On the Parameterization and Design of an Extended Kalman Filter Frequency Tracker

Sergio Bittanti and Sergio M. Savaresi

***Abstract*—In this paper, the problem of estimating the frequency of a harmonic signal embedded in broad-band noise is considered. The paper focuses on the extended Kalman filter frequency tracker, which is the application of the extended Kalman filter (EKF) framework to the frequency estimation problem. The EKF frequency tracker recently proposed in the literature is characterized by a vector of three design parameters $\{q,\ r,\ \varepsilon\}$, whose role and tuning is still a controversial and unclear issue. In this paper it is shown that a wise parameterization of the extended Kalman frequency tracker is characterized by just one parameter: as a matter of fact ε must be set to zero to achieve the basic property of unbiasedness in a noise-free setting; moreover, the performances of the tracker are not influenced independently by q and r; what really matters is the ratio $\lambda = r/q$ only. The proposed simplification of the extended Kalman filter frequency tracker allows an easier and more transparent tuning of its tracking behavior.**

***Index Terms*—Extended Kalman filter, frequency tracking, harmonic analysis, parameter estimation.**

I. Introduction

This paper deals with the problem of estimating the frequency ω of an harmonic signal $s(t) = A\cos(\omega t + \varphi)$, given its noisy measurement $y(t) = s(t) + n(t)$, where $n(t)$ is a broad-band stationary signal (e.g., a white noise). This problem is frequently encountered in real-world applications, especially in the fields of adaptive control and signal processing, and numerous techniques have been developed for its treatment (see, e.g., [4], [7]–[8]). This paper focuses on the use of the extended Kalman filter (EKF), an approach which has been recently proposed by Bitmead and co-authors in [9]–[11] for the design of a frequency tracker. In particular, in [9] the EKF frequency tracker is proposed, and some heuristic guidelines for the tuning of its parameters are proposed. In [11] its stability analysis is developed, whereas the case of high-noise environments is considered in [10].

The basic idea around which papers [9]–[11] evolve is to model the measured signal $y(t) = s(t) + n(t)$ by means of a third-order state space model, as follows:

$$\begin{cases} x_1(t+1) = \cos(x_3(t))x_1(t) - \sin(x_3(t))x_2(t) \\ x_2(t+1) = \sin(x_3(t))x_1(t) + \cos(x_3(t))x_2(t) \\ x_3(t+1) = (1-\varepsilon)x_3(t) + w(t) \\ \quad y(t) = x_1(t) + v(t). \end{cases} \tag{1}$$

Here $w(t)$ and $v(t)$ are zero-mean uncorrected white noises, having variances q and r, respectively, $w(t) \approx WN(0, q)$ and $v(t) \approx WN(0, r)$. The parameter ε (where $\varepsilon \geq 0$ and, typically, $\varepsilon \ll 1$) has the goal of enforcing stability into the third equation of (1). The state variable $x_3(t)$ represents the unknown frequency ω.

Given the state-space model (1), the EKF technique can be applied straightforwardly (see, e.g., [1] and [15] for a general description of EKF). The complete set of equations of the EKF frequency tracker derived from (1) is here recalled (see [9]–[11] for more details)

$$\begin{cases} \hat{x}(t/t) = f(\hat{x}(t-1/t-1)) \\ \qquad\qquad + K(t)(y(t) - Hf(\hat{x}(t-1/t-1))) \\ \hat{\omega}(t) = C\hat{x}(t/t) \\ K(t) = P(t)H^T\left(HP(t)H^T + r\right)^{-1} \\ P(t+1) = F(t)\left[P(t) - P(t)H^T\left(HP(t)H^T + r\right)^{-1}HP(t)\right] \\ \qquad\qquad \cdot F^T(t) + qI_3 \end{cases} \tag{2a}$$

where

$$f(\hat{x}(t/t)) = \begin{bmatrix} \cos(\hat{x}_3(t/t))\hat{x}_1(t/t) - \sin(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ \sin(\hat{x}_3(t/t))\hat{x}_1(t/t) + \cos(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ (1-\varepsilon)\hat{x}_3(t/t) \end{bmatrix}$$
$$H = [1 \quad 0 \quad 0], \quad C = [0 \quad 0 \quad 1] \tag{2b}$$

and

$$\hat{x}(t/t) = \begin{bmatrix} \hat{x}_1(t/t) \\ \hat{x}_2(t/t) \\ \hat{x}_3(t/t) \end{bmatrix}, \quad F(t) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\hat{x}(t-1/t-1)}$$
$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{12}(t) & p_{22}(t) & p_{23}(t) \\ p_{13}(t) & p_{23}(t) & p_{33}(t) \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recursive equations in (2) are suitable for on-line frequency tracking. In particular, the fourth equation in (2a), giving a recursion for the auxiliary matrix $P(t)$ of dimension 3×3, is the celebrated *Riccati equation* [3].

The structure of the EKF frequency tracker is diagrammatically depicted in Fig. 1. As with any frequency tracker, from an external point of view it can be seen as a device receiving the measured signal $y(t)$ as input, and delivering the estimated frequency $\hat{\omega}(t) = \hat{x}_3(t/t)$ as output. The internal structure is constituted by two main sub-blocks: the first performs the updating of the state equations, according to the first expression in (2a); the second has the task of providing the gain vector at each time step and is based on the third and fourth equations of (2a).

Notice that the EKF is parameterized in $\{q,\ r,\ \varepsilon\}$. These three parameters are the "knobs," by means of which the designer can obtain suitable tracking performance.

In [9] and [11] a preliminary analysis of the roles played by the parameters $\{q,\ r,\ \varepsilon\}$ is given, and some guidelines for their tuning are proposed. Such analysis, having the objective of explaining the effects of each parameter on the EKF performance, is mainly based upon Monte Carlo simulation trials. Even though some useful and interesting indications have been obtained in this way, the tuning of $\{q,\ r,\ \varepsilon\}$ still remains a difficult task due to the unclear cross-relationships between such parameters.

Moving from the seminal works [9]–[11], the goal of this paper is to go a step further in the analysis of the EKF frequency tracker, in order to gain more insight in the relationships between its behavior and its tuneable parameters $\{q,\ r,\ \varepsilon\}$. Precisely, two simple but fundamental results will be established.

1) Even in a noise-free setting $(n(t) = 0\ \forall t)$, if $\varepsilon > 0$ the EKF frequency tracker (2) provides an asymptotically biased estimate of the unknown frequency ω. The bias is zero only for $\varepsilon = 0$.

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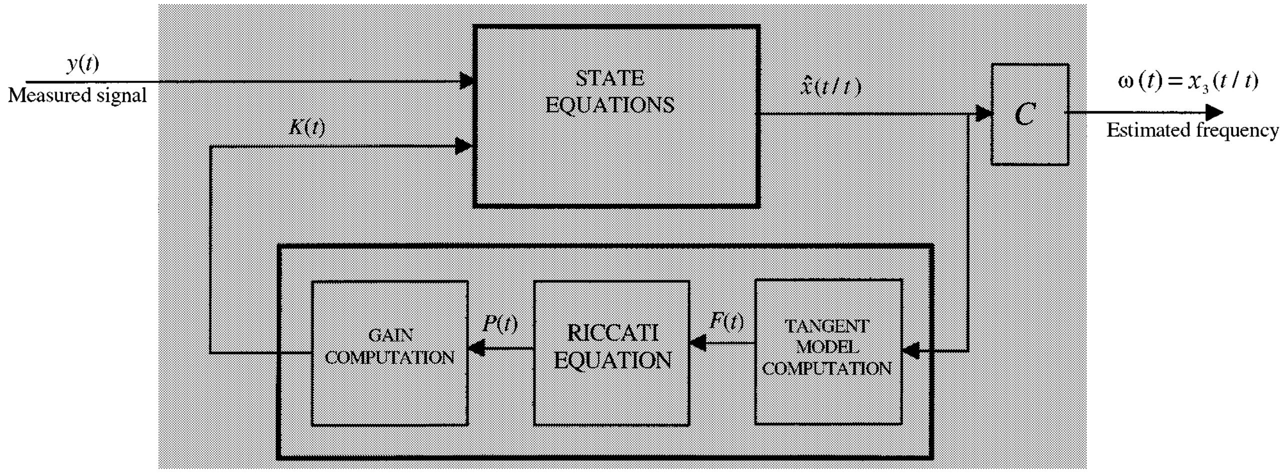


Fig. 1. A representation of the EKF frequency tracker.

2) The performance of the EKF frequency tracker depends on the ratio $\lambda = r/q$ only.

The main outcome of these statements is that a wise parameterization of (2) corresponds to set $\varepsilon = 0$ from the very beginning and then using $\lambda = r/q$ as the only tuning knob. This greatly simplifies the designer's task.

The outline of the paper is as follows: in Section II a bias-analysis of (2) is given (result 1)), whereas Section III is devoted to the analysis of the dependence of (2) upon r and q (result 2)). The results 1) and 2) are both illustrated with a numerical example at the end of Sections II and III, respectively.

II. Bias Analysis of the EKF Frequency Tracker

A basic feature (and minimal requirement) for a good frequency tracker is the so-called **unbiasedness in a noise-free setting**. This means that if the measure $y(t)$ of the harmonic signal $s(t) = A\cos(\omega t)$ is not corrupted by noise ($n(t) = 0\ \forall t$), and the frequency ω is constant, the frequency tracker is expected to provide an estimate $\hat{\omega}(t)$ which, at steady state, exactly coincides with ω. Most of the frequency trackers typically used in real-world applications (see, e.g., [2], [5]–[8], and [12]–[14]) obviously have the above simple property. In the subsequent result we will show that the EKF frequency tracker (2) does not have this property if $\varepsilon > 0$.

Proposition 1: Consider the EKF frequency tracker (2). Assume that $n(t) = 0,\ \forall t$ and that the frequency ω of the harmonic signal $s(t) = A\cos(\omega t)$ is constant. At steady state, the frequency estimate $\hat{\omega}(t) = \hat{x}_3(t/t)$ provided by (2) is unbiased if and only if $\varepsilon\omega = 0$.

Proof: Consider the state equations of the EKF (2a), under the assumption that the frequency estimate has already reached its steady-state value, namely that $\hat{x}_3(t/t) = \hat{x}_3(t-1/t-1) = \overline{x}_3$. Then, denoting by $\{K_i(t)\}_{i=1,2,3}$ the three components of $K(t)$, the original equations of (2) can be written as

$$\begin{cases} \hat{x}_1(t/t) = \cos(\overline{x}_3)\hat{x}_1(t-1/t-1) \\ \qquad - \sin(\overline{x}_3)\hat{x}_2(t-1/t-1) + K_1(t) \\ \qquad \cdot [y(t) - \cos(\overline{x}_3)\hat{x}_1(t-1/t-1) \\ \qquad + \sin(\overline{x}_3)\hat{x}_2(t-1/t-1)] \\ \hat{x}_2(t/t) = \sin(\overline{x}_3)\hat{x}_1(t-1/t-1) \\ \qquad + \cos(\overline{x}_3)\hat{x}_2(t-1/t-1) + K_2(t) \\ \qquad \cdot [y(t) - \cos(\overline{x}_3)\hat{x}_1(t-1/t-1) \\ \qquad + \sin(\overline{x}_3)\hat{x}_2(t-1/t-1)] \\ \overline{x}_3 = (1-\varepsilon)\overline{x}_3 + K_3(t) \\ \qquad \cdot [y(t) - \cos(\overline{x}_3)\hat{x}_1(t-1/t-1) \\ \qquad + \sin(\overline{x}_3)\hat{x}_2(t-1/t-1)]. \end{cases} \tag{3}$$

Note that the dynamics of (3) is confined into the first two equations. Thus, $\hat{x}_1(t/t)$ and $\hat{x}_2(t/t)$ are the only state variables of (3). $\overline{x}_3$ can be seen as its output, whereas $y(t)$ plays the role of input signal. Interestingly enough, while (3) is nonlinear, the system obtained by setting $\hat{x}_3(t/t) =$ const. is linear. So, one can compute the transfer function from $y(t)$ to $\overline{x}_3$. To this purpose, using the first two equations of (3), by cumbersome but simple computations, $\hat{x}_1(t/t)$ and $\hat{x}_2(t/t)$ can be given an input–output representation as a function of $y(t)$ (z is the one-step-ahead shift operator), as shown in (4) at the bottom of the page.

Plugging (4) in the third equation of (3), one obtains

$$\varepsilon\overline{x}_3 = K_3(t)G(z;\ K_1(t),\ K_2(t),\ \overline{x}_3)y(t) \tag{5}$$

as shown in (6) at the bottom of the next page. Equation (5) expresses the relationship between the input $y(t)$ and the output $\varepsilon\overline{x}_3$. Under the assumption of noise-free setting, the input signal of (5) is a pure sinusoid: $y(t) = s(t) = A\cos(\omega t)$. Therefore we can resort to the

$$\begin{cases} \hat{x}_1(t/t) = \dfrac{z(zK_1(t) - K_1(t)\cos(\overline{x}_3) - K_2(t)\sin(\overline{x}_3))}{z^2 - z(K_2(t)\sin(\overline{x}_3) + 2\cos(\overline{x}_3)) + (K_1(t)\cos(\overline{x}_3) - K_1(t) + 1)}\, y(t) \\ \hat{x}_2(t/t) = \dfrac{z(zK_2(t) + K_1(t)\sin(\overline{x}_3) - K_2(t)\cos(\overline{x}_3))}{z^2 - z(K_2(t)\sin(\overline{x}_3) + 2\cos(\overline{x}_3)) + (K_1(t)\cos(\overline{x}_3) - K_1(t) + 1)}\, y(t) \end{cases} \tag{4}$$

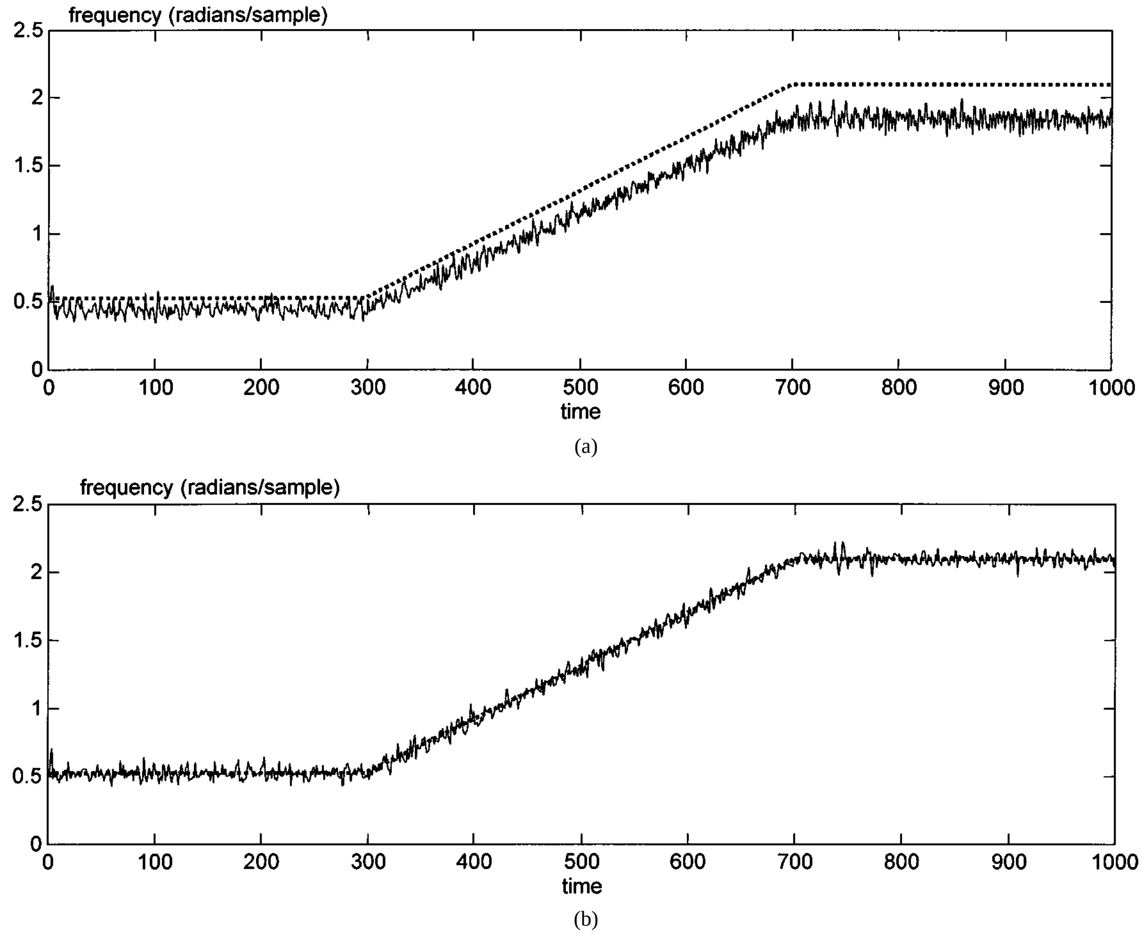


Fig. 2. (a) Frequency tracking of the EKF frequency tracker using $\varepsilon = 0.05$ (the dashed line indicates the true frequency). (b) Frequency tracking of the EKF frequency tracker using $\varepsilon = 0$ (the dashed line indicates the true frequency).

frequency-response theorem to determine the steady-state value of the output. Equation (5) hence can be rewritten as

$$\varepsilon\overline{x}_3 = K_3(t)\Big|G(e^{j\omega};\ K_1(t),\ K_2(t),\ \overline{x}_3)\Big| \cdot \cos\Big(\omega t + \angle G(e^{j\omega};\ K_1(t),\ K_2(t),\ \overline{x}_3)\Big). \tag{7}$$

Equation (7) is peculiar in that it expresses how a *sinusoidal* signal determines a *constant* signal. As a matter of fact, what really matters is when the bias $\overline{x}_3 - \omega$ is zero. Now from (6) it is apparent that the numerator evaluated in $e^{j\omega}$ is zero if and only if $\overline{x}_3 = \omega$. Thus, from (7) it is possible to conclude that the frequency estimation bias is zero if and only if $\varepsilon\omega = 0$. ■

The above Proposition 1 states a very simple but interesting twofold result.

- At steady state, the EKF frequency tracker (2) is unbiased (for every value of ω) in a noise-free setting only if the parameter ε is set to zero.
- Since $|G(e^{j\omega};\ K_1(t),\ K_2(t),\ \overline{x}_3)| = 0\ \forall t$ if and only if $\overline{x}_3 = \omega$, (5) points out that the bias must depend on the value $\overline{x}_3$ of the sinusoidal signal frequency. Specifically, since the first member of (5) grows as $\overline{x}_3$ increases, the bias is expected to grow with $\overline{x}_3$. Such a dependence of the bias upon the estimated frequency was already empirically pointed out in [9] (see the simulation results depicted in [9, Figs. 1–4]). Proposition 1 now provides a simple explanation for this fact.

A straightforward consequence of the above Proposition 1 is that the best choice for ε is to set $\varepsilon = 0$; as a matter of fact $\varepsilon > 0$ may lead to a strong estimation bias which (unless ω is very close to zero) dominates the mean square error of the frequency estimate (this is confirmed by the heuristic analysis proposed in [9]—see also Example 1 below), without providing any clear benefit. In this regard, it is interesting to notice that the condition $\varepsilon > 0$ was introduced in [9]–[11] with the goal of having the third equation of model (1) asymptotically stable. As is well known, however, the stability of the model is not required in order to come out with a stable Kalman Filter: specifically, it is easy to check by simulation that the stability properties of the EKF frequency tracker using $\varepsilon = 0$ or $\varepsilon > 0$ (with $\varepsilon \ll 1$) are indistinguishable (see Example 1 below). This is confirmed by the fact that, in the stability analysis of the EKF proposed in [11], the assumption $\varepsilon > 0$ does not play a crucial role.

$$G(z;\ K_1(t),\ K_2(t),\ \overline{x}_3) = \frac{z(z^2 - 2z\ \cos(\overline{x}_3) + 1)}{z^2 - z(K_2(t)\sin(\overline{x}_3) + 2\ \cos(\overline{x}_3)) + (K_1(t)\cos(\overline{x}_3) - K_1(t) + 1)} \tag{6}$$

On the basis of the above result, in the rest of the paper the EKF frequency tracker (2) will be replaced by its simplified version obtained by substituting $\varepsilon = 0$ in (2), namely,

$$\begin{cases} \hat{x}(t/t) = f(\hat{x}(t-1/t-1)) \\ \qquad + K(t)(y(t) - Hf(\hat{x}(t-1/t-1))) \\ \hat{\omega}(t) = C\hat{x}(t/t) \\ K(t) = P(t)H^T\left(HP(t)H^T + r\right)^{-1} \\ P(t+1) = F(t)\left[P(t) - P(t)H^T\left(HP(t)H^T + r\right)^{-1} HP(t)\right] \\ \qquad \cdot F^T(t) + qI_3 \end{cases} \tag{8a}$$

where

$$f(\hat{x}(t/t)) = \begin{bmatrix} \cos(\hat{x}_3(t/t))\hat{x}_1(t/t) - \sin(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ \sin(\hat{x}_3(t/t))\hat{x}_1(t/t) + \cos(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ \hat{x}_3(t/t) \end{bmatrix}$$
$$H = [1 \quad 0 \quad 0], \quad C = [0 \quad 0 \quad 1]. \tag{8b}$$

In this way, the vector of "tuneable" parameters reduces to two parameters only, q and r. We conclude this section with a simulation example, having the goal of showing the effect of setting $\varepsilon = 0$ versus $\varepsilon \neq 0$ in (2).

Example 1: The signal considered in this numerical example is an harmonic signal embedded in white noise, characterized by a time-varying frequency. Specifically, $y(t) = s(t) + n(t)$, $t = 1, 2, \cdots, 1000$, where $n(t) \approx WN(0, 0.01)$, $s(t) = \cos((\pi/6)t)$ for $t = 1, 2 \cdots 300$, and $s(t) = \cos((2\pi/3)t)$ for $t = 701, 702 \cdots 1000$; for $t = 301, 302 \cdots 700$ $s(t)$ is characterized by a time-varying frequency which increases linearly from $\pi/6$ to $2\pi/3$.

In order to estimate the frequency of the harmonic signal embedded in $y(t)$, the EKF frequency tracker (2), with $r = 0.01$ and $q = 0.005$, has been used. First the parameter ε has been set to $\varepsilon = 0.05$; the results of the so-obtained frequency estimation are displayed in Fig. 2(a). As a second trial, ε is set to zero; the results obtained in this case are in Fig. 2(b). These diagrams clearly show the main conclusions which can be drawn from Proposition 1.

- If $\varepsilon > 0$, the estimation is strongly biased even for small values of ε.
- The bias increases as the value of the unknown frequency increases.
- The dynamic properties (stability and noise rejection) of the EKF frequency tracker in the case $\varepsilon = 0.05$ and $\varepsilon = 0$ are similar (the only clearly visible difference is just the presence of bias if $\varepsilon > 0$).

III. Parameterization Analysis

In the previous section it has been shown that, in order for the EKF frequency tracker to be asymptotically unbiased in a noise-free setting, the parameter ε must be set to zero. Another issue—even more important and interesting—is the role played by q and r in determining the tracker performance. Some indications and guidelines are given in [9] and [11]. The issue, however, remains somewhat controversial and unclear.

In this section it will be shown that the performances of the EKF frequency tracker are not influenced independently by q and r, but they only depend on the ratio $\lambda = r/q$. This result is particularly interesting since it allows a further simplification of the tracker (8).

The result is formally stated and proven in the following Proposition 2.

Proposition 2: Given a measured signal $y(t) = A\cos(\omega t) + n(t)$, the estimated frequency $\hat{\omega}(t) = \hat{x}_3(t/t)$ obtained using the EKF frequency tracker (8) depends upon the ratio $\lambda = r/q$ only.

Proof: Consider the time series $\hat{x}(t/t)$, $F(t)$, $P(t)$, and $K(t)$ obtained by means of the EKF frequency tracker (8) when using the parameters q and r (and fed with the measured signal $y(t)$). We are now interested in understanding the relationships between $\hat{x}(t/t)$, $F(t)$ $P(t)$, and $K(t)$ and the corresponding time series, say $\tilde{\hat{x}}(t/t)$, $\tilde{F}(t)$, $\tilde{P}(t)$, and $\tilde{K}(t)$, obtained when using $\tilde{q} = \alpha q$ and $\tilde{r} = \alpha r$ $(\alpha > 0)$ instead of q and r. In other words, we want to check the effect of scaling the two design parameters q and r by the same multiplicative factor α.

To this end, consider the Riccati Equation used in (8), given by:

$$P(t+1) = F(t)\left[P(t) - P(t)H^T\left(HP(t)H^T + r\right)^{-1} HP(t)\right] \cdot F^T(t) + qI_3 \tag{9}$$

and plug in (9) $\alpha P(t+1)$ in place of $P(t+1)$, $\alpha P(t)$ in place of $P(t)$, αq in place of q, and αr in place of r

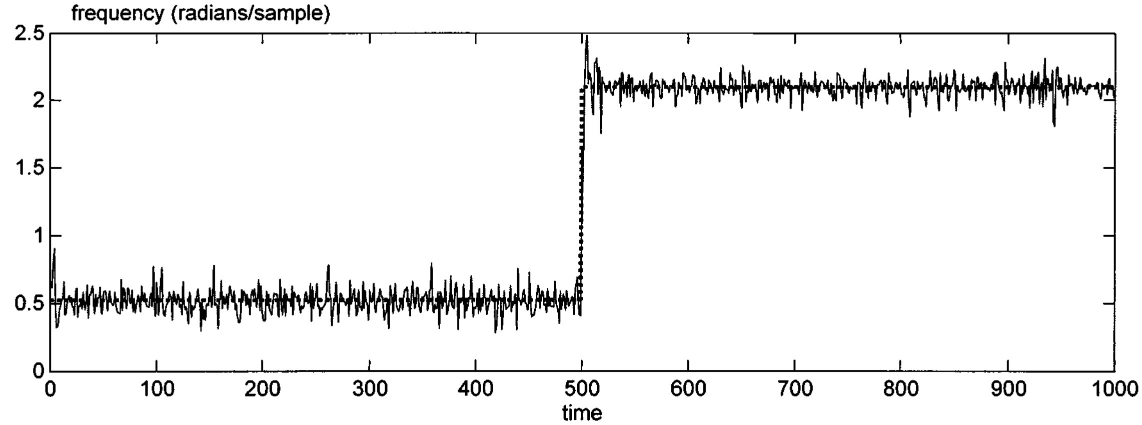
$$\begin{aligned} &\alpha P(t+1) \\ &\quad = F(t)\left[\alpha P(t) - \alpha P(t)H^T\left(H\alpha P(t)H^T + \alpha r\right)^{-1} H\alpha P(t)\right] \\ &\quad\ \cdot F^T(t) + \alpha q I_3. \end{aligned} \tag{10}$$

After some simple manipulations of (10), we come to the Riccati equation

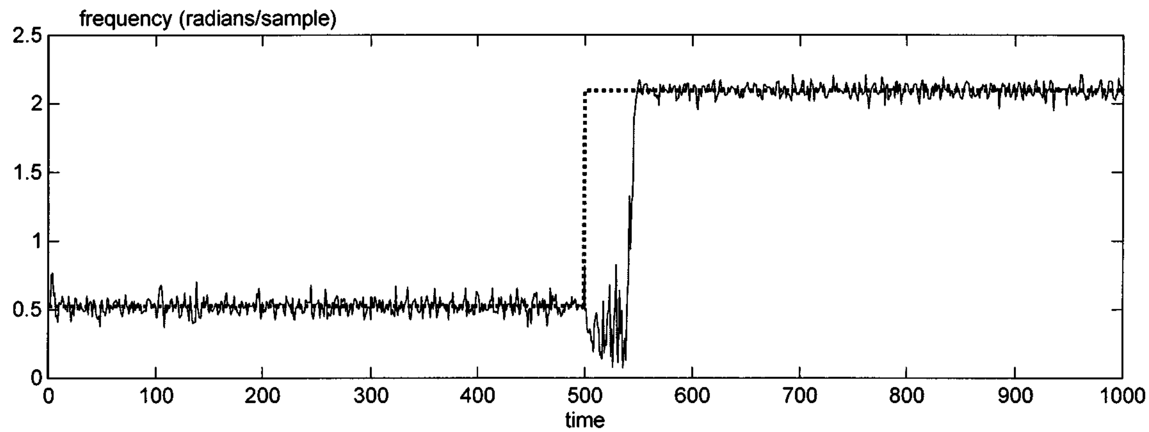
$$P(t+1) = F(t)\left[P(t) - P(t)H^T\left(HP(t)H^T + r\right)^{-1} HP(t)\right] \cdot F^T(t) + qI_3 \tag{11}$$

which actually coincides with (9). This means that if $\tilde{q} = \alpha q$ and $\tilde{r} = \alpha r$ $(\alpha > 0)$ are used instead of q and r, the resulting solution $\tilde{P}(t)$ is equal to $\alpha P(t)$. It is important to stress, however, that this is true under the assumption that $F(t)$ does not change if q and r are both scaled by the same factor α, namely that $\tilde{F}(t) = F(t)$. By inspecting the structure of the EKF frequency tracker (8), as shown at the bottom of the next page, it is easy to see that $\tilde{F}(t) = F(t)$ if $\tilde{\hat{x}}(t/t) = \hat{x}(t/t)$ $\forall t$. Moreover, $\tilde{K}(t) = K(t)$ implies that $\tilde{\hat{x}}(t/t) = \hat{x}(t/t)$. This can be easily proven since $\hat{x}(t/t) = f(\hat{x}(t-1/t-1)) + K(t)(y(t) - Hf(\hat{x}(t-$

$$\begin{aligned} F(t) &= \left.\frac{\partial f(x)}{\partial x}\right|_{x=\hat{x}(t-1/t-1)} \\ &= \begin{bmatrix} \cos(\hat{x}_3(t-1/t-1)) & -\sin(\hat{x}_3(t-1/t-1)) & -\sin(\hat{x}_3(t-1/t-1))\hat{x}_1(t-1/t-1) - \cos(\hat{x}_3(t-1/t-1))\hat{x}_2(t-1/t-1) \\ \sin(\hat{x}_3(t-1/t-1)) & \cos(\hat{x}_3(t-1/t-1)) & \cos(\hat{x}_3(t-1/t-1))\hat{x}_1(t-1/t-1) - \sin(\hat{x}_3(t-1/t-1))\hat{x}_2(t-1/t-1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$



(a)



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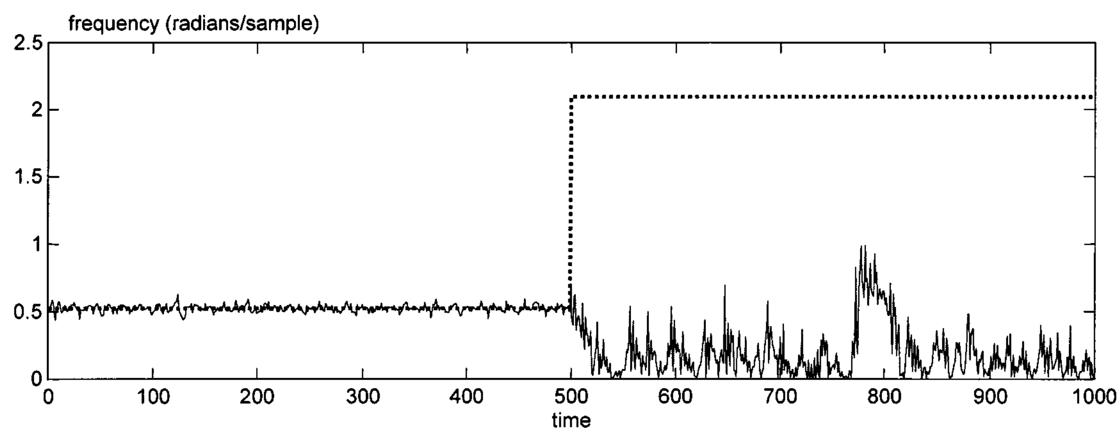

(c)

Fig. 3. (a) Frequency tracking of the EKF frequency tracker using $\lambda = 0$ (the dashed line indicates the true frequency). (b) Frequency tracking of the EKF frequency tracker using $\lambda = 1$ (the dashed line indicates the true frequency). (c) Frequency tracking of the EKF frequency tracker using $\lambda = 10$ (the dashed line indicates the true frequency).

$1/t-1)))$. From this expression, it is apparent that, if $\tilde{K}(t) = K(t)$ and $\tilde{\hat{x}}(t-1/t-1) = \hat{x}(t-1/t-1)$, then $\tilde{\hat{x}}(t/t) = \hat{x}(t/t)$. Since the initial condition of the filter are chosen by the designer at a value say $\overline{x}(0)$, we have to take $\tilde{\hat{x}}(0) = \hat{x}(0) = \overline{x}(0)$, and then, by induction, we can conclude that $\tilde{K}(t) = K(t)$ implies $\tilde{\hat{x}}(t/t) = \hat{x}(t/t)\ \forall t$. Thus, the following chain of implications holds true:

$$\tilde{K}(t) = K(t) \Rightarrow \tilde{\hat{x}}(t/t) = \hat{x}(t/t) \Rightarrow \tilde{F}(t)$$
$$= F(t) \Rightarrow \tilde{P}(t) = \alpha P(t)\ \forall t.$$

So far, we have to analyze the gain variation against the scaling effect. To this end, recall that

$$K(t) = P(t)H^T\left(HP(t)H^T + r\right)^{-1} \tag{12}$$

By plugging in (12) $\alpha P(t)$ in place of $P(t)$, and αr in place of r, we obtain

$$K(t) = \alpha P(t) H^T \left(H \alpha P(t) H^T + \alpha r \right)^{-1} \tag{13}$$

which, after some manipulations becomes

$$K(t) = P(t) H^T \left(H P(t) H^T + r \right)^{-1}. \tag{14}$$

The fact that (14) coincides with (12) means that $\tilde{P}(t) = \alpha P(t) \Rightarrow \tilde{K}(t) = K(t)\ \forall t$. Therefore, the following set of circular implications holds:

$$\begin{aligned}\tilde{K}(t) = K(t) &\Rightarrow \tilde{\hat{x}}(t/t) = \hat{x}(t/t) \Rightarrow \tilde{F}(t)\\ &= F(t) \Rightarrow \tilde{P}(t) = \alpha P(t) \Rightarrow \tilde{K}(t) = K(t)\ \forall t.\end{aligned}$$

By inspecting the structure of the EKF (see (8) and Fig. 1) we can conclude that, if the initialization of the filter is the same, scaling the parameters q and r by the same positive real number α has no effects on the state estimation (in particular, on the frequency estimate $\hat{\omega}(t) = C\hat{x}(t/t) = \hat{x}_3(t/t)$). This means that the estimated frequency $\hat{\omega}(t)$ obtained using the EKF frequency tracker (8) depends on the ratio $\lambda = r/q$ only. ■

The above Proposition 2 is interesting since it rigorously allows a reparameterization of the EKF frequency tracker (8). As a matter of fact, by conventionally setting $q = 1$ (hence $\lambda = r$), we obtain

$$\begin{cases} \hat{x}(t/t) = f(\hat{x}(t-1/t-1)) \\ \qquad\qquad + K(t)(y(t) - Hf(\hat{x}(t-1/t-1))) \\ \hat{\omega}(t) = C\hat{x}(t/t) \\ K(t) = P(t)H^T\left(HP(t)H^T + \lambda\right)^{-1} \\ P(t+1) = F(t)\left[P(t) - P(t)H^T\left(HP(t)H^T + \lambda\right)^{-1} HP(t)\right] \\ \qquad\qquad \cdot F^T(t) + I_3 \end{cases} \tag{15a}$$

where

$$f(\hat{x}(t/t)) = \begin{bmatrix} \cos(\hat{x}_3(t/t))\hat{x}_1(t/t) - \sin(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ \sin(\hat{x}_3(t/t))\hat{x}_1(t/t) + \cos(\hat{x}_3(t/t))\hat{x}_2(t/t) \\ \hat{x}_3(t/t) \end{bmatrix}$$
$$H = [1 \quad 0 \quad 0], \quad C = [0 \quad 0 \quad 1]. \tag{15b}$$

Tracker (15) is a significant simplification of (8); the vector of the "tuneable" parameters of (15) is now constituted by a single parameter λ. In conclusion, the task of tuning the design parameters of the EKF frequency tracker (2) (parameterized with $\{q, r, \varepsilon\}$) boils down to that of tuning the tracker (15), where a single parameter $\{\lambda\}$ has to be chosen.

We conclude this section by presenting a numerical example, which qualitatively shows the influence of the parameter λ on the tracker behavior.

Example 2: The signal considered in this example is an harmonic signal embedded in white noise, characterized by a time-varying frequency. Specifically, $y(t) = s(t) + n(t)$, $t = 1, 2, \cdots, 1000$, where $n(t) \approx WN(0, 0.01)$ and $s(t) = \cos((\pi/6)t)$ for $t = 1, 2 \cdots 500$, $s(t) = \cos((2\pi/3)t)$ for $t = 501, 502 \cdots 1000$.

In order to estimate the frequency of the harmonic signal embedded in $y(t)$, the EKF frequency tracker (15), is used. To assess the effect of λ on the tracker behavior, three different values of λ have been used: $\lambda = 0$, $\lambda = 1$, and $\lambda = 10$; the corresponding frequency estimation results are displayed in Fig. 3(a)–(c), respectively. From Fig. 3 some useful indications on the tracker behavior can be drawn. In particular, as for every on-line frequency tracker (see, e.g., [5], [8], and [14]), there is a tradeoff between the variance error of the estimated frequency and the ability of the tracker to follow frequency variations (in the case considered herein we have considered the most demanding situation, given by a large and sudden jump of the frequency to be tracked). Specifically:

- when λ increases the variance error diminishes, as can be seen by observing the decrement in the fluctuations of the estimated frequency up to time 500; however, the ability of the tracker to follow rapidly varying frequency changes deteriorates;
- when λ decreases the variance error grows, but the tracker is more robust toward frequency variations.

In particular notice from Fig. 3(c) that if $\lambda = 10$ the frequency is accurately estimated at steady-state, but the EKF gets completely lost after the frequency jump. This problem can be overcome using a lower value of λ, at the price of a worse noise rejection.

The correct choice of λ clearly depends on the features of the measured signal: if the frequency of the harmonic signal is constant or slowly varying a large value of λ should be used; on the other hand, if the frequency of the signal is subject to large and sudden variations, a small value of λ should be used, in order to guarantee robustness of the tracker.

IV. Conclusions

In this work the problem of estimating the frequency of an harmonic signal in noise, by means of an EKF frequency tracker, has been considered. The EKF frequency tracker recently proposed in literature is characterized by a vector of three design parameters $\{q, r, \varepsilon\}$. In this paper it is shown that:

- even in a noise-free setting, if $\varepsilon > 0$ the EKF frequency tracker (2) provides a biased estimate of ω;
- the performance of the EKF frequency tracker only depend on the ratio $\lambda = r/q$.

These results allow a simplification of the parameterization of the EKF frequency tracker: with no loss of generality the tracker can be parameterized with just one parameter. This makes the problem of parameters tuning easier and transparent.

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On Impulsive Control of a Periodically Forced Chaotic Pendulum System

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***Abstract*—In this paper, we consider impulsive control of a periodically forced pendulum system which has rich chaos and bifurcation phenomena. A new impulsive control method for chaos suppression of this pendulum system is developed. Some simple sufficient conditions for driving the chaotic state to its zero equilibrium are presented, and some criteria for eventually exponentially asymptotical stability are established. This work provides a rigorous theoretical analysis to support some early experimental observations on controlling chaos in the periodically forced pendulum system.**

***Index Terms*—Chaos, impulsive control, stabilization.**

I. Introduction

Chaos in control systems and controlling chaos in dynamical systems have both attracted increasing attention in recent years [1]. It has been observed that a dynamical system driven by impulses can display rich complex dynamics such as chaos [2]. On the other hand, it has also been demonstrated that chaos in many complex dynamical systems can be suppressed by impulsive control inputs (see [3] and the references therein). This paper addresses an interesting issue of a combination of these two aspects: using impulsive control to suppress chaos in a periodically forced pendulum system driven by impulses. Some simple and easily verified sufficient conditions are derived for such stabilization control applications. Finally, an illustrative example, along with computer simulation results, is included to visualize the satisfactory control performance.

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II. Problem Formulation

Consider the following periodically forced pendulum:

$$\begin{aligned} \dot{x} &= f(x, y) + \frac{\pi}{2} h \sum_{k=1}^{\infty} \delta(t - k\tau) \triangleq y + \frac{\pi}{2} h \sum_{k=1}^{\infty} \delta(t - k\tau) \\ \dot{y} &= g(x, y) \triangleq -by - \sin x \end{aligned} \tag{1}$$

where $(x, y)^\top \triangleq X \in R^2$ is the system state, with $x \in S^1 = \{x \in R \bmod 2\pi\}$, and $b > 0$ is a damping coefficient, h a constant, $\tau > 0$ the impulse interval, and $\delta(t)$ the Dirac delta function.

It has been shown that this pendulum system is chaotic with rich bifurcation phenomena [2]. Fig. 1 shows its chaotic behavior, with $b = 0.2$, $h = 2.0$, and $\tau = 4.32$. In this paper, we develop an impulsive control method for chaos suppression for this pendulum system. The main idea follows from some fundamental theories and techniques of impulsive differential equations and their control [4]–[7].

More precisely, in this study, an impulsive control of the form

$$u(t) = -\frac{\pi}{2} \sum_{k=1}^{\infty} g_k \delta(t - k\tau) \tag{2}$$

is added to the right-hand side of the pendulum (1), which simply means that a pulse, g_k, is input to the system by the controller at $t = k\tau$, $k = 1, 2, \cdots$, where many of them may actually be very small or even zero and the controller itself can be a very simple pulse generator. The controlled system is

$$\begin{aligned} \dot{x} &= f(x, y) + \frac{\pi}{2} h \sum_{k=1}^{\infty} \delta(t - k\tau) + u(t) \\ &= y + \frac{\pi}{2} \sum_{k=1}^{\infty} (h - g_k) \delta(t - k\tau) \\ \dot{y} &= g(x, y) = -by - \sin x. \end{aligned} \tag{3}$$

The objective is to find some (sufficient) conditions on the constant control gains, g_k, such that the chaotic state of the pendulum is driven to its zero equilibrium. Moreover, the eventually exponentially asymptotical stability of the controlled system is investigated.

Note that it has been experimentally verified that a suitable choice of $(h - g_k)$ can indeed suppress chaos [2]. This study is to provide a rigorous theoretical analysis to support that observation, and to provide some guidelines for the impulsive controller design.

III. Preliminaries

Let $R^+ = [0, +\infty)$ and R^n denote the n-dimensional Euclidean space with the norm $\|X\| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$, where $X = (x_1, \cdots, x_n)^\top \in R^n$.

System (3) can be rewritten as

$$\begin{cases} Dx = y + Dw \\ Dy = -by - \sin x \end{cases} \tag{4}$$

with

$$w(t) = \sum_{k=1}^{\infty} a_k H_k(t), \qquad a_k = \frac{\pi}{2}(h - g_k)$$

and