie in different connected components. We obtain set Y' from Y by adding k – 1 – $|Y|$ vertices to Y from V \{x, y}. Then G – Y' ⊃ {x, y} is a disconnected graph and $|Y'|$ = k – 1. This is a contradiction.