# Transverse Vibration Analysis of Euler-Bernoulli Beam using Finite Difference Method

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**Abstract**

The purpose of this project report is to present the application of the finite difference method to solve the governing equations for transverse vibration of the Euler-Bernoulli uniform beam and verify the efficiency and accuracy of the approach by comparing it with the theoretical results.von Neumann stability analysis is done to develop the stability requirements of the solution of finite difference equations. A Python script is implemented for computation of natural frequencies of vibration, mode shapes, and vibration response of the elastic beam for different boundary conditions, viz. fixed-fixed, fixed-pinned (propped-cantilever), pinned-pinned (simply-supported), and fixed-free (cantilever).

## 1 Introduction

In general, mechanical systems are comprised of structural components with distributed mass and elasticity. Rods, beams, plates, and shells are examples of structural components. In this report, the transverse free vibration analysis of beam is discussed. A beam is a slender structural member that resists lateral loads by developing bending moments and shear forces. It can be found as supporting members in high-rise buildings, trains, long-span bridges, flexible satellites, rifle barrels, robot arms, and aircraft wings, among other things. Beams are subjected to dynamic loads in many engineering applications, which can cause structural vibrations in the beam.

A beam is regarded a continuous system for vibration analysis, with a continuous distribution of mass, damping, and elasticity, and it is assumed that each of the system's infinite number of points can vibrate. This is why a continuous system is also called a system of infinite degrees of freedom. The vibration of such systems is governed by partial differential equations which involve variables that depend on time as well as the spatial coordinates.

When the differential equation governing the free or forced vibration of a system cannot be integrated in closed form, a numerical approach is to be used for the vibration analysis. The finite difference method, which is based on the approximation of the derivatives appearing in the equation of motion and the boundary conditions, is presented. The transverse vibration solution of elastic beam, a continuous system, is considered using the finite difference method with different boundary conditions.

The main idea in the finite difference method is to use approximations to derivatives. Thus the governing differential equation of motion and the associated boundary conditions, if applicable, are replaced by the corresponding finite difference equations. The central difference formulas are considered in this project, since they are more accurate that forward and backward formulas.

## 2 Mathematical Formulation

### 2.1 Governing Equations

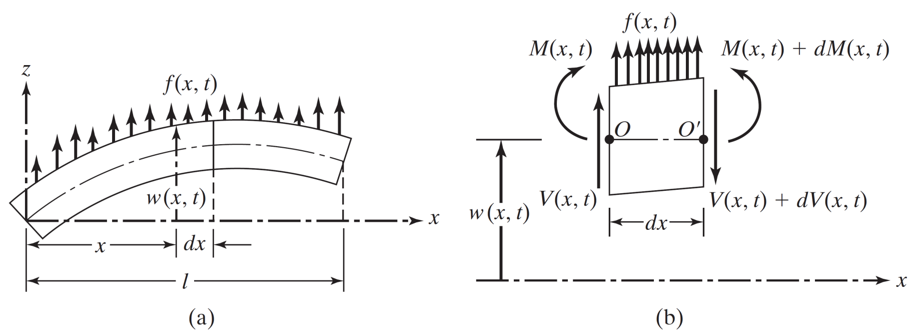


Figure 1: A beam in bending

Consider the free-body diagram of an element of a beam shown in Figure 1 where $M(x,t)$ is the bending moment, $V(x,t)$ is the shear force, $f(x,t)$ is the external force per unit length of the beam and $w(x,t)$ is the deflection in the beam. The governing differential equation for the transverse vibration of a uniform beam is given by Equation (1).

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t) \tag{1}$$

From the elementary theory of bending of beams (also known as the Euler-Bernoulli or elastic beam theory), the relationship between bending moment and deflection can be expressed as

$$M(x,t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t) \tag{2}$$

where $E$ is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the $y$-axis. For free vibration, $f(x,t) = 0$, and so the equation of motion becomes

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t) = 0 \tag{3}$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$

The free-vibration solution can be found using the method of separation of variables as

$$w(x,t) = W(x)T(t) \tag{4}$$

Substituting Equation (4) into Equation (3) and rearranging leads to

$$\frac{c^2}{W(x)}\frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = a = \omega^2 \tag{5}$$

where $a = \omega^2$ is a positive constant. Equation (5) can be written as two equations:

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0 \tag{6}$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 \tag{7}$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A\omega^2}{EI} \tag{8}$$

The solution of Equation (7) can be expressed as

$$T(t) = A\sin\omega t + B\cos\omega t \tag{9}$$

where $A$ and $B$ are constants that can be found from the initial conditions.The solution of Equation (6) can be expressed as

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x \tag{10}$$

The constants $C_1$, $C_2$, $C_3$ and $C_4$ can be found from appropriate boundary conditions.

The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and $\omega$ is called the *natural frequency* of vibration.

## 2.2 Discretization of the Governing Equations

In this section, equation governing Euler-Bernoulli beam transverse vibrations is discretized. In order to apply finite difference method (FDM), it is necessary to introduce a set of grid points in the set $\Omega = [0,L]\times[0,T]$, where $L$ is the length of the beam and $T$ is the time of simulation.

In spatial dimension, $n_x$ grid points are introduced with length of each interval between two points $\Delta x = \frac{L}{n_x - 1}$. In time dimension, $n_t$ discrete points are considered with each time step $\Delta t = \frac{T}{n_t - 1}$. In calculation, it is necessary to find $n_x \times n_t$ values of the solution $w_{j,k}$ at each $(x_j, t_k)$ in $\Omega$.

Using central difference numerical scheme, the second order partial derivative with respect to time and fourth order partial derivative with respect to space can be given as:

$$\left(\frac{\partial^2 w}{\partial t^2}\right)_{j,k} \approx \frac{w_{j,k+1} - 2w_{j,k} + w_{j,k-1}}{\Delta t^2} \tag{11}$$

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{j,k} \approx \frac{w_{j+2,k} - 4w_{j+1,k} + 6w_{j,k} - 4w_{j-1,k} + w_{j-2,k}}{\Delta x^4} \tag{12}$$

Substituting above finite difference approximations in Equation (3) to obtain the **explicit** finite difference scheme as

$$\frac{w_{j,k+1} - 2w_{j,k} + w_{j,k-1}}{\Delta t^2} + c^2\left(\frac{w_{j+2,k} - 4w_{j+1,k} + 6w_{j,k} - 4w_{j-1,k} + w_{j-2,k}}{\Delta x^4}\right) = 0 \tag{13}$$

or

$$w_{j,k+1} = 2w_{j,k} - w_{j,k-1} - r^2\left(w_{j+2,k} - 4w_{j+1,k} + 6w_{j,k} - 4w_{j-1,k} + w_{j-2,k}\right) \tag{14}$$

where

$$r = \frac{c\Delta t}{\Delta x^2}$$

Using above formula, the value $w_{j,k+1}$ can be explicitly described at the next moment of time in terms of the previous values. The derived finite difference scheme is second order accurate ($\mathcal{O}(\Delta t^2 + \Delta x^2)$) in time and space.

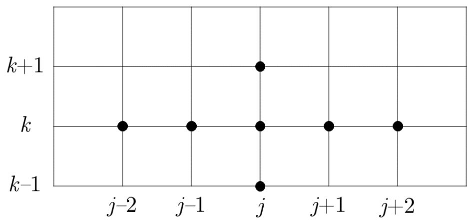


Figure 2: Finite difference 7-point 2D stencil.

## 2.3 Initial and Boundary Conditions

**Initial Conditions**

*At* $t_k = t_1 = 0$

$$w(x,0) = w^*(x) \implies w_{j,0} = w_j^* \tag{15}$$

$$\frac{\partial w(x,0)}{\partial t} = v^*(x) \implies \frac{w_{j,1} - w_{j,0}}{\Delta t} = v_j^* \tag{16}$$

**Boundary Conditions**

*Fixed end at* $x_j = x_\alpha$

$$w(x_\alpha,t) = 0 \implies w_{\alpha,k} = 0 \tag{17}$$

$$\frac{\partial w(x_\alpha,t)}{\partial x} = 0 \implies \frac{w_{\alpha+1,k} - w_{\alpha-1,k}}{2\Delta x} = 0 \tag{18}$$

*Pinned end at* $x_j = x_\alpha$

$$w(x_\alpha,t) = 0 \implies w_{\alpha,k} = 0 \tag{19}$$

$$\frac{\partial^2 w(x_\alpha,t)}{\partial x^2} = 0 \implies \frac{w_{\alpha+1,k} - 2w_{\alpha,k} + w_{\alpha-1,k}}{\Delta x^2} = 0 \tag{20}$$

*Free end at* $x_j = x_\alpha$

$$\frac{\partial^2 w(x_\alpha,t)}{\partial x^2} = 0 \implies \frac{w_{\alpha+1,k} - 2w_{\alpha,k} + w_{\alpha-1,k}}{\Delta x^2} = 0 \tag{21}$$

$$\frac{\partial^3 w(x_\alpha,t)}{\partial x^3} = 0 \implies \frac{w_{\alpha+2,k} - 2w_{\alpha+1,k} + 2w_{\alpha-1,k} - w_{\alpha-1,k}}{2\Delta x^3} = 0 \tag{22}$$

In this project, the different boundary conditions for beam vibration are considered at $x_\alpha = x_1 = 0$ or $x_\alpha = x_{n_x} = L$.

### 2.4 Natural Frequencies and Mode Shapes

To find the natural frequencies and mode shapes of vibration, Equation (6) is used. The system of finite difference approximation (Equation (23)) equations at interior mesh points along with appropriate boundary conditions yields an eigenvalue-eigenvector problem.

$$W_{j+2} - 4W_{j+1} + (6-\lambda)W_j - 4W_{j-1} + W_{j-2} = 0 \tag{23}$$

where

$$\lambda = (\beta \Delta x)^4$$

The solution of the aforementioned system of equations yields eigenvalues, which can be used to calculate natural frequencies, and eigenvectors, which are the mode shapes.

### 2.5 von Neumann Stability Analysis

The von Neumann method is based on the decomposition of the errors into Fourier series. It provides an uniform way of verifying if a finite difference scheme is stable. Consider the finite difference scheme for the governing equation of transverse vibration of beam given in Equation (14). To apply von Neumann stability analysis, substitute

$$w_{j,k} = e^{ij\Delta x\xi} \tag{24}$$

and then expect that

$$w_{j,k+1} = G(\xi)e^{ij\Delta x\xi} \tag{25}$$

where $G(\xi)$ is the amplification factor at wave number $\xi$. Inserting Equation (24) and Equation (25) in Equation (14) gives

$$G(\xi)e^{ij\Delta x\xi} = 2e^{ij\Delta x\xi} - \frac{1}{G(\xi)}e^{ij\Delta x\xi} - r^2\left(e^{i(j+2)\Delta x\xi} - 4e^{i(j+1)\Delta x\xi} + 6e^{ij\Delta x\xi} - 4e^{i(j-1)\Delta x\xi} + e^{i(j-2)\Delta x\xi}\right) \tag{26}$$

or

$$G(\xi) = 2 - \frac{1}{G(\xi)} - 4r^2\left(\cos^2(\xi\Delta x) - \cos(\xi\Delta x) + 1\right) \tag{27}$$

let

$$\mu = r^2\left(\cos^2(\xi\Delta x) - \cos(\xi\Delta x) + 1\right) \geq 0 \tag{28}$$

Equation (27) becomes

$$G(\xi) = 2 - \frac{1}{G(\xi)} - 4\mu \tag{29}$$

The solution of quadratic equation in $G(\xi)$ can be given as

$$G(\xi)_{\pm} = 1 - 2\mu \pm 2\sqrt{\mu^2 - \mu} \tag{30}$$

For the finite difference scheme to be stable, $|G(\xi)| \leq 1$. On solving the modulus inequality, the stability condition is obtained as

$$r = \frac{c\Delta t}{\Delta x^2} \leq \frac{1}{\sqrt{3}} \tag{31}$$

Hence, the explicit finite difference scheme used for approximate solution of transverse vibrations of elastic beam is conditionally stable.

## 3 Numerical Example

Consider a simply-supported behaving square cross section with following numerical data $E = 1N/m^2$, $A = 1m^2$, $I = 1m^4$, $\rho = 1Kg/m^3$, $L = 1m$, $T = 1s$, $n_x = 10$ and $n_t = 500$. The analytical solution of vibration of aforementioned beam is given in Equation (32)

$$w(x,t) = (\cos\omega t + \frac{1}{\omega}\sin\omega t)\sin\frac{\pi x}{L} \tag{32}$$

where $\omega = \pi^2\sqrt{\frac{EI}{\rho AL^4}}$ is the fundamental frequency of vibration. The comparison of theoretical and approximate is shown.

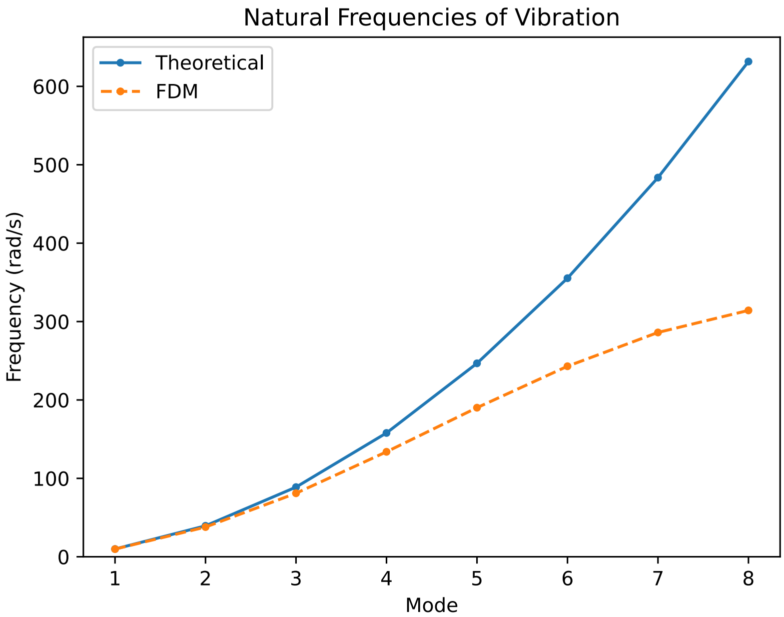


Figure 3: Comparison of natural frequencies of vibration from theoretical and FDM solution.

Figure 3 shows the comparison of natural frequencies of the vibration from analytical and FDM solution. Since beam is a continuous system, it has infinite natural frequencies but in case of FDM solution, the number of natural frequencies is equal to the number of vibrating nodes. In order to achieve more accuracy, number of mesh points should be increased.

Figure 4 shows the corresponding mode shapes of transverse vibration of the beam. The first mode is called the fundamental mode of vibration.

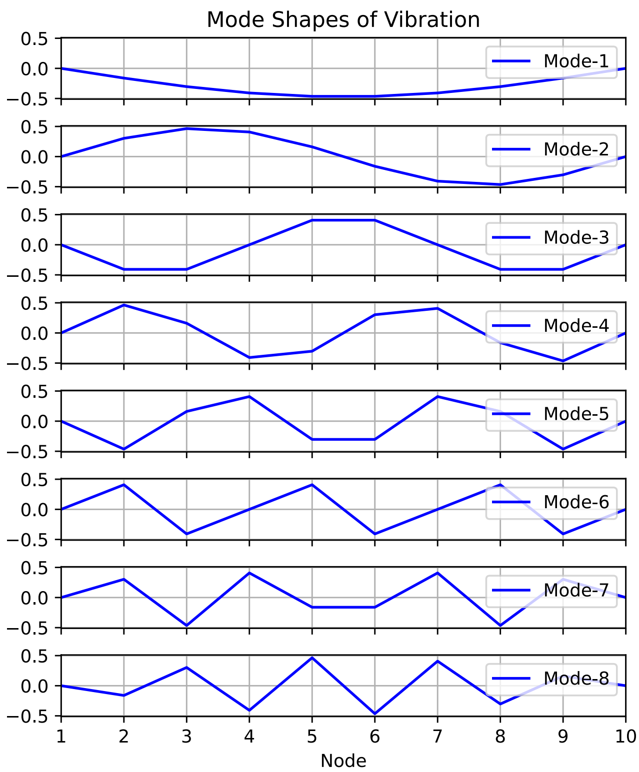


Figure 4: Mode shapes of vibration using FDM.

Figure 5 shows the contour plot of analytical solution of transverse vibration of beam. The colorbar on the right gives the displacement from the mean position.

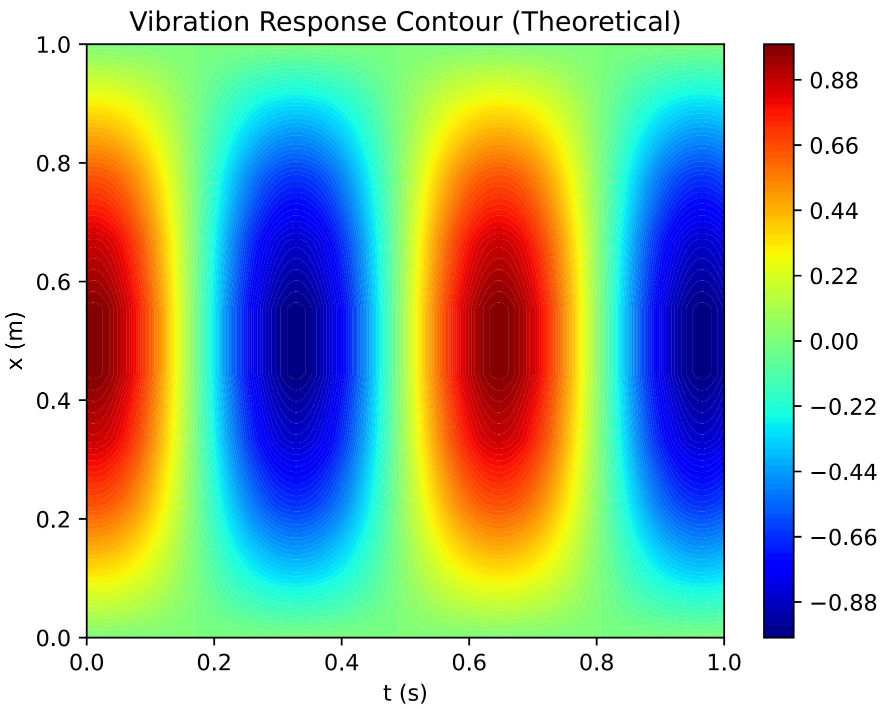


Figure 5: Contour plot of analytical vibration response.

Approximate solution of transverse vibration response of beam using FDM with stable and unstable conditions is shown in Figure 6 and Figure 7.

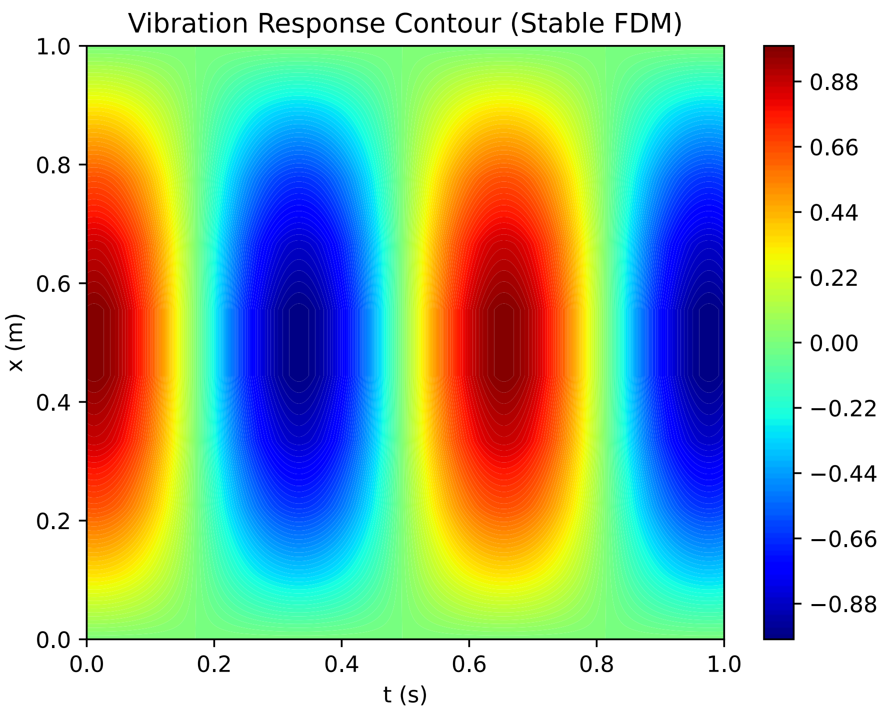


Figure 6: Contour plot of vibration response using FDM (Stable).

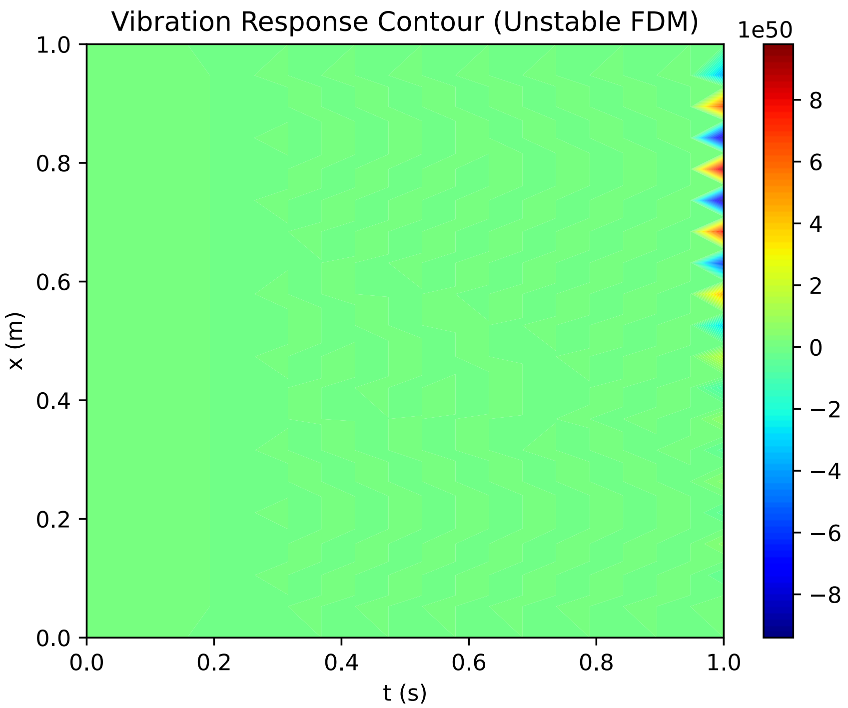


Figure 7: Contour plot of vibration response using FDM (Unstable).

Figure 8 shows a snapshot from the animation of transverse vibrations of a simply-supported beam. Animation can be viewed by running the Python script attached.

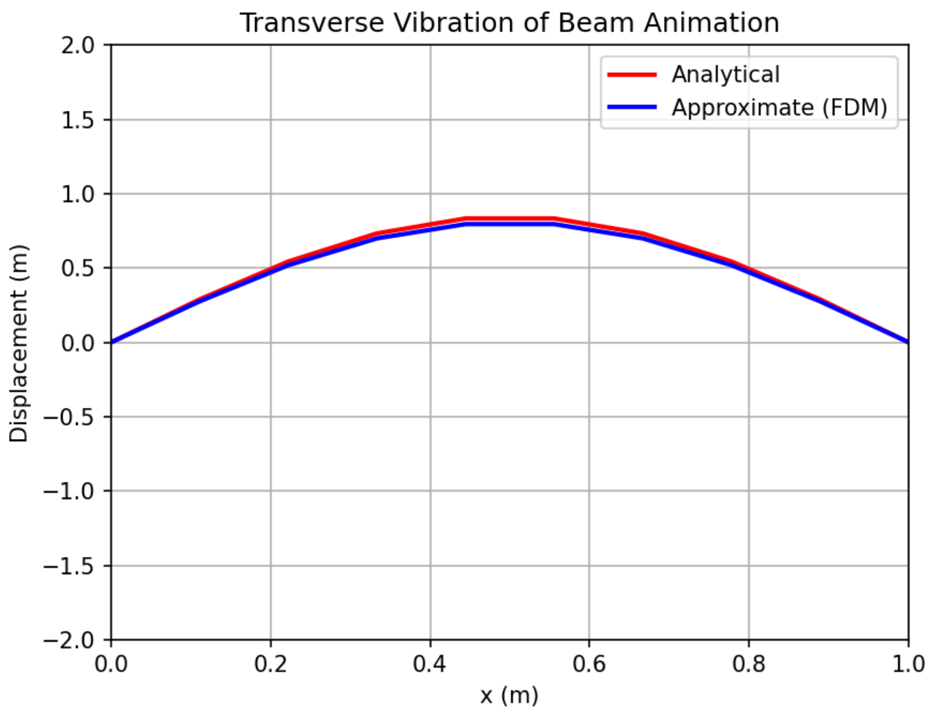


Figure 8: Snapshot of animation of beam vibration.

# 4 Conclusion

In this project, the transverse vibration analysis of Euler-Bernoulli beam using finite difference method was done. The equations governing the transverse vibration of the beam were dicretized using central difference scheme. Finite difference equations were established for appropriate boundary conditions, viz. fixed-fixed, fixed-pinned (propped-cantilever), pinned-pinned (simply-supported), and fixed-free (cantilever).

Natural frequencies and mode shapes of vibration were found by computationally solving eigenvalue-eigenvector problem obtained from PDE for characteristic function.

Transverse vibration response of the Euler-Bernoulli beam was found by solving explicit finite difference scheme, the results of which were compared with the analytical vibration response through a numerical example. The percentage root-mean-squared error in approximate vibration response was found to be 3.05%, which can further be reduced by refining the mesh grid.

von Neumann stability analysis was done to establish the stability requirements of the solution of the finite difference equation. The explicit finite difference scheme for transverse vibrations of beam was found to be conditionally stable. At the accuracy and stability of the explicit finite difference method is not always acceptable, thus implicit finite difference scheme can be used.