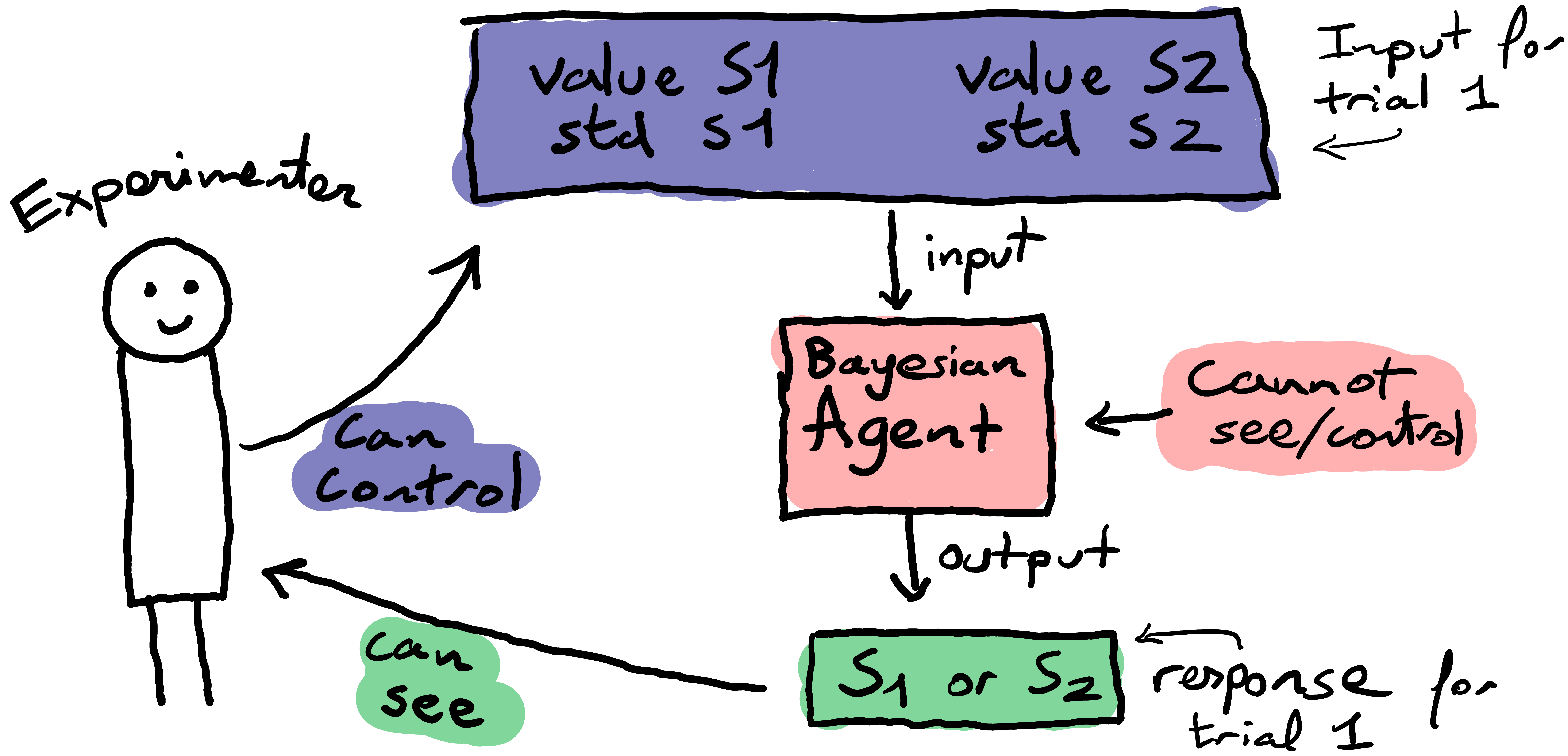
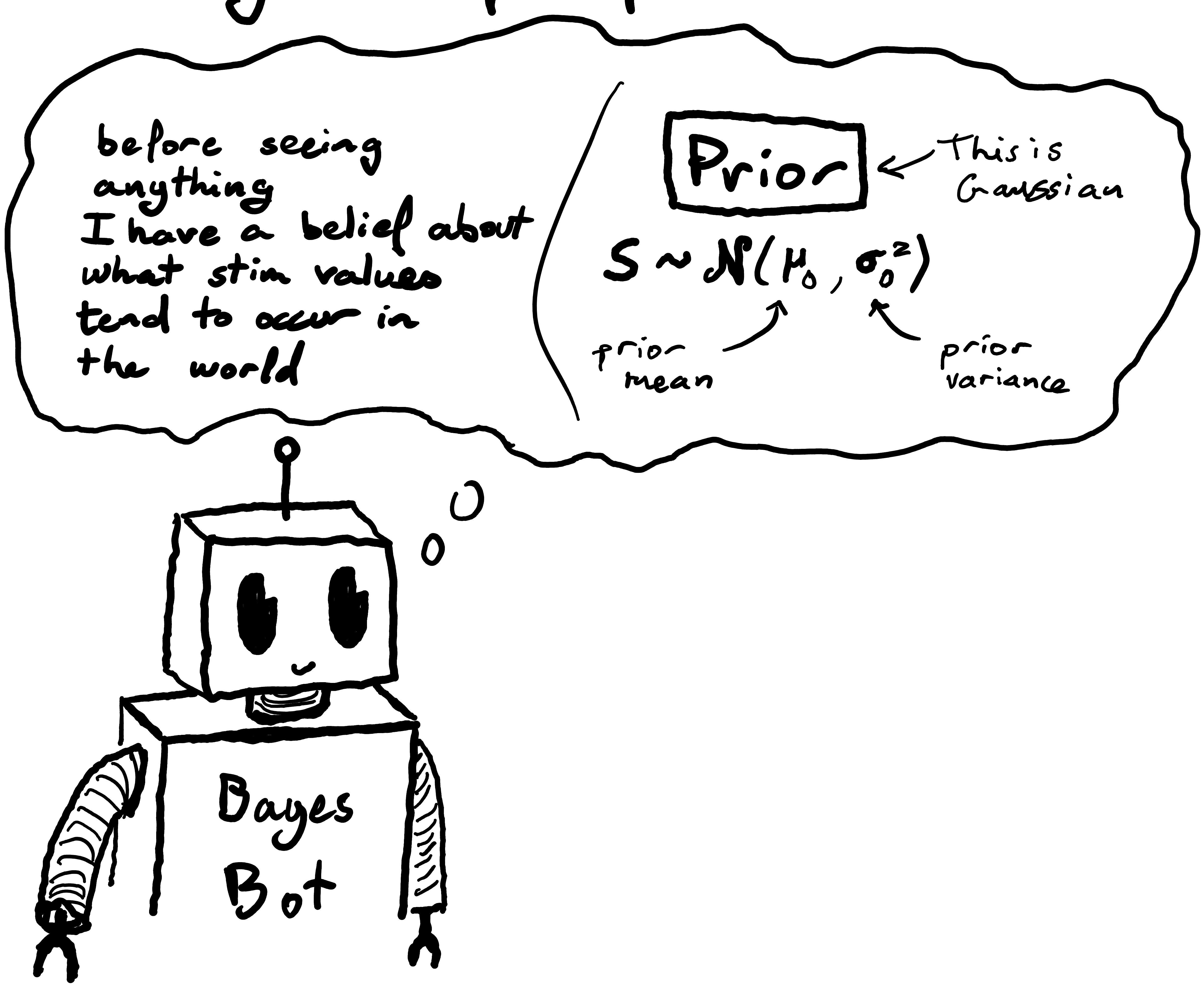
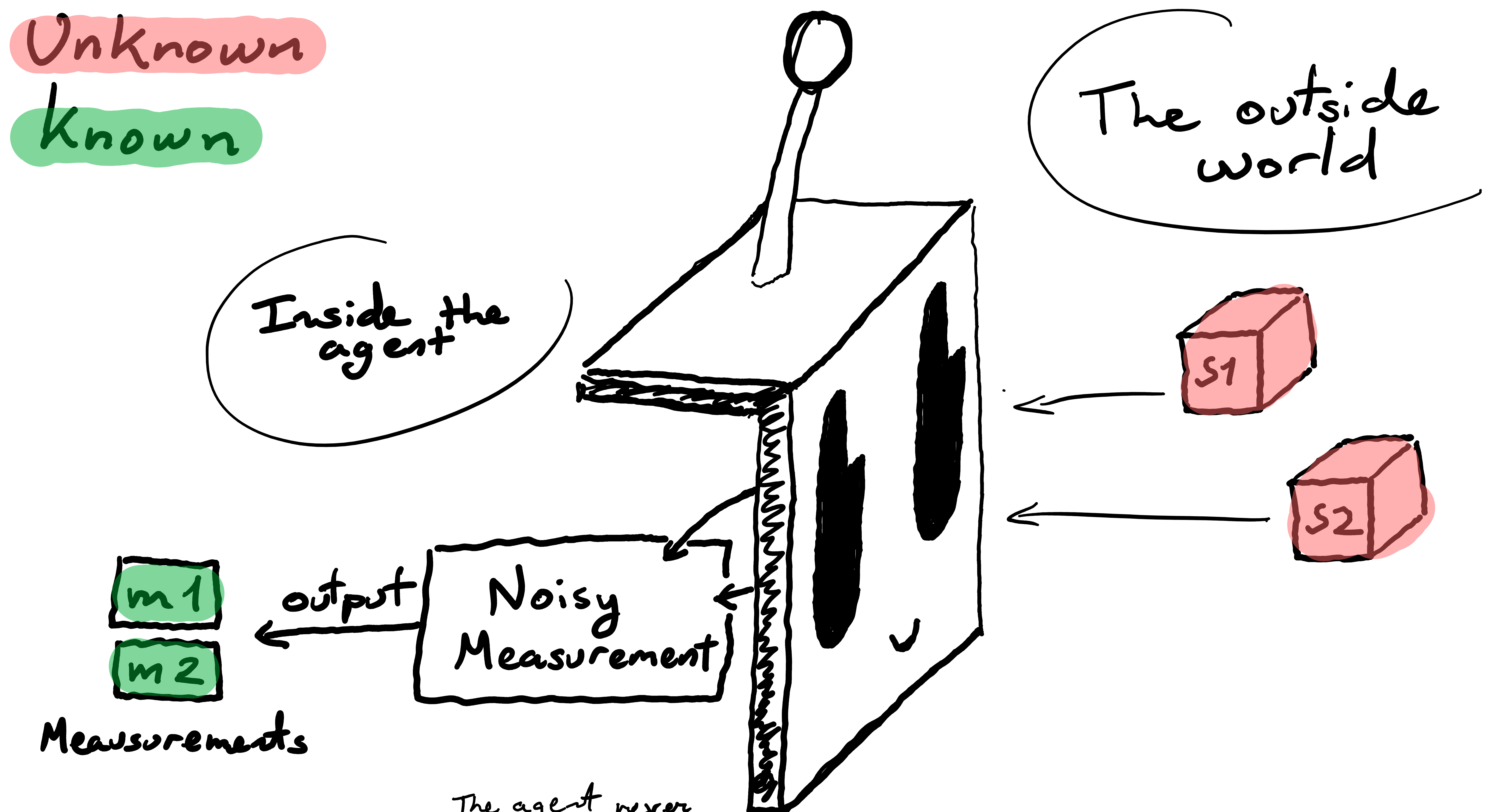
The experimenters perspective



The agents perspective





The agent never sees these

$$m1 = \text{value } S1 = \text{True } S1 + \text{Noise } S1$$

$$\text{Noise } S1 \sim \mathcal{N}(0, \text{std } S1^2)$$

this is the known std of the sensory noise

"if I measured the same S1 again, my measurement wld fluctuate around the true S1 with this std"

$$m2 = \text{value } S2 = \text{True } S2 + \text{Noise } S2$$

$$\text{Noise } S2 \sim \mathcal{N}(0, \text{std } S2^2)$$

Using the notation we get:

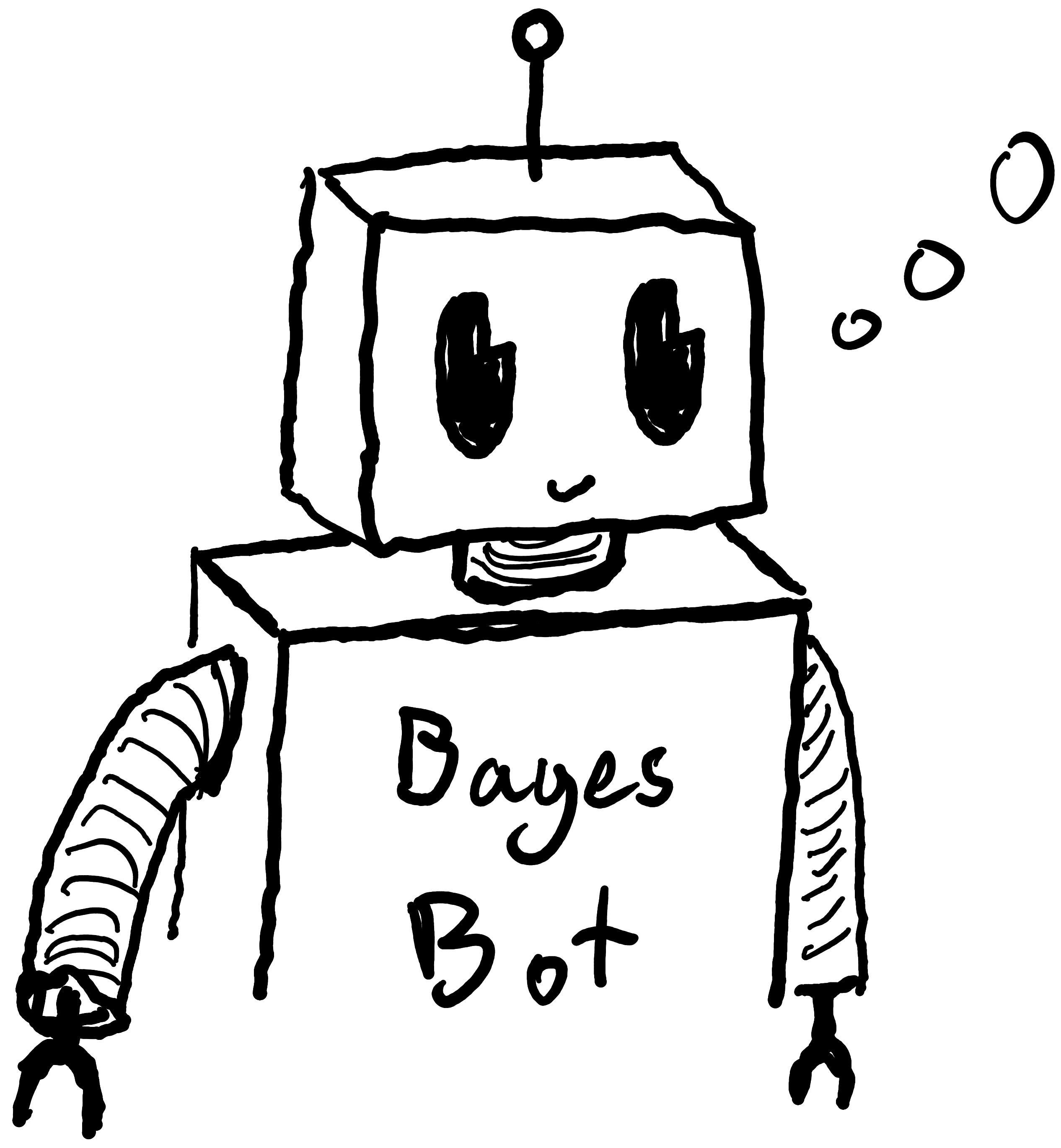
$$M_1 | S_1 \sim \mathcal{N}(S_1, \sigma_1^2)$$

$$M_2 | S_2 \sim \mathcal{N}(S_2, \sigma_2^2)$$

This is the notation I will use from now

- std S1 = σ_1 — known sensory noise for S1
- std S2 = σ_2
- value S1 = m_1 — measurement
- value S2 = m_2
- True S1 = S_1 — true value of S1
- True S2 = S_2

Now we have the following informat°:



Prior

m1

m2

$M_1 | S_1 \sim \mathcal{N}(S_1, \sigma_1^2)$

$M_2 | S_2 \sim \mathcal{N}(S_2, \sigma_2^2)$

The aim is to use this informat° to generate a posterior for each stimulus

Known

Unknown

Calculating the Posterior for stim 1:

we have : a measurement → m_1

a prior → $\mathcal{N} \sim (\mu_0, \sigma_0^2)$

a known std of sensory noise → σ_1^2

what we want

prior mean

measurement

posterior mean for S1 $= \mu_{post_1} = \dfrac{\frac{1}{\sigma_0^2} \cdot \mu_0 + \frac{1}{\sigma_1^2} \cdot m_1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2}}$

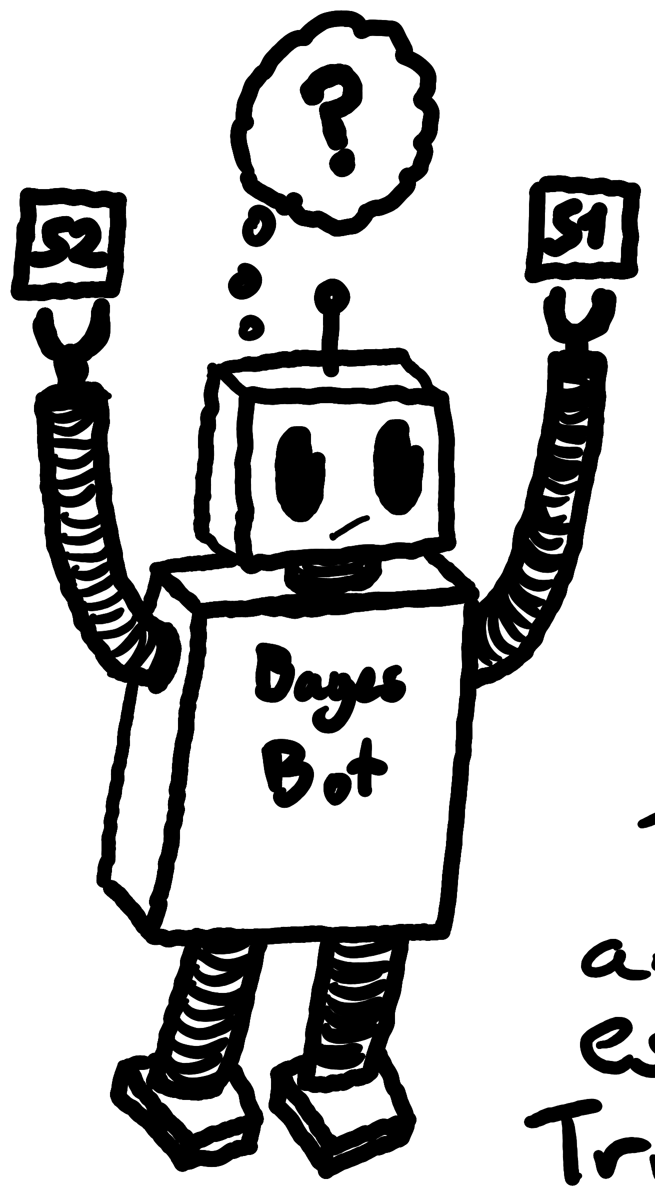
This is the MAP formula From Lecture 3 but with only one measure

posterior variance for S1 $= \sigma^2_{post\,1} = \dfrac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2}}$

Same for stim 2

↳ gives: μ_{post_2} and σ^2_{post2}

Time to make a decision!



we now have our two posteriors:

$$\hat{S}_1 \sim \mathcal{N}(\mu_{post1}, \sigma^2_{post1})$$

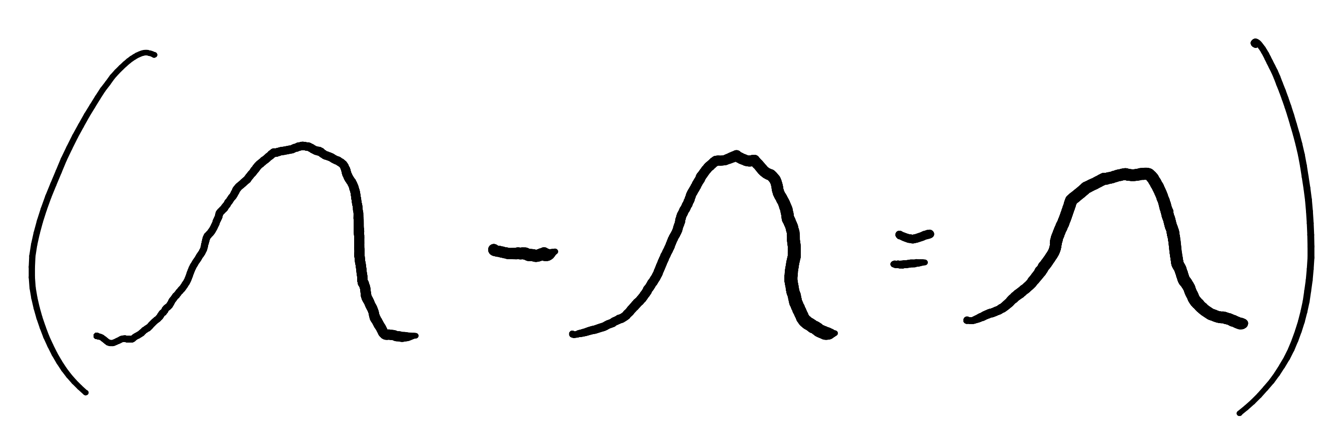
$$\hat{S}_2 \sim \mathcal{N}(\mu_{post2}, \sigma^2_{post2})$$

These are the estimated True S values

⚠ $\hat{S}_1$ and $\hat{S}_2$ are Gaussian distributions, not samples from the distribution

We want to compare $\hat{S}_1$ and $\hat{S}_2$ so we compute the difference:

$$\text{Difference} = D = \hat{S}_1 - \hat{S}_2$$



These are two Gaussians so D is also Gaussian

$$D \sim \mathcal{N}(\mu_D, \sigma^2_D)$$

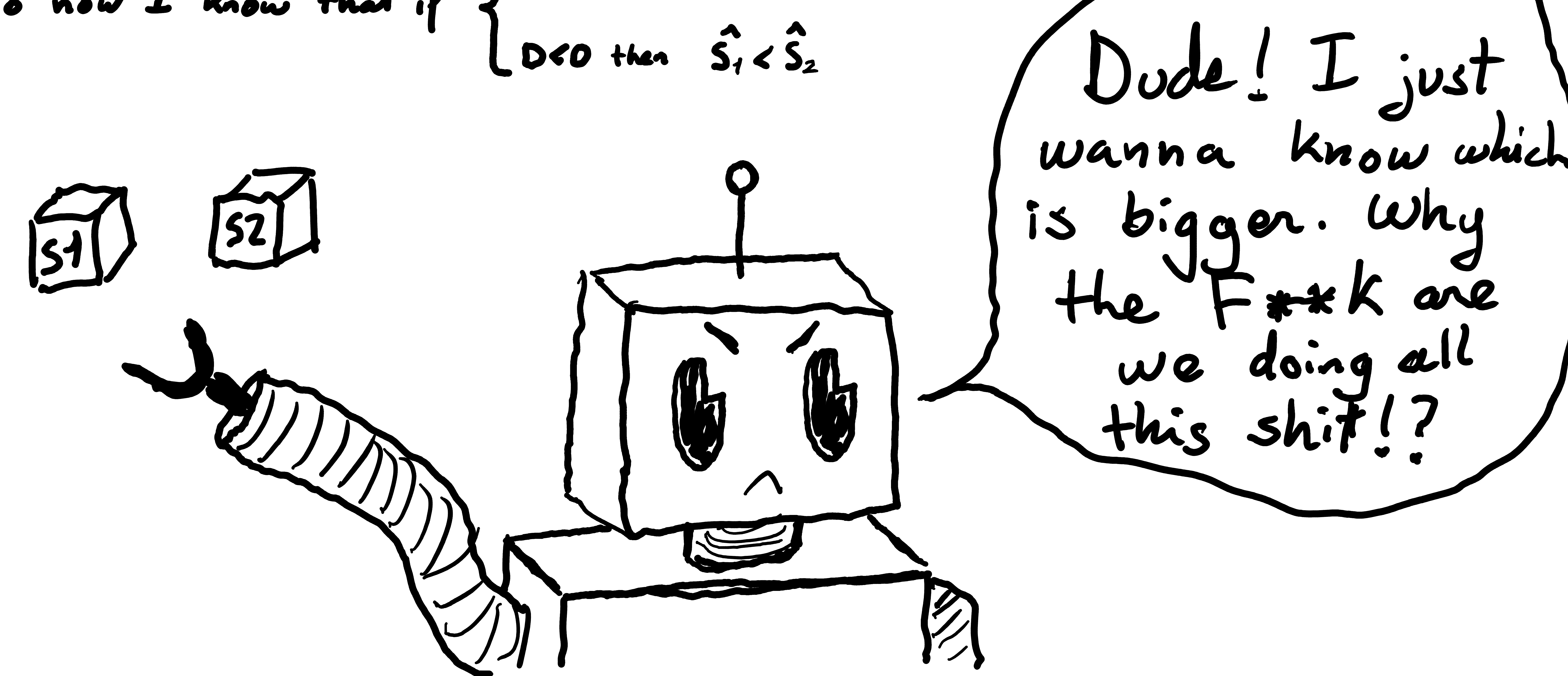
with:

$$\mu_D = \mathbb{E}[D] = \mathbb{E}[\hat{S}_1 - \hat{S}_2] = \mu_{post1} - \mu_{post2}$$

$$\sigma^2_D = V[\hat{S}_1 - \hat{S}_2] = V[\hat{S}_1] + V[\hat{S}_2] = \sigma^2_{post1} + \sigma^2_{post2}$$

This comes from the Lecture 3 pdf

So now I know that if $\begin{cases} D > 0 \text{ then } \hat{S}_1 > \hat{S}_2 \\ D < 0 \text{ then } \hat{S}_1 < \hat{S}_2 \end{cases}$



What we really want to know is

$$p(\text{true } S_1 > \text{true } S_2)$$ In the outside world

but we can't! We <u>never</u> see this.

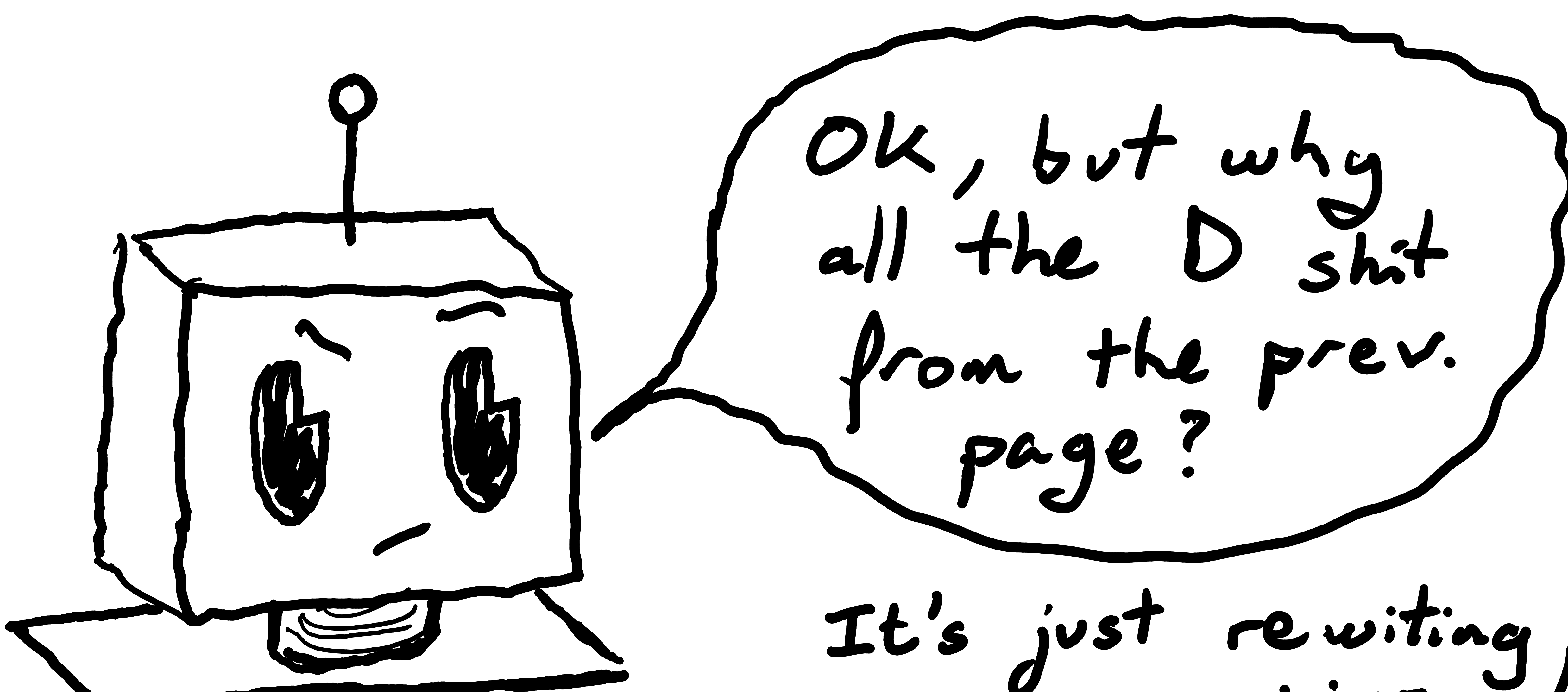
The best we can do is guess based on what we <u>Do</u> know.

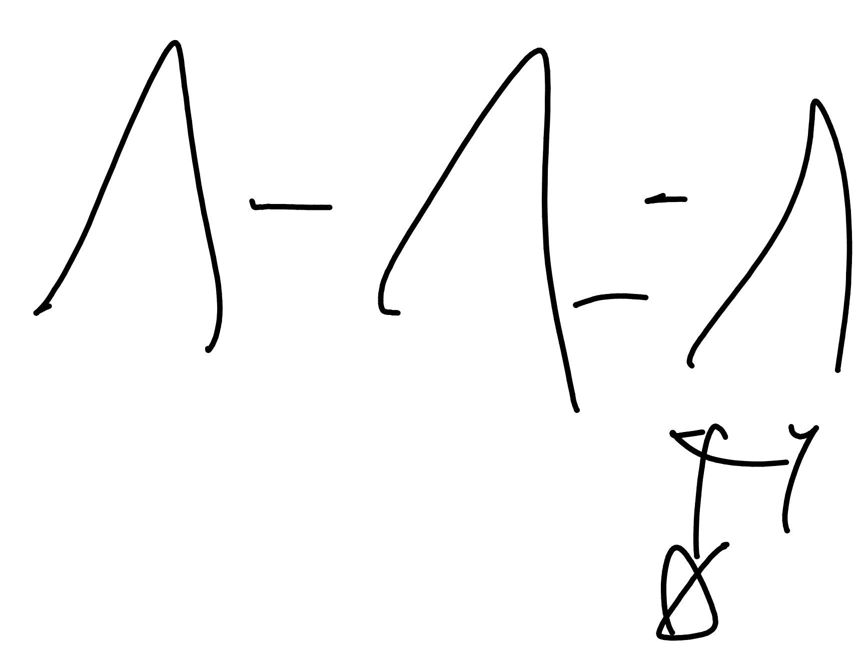
In other words: $p(\text{true } S_1 > \text{true } S_2 \mid \text{what I know})$

and that is exactly what $\hat{S_1}$ and $\hat{S_2}$ are

so...

$$p(\text{true } S_1 > \text{true } S_2 \mid \text{what I know}) = p(\hat{S_1} > \hat{S_2})$$





It's just rewiting the same thing in an easier form.

we define $D = \hat{S_1} - \hat{S_2}$

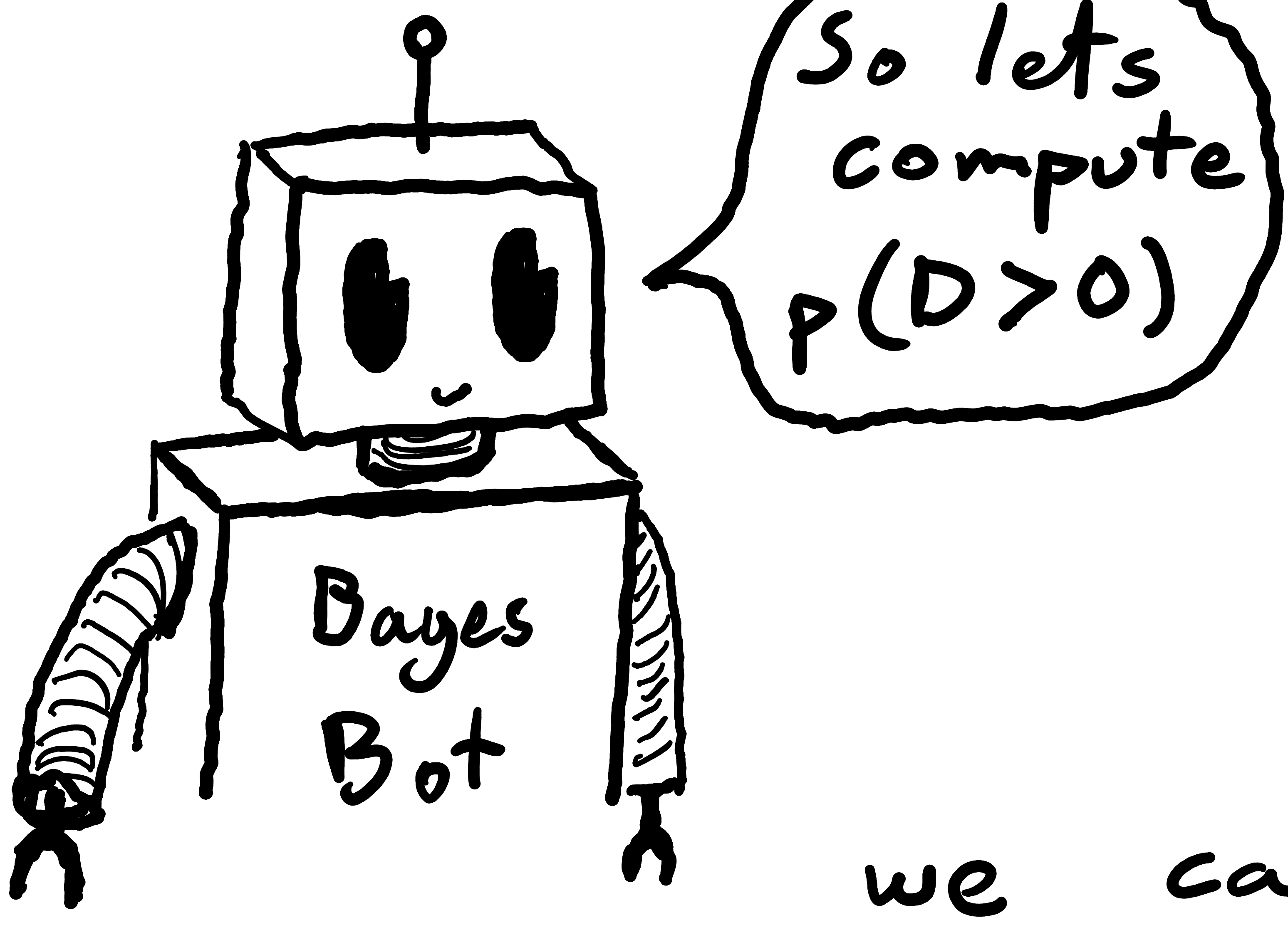
then we get $p(\hat{S_1} > \hat{S_2}) = p(\hat{S_1} - \hat{S_2} > 0)$

which is the same as

$$p(D > 0)$$

→ And this is just asking: "whats the probability that the Gaussian random variable D is bigger than 0"

And this ↗ we can compute!

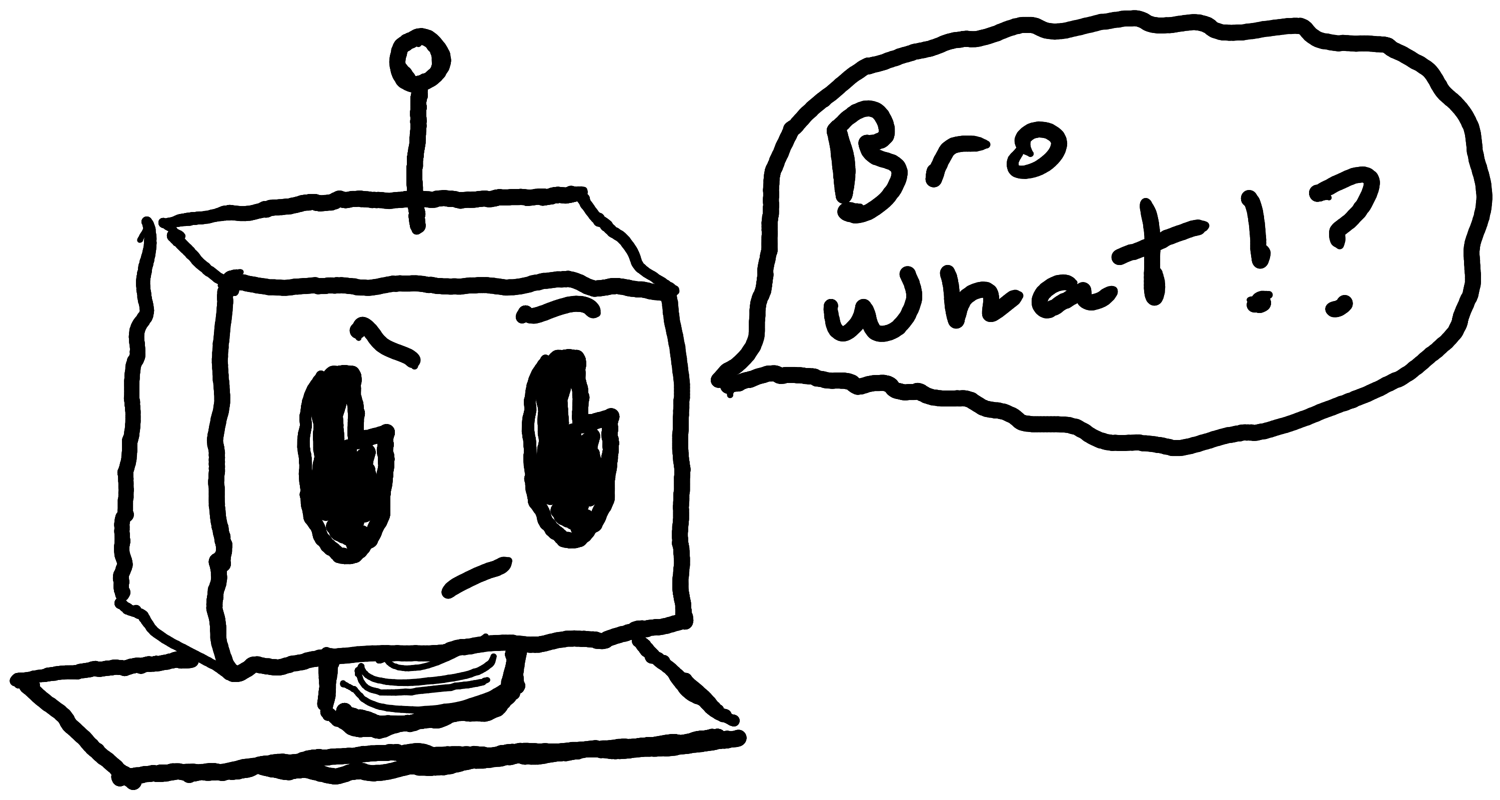


For this we have

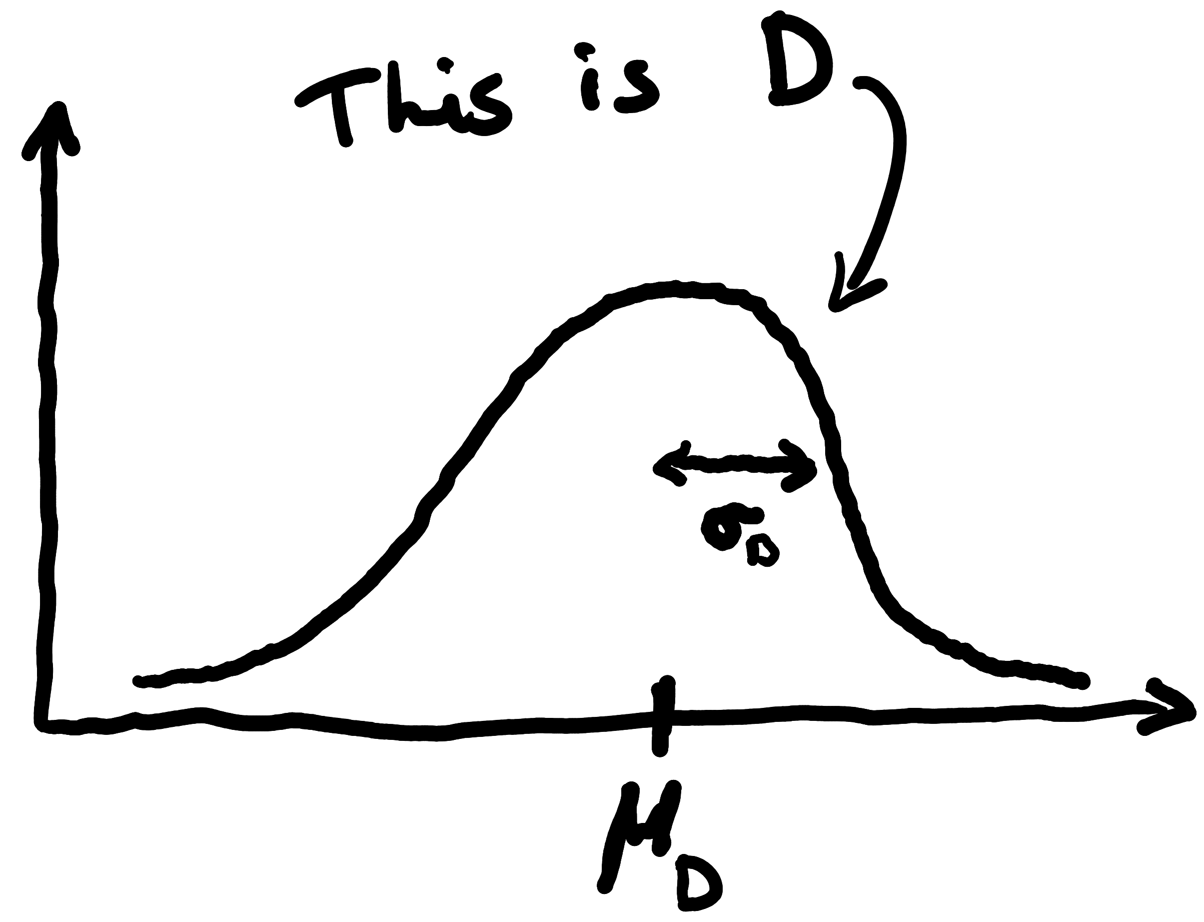
D a Gaussian rdm variable

$$D \sim \mathcal{N}(\underbrace{\mu_D}_{\text{mean}}, \underbrace{\sigma_D^2}_{\text{var}})$$

we can use the Standard Normal Cummulative Density Function (CDF)



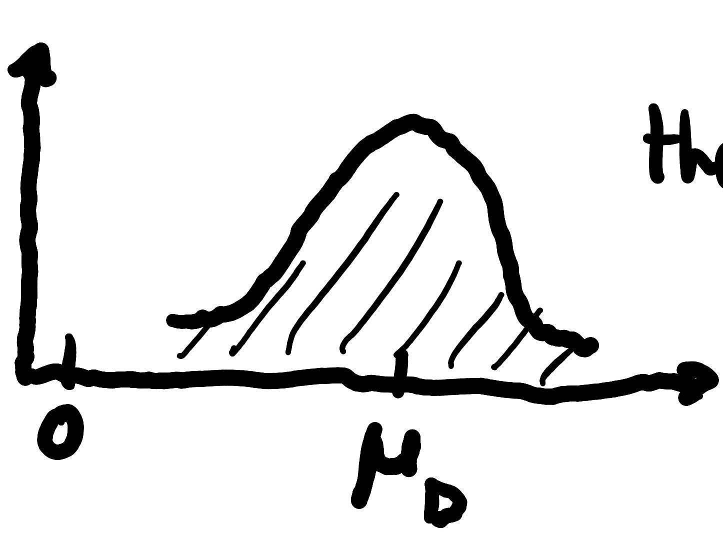
Yh, that's fair, let's be more explicit!



we want to know the area under the curve for $D>0$) Thats the anwer to what's $p(D>0)$

in math: $p(D>0) = \int_0^{+\infty} \mathcal{N}(d; \mu_D, \sigma_D^2)\, dd$

but what if μ_D is way above/bellow 0?



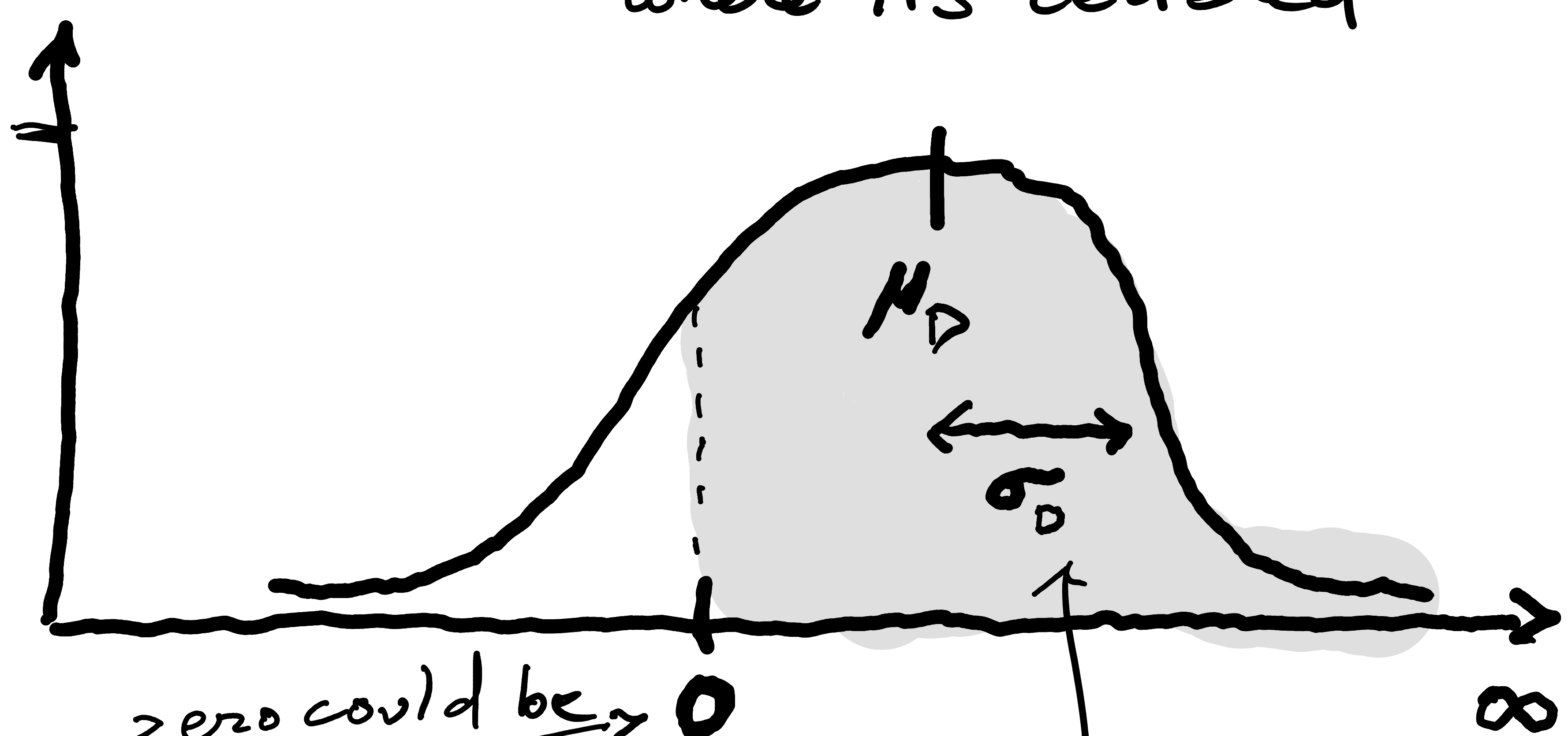
then the area wld be ~ 1 or ~ 0) i.e. the agent is very confident!

→ This is a pain to compute which is why we first standardize the Gaussian →

Standardizing $D \sim \mathcal{N}(\mu_D, \sigma_D^2)$

how wide it is

where its centered



zero could be anywhere

we want $\int_0^{+\infty} \mathcal{N}(\mu_D, \sigma_D^2)\, dd = p(D>0)$

In its current form this gets messy:

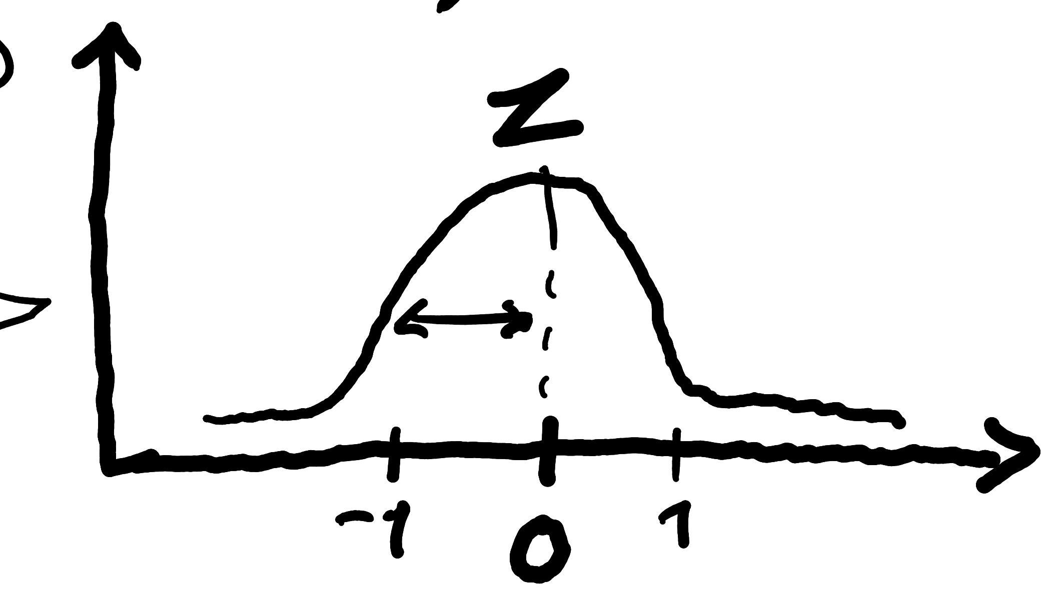
$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_D} \exp\left(-\frac{(d-\mu_D)^2}{2\sigma_D^2}\right) dd$$

To simplify we use the standard normal CDF $= \phi(z)$

This only works on a standard normal

$Z \sim \mathcal{N}(0,1)$

centered on 0 with a std of 1



So let's convert D into a standard normal

$$D' = D - \mu_D$$

this shifts the curve left/right so that it is centered on 0

now $D' \sim \mathcal{N}(0, \sigma_D^2)$

Then we rescale D' so that the std = 1

$$Z = \frac{D'}{\sigma_D} = \frac{D - \mu_D}{\sigma_D}$$

Now $Z \sim \mathcal{N}(0,1)$

We can now use the CDF $= \phi(z)$

we convert the event $p(D>0)$ into a statement about Z

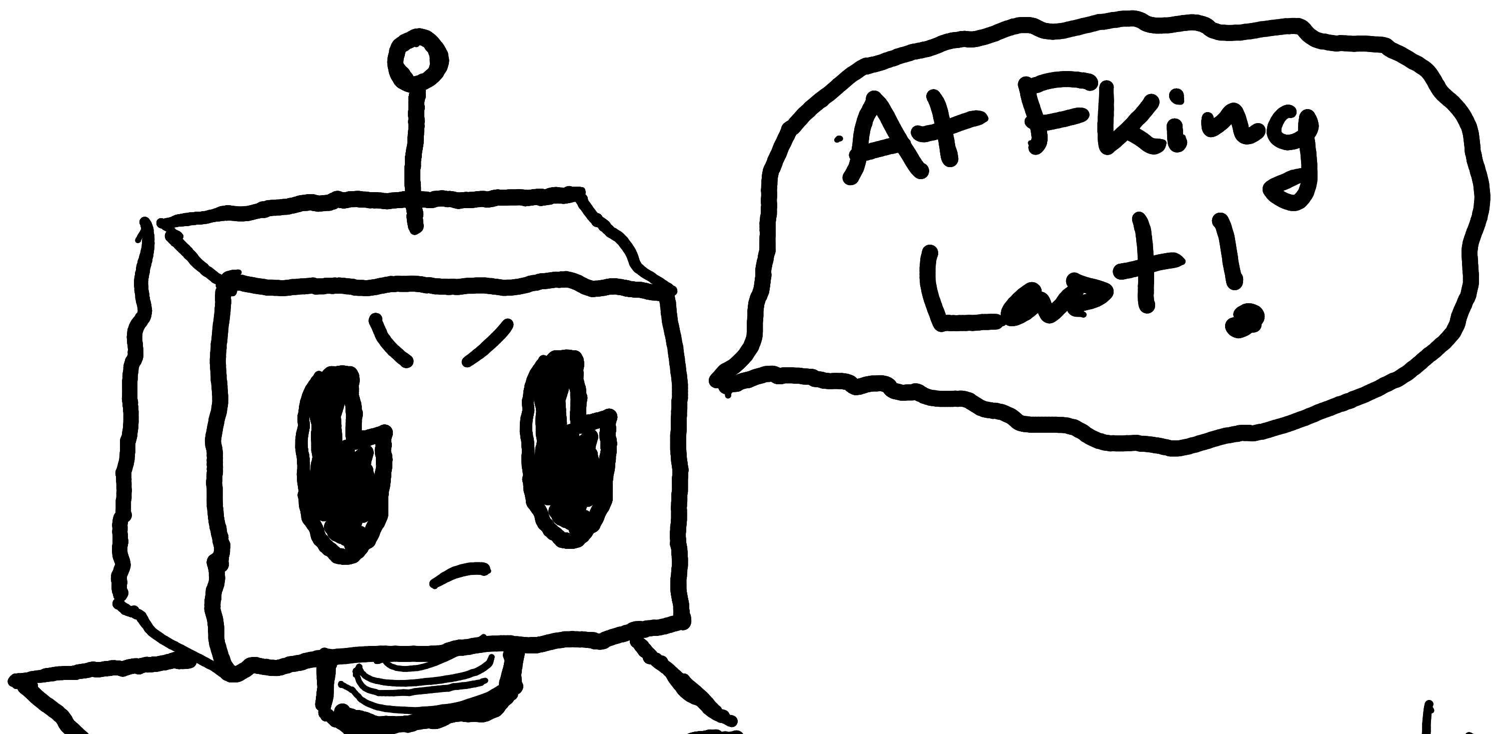
$$Z = \frac{D - \mu_D}{\sigma_D}$$

solve for D: $D = \mu_D + \sigma_D \cdot Z$

$$D > 0 = \mu_D + \sigma_D \cdot Z > 0$$
$$= Z > -\frac{\mu_D}{\sigma_D}$$

Thus $p(D>0) = p\left(Z > -\frac{\mu_D}{\sigma_D}\right)$ ← This is easy to get!

Now lets get $p(D>0)$



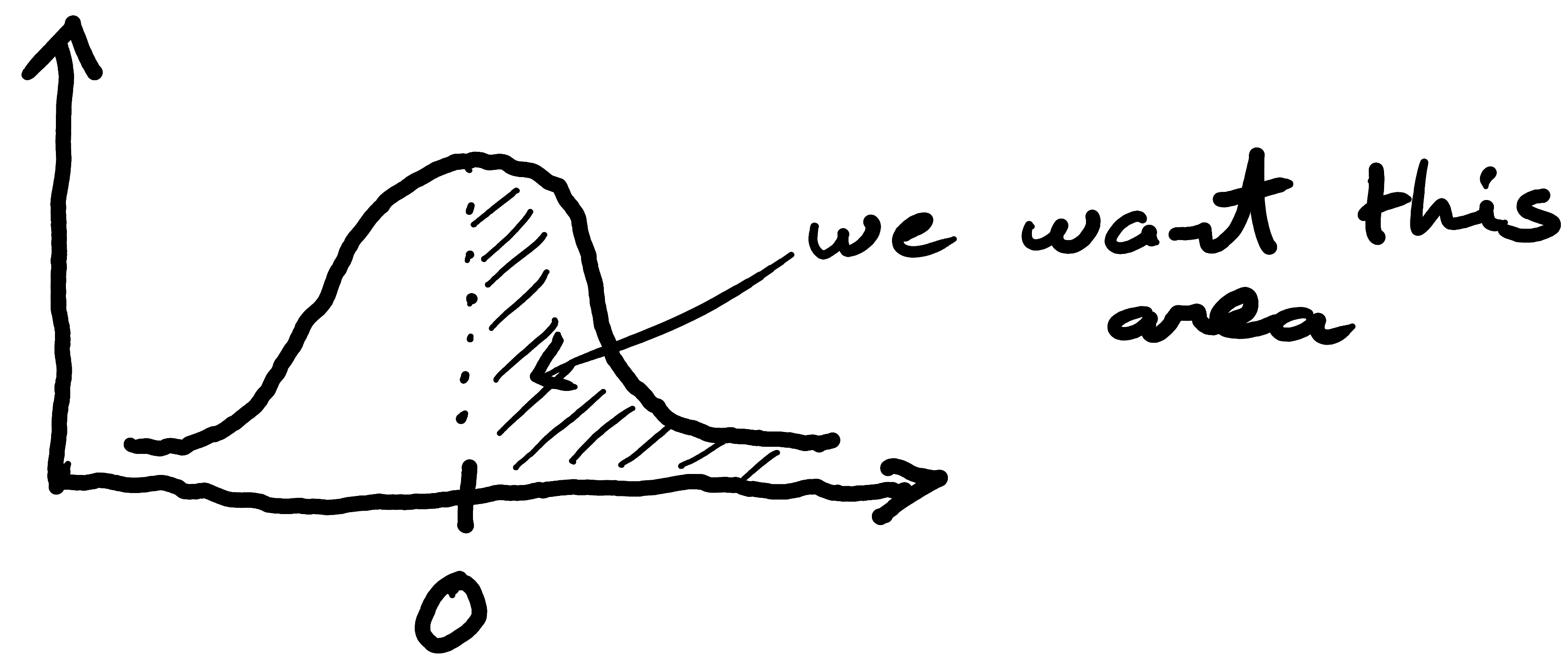
By definito the std nrml CDF:

$$\phi(a) = p(Z \leq a)$$

But we want $p(Z > a)$

which is $= 1 - p(Z \leq a)$ ← true for any probability distributo

$= 1 - \phi(a)$



so we need to Fk around a bit...

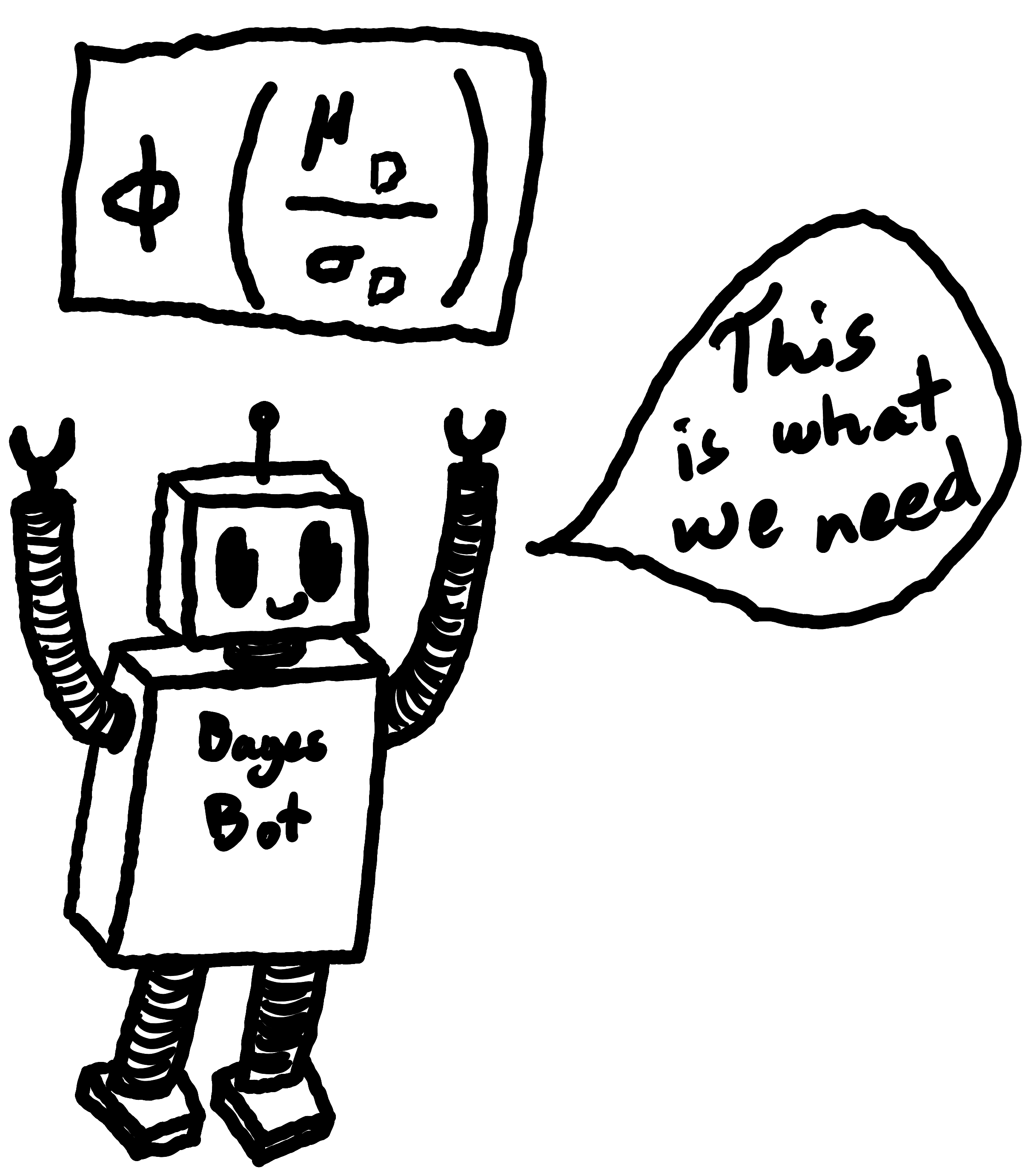
$$p(Z>a) = 1 - p(Z \leq a) = 1 - \phi(a)$$

replacing with the stuff we found on the last page

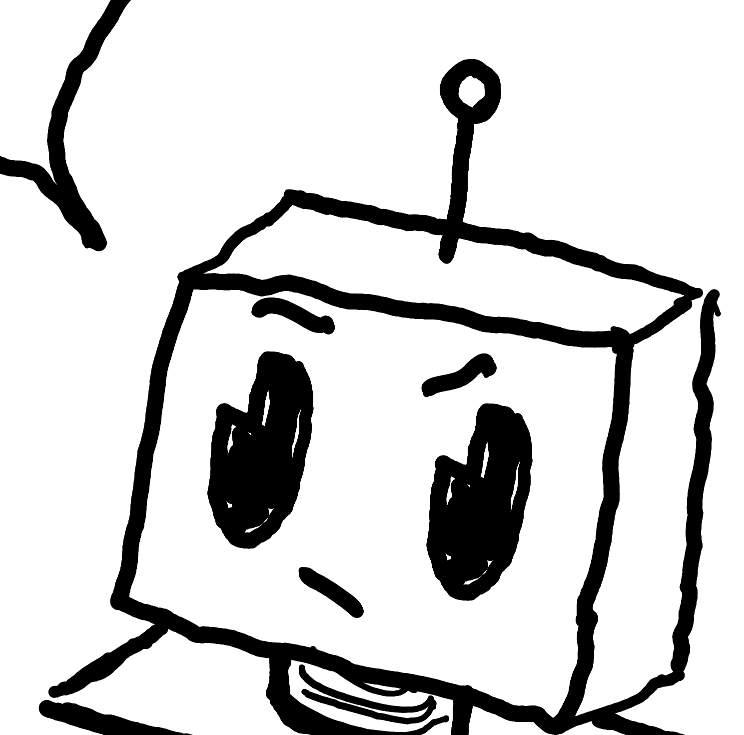
$$p(D>0) = 1 - \phi\left(-\frac{\mu_D}{\sigma_D}\right) = \phi\left(\frac{\mu_D}{\sigma_D}\right)$$

because $\phi(-z) = 1 - \phi(z)$

is the same as $1 - \phi(-z) = \phi(z)$

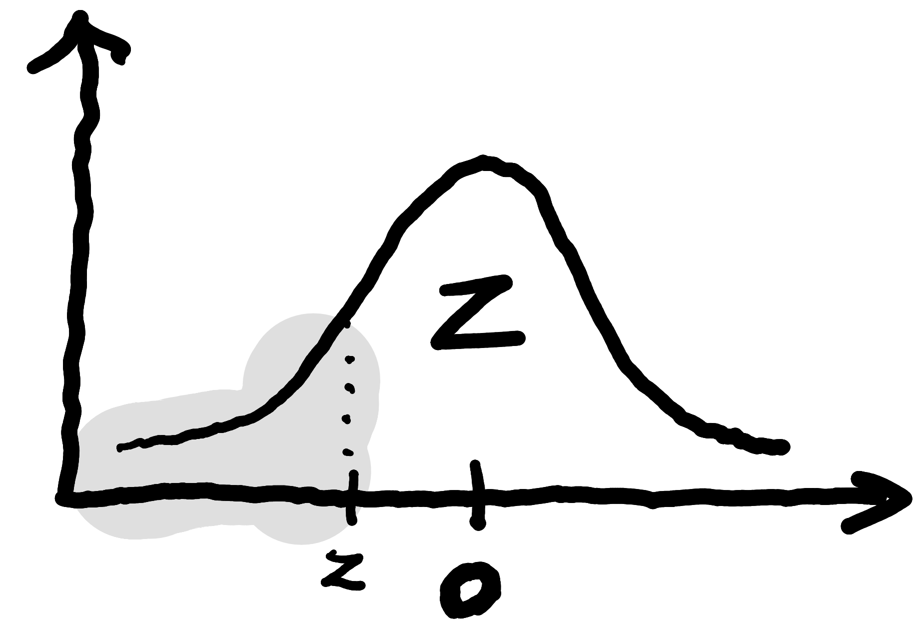


But wtf is ϕ?



well it doesnt rly matter but lets go into that.

What is $\phi(z)$? It's the fn that gives you the area under the curve from $(-\infty, z)$.



explicitly, in our case where we want the upper part:

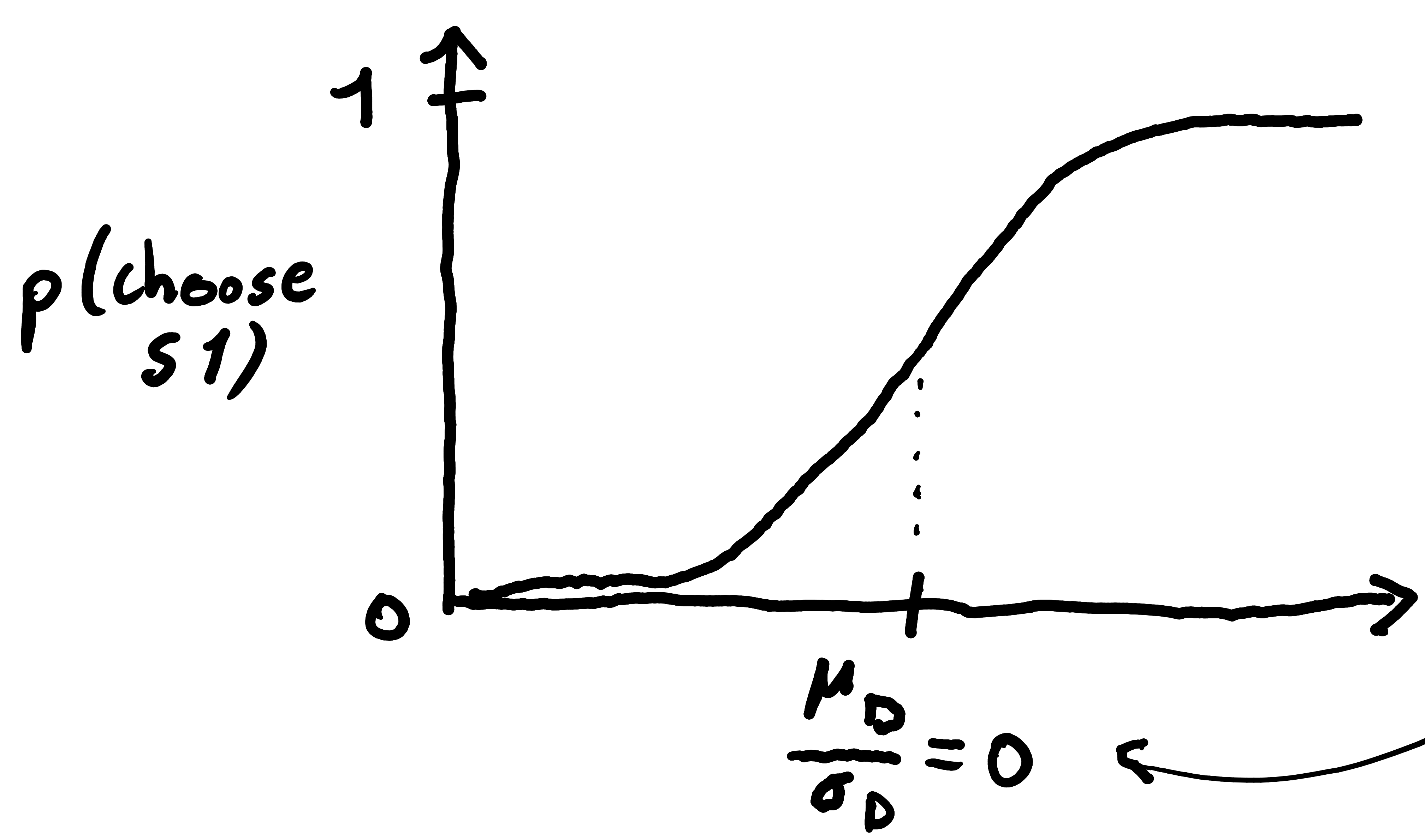
$$p(\text{choose } S1) = \int_{-\frac{\mu_D}{\sigma_D}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

To recap:

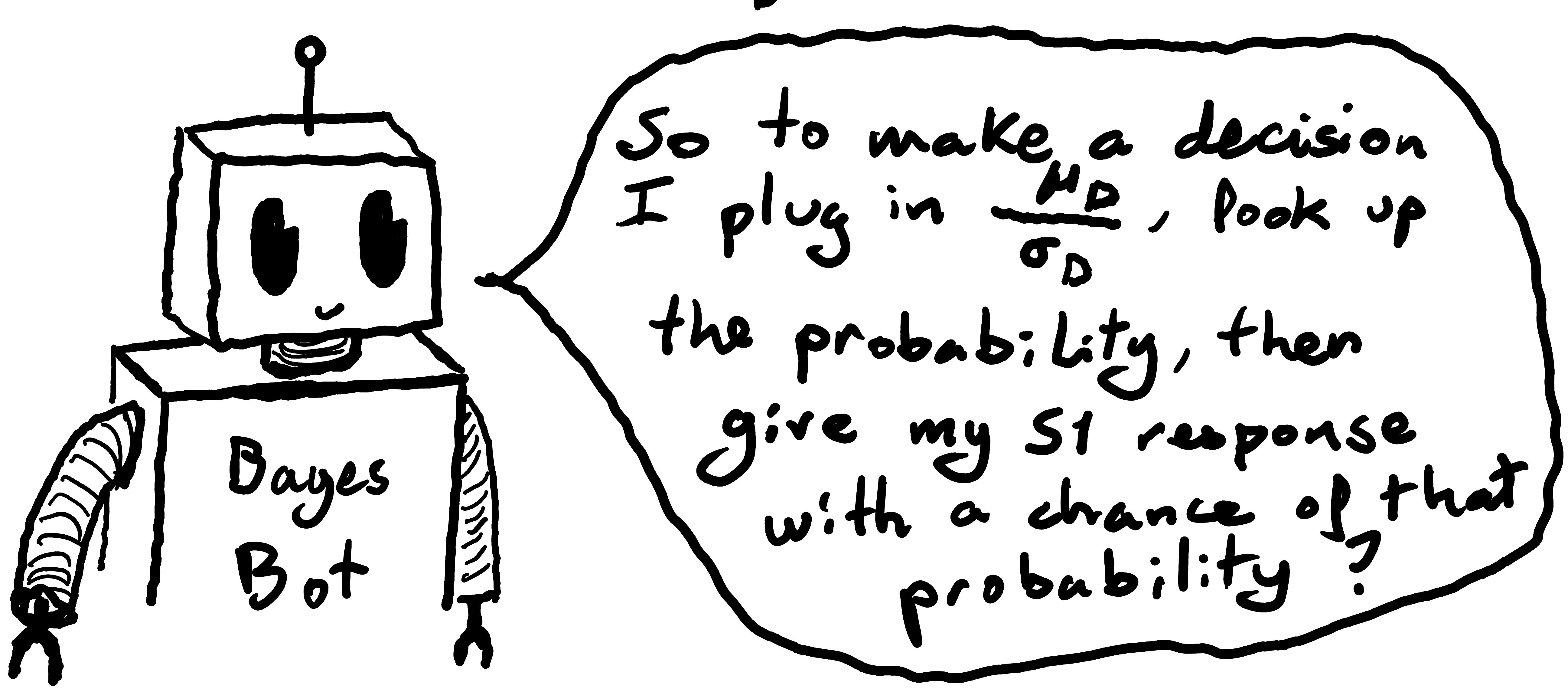
$$\hat{S}_1 - \hat{S}_2 \longrightarrow D \longrightarrow z = \frac{D - \mu_D}{\sigma_D}$$

and $p(D>0) \longrightarrow \phi\left(\frac{\mu_D}{\sigma_D}\right) = p(\text{choose } S1)$

$$p(\text{choose } S1) = \phi\left(\frac{\mu_D}{\sigma_D}\right)$$ plotted is:



centered wherever $\frac{\mu_D}{\sigma} = 0$



Well...
That depends
...

There are actually 2 posibilities:

1. The agent is a "normative Bayesian" observer

   Then the rule is

   if $p(\text{choose } S1) > 0.5 \longrightarrow S1$

   if $p(\text{choose } S1) < 0.5 \longrightarrow S2$

   What if $p = 0.5$? multiple options:

   1. tie breaking rule

      if $p = 0.5 \longrightarrow S1$

      or

      if $p = 0.5 \longrightarrow S2$ ← most common

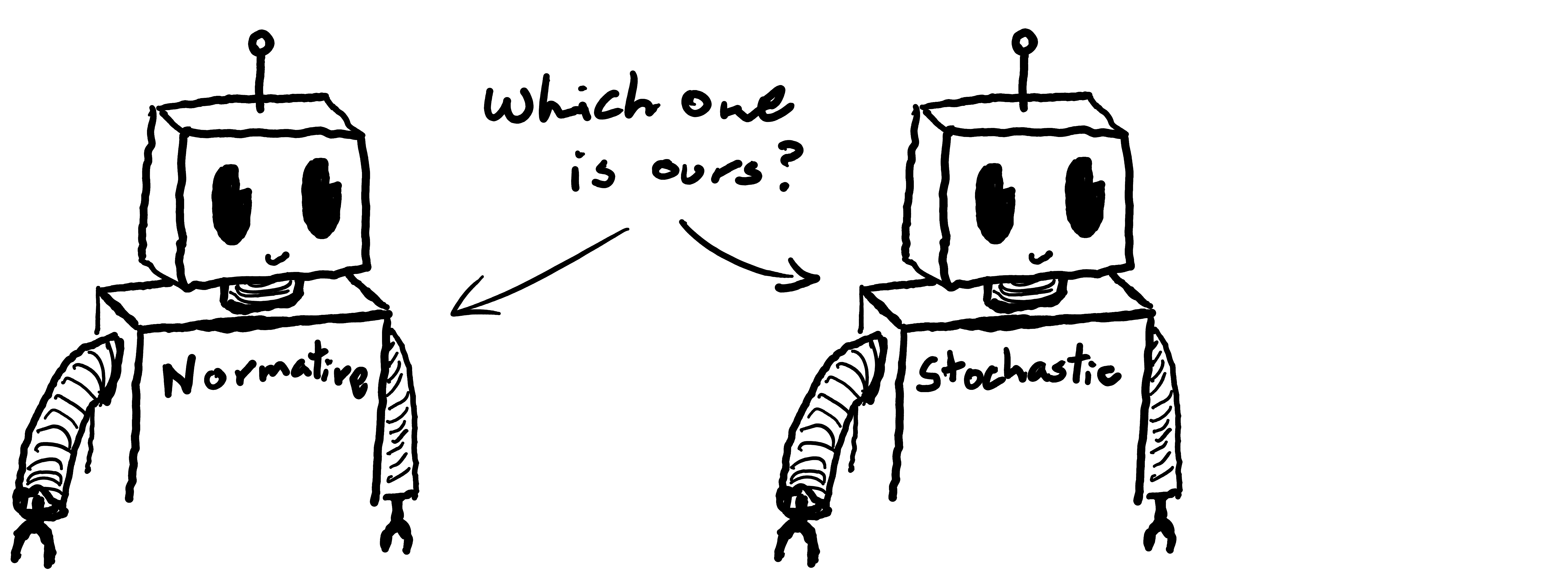
   2. random choice between S1 & S2

2. The agent has a "stochastic response" (used to fit humans)

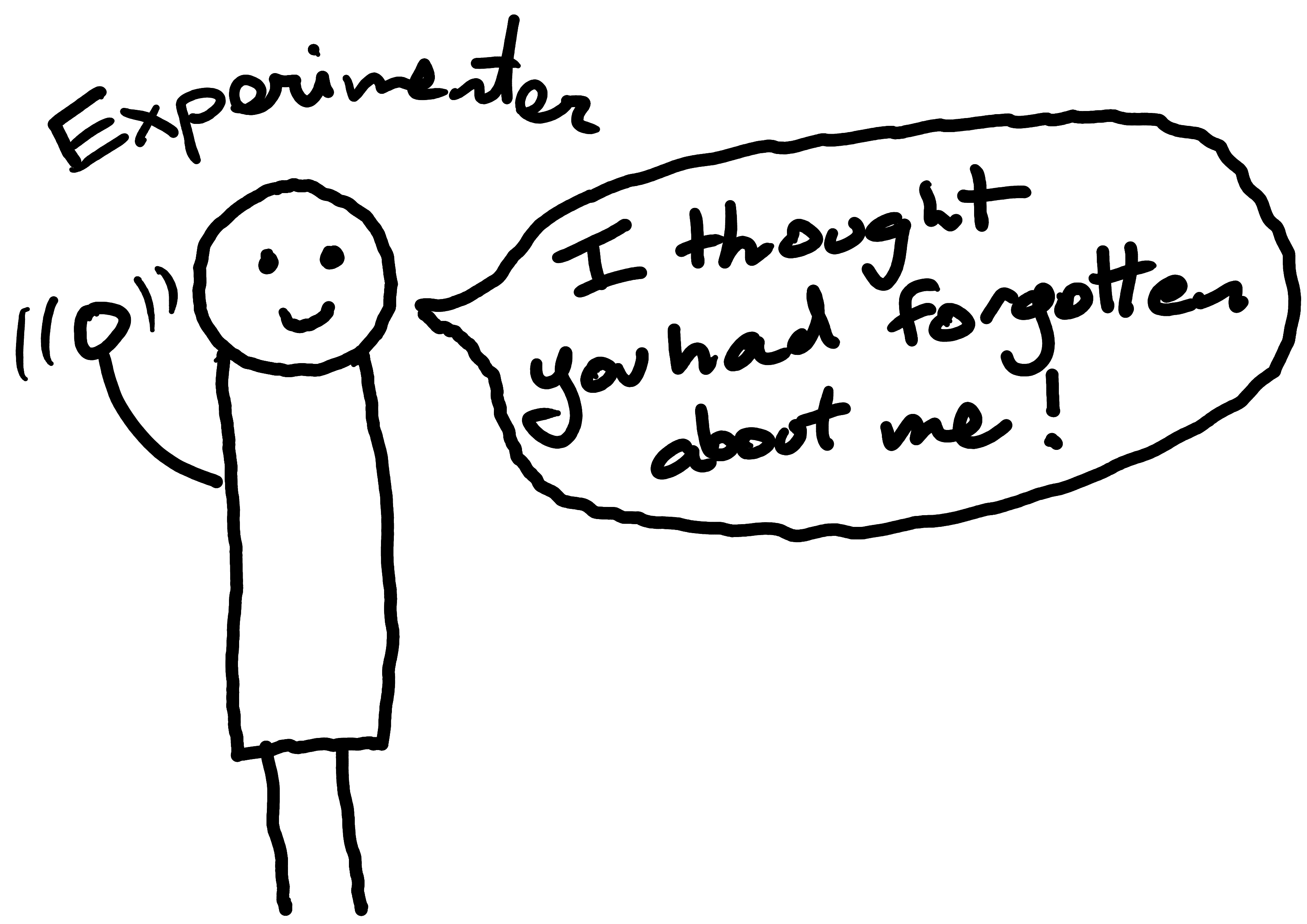
   Then the rule is:

   $$p = \phi\left(\frac{\mu_D}{\sigma_D}\right)$$

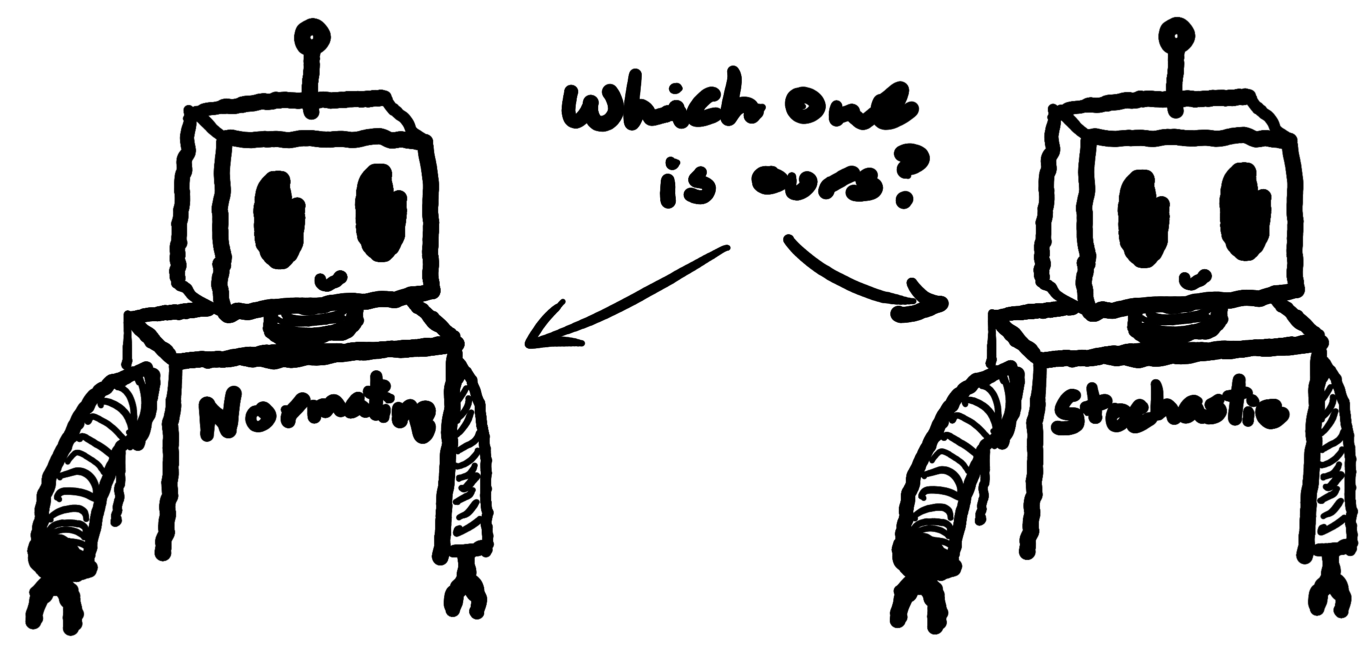
   $$\text{Response} = \begin{cases} S1 & \text{with probability } p \\ S2 & \text{with probability } 1-p \end{cases}$$



Now we can return to the experimenter!



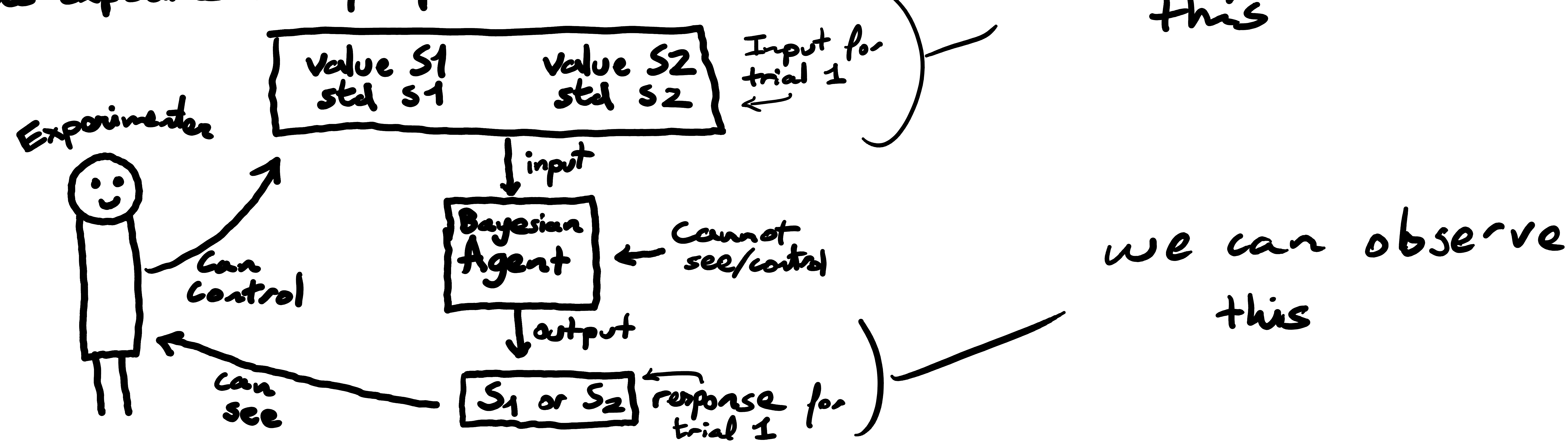
How can the experimenter find out if the agent is normative or stochastic?



we need to explore the behaviour around the decision boundary.

Recal:

The experimenters perspective



We can vary the difference between S1 and S2 in increments around

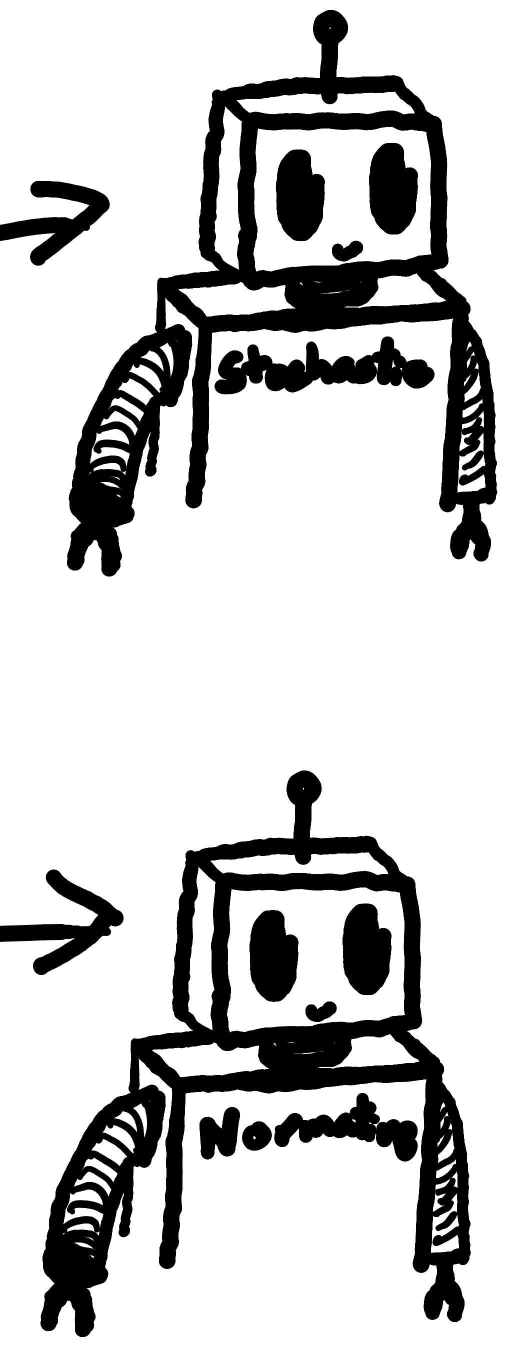
diff $\rightarrow \Delta = 0$

And observe the pattern of the responses.

Test input:

Trial	S1	std_S1	S2	std_S2
1	4.96	0	5	0
2	4.97	0	5	0
3	4.98	0	5	0
4	4.99	0	5	0
5	5	0	5	0
6	5.01	0	5	0
7	5.02	0	5	0
8	5.03	0	5	0
9	5.04	0	5	0

Mutiple times



Stochastic output:

Trial nº → / Iteratº ↓	1	2	3	4	5	6	7	8	9
1	S2	S1	S2	S2	S1	S1	S2	S1	S1
2	...	...	...	...	...	...	...	...	...
3	...	...	...	...	...	...	...	...	...
4	...	...	...	...	...	...	...	...	...
Total per trial %S1	0	0.2	8	24	50	76	92	99.8	100
%S2	100	99.8	92	76	50	24	8	0.2	0

Normative output:

Trial nº → / Iteratº ↓	1	2	3	4	5	6	7	8	9
1	S2	S2	S2	S2	S2	S1	S1	S1	S1
2	...	...	...	...	...	...	...	...	...
3	...	...	...	...	...	...	...	...	...
4	...	...	...	...	...	...	...	...	...
Total per trial %S1	0	0	0	0	0	100	100	100	100
%S2	100	100	100	100	100	0	0	0	0

$$\frac{1}{\text{slope}} = \sigma_{S1}^2 + \sigma_{S2}^2$$

$$\sigma_{S1}^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_{S1}^2}} \qquad \sigma_{S2}^2 = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_{S2}^2}}$$

prior var

input std for S1

$$\frac{1}{\text{slope}} = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_{S1}^2}} + \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_{S2}^2}}$$

known

known

Let $s = \frac{1}{\text{slope}}$ $\quad S1 = \frac{1}{\sigma_{S1}^2}$ $\quad S2 = \frac{1}{\sigma_{S2}^2}$

$x = \frac{1}{\sigma_0^2}$

$$S = \frac{1}{x+S1} + \frac{1}{x+S2} = \frac{(x+S2)+(x+S1)}{(x+S1)(x+S2)}$$

$$= \frac{2x+S1+S2}{(x+S1)(x+S2)}$$

so $S(x+S1)(x+S2) = 2x+S1+S2$

$S(x^2 + x(S1+S2) + S1 \cdot S2 = 2x+S1+S2$

$Sx^2 + S(S1+S2)x + S \cdot S1 \cdot S2 = 2x+S1+S2$

$Sx^2 + (S(S1+S2)-2)x + (S \cdot S1 \cdot S2 - (S1+S2) = 0$

solve for $x = \frac{1}{\sigma_0^2} = \frac{2-S(S1+S2)+\sqrt{S^2(S1-S2)^2+4}}{2S}$

invert for $\sigma_0^2 = \frac{1}{x} = \frac{2S}{2-S(S1+S2)+\sqrt{S^2(S1-S2)^2+4}}$

We have : $\sigma_0^2 = \frac{1}{x} = \frac{2s}{2 - s(s1 + s2) + \sqrt{s^2(s1 - s2)^2 + 4}}$

$$\text{so } \sigma_0^2 = \frac{2 \cdot \frac{1}{slope}}{2 - \frac{1}{slope}\left(\frac{1}{\sigma_{s1}^2} + \frac{1}{\sigma_{s2}^2}\right) + \sqrt{\left(\frac{1}{slope}\right)^2 \cdot \left(\frac{1}{\sigma_{s1}^2} - \frac{1}{\sigma_{s2}^2}\right)^2 + 4}}$$