SE 380

Feedback Control

Day 1

Example

> There are two cars on a linear track. There is a follower and a leader. There are the following constraints:

> - The distinace from the beginning of the track to the follower's center is x_f
> - The distinace from the beginning of the track to the leader's center is x_l

> - The distance between the follower and the leader is $x_l - x_f$
> - leader is driven by a human
> - follower is driven by a computer which is able to assign the velocity of the vehicle

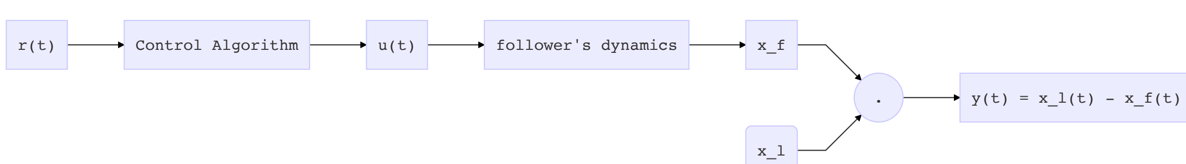
> Objective: Write a program that decides the appropriate speed of the follower in order to maintain a given (safe) inter-vehicle disance

> Assumptions:
>
> - 1-d motion only
> - $\frac{dx_f(t)}{dt} - u(t)$ where u is the follower's velocity which we pick
> - The leader's speed is unknown and beyond our control but they don't drive too wildly, ie $\frac{dx_l(t)}{dt} \approx C$ for some constant C

Option 1: Open-Loop

- don't equip the follower with sensors (save money!)
- algorithm to decide velocity only has access to the desired inter-vehicle (denoted by $r(t)$)

Control Diagram:



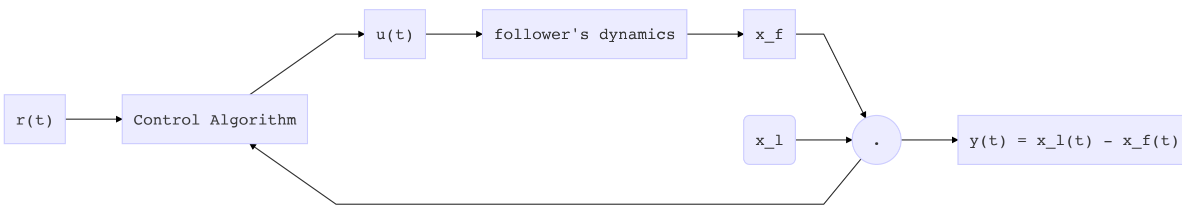
Problem:

- since the controller has no idea what $y(t)$ is, it cannot make a ood decision
- open-loop will not work well in practice

Option 2: Closed-Loop

- equip the follower with a stereo camera or LIDAR sensors to measure $y(t) = x_l(t) - x_f(t)$
- same control diagram as before bu thte control algo now receives $y(t)$

Control Diagram:



Now we choose a control algo!

Choice 1: simplest control algorithm is "on-off" control

$$u(t) = \begin{cases} \bar{u}, & r(t) - y(t) < 0 \\ u, & r(t) - y(t) > 0 \end{cases}$$

This algorithm has problems

1. $\bar{u}$ must be picked to be greater than the leader's velocity , same with u
2. the resulting motion is uncomfortable

Choice 2: proportional error feedback

We pick some K_p such that $K_p > 0$ and have,

$$u(t) = -K_p(r(t) - y(t))$$

We'll learn that an even better controller is proportional-integral error feedback.

$$u(t) = -K_p(r(t) - y(t)) - K_i \int_0^t r(\tau) - y(\tau) d\tau$$

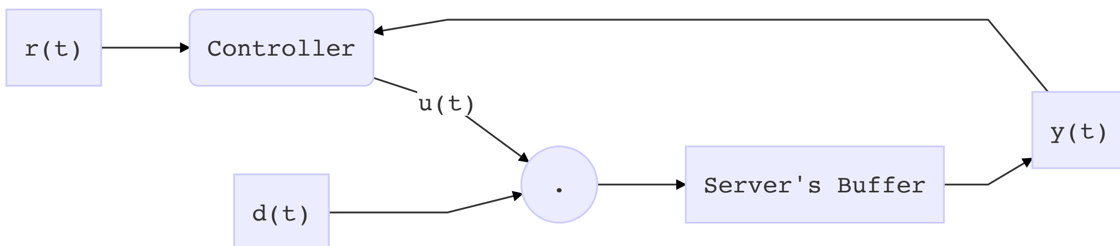
Day 2

Example 1.4.1

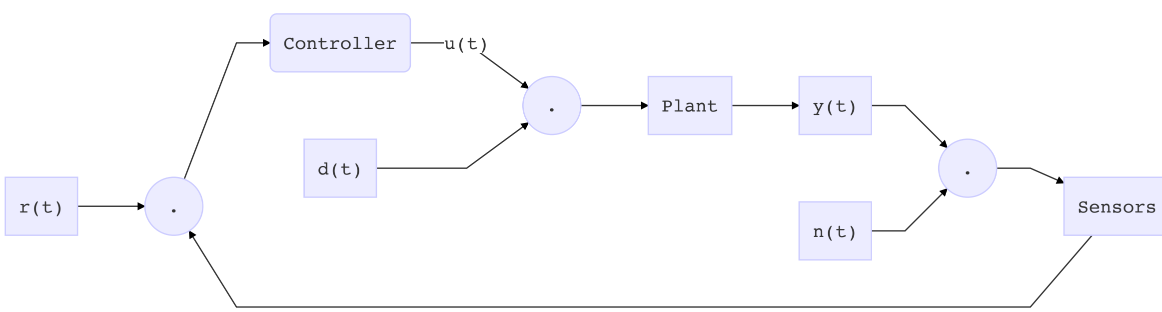
> Consider a webserver that responds to queries from browsers
>
> - The server contains a buffer of incoming requests, so that if it's busy and a request comes then it doesn't loose track of the request
> - Let $a(t)$ be the total requests received at time t

> - Let $b(t)$be the total served requests at time t
> - Let $y(t)$ be the size of the buffer at time t
> - We then have $y(t) = a(t) - b(t)$
>
> Objective: Keep the buffer size $y(t)$ at a constant non-zero value $r(t)$
>
> - Constraint: the server's service rate is unknown as it depends on the number of clients. Therefore we model the service rate as an unknown disturbance
> - Control objective: decide on the request rate input $u(t)$ so that $y(t) \approx r(t)$

We then have $\frac{dy}{dt} = u(t) - d(t)$ for demand $d(t)$ in order for the buffer size to stay constant, since if $d(t) >> u(t)$ then the buffer will get very large and so we hope to increase $u(t)$ such that the rate that the buffer is increasing, $\frac{dy}{dt}$, decreases.



Control engineering attempts to change the behaviour of a system (Plant) in a useful way despite the presence of external influences ("distrubances") and *model uncertainty*.



Signals

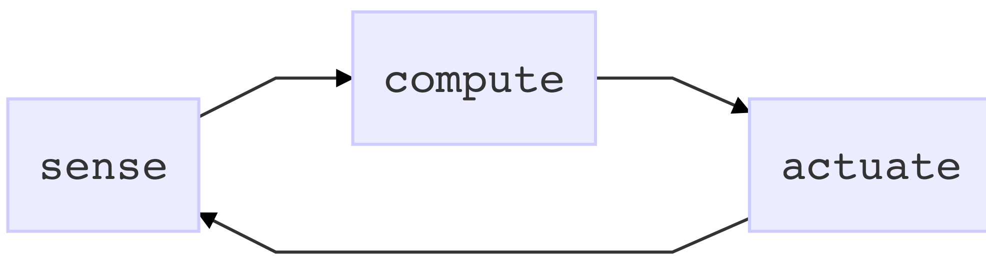
- $r(t)$: reference
- $y(t)$: plant output
- $u(t)$ controller output
- $a(t)$: distrubance
- $n(t)$: noise

Systems

- Controller
- Plant
- Sensors

Note: In the course, often we will be given the plant but we must decide on the sensors and controller!

Control Cycle:



The controllers we develop differ from traditional software because of the control role that *system dynamics* play and the real-time nature of implementation

1.3 Control Engineering Design Cycle

1. Study the system that is to be controlled and then decide on sensors & actuators
   - e.g. camera or LIDAR, type of motor
2. Model the resulting system
   - a mathematical model
   - often one or more differential equations
     - e.g. follower in last day's example: $\frac{dx_f}{dr} = u(t)$
3. Simplify the model if necessary
   - classical control requires that the plant have a *transfer function*, ie it is LTI and is an input-output model
   - e.g. follower example: we take the Laplace transform of both sides
     - $L\{\frac{dx_f}{dr}\} = L\{u(t)\}$
     - $X_f(s) - x_f(0) = U(s)$

        - $X_f(s) = \frac{1}{s}U(s) + \frac{1}{s}x_f(0)$
        - The fraction $\frac{1}{s}$ in the term $\frac{1}{s}U(s)$ is the transfer function!
4. Analyze the model
5. Determine design specifications
    - Stability (the most important specification)
        - ex. The follower does not stop or keep accelerating as time goes on
    - Good steady-state behaviour
        - ex. The follower is the right distance away from the leader car
    - Robustness
        - ex. The system consistent with different parameters
    - Good transient behaviour
        - ex. We don't overshoot or undershoot too much, so as to make the acceleration pleasant
6. Decide on type of controller
7. Design the controller
    - In this course we will limit ourselves to LTI controllers, has a transfer function
    - e.g. the follower's P1 controller
        - $\frac{U(S)}{E(s)} = K_p + \frac{1}{s}K_i$
8. Simulate a closed-loop system
    - ex. use MATLAB to run the system and plot system variables, error, etc
9. Return to step 1 if necessary
10. Implement the controller on an actual system!

Note: we can build circuits that have the transfer function from step 7

- More realistically, the ODE from step 7 is approximated as a *difference equation* so as to perform computation over discretized time
- This approximation works well so long as the computer runs fast enough
    - Ex. the control system could have very fast dynamics which would require faster compute to run digitally

Tutorial 1

Complex Numbers

A complex number is a pair of real numbers a and b put into the form $a + bj$ such that $j^2 = -1$.

Addition:

$$a + bj + c + dj \\ = (a + c) + (b + d)j$$

Multiplication:

$$(a + jb)(c + jd) \\ = ac + jbc + jad + bdj^2 \\ = (ac - bd) + (bc + ad)j$$

We may also plot the complex numbers on a plane where $y = Im(a + jb) = b \wedge x = Re(a + jb) = a$

We denote the length as the *modulus*, ex. $|1 + \sqrt{3}j| = \sqrt{(1)^2 + (\sqrt{3})^2}$. We denote the angle as the *argument*, ex. $Arg(1 + \sqrt{3}j) = \frac{\pi}{3}$. We denote this form as the *rectangular* or *cartesian* form.

The *polar* form of a complex number is $re^{j\Theta}$ such that $r > 0$ and $\Theta \in (-\pi, \pi]$. Because of Euler's identity we have $e^{j\Theta} = cos\Theta + jsin\Theta$ with the following properties.

1. $|c| = r$
2. $Arg(c) = \Theta$

Ex. Compute $\frac{1}{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})^4}$

$$\frac{1}{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})^4} \\ = \frac{1}{(e^{\frac{j\pi}{4}})^4} \\ = \frac{1}{e^{j\pi}} \\ = e^{-j\pi} \\ = -1$$

Laplace Transforms

Recall:

- say $f : R -> R, f \in O(s^+)$
- Then the Laplace transform of f, denoted F, is $F(s) = \int_0^\infty f(\tau)e^{-st}d\tau$
- Eg: The Laplace transform of $f(t) = e^{3t}$ is $F(s) = \frac{1}{s-3}$

Recall:

- $L\{f + g\} = L\{f\} + L\{g\}$
- $L\{cf\} = cL\{f\}$

$$y' + 3y = 0$$
$$sY(s) - y(0) + 3Y(1) = 0$$

- Derivative Rule:

$$L\{\frac{d^n}{dt^n}\}$$
$$= s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \ldots - 1y^{n-1}(0)$$

- Convolution: $L\{f * g\} = F(s)G(s)$
- Frequency Diff: $L\{e^{at}f(t)\} = F(s-a)$

Ex. Find the transfer function

$$y''(t) + y'(t) = u(t)$$
$$y(0) = y'(0) = 0$$
$$s^2Y(s) + sY(s) = U(s)$$
$$Y(s) = \frac{1}{s^2+s}U(s)$$

Here, $\frac{1}{s^2+s}$ is the transfer function since it multiples the input to get the output in the frequency domain!

Linear Algebra

Let $x = (x_1, x_2)^T \in R^2$ be a vector

$$f(x) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3(1) + 1(1) \\ 0(1) + 2(1) \end{bmatrix}$$
$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3(3) + 1(1) & 3(0) + 1(1) & 3(0) + 1(1) \\ 0(3) + 2(1) & 0(0) + 2(1) & 0(0) + 2(1) \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Lets say that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and we know y_1 and y_2.

Eg:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For a 2x2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define the determinant as $det(M) = ad - bc$. If $det(M) \neq 0, \exists M^{-1}, M^{-1}M = I$ then we have $M^{-1} = \frac{1}{det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Day 3

Mathematical Models of Systems

- A *model* is a set of equations that describe how a system transfers inputs to outputs
- No model is perfect!
- There is a compelxity trade-off between accuracy and complexity