SE 380

Feedback Control

Day 1

Example

> There are two cars on a linear track. There is a follower and a leader. There are the following constraints:

> - The distinace from the beginning of the track to the follower's center is x_f
> - The distinace from the beginning of the track to the leader's center is x_l

> - The distance between the follower and the leader is $x_l - x_f$
> - leader is driven by a human
> - follower is driven by a computer which is able to assign the velocity of the vehicle

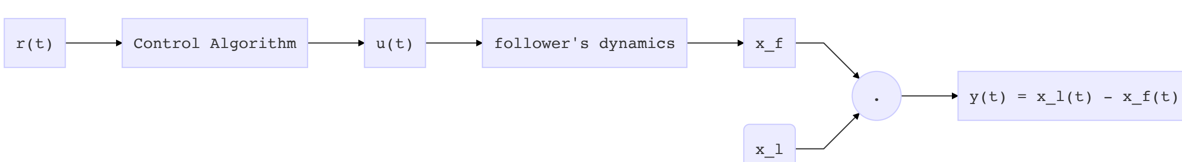
> Objective: Write a program that decides the appropriate speed of the follower in order to maintain a given (safe) inter-vehicle disance

> Assumptions:
>
> - 1-d motion only
> - $\frac{dx_f(t)}{dt} - u(t)$ where u is the follower's velocity which we pick
> - The leader's speed is unknown and beyond our control but they don't drive too wildly, ie $\frac{dx_l(t)}{dt} \approx C$ for some constant C

Option 1: Open-Loop

- don't equip the follower with sensors (save money!)
- algorithm to decide velocity only has access to the desired inter-vehicle (denoted by $r(t)$)

Control Diagram:



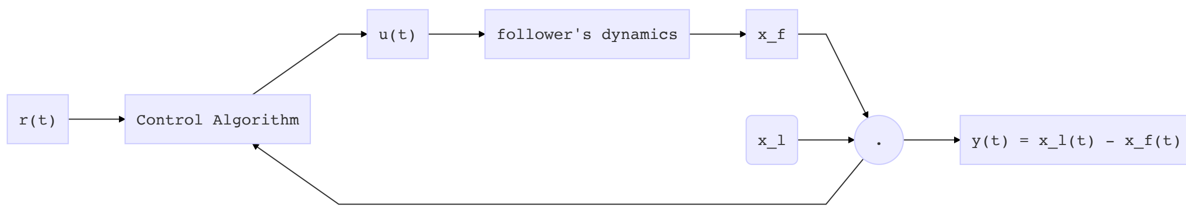
Problem:

- since the controller has no idea what $y(t)$ is, it cannot make a ood decision
- open-loop will not work well in practice

Option 2: Closed-Loop

- equip the follower with a stereo camera or LIDAR sensors to measure $y(t) = x_l(t) - x_f(t)$
- same control diagram as before bu thte control algo now receives $y(t)$

Control Diagram:



Now we choose a control algo!

Choice 1: simplest control algorithm is "on-off" control

$$u(t) = \begin{cases} \bar{u}, & r(t) - y(t) < 0 \\ u, & r(t) - y(t) > 0 \end{cases}$$

This algorithm has problems

1. $\bar{u}$ must be picked to be greater than the leader's velocity , same with u
2. the resulting motion is uncomfortable

Choice 2: proportional error feedback

We pick some K_p such that $K_p > 0$ and have,

$$u(t) = -K_p(r(t) - y(t))$$

We'll learn that an even better controller is proportional-integral error feedback.

$$u(t) = -K_p(r(t) - y(t)) - K_i \int_0^t r(\tau) - y(\tau) d\tau$$

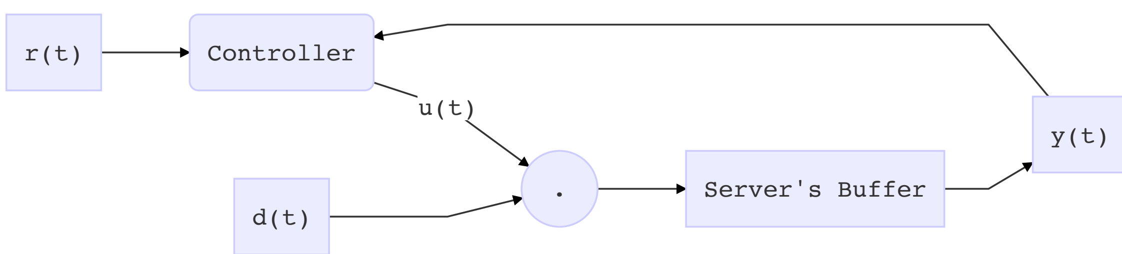
Day 2

Example 1.4.1

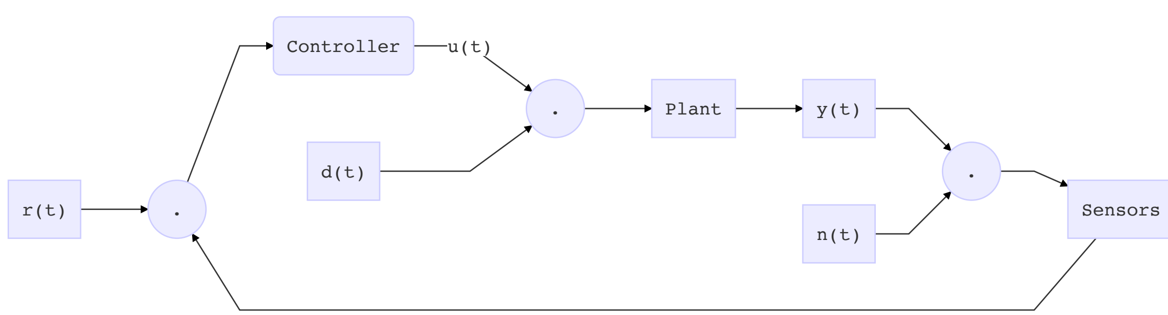
> Consider a webserver that responds to queries from browsers
>
> - The server contains a buffer of incoming requests, so that if it's busy and a request comes then it doesn't loose track of the request
> - Let $a(t)$ be the total requests received at time t

> - Let $b(t)$be the total served requests at time t
> - Let $y(t)$ be the size of the buffer at time t
> - We then have $y(t) = a(t) - b(t)$
>
> Objective: Keep the buffer size $y(t)$ at a constant non-zero value $r(t)$
>
> - Constraint: the server's service rate is unknown as it depends on the number of clients. Therefore we model the service rate as an unknown disturbance
> - Control objective: decide on the request rate input $u(t)$ so that $y(t) \approx r(t)$

We then have $\frac{dy}{dt} = u(t) - d(t)$ for demand $d(t)$ in order for the buffer size to stay constant, since if $d(t) >> u(t)$ then the buffer will get very large and so we hope to increase $u(t)$ such that the rate that the buffer is increasing, $\frac{dy}{dt}$, decreases.



Control engineering attempts to change the behaviour of a system (Plant) in a useful way despite the presence of external influences ("distrubances") and *model uncertainty*.



Signals

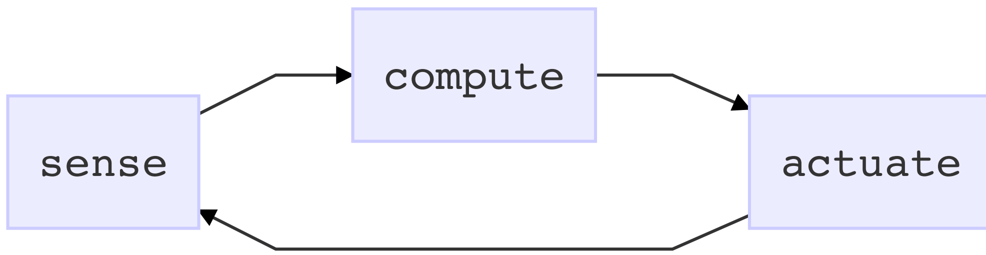
- $r(t)$: reference
- $y(t)$: plant output
- $u(t)$ controller output
- $a(t)$: distrubance
- $n(t)$: noise

Systems

- Controller
- Plant
- Sensors

Note: In the course, often we will be given the plant but we must decide on the sensors and controller!

Control Cycle:



The controllers we develop differ from traditional software because of the control role that *system dynamics* play and the real-time nature of implementation

1.3 Control Engineering Design Cycle

1. Study the system that is to be controlled and then decide on sensors & actuators
    - e.g. camera or LIDAR, type of motor
2. Model the resulting system
    - a mathematical model
    - often one or more differential equations
        - e.g. follower in last day's example: $\frac{dx_f}{dr} = u(t)$
3. Simplify the model if necessary
    - classical control requires that the plant have a *transfer function*, ie it is LTI and is an input-output model
    - e.g. follower example: we take the Laplace transform of both sides
        - $L\{\frac{dx_f}{dr}\} = L\{u(t)\}$
        - $X_f(s) - x_f(0) = U(s)$

        - $X_f(s) = \frac{1}{s}U(s) + \frac{1}{s}x_f(0)$
        - The fraction $\frac{1}{s}$ in the term $\frac{1}{s}U(s)$ is the transfer function!
4. Analyze the model
5. Determine design specifications
    - Stability (the most important specification)
        - ex. The follower does not stop or keep accelerating as time goes on
    - Good steady-state behaviour
        - ex. The follower is the right distance away from the leader car
    - Robustness
        - ex. The system consistent with different parameters
    - Good transient behaviour
        - ex. We don't overshoot or undershoot too much, so as to make the acceleration pleasant
6. Decide on type of controller
7. Design the controller
    - In this course we will limit ourselves to LTI controllers, has a transfer function
    - e.g. the follower's P1 controller
        - $\frac{U(S)}{E(s)} = K_p + \frac{1}{s}K_i$
8. Simulate a closed-loop system
    - ex. use MATLAB to run the system and plot system variables, error, etc
9. Return to step 1 if necessary
10. Implement the controller on an actual system!

Note: we can build circuits that have the transfer function from step 7

- More realistically, the ODE from step 7 is approximated as a *difference equation* so as to perform computation over discretized time
- This approximation works well so long as the computer runs fast enough
    - Ex. the control system could have very fast dynamics which would require faster compute to run digitally

Tutorial 1

Complex Numbers

A complex number is a pair of real numbers a and b put into the form $a + bj$ such that $j^2 = -1$.

Addition:

$$a + bj + c + dj$$
$$= (a + c) + (b + d)j$$

Multiplication:

$$(a + jb)(c + jd)$$
$$= ac + jbc + jad + bdj^2$$
$$= (ac - bd) + (bc + ad)j$$

We may also plot the complex numbers on a plane where
$y = Im(a + jb) = b \wedge x = Re(a + jb) = a$

We denote the length as the *modulus*, ex. $|1 + \sqrt{3}j| = \sqrt{(1)^2 + (\sqrt{3})^2}$. We denote the angle as the *argument*, ex. $Arg(1 + \sqrt{3}j) = \frac{\pi}{3}$. We denote this form as the *rectangular* or *cartesian* form.

The *polar* form of a complex number is $re^{j\Theta}$ such that $r > 0$ and $\Theta \in (-\pi, \pi]$. Because of Euler's identity we have $e^{j\Theta} = cos\Theta + jsin\Theta$ with the following properties.

1. $|c| = r$
2. $Arg(c) = \Theta$

Ex. Compute $\frac{1}{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})^4}$

$$\frac{1}{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})^4}$$
$$= \frac{1}{(e^{\frac{j\pi}{4}})^4}$$
$$= \frac{1}{e^{j\pi}}$$
$$= e^{-j\pi}$$
$$= -1$$

Laplace Transforms

Recall:

- say $f : R -> R, f \in O(s^+)$
- Then the Laplace transform of f, denoted F, is $F(s) = \int_0^\infty f(\tau)e^{-st}d\tau$
- Eg: The Laplace transform of $f(t) = e^{3t}$ is $F(s) = \frac{1}{s-3}$

Recall:

- $L\{f + g\} = L\{f\} + L\{g\}$
- $L\{cf\} = cL\{f\}$

$$y^{'} + 3y = 0$$
$$sY(s) - y(0) + 3Y(1) = 0$$

- Derivative Rule:

$$L\{\frac{d^n}{dt^n}\}$$
$$= s^nY(s) - s^{n-1}y(0) - s^{n-2}y^{'}(0) - s^{n-3}y^{''}(0) - \ldots - 1y^{n-1}(0)$$

- Convolution: $L\{f * g\} = F(s)G(s)$
- Frequency Diff: $L\{e^{at}f(t)\} = F(s-a)$

Ex. Find the transfer function

$$y^{''}(t) + y^{'}(t) = u(t)$$
$$y(0) = y^{'}(0) = 0$$
$$s^2Y(s) + sY(s) = U(s)$$
$$Y(s) = \frac{1}{s^2+s}U(s)$$

Here, $\frac{1}{s^2+s}$ is the transfer function since it multiples the input to get the output in the frequency domain!

Linear Algebra

Let $x = (x_1, x_2)^T \in R^2$ be a vector

$$f(x) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3(1)+1(1) \\ 0(1)+2(1) \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3(3)+1(1) & 3(0)+1(1) & 3(0)+1(1) \\ 0(3)+2(1) & 0(0)+2(1) & 0(0)+2(1) \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Lets say that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and we know y_1 and y_2.

Eg:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For a 2x2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define the determinant as $det(M) = ad - bc$. If $det(M) \neq 0, \exists M^{-1}, M^{-1}M = I$ then we have $M^{-1} = \frac{1}{det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Day 3

Mathematical Models of Systems

- A *model* is a set of equations that describe how a system transfers inputs to outputs
- No model is perfect!
- There is a compelxity trade-off between accuracy and complexity

> How do you come up with a model?
>
> 1. first principles
>   - there is some phenomena that already describes the system
> 2. system indentification
>   - run some experiments on the system and identify its behavior

First principles / system identification $\xrightarrow{\text{coupled DEs}}$ Simplify, linearize model near a pt $\xrightarrow{\text{linear ODEs}}$ Take LTs with zero init cond. $\longrightarrow$ express I/O $\longrightarrow$ TF

2.3 Modelling based on 1st principles

2.3.1 Mechanical Systems

Ex 2.11. mass-spring-damper

- u - input, applied force to the mass pulling it away from the spring
- M - mass in kg
- $q(t) \in R$ - position, output

By Newton's second law, we have $Mq^{''}(t) = \sum F$. We note that the 3 forces acting on the mass are the spring and damper which resist the applied force u. By Hooke's law, we know that the force from the spring is linear, Kq. We also know the force due to the damper is possible non linear, $c(q^{'}), c : R \implies R$. Therefore our free body diagram equation yields the following.

$$Mq^{''}(t) = u(t) - Kq(t) - c(q(t))$$

We now have a non-linear ODE, we may make the assumption that the force due to friction is linear, which implies $c(q^{'}) = bq^{'}$, and the model becomes linear, and we have the following.

$$Mq^{''}(t) = u(t) - Kq(t) - bq^{'}(t)$$

Ex. 2 masses connected by a spring and damper

- u - input, applied force to M_1 pushing it in the direction of M_2
- $M_1 \wedge M_2$ are the masses of each block in kg
- $q_1 \wedge q_2$ are the positions of $M_1 \wedge M_2$
- we assume that when $q_1 = q_2$ the spring is at rest

We then have,

$$M_1 q_1^{''}(t) = u(t) - Kq_1(t) + Kq_2(t) - bq_1^{'}(t) + bq_2^{'}(t)$$
$$M_2 q_2^{''}(t) = Kq_1(t) - Kq_2(t) + bq_1^{'}(t) - bq_2^{'}(t)$$

2.3.2 Electric Systems

Ex. Simple Circuit

- We have an applied voltage $u(t)$
- We have a voltage accross a capacitor C, $y(t)$
- The above 2 components are connected in series with a non-linear resistor V_r
- $V_r = h(i)$ may in fact be non-linear, even though in the past we have seen this be linear
- We then have $V_r(t) = h(i(t))$ for some possible non-linear $h : R \implies R$.

We then use KVL, we simply pick if whether positive to negative is negative or positive. The other will be the opposite sign.

$$-u(t) + V_r(t) + y(t) = 0$$
$$-u(t) + h(Cy^{'}(t)) + y(t) = 0$$

If we assume $h(i(t))$ is linear, we have $V_r(t) = Ri(t)$, and our system is the following linear model.

$$-u(t) + RCy^{'}(t) + y(t) = 0$$

Our goal now is to unify all these models in a single framework, for a class of models call *state-space models*.

State-Space Models

A way to express mathematical models in a standard form.

Ex. 2.4.1

- We have some mass M moving linearly on the ground
- We have an applied force on M to the right called $u(t)$
- We have a force that opposes $u(t)$, the air resistance which may be non-linear as $D(y^{'})$
- Lastly the output of the system is the position of M to the right of a starting relative position

We now apply Newton's second law,

$$My^{''}(t) = u(t) - D(y^{'}(t))$$

Now we put the model into standard form by defining two so-called state variables.

1. $x_1 := y$ as the position
2. $x_2 := y^{'}$ as the velocity

We may then write the following *static equations*

$$x_1^{'} = x_2$$
$$x_2^{'} = y^{''}(t)$$
$$x_2^{'} = \frac{1}{M}u - \frac{1}{M}D(x_2))$$

We now write our *output equations*, the algebraic part.

$$y = x_1$$

These equations have the general form $x^{'} = f(x, u)$ and $y = h(x)$, for a non-linear state-space model. This will be our standard way of expressing models. In this example, we have $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^2$ and we also have $f(x, u) = \begin{bmatrix} x_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2)) \end{bmatrix}$. The function $f : R^2xR \implies R^2$ is linear if $D(x_2)$ is linear, $h : R^2 \implies R$ is linear.