Birla Institute of Technology & Science, Pilani

Department of Mathematics

Second Semester 2021-2022

MATH F243 **ProblemSheet-2**

Q.1 Let T be a tree:

a) Use the fact that there is a unique path between any pair of vertices in a tree to show that if v is an internal vertex of T then T - v is not connected.

b) If v is a leaf in a tree T show that T - v is connected.

Ans: a) Use the fact that there is a unique path between any pair of vertices in a tree to show that if v is an internal vertex of T then T - v is not connected. By the definition of internal d(v) > 1, so v has at least two neighbors in T, u1 and u2 say. But since T is a tree there is a unique path from u1 to u2, which must be { u1, v, u2 } Now T - v has no u1 u2–path, and so is disconnected.

b) Show that if v is a leaf in T then T - v is connected. Let T be a tree and v a leaf of T. By the definition of leaf d(v) = 1, so v has exactly one neighbor, u say. Let x and y be any pair of vertices in T - v. Since T is a tree there is a unique xy–path in T, P say. The degree of an internal vertex of a path must be at least 2, but the degree of v is one, so v does not appear as an internal vertex of P. Thus P only contains vertices in T - v and so connects x and y in T - v.

Q.2 Let T, T' be two spanning trees of a connected graph G. Prove that there is an edge e' ∈ E(T)E(T) such that T'+e−e' and T+e'−e are both spanning trees of G ?

Ans:

Let H and H' be the two components of T e and let $F \subset E(T')$ consist of the edges of T' with one endpoint in V (H), the other in V (H'). Since T' is connected, $F \neq \phi$. Furthermore, since T has the unique edge e joining H and H', $F \subset E(T')nE(T)$. T' + e contains a unique cycle C of which e is an edge. C leaves H and enters H0 via e. In order to complete the cycle, one must use one edge e' of E(T') to come back from H' to H. But then e' ∈ F. It is now clear that for this e' both T – e + e and T' + e – e' are spanning trees of G. (Note that the cycle C, after coming back to H, may again enter H' and subsequently return back to H. Every time it does so, it has to use two new edges from F. That is, the choice of e' is not always unique.)

Q.3 Consider a graph G = (V, E), where V = v_1, v_2, . . . , v_{100}, E = (v_i, v_j) 1 ≤ i ≤ j ≤ 100, and weight of the edge (v_i, v_j) is $|i - j|$. The weight of the minimum spanning tree of G is ?

Ans: 99

If there are n vertices in the graph, then each spanning tree has n – 1 edges. N =100. Edge weight is $|i - j|$ for Edge (v_i, v_j) 1 ≤ i ≤ j ≤ 100. Ex- The weight of edge(v_1, v_2) is 1 , edge(v_5, v_6) is 1. So 99 edges of weight is 99

Q.4 Let G be a complete undirected graph on 4 vertices, having 6 edges with weights being 1, 2, 3, 4, 5, and 6. The maximum possible weight that a minimum weight spanning tree of G can have is ?

Ans: 7

Let G be a complete undirected graph with 4 vertices 6 edges so according to graph theory, if we use Prim's / Kruskal's algorithm, the graph looks like

Now consider vertex A to make Minimum spanning tree with Maximum weights. As weights are 1, 2, 3, 4, 5, 6. AB, AD, AC takes maximum weights 4, 5, 6 respectively. Next consider vertex B, BA = 4, and minimum spanning tree with maximum weight next is BD BC takes 2, 3 respectively. And the last edge CD takes 1. So, 1+2+4 in our graph will be the Minimum spanning tree with maximum weights.

Q.5 Let G = (V,E) be an undirected simple graph in which each edge has a distinct weight, and e is a particular edge of G. Is the following statement about the minimum spanning trees (MSTs) of G TRUE?

I. If e is the lightest edge of some cycle in G, then every MST of G includes e

Ans: False

The MSTs of G may or may not include the lightest edge. Take rectangular graph labelled with P,Q,R,S. Connect with P-Q = 5, Q-R = 6, R-S = 8, S-P = 9, P-R = 7. When we are forming a cycle R-S-P-R. P-R is the lightest edge of the cycle. The MST abcd with cost 11 P-Q + Q-R + R-S does not include it.

Q.6 Let G be a connected undirected graph of 100 vertices and 300 edges. The weight of a minimum spanning tree of G is 500. When the weight of each edge of G is increased by five, the weight of a minimum spanning tree becomes?

Ans: 995

G has 100 vertices spanning tree contain 99 edges given, weight of a minimum spanning tree of G is 500 since, each edge of G is increased by five. Weight of a minimum spanning tree becomes $500 + 5 \cdot 99 = 995$

Q.7 Consider a tree T having 50 leaves and equal number of vertices of degree 2, 3, 4 and 5 and it has no vertices of degree greater than 5. Compute the order of T.

Ans: 82

On solving equations 50+14x=2n-2 and 50+4x = n, we get n = 82

Q.8 **Ans:** 10

Here the point to be noted is that vertex 0 is a leaf node. So degree of vertex 0 cannot be more than or equal to 2. So, the edges of the spanning tree given that vertex 0 is a leaf node in the tree, will be between vertices 0 and 1, 1 and 3, 3 and 4, 2 and 4. So, the minimum possible weight of spanning tree will be = 1 + 4 + 2 + 3 = 10

Q.9 For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

Consider a complete undirected graph with vertex set {0, 1, 2, 3, 4}. Entry W_{ij} in the matrix W below is the weight of the edge {i, j}.

$$W = \begin{pmatrix} 0 & 1 & 8 & 1 & 4 \\ 1 & 0 & 12 & 4 & 9 \\ 8 & 12 & 0 & 7 & 3 \\ 1 & 4 & 7 & 0 & 2 \\ 4 & 9 & 3 & 2 & 0 \end{pmatrix}$$

What is the minimum possible weight of a spanning tree T in this graph such that vertex 0 is a leaf node in the tree T?

a. (4,1,1,1,1)
b. (3,3,2,1,1)
c. (2,2,2,1,1)
d. (4,4,3,3,3,2,2,1,1,1,1,1,1,1)

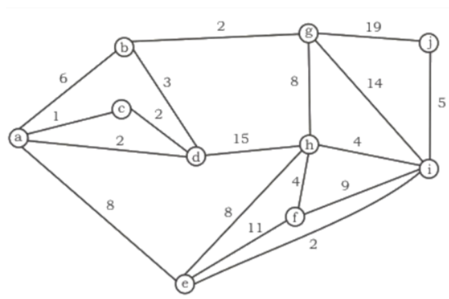
Ans:

a. This must be the degree sequence for a tree. This is because the vertex of degree 4 must be adjacent to the four vertices of degree 1 (there are no other vertices for it to be adjacent to), and thus we get a star.

b. This cannot be a tree. Each degree 3 vertex is adjacent to all but one of the vertices in the graph. Thus each must be adjacent to one of the degree 1 vertices (and not the other). That means both degree 3 vertices are adjacent to the degree 2 vertex, and to each other, so that means there is a cycle. Alternatively, count how many edges there are!

c. This might or might not be a tree. The length 4 path has this degree sequence (this is a tree), but so does the union of a 3-cycle and a length 1 path (which is not connected, so not a tree).

d. This cannot be a tree. The sum of the degrees is 28, so there are 14 edges. But there are 14 vertices as well, so we don't have v=e+1,v=e+1, meaning this cannot be a tree.



Q.10 What is the weight of a minimum spanning tree of the given graph ?
Ans: 31
1+2+3+2+8+4+4+2+5 =31

Q.11 For $2 \leq k \leq$ n-1, the n-vertex graph formed by adding one vertex adjacent to every vertex of P_{n-1} has a spanning tree with diameter k.
Ans: Let v1, . . . , v_{n-1} be the vertices of the path in order, and let x be the vertex adjacent to all of them. The spanning tree consisting of the path v1, . . . , vk1 and the edges x-vk1, . . . , x-v_{n-1} has diameter k.

Q.12 Suppose that G is a forest with n vertices and c components. Prove that G has n c edges.
Ans: Let G1, . . . , Gc be all the components of G. Each component is then connected and with no cycle, i.e., every Gi is a tree. If we denote by ni the number of vertices in component Gi, then we have e(Gi) = ni – 1 and n = n1 + n2 + · · · + nc. As we have no edges between two components, we also have e(G) = e(G1) + e(G2) + · · · + e(Gc), and therefore e(G) = e(G1) + e(G2) + · · · + e(Gc) = (n1 – 1) + (n2 – 1) + · · · + (nc – 1) = n – c.

Q.13 Let G be a graph. Prove that G is a tree if and only if for every pair of vertices u and v, there is a unique path between u and v.

Q.14 If $n \geq 2$ and $d_1, \ldots, d_n$ are positive integers, then there exists a tree with these as its vertex degrees if and only if $d_n = 1$ and $\sum_{i=1}^{n} d_i = 2(n{-}1)$.
Ans: Necessity: Every n-vertex tree is connected and has n1 edges, so every vertex has degree at least 1 (when $n \geq 2$) and the total degree sum is 2(n – 1). Sufficient: Basis step (n = 2): The only such list is (1, 1), which is the degree list of the only tree on two vertices. Induction step (n > 2): Consider d1, . . . , dn satisfying the conditions. Since sum(di) > n, some element exceeds 1. Since sum(di) < 2n, some element is at most 1. Let d0 be the list obtain by subtracting 1 from the largest element of d and deleting an element that equals 1. The total is now 2(n –2), and all elements are positive, so by the induction hypothesis there is a tree on n –1 vertices with d0 as its vertex degrees. Adding a new vertex and an edge from it to the vertex whose degree is the value that was reduced by 1 yields a tree with the desired vertex degrees.