The Time Value of Money

McGraw-Hill/Irwin

Chapter Outline

- Time value associated with money
- Determining future value at given interest rate
- Present value based on current value of funds to be received
- Determining Yield on an Investment.
- Compounding or discounting occurring on a less than annual basis

Relationship to The Capital Outlay Decision

- The time value of money is used to determine whether future benefits are sufficiently large to justify current outlays
- Mathematical tools of the time value of money are used in making capital allocation decisions

Future Value – Single Amount

- Measuring value of an amount that is allowed to grow at a given interest over a period of time
  - Assuming that the worth of $1,000 needs to be calculated after 4 years at a 10% interest per year, we have:

1st year……$1,000 X 1.10 = $1,100

2nd year……$1,100 X 1.10 = $1,210

3rd year……$1,210 X 1.10 = $1,331

4th year……$1,331 X 1.10 = $1,464

Future Value – Single Amount (Cont'd)

A generalized formula for Future Value:

$$FV = PV(1 + i)^n$$

Where

FV = Future value

PV = Present value

i = Interest rate

n = Number of periods;

In the previous case, PV = $1,000, i = 10%, n = 4, hence;

$$FV = \$1,000(1.10)^4, \text{ or } \$1,000 \times 1.464 = \$1,464$$

Future Value of $1($FV_{IF}$)

Table 9–1

Periods	1%	2%	3%	4%	6%	8%	10%
1	1.010	1.020	1.030	1.040	1.060	1.080	1.100
2	1.020	1.040	1.061	1.082	1.124	1.166	1.210
3	1.030	1.061	1.093	1.125	1.191	1.260	1.331
4	1.041	1.082	1.126	1.170	1.262	1.360	1.464
5	1.051	1.104	1.159	1.217	1.338	1.469	1.611
10	1.105	1.219	1.344	1.480	1.791	2.159	2.594
20	1.220	1.486	1.806	2.191	3.207	4.661	6.727

Future Value – Single Amount (Cont'd)

- In determining future value, the following can be used:

$$FV = PV \times FV_{IF}$$

Where FV_{IF} = the interest factor

- If $10,000 were invested for 10 years at 8%, the future value would be:

$$FV = PV \times FV_{IF}\,(n = 10, i = 8\%)$$
$$FV = \$10{,}000 \times 2.159 = \$21{,}590$$

Present Value – Single Amount

- A sum payable in the future is worth less today than the stated amount
  - The formula for the present value is derived from the original formula for future value:

$$FV = PV(1 + i)^n \quad \text{Future value}$$

$$PV = FV\left[\frac{1}{(1 + i)^n}\right] \quad \text{Present value}$$

  - The present value can be determined by solving for a mathematical solution to the formula above, thus restating the formula as:
  - Assuming

$$PV = FV \times PV_{IF}$$

$$PV = FV \times PV_{IF} \quad (n = 4, i = 10\%) \text{ [Table 9–2]}$$

$$PV = \$1{,}464 \times 0.683 = \$1{,}000$$

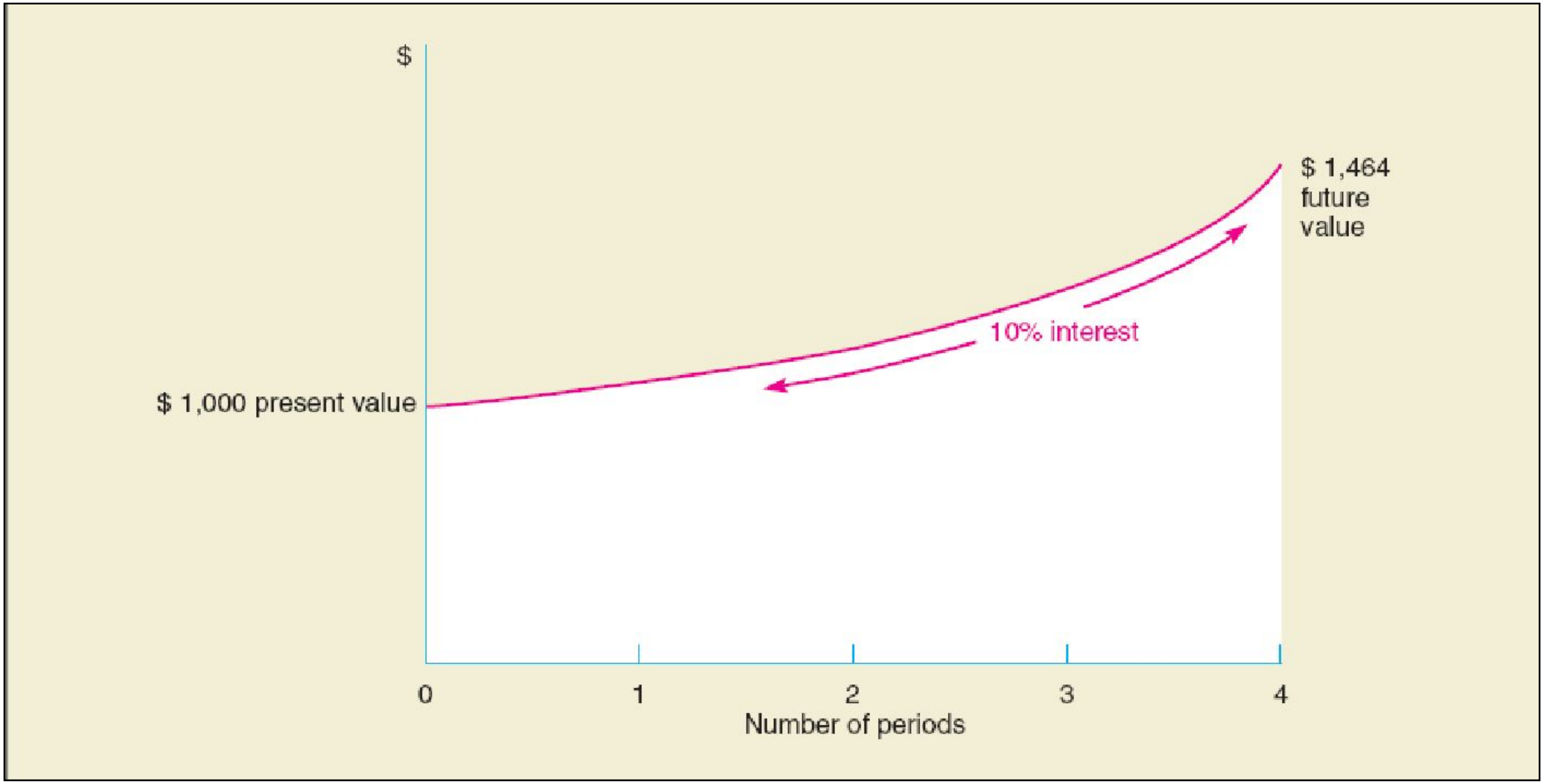
Present Value of $1 ($PV_{IF}$)

Table 9–2

Periods	1%	2%	3%	4%	6%	8%	10%
1	0.990	0.980	0.971	0.962	0.943	0.926	0.909
2	0.980	0.961	0.943	0.925	0.890	0.857	0.826
3	0.971	0.942	0.915	0.889	0.840	0.794	0.751
4	0.961	0.924	0.888	0.855	0.792	0.735	0.683
5	0.951	0.906	0.863	0.822	0.747	0.681	0.621
10	0.905	0.820	0.744	0.676	0.558	0.463	0.386
20	0.820	0.673	0.554	0.456	0.312	0.215	0.149

An expanded table is presented in Appendix B.

Relationship of Present and Future Value



Future Value – Annuity

- Annuity:
  - A series of consecutive payments or receipts of equal amount
- Future Value of an Annuity:
  - Calculated by compounding each individual payment into the future and then adding up all of these payments

Future Value – Annuity (cont'd)

- A generalized formula for Future Value of Annuity:

  $FV_A = A \times FV_{IFA}$

  Where:

  FV_A = Future value of the Annuity

  FV_{IFA} = Annuity Factor = $\{[(1+i)^n - 1] \div i\}$

  A = Annuity value

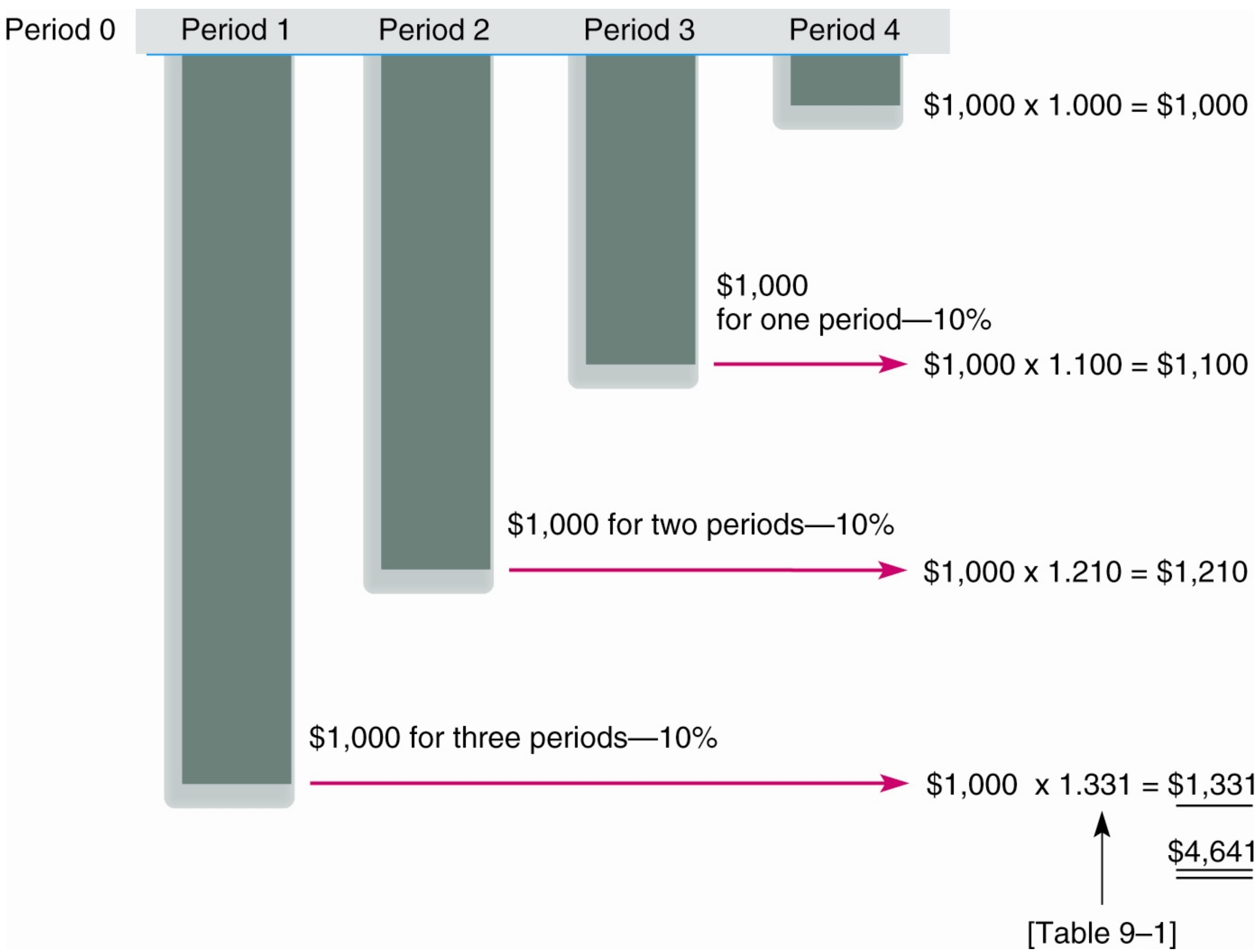
  i = Interest rate

  n = Number of periods;
- Assuming, A = \$1,000, n = 4, and i = 10%

$$FV_A = A \times FV_{IFA} \quad (n = 4, i = 10\%)$$

$$FV_A = \$1{,}000 \times 4.641 = \$4{,}641$$

Compounding Process for Annuity



Future Value of an Annuity of \$1 (FV_{IFA})

Table 9–3

Periods	1%	2%	3%	4%	6%	8%	10%
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	2.010	2.020	2.030	2.040	2.060	2.080	2.100
3	3.030	3.060	3.091	3.122	3.184	3.246	3.310
4	4.060	4.122	4.184	4.246	4.375	4.506	4.641
5	5.101	5.204	5.309	5.416	5.637	5.867	6.105
10	10.462	10.950	11.464	12.006	13.181	14.487	15.937
20	22.019	24.297	26.870	29.778	36.786	45.762	57.275
30	34.785	40.588	47.575	56.085	79.058	113.280	164.490

An expanded table is presented in Appendix C.

Present Value – Annuity

- Calculated by discounting each individual payment back to the present and then adding up all of these payments
- A generalized formula for Present Value of Annuity:

$$PV_A = A \times PV_{IFA}$$

Where:

PV_A = Present value of the Annuity

PV_{IFA} = Annuity Factor = $\{1 - [1 \div (1+i)^n] \div i\}$

A = Annuity value

i = Interest rate

n = Number of periods

Present Value of an Annuity of $1 ($PV_{IFA}$)

Assuming that A = $1,000, n = 4, i = 10%, we have:

$$PV_A = A \times PV_{IFA} \quad (n = 4, i = 10\%)$$

$$PV_A = \$1{,}000 \times 3.170 = \$3{,}170$$

Table 9–4

Periods	1%	2%	3%	4%	6%	8%	10%
1	0.990	0.980	0.971	0.962	0.943	0.926	0.909
2	1.970	1.942	1.913	1.886	1.833	1.783	1.736
3	2.941	2.884	2.829	2.775	2.673	2.577	2.487
4	3.902	3.808	3.717	3.630	3.465	3.312	3.170
5	4.853	4.713	4.580	4.452	4.212	3.993	3.791
8	7.652	7.325	7.020	6.773	6.210	5.747	5.335
10	9.471	8.983	8.530	8.111	7.360	6.710	6.145
20	18.046	16.351	14.877	13.590	11.470	9.818	8.514
30	25.808	22.396	19.600	17.292	13.765	11.258	9.427

An expanded table is presented in Appendix D.

Time Value Relationships

- Comparisons include:
  - The relationship between present value and future value
    - Inverse relationship exists between the present value and future value of a single amount
  - The relationship between the Present Value of a single amount and the Present Value of an Annuity
    - The Present Value of an Annuity is the sum of the present values of single amounts payable at the end of each period
  - The relationship between the Future Value and Future Value of Annuity
    - The Future Value of an Annuity is the sum of the future values of single amounts receivable at the end of each period

Determining the Annuity Value

- A re-look at the variables involved in time value of money:
  1. FV/PV : Future/Present value of money
  2. N : no. of years
  3. I : Interest or *YIELD*
  4. A : Annuity Value / payment per period in an annuity

- Given the first three variables, and determining the fourth variable “A” (unknown).

Annuity Equaling a Future Value

– Assuming that at a 10% interest rate, after 4 years, an amount of $4,641 needs to accumulated:

$$FV_A = A \times FV_{IFA}$$

$$A = \frac{FV_A}{FV_{IFA}}$$

– For n = 4, and i = 10%, FV_{IFA} is 4.641. Thus, A equals $1,000 as below :

$$A = \frac{FV_A}{FV_{IFA}} = \frac{\$4{,}641}{4.641} = \$1{,}000$$

Annuity Equaling a Present Value

– Determining what size of an annuity can be equated to a given amount:

$$PV_A = A \times PV_{IFA}$$

$$A = \frac{PV_A}{PV_{IFA}}$$

– Assuming n = 4, i = 6%:

$$A = \frac{PV_A}{PV_{IFA}} \quad (n = 4, i = 6\%)$$

$$A = \frac{\$100{,}000}{3.465} = \$2{,}886$$

Relationship of Present Value to Annuity

Annual interest is based on the beginning balance for each year as shown in the following table that shows flow of funds:

Table 9–5

Year	Beginning Balance	Annual Interest (6%)	Annual Withdrawal	Ending Balance
1	$10,000.00	$600.00	$2,886.00	$7,714.00
2	7,714.00	462.84	2,886.00	5,290.84
3	5,290.84	317.45	2,886.00	2,722.29
4	2,722.29	163.71	2,886.00	0

Loan Amortization

- A mortgage loan to be repaid over 20 years at 8% interest:

$$A = \frac{PV_A}{PV_{IFA}} \quad (n = 20,\ i = 8\%)$$

$$A = \frac{\$80{,}000}{9.818} = \$8{,}148$$

Loan Amortization Table

- In such a case the part of the payments to the mortgage company will go toward the payment of interest, with the remainder applied to debt reduction, as indicated in the following table:

Table 9–6

Period	Beginning Balance	Annual Payment	Annual Interest (8%)	Repayment on Principal	Ending Balance
1	$80,000	$8,148	$6,400	$1,748	$78,252
2	78,252	8,148	6,260	1,888	76,364
3	76,364	8,148	6,109	2,039	74,325

Six Formulas

		Formula	Table	Appendix
Future value—single amount	(9–1)	$FV = PV \times FV_{IF}$	9–1	A
Present value—single amount	(9–2)	$PV = FV \times PV_{IF}$	9–2	B
Future value—annuity	(9–3)	$FV_A = A \times FV_{IFA}$	9–3	C
Present value—annuity	(9–4)	$PV_A = A \times PV_{IFA}$	9–4	D
Annuity equaling a future value	(9–5)	$A = \frac{FV_A}{FV_{IFA}}$	9–3	C
Annuity equaling a present value	(9–6)	$A = \frac{PV_A}{PV_{IFA}}$	9–4	D

Determining the Yield on Investment

- Determining the unknown variable " i ", given the following variables :
  1. FV/PV : Future/Present value of money
  2. N : no. of years
  3. A : Annuity Value / payment per period in an annuity

Yield – Present Value of a Single Amount

- To calculate the yield on an investment producing \$1,464 after 4 years having a present value of \$1,000:

$$PV = FV \times PV_{IF}$$

$$PV_{IF} = \frac{PV}{FV} = \frac{\$1{,}000}{\$1{,}464} = 0.683$$

Periods	1%	2%	3%	4%	5%	6%	8%	10%
2	0.980	0.961	0.943	0.925	0.907	0.890	0.857	0.826
3	0.971	0.942	0.915	0.889	0.864	0.840	0.794	0.751
4	0.961	0.924	0.888	0.855	0.823	0.792	0.735	0.683

- We see that for n = 4 and PV_{IF} = 0.683, the interest rate or yield is 10%

Yield – Present Value of a Single Amount (Cont'd)

- Interpolation may also be used to find a more precise answer

PV_{IF} at 5%	0.864
PV_{IF} at 6%	0.840
	0.024

- Difference between the PV_{IF} value at the lowest interest rate and the designated PV_{IF} value

PV_{IF} at 5%.	0.864
PV_{IF} designated	0.861
	0.003

- The exact value can be determined as:

$$5\% + \frac{0.003}{0.024}(1\%) =$$

$$5\% + 0.125\ (1\%) =$$

$$5\% + 0.125\% \quad = 5.125\%$$

Yield – Present Value of an Annuity

- To calculate the yield on an investment of $10,000, producing $1,490 per annum for 10 years:

$$PV_A = A \times PV_{IFA}$$

$$PV_{IFA} = \frac{PV_A}{A}$$

- Hence:

$$PV_{IFA} = \frac{PV_A}{A} = \frac{\$10,000}{\$1,490} = 6.710$$

Yield – Present Value of an Annuity (Cont'd)

- Flip back to the table containing the Present Value-Annuity factors on Slide 9-16
- Read across the columns for n = 10 periods, one can see that the yield is 8 percent
- Interpolation applied to a single amount can also be applied here for a more precise answer

Special Considerations in Time Value Analysis

- Compounding frequency
  - Certain contractual agreements may require semiannual, quarterly, or monthly compounding periods
  - In such cases,

    N = *No. of years* × *No. of compounding periods during the year*

    I = *Quoted annual interest / No. of compounding periods during the year*

Special Considerations in Time Value Analysis

- Patterns of Payment
  - Problems may evolve around a number of different payment or receipt patterns
  - Not every situation involves a single amount or an annuity
  - A contract may call for the payment of a different amount each year over the stated period or period of annuity

Compounding frequency : Cases

- Case 1: Determine the future value of a $1,000 investment after 5 years at 8% annual interest compounded semiannually
  - Where, n = 5 × 2 = 10; i = 8% / 2 = 4% (using Table 9–1 FV_{IF} = 1.480)

$$FV = PV \times FV_{IF}$$
$$FV = \$1{,}000 \times 1.480 = \$1{,}480$$

- Case 2: Determine the present value of 20 quarterly payments of $2,000 each to be received over the next 5 years, where i = 8% per annum
  - Where, n = 20; i = 2%

$$PV_A = A \times PV_{IFA} \quad (n = 20, i = 2\%) \text{ [Table 9–4]}$$
$$PV_A = A \times \$2{,}000 \times 16.351 = \$32{,}702$$

Patterns of Payment : Cases

- Assume a contract involving payments of different amounts each year for a three-year period
- To determine the present value, each payment is discounted to the present and then totaled

(Assuming 8% discount rate)

1.	$1{,}000 \times 0.926 =$	$	926
2.	$2{,}000 \times 0.857 =$		1,714
3.	$3{,}000 \times 0.794 =$		2,382
			$5,022

Deferred Annuity

- Situations involving a combination of single amounts and an annuity.
- When annuity is paid sometime in the future

Deferred Annuity : Case

- Assuming a contract involving payments of different amounts each year for a three year period :
  - An annuity of $1,000 is paid at the end of each year from the fourth through the eighth year
  - To determine the present value of the cash flows at 8% discount rate

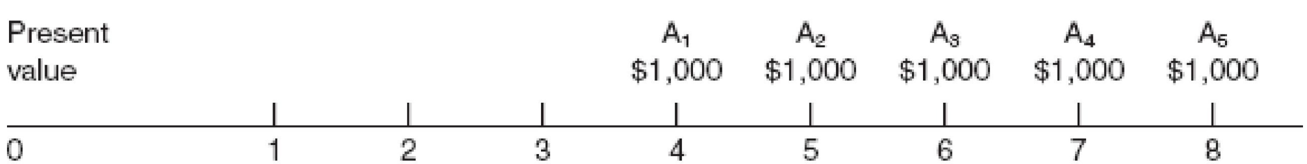
1. $1,000
2. 2,000
3. 3,000

Present value = $5,022

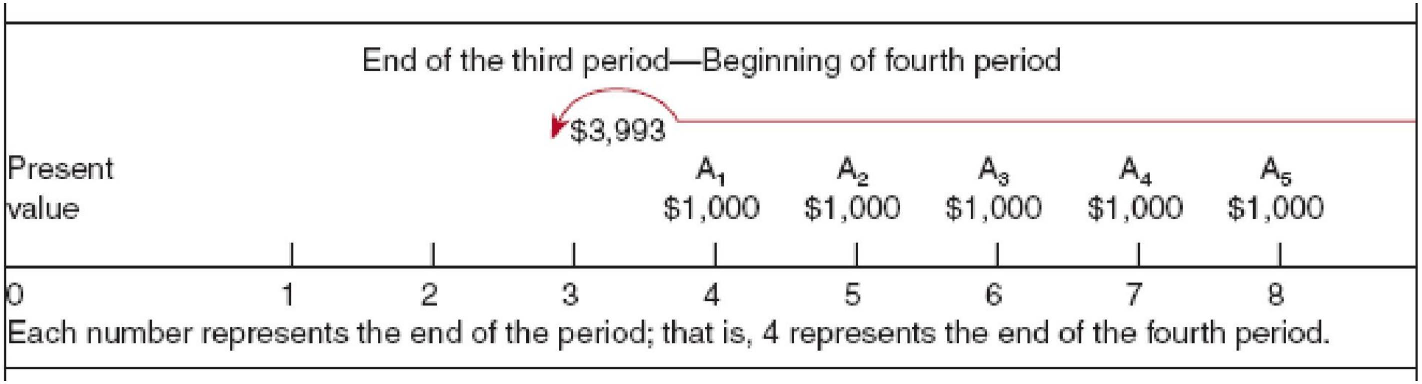
4. 1,000
5. 1,000
6. 1,000
7. 1,000
8. 1,000

Five-year annuity

- To determine the annuity



Deferred Annuity : Case (Cont'd)

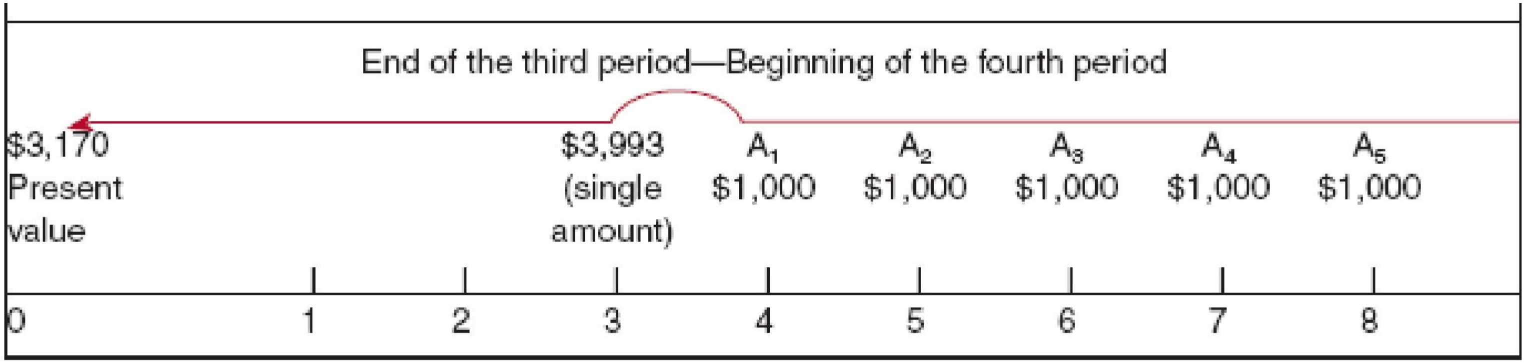


- To discount the $3,993 back to the present, which falls at the beginning of the fourth period, in effect, the equivalent of the end of the third period, it is discounted back three periods, at 8% interest rate

$$PV = FV \times PV_{IF} \qquad (n = 3, i = 8\%)$$

$$PV = \$3{,}993 \times 0.794 = \$3{,}170 \text{ (actual present value)}$$

Deferred Annuity : Case (Cont'd)



Alternate Method to Compute Deferred Annuity

1. Determine the present value factor of an annuity for the total time period, where n = 8, i = 8%, the PV_{IFA} = 5.747

2. Determine the present value factor of an annuity for the total time period (8) minus the deferred annuity period (5). Here, 8 – 5 = 3; n = 3; i = 8%. Thus the value is 2.577

3. Subtracting the value in step 2 from the value of step 1, and multiplying by A;

$$\begin{array}{r} 5.747 \\ -\ 2.577 \\ \hline 3.170 \end{array}$$

$3.170 \times \$1{,}000 = \$3{,}170$ (present value of the annuity)

Alternate Method to Compute Deferred Annuity (Cont'd)

4. $3,170 is the same answer for the present value of the annuity as that reached by the first method

5. The present value of the five-year annuity is added up to the present value of the inflows over the first three years to arrive at:

$5,022	Present value of first three period flows
+3,170	Present value of five-year annuity
$8,192	Total present value