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4 Stocks: 1. COST
2. JNJ
3. JPM
4. AAPL

> The calculation and checking 4 conditions for each case are conducted in the attached Excel file. This document simply illustrates algebra operation and results of variables. The Excel file also shows which case has the lowest variance

Minimize $w'\Sigma w$ st. $w'\vec{1} = 1$; $w_i \geq 5\%$

→ Constraint functions:

$w_1 + w_2 + w_3 + w_4 = 100$ (The numbers are in %)

$\sum_{i=1}^{4} w_i \geq 5$

$\Rightarrow \mathcal{L} = -w'\Sigma w + \lambda(100 - w_1 - w_2 - w_3 - w_4) + \delta_1(w_1 - 5) + \delta_2(w_2 - 5) + \delta_3(w_3 - 5) + \delta_4(w_4 - 5)$

where: $w'\Sigma w = w_1^2\sigma_{11} + w_2^2\sigma_{22} + w_3^2\sigma_{33} + w_4^2\sigma_{44} + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_1w_4\sigma_{14} + 2w_2w_3\sigma_{23} + 2w_2w_4\sigma_{24} + 2w_3w_4$

① $\frac{\partial \mathcal{L}}{\partial w_1} = -2w_1\sigma_{11} - 2w_2\sigma_{12} - 2w_3\sigma_{13} - 2w_4\sigma_{14} - \lambda + \delta_1 = 0$

② $\frac{\partial \mathcal{L}}{\partial w_2} = -2w_2\sigma_{22} - 2w_1\sigma_{21} - 2w_3\sigma_{23} - 2w_4\sigma_{24} - \lambda + \delta_2 = 0$

③ $\frac{\partial \mathcal{L}}{\partial w_3} = -2w_3\sigma_{33} - 2w_1\sigma_{31} - 2w_2\sigma_{32} - 2w_4\sigma_{34} - \lambda + \delta_3 = 0$

④ $\frac{\partial \mathcal{L}}{\partial w_4} = -2w_4\sigma_{44} - 2w_1\sigma_{41} - 2w_2\sigma_{42} - 2w_3\sigma_{43} - \lambda + \delta_4 = 0$

⑤ $\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - w_1 - w_2 - w_3 - w_4 = 0$

⑥ $\frac{\partial \mathcal{L}}{\partial \delta_1} = w_1 - 5 \geq 0$ ⑩ $\delta_1 \cdot \frac{\partial \mathcal{L}}{\partial \delta_1} = 0$

⑦ $\frac{\partial \mathcal{L}}{\partial \delta_2} = w_2 - 5 \geq 0$ ⑪ $\delta_2 \cdot \frac{\partial \mathcal{L}}{\partial \delta_2} = 0$

⑧ $\frac{\partial \mathcal{L}}{\partial \delta_3} = w_3 - 5 \geq 0$ ⑫ $\delta_3 \cdot \frac{\partial \mathcal{L}}{\partial \delta_3} = 0$

⑨ $\frac{\partial \mathcal{L}}{\partial \delta_4} = w_4 - 5 \geq 0$ ⑬ $\delta_4 \cdot \frac{\partial \mathcal{L}}{\partial \delta_4} = 0$

16 Cases:

1. $w_1 = w_2 = w_3 = w_4 = 5$
2. $w_1 > 5$; $w_2 = w_3 = w_4 = 5$
3. $w_2 > 5$; $w_1 = w_3 = w_4 = 5$
4. $w_3 > 5$; $w_1 = w_2 = w_4 = 5$
5. $w_4 > 5$; $w_1 = w_2 = w_3 = 5$
6. $w_1 > 5$, $w_2 > 5$; $w_3 = w_4 = 5$
7. $w_1 > 5$, $w_3 > 5$; $w_2 = w_4 = 5$
8. $w_1 > 5$, $w_4 > 5$; $w_2 = w_3 = 5$
9. $w_2 > 5$, $w_3 > 5$; $w_1 = w_4 = 5$
10. $w_2 > 5$, $w_4 > 5$; $w_1 = w_3 = 5$
11. $w_3 > 5$; $w_4 > 5$; $w_1 = w_2 = 5$
12. $w_1 > 5$; $w_2 > 5$; $w_3 > 5$; $w_4 = 5$
13. $w_1 > 5$; $w_2 > 5$; $w_4 > 5$; $w_3 = 5$
14. $w_1 > 5$; $w_3 > 5$; $w_4 > 5$; $w_2 = 5$
15. $w_2 > 5$; $w_3 > 5$; $w_4 > 5$; $w_1 = 5$
16. $w_1 > 5$; $w_2 > 5$; $w_3 > 5$; $w_4 > 5$

Case 1: $w_1 = w_2 = w_3 = w_4 = 5$

→ ⑤ : $100 - w_1 - w_2 - w_3 - w_4 = 80 \neq 0$ ⇒ Not feasible

Case 2: $w_1 > 5$; $w_2 = w_3 = w_4 = 5$

→ ⑤ : $w_1 = 100 - 15 = 85$ → ⑥ $\frac{\partial \mathcal{L}}{\partial \delta_1} = w_1 - 5 = 80 > 0$

→ ⑩ : $\delta_1 \cdot \frac{\partial \mathcal{L}}{\partial \delta_1} = 0$ → $\delta_1 = 0$

→ ① :

$$2w_1\sigma_{11} + 2w_2\sigma_{12} + 2w_3\sigma_{13} + 2w_4\sigma_{14} = -\lambda$$
$$2w_1\sigma_{12} + 2w_2\sigma_{22} + 2w_3\sigma_{23} + 2w_4\sigma_{24} = \delta_2 - \lambda$$
$$2w_1\sigma_{31} + 2w_2\sigma_{32} + 2w_3\sigma_{33} + 2w_4\sigma_{43} = \delta_3 - \lambda$$
$$2w_1\sigma_{41} + 2w_2\sigma_{42} + 2w_3\sigma_{43} + 2w_4\sigma_{44} = \delta_4 - \lambda$$

Or: $2w'\Sigma = \begin{bmatrix} -\lambda \\ \delta_2 - \lambda \\ \delta_3 - \lambda \\ \delta_4 - \lambda \end{bmatrix}$

⇒ $\lambda = -75.118$
$\delta_2 = -55.604$
$\delta_3 = -41.711$
$\delta_4 = -12.44$

Case 3: $w_2 > 5$; $w_1 = w_3 = w_4 = 5$

(Similar process as Case 2)

$w_2 = 85$; $\delta_2 = 0$

$$\Rightarrow 2w'\Sigma = \begin{bmatrix} \delta_1 - \lambda \\ -\lambda \\ \delta_3 - \lambda \\ \delta_4 - \lambda \end{bmatrix}$$

$\Rightarrow$ $\delta_1 = -29.524$, $\lambda = -46.858$, $\delta_3 = -11.611$, $\delta_4 = -5.7$

Case 4: $w_3 > 5$; $w_1 = w_2 = w_4 = 5$

(Similar process as Case 2)

$w_3 = 85$; $\delta_3 = 0$

$$\Rightarrow 2w'\Sigma = \begin{bmatrix} \delta_1 - \lambda \\ \delta_2 - \lambda \\ -\lambda \\ \delta_4 - \lambda \end{bmatrix}$$

$\Rightarrow$ $\delta_1 = -82.545$, $\delta_2 = -83.525$, $\lambda = -113.775$, $\delta_3 = -63.481$

Case 5: $w_4 > 5$; $w_1 = w_2 = w_3 = 5$

(Similar process as Case 2)

$w_4 = 85$; $\delta_4 = 0$

$$\Rightarrow 2w'\Sigma = \begin{bmatrix} \delta_1 - \lambda \\ \delta_2 - \lambda \\ \delta_3 - \lambda \\ -\lambda \end{bmatrix}$$

$\Rightarrow$ $\delta_1 = -64.952$, $\delta_2 = -89.292$, $\delta_3 = -75.159$, $\lambda = -123.11$

Case 6: $w_1 > 5$; $w_2 > 5$; $w_3 = w_4 = 5$

⑤ $\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - w_1 - w_2 - 10 = 0 \Rightarrow w_1 + w_2 - 90 = 0$

Since ⑥ $\frac{\partial \mathcal{L}}{\partial \delta_1} = w_1 - 5 > 0$; ⑩ : $\delta_1 \cdot \frac{\partial \mathcal{L}}{\partial \delta_1} = 0 \Rightarrow \delta_1 = 0$

Similarly, from ⑦ and ⑪ : $\delta_2 = 0$

$\Rightarrow$ A system of 3 equations to solve for w_1 ; w_2 ; λ

①: $2w_1\sigma_{11} + 2w_2\sigma_{12} + \lambda = -2w_3\sigma_{13} - 2w_4\sigma_{14}$

②: $2w_1\sigma_{12} + 2w_2\sigma_{22} + \lambda = -2w_3\sigma_{23} - 2w_4\sigma_{24}$

$w_1 + w_2 = 90$

Or:
$$\begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & 1 \\ 2\sigma_{12} & 2\sigma_{22} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_3\sigma_{13} - 2w_4\sigma_{14} \\ -2w_3\sigma_{23} - 2w_4\sigma_{24} \\ 90 \end{bmatrix}$$

$\Rightarrow$ $w_1 = 33.1142$, $w_2 = 56.8858$, $\lambda = -40.88376$

System of 2 equation from ③ and ④

$\delta_3 = 2w_1\sigma_{13} + 2w_2\sigma_{23} + 2w_3\sigma_{33} + 2w_4\sigma_{34} + \lambda$

$\delta_4 = 2w_1\sigma_{14} + 2w_2\sigma_{24} + 2w_3\sigma_{34} + 2w_4\sigma_{44} + \lambda$

Or:
$$\begin{bmatrix} \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{43} & 1 \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix}$$

$\Rightarrow$ $\delta_3 = -6.2834$ $\delta_4 = 7.837$

Case 7: $w_1 > 5$; $w_3 > 5$; $w_2 = w_4 = 5$

(Similar process as Case 6): $\delta_2 = \delta_3 = 0$

$$\begin{bmatrix} 2\sigma_{11} & 2\sigma_{13} & 1 \\ 2\sigma_{31} & 2\sigma_{33} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_2\sigma_{12} - 2w_4\sigma_{14} \\ -2w_2\sigma_{23} - 2w_4\sigma_{34} \\ 90 \end{bmatrix}$$

$\Rightarrow$ $w_1 = 58.1451$, $w_3 = 31.855$, $\lambda = -60.385$

$$\begin{bmatrix} \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & 1 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_4 \end{bmatrix}$$

$\Rightarrow \delta_2 = -37.2675$

$\delta_4 = -1.8645$

Case 8: $w_1 > 5$; $w_4 > 5$; $w_2 = w_3 = 5$

$$\begin{bmatrix} 2\sigma_{11} & 2\sigma_{14} & 1 \\ 2\sigma_{41} & 2\sigma_{44} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_2\sigma_{12} - 2w_3\sigma_{13} \\ -2w_2\sigma_{24} - 2w_3\sigma_{34} \\ 90 \end{bmatrix}$$

$\Rightarrow w_1 = 72.141$

$w_4 = 17.859$ $\quad \delta_1 = \delta_4 = 0$

$\lambda = -72.392$

$$\begin{bmatrix} \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & 1 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

$\Rightarrow \delta_2 = -50.57862$

$\delta_3 = -36.697$

Case 9: $w_2 > 5$; $w_3 > 5$; $w_1 = w_4 = 5$

(Similar process as Case 6) $\quad \delta_2 = \delta_3 = 0$

$$\begin{bmatrix} 2\sigma_{22} & 2\sigma_{23} & 1 \\ 2\sigma_{23} & 2\sigma_{33} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_1\sigma_{21} - 2w_4\sigma_{24} \\ -2w_1\sigma_{31} - 2w_4\sigma_{34} \\ 90 \end{bmatrix}$$

$\Rightarrow w_2 = 75.236$

$w_3 = 14.764$

$\lambda = -44.831$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & 1 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_4 \end{bmatrix}$$

$\Rightarrow \delta_1 = -21.411$

$\delta_4 = -2.558$

Case 10: $w_2 > 5$; $w_4 > 5$; $w_1 = w_3 = 5$

(Similar process as Case 6) $\quad \delta_2 = \delta_4 = 0$

$$\begin{bmatrix} 2\sigma_{22} & 2\sigma_{24} & 1 \\ 2\sigma_{42} & 2\sigma_{44} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_1\sigma_{21} - 2w_3\sigma_{23} \\ -2w_1\sigma_{41} - 2w_3\sigma_{43} \\ 90 \end{bmatrix}$$

$\Rightarrow w_2 = 80.2$

$w_4 = 9.8$

$\lambda = -46.076$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & 1 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_3 \end{bmatrix}$$

$\Rightarrow \delta_1 = -21.59$

$\delta_3 = -10.066$

Case 11: $w_3 > 5$; $w_4 > 5$; $w_1 = w_2 = 5$

(Similar process as Case 6) $\quad \delta_3 = \delta_4 = 0$

$$\begin{bmatrix} 2\sigma_{33} & 2\sigma_{34} & 1 \\ 2\sigma_{43} & 2\sigma_{44} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_3 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2w_1\sigma_{31} - 2w_2\sigma_{32} \\ -2w_1\sigma_{41} - 2w_2\sigma_{42} \\ 90 \end{bmatrix}$$

$\Rightarrow w_3 = 48.37$

$w_4 = 41.63$

$\lambda = -83.635$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & 1 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & 1 \end{bmatrix} \begin{bmatrix} 2w_1 \\ 2w_2 \\ 2w_3 \\ 2w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$\Rightarrow \delta_1 = -40.075$

$\delta_4 = -51.75$

Case 12: $w_1 > 5$; $w_2 > 5$; $w_3 > 5$; $w_4 = 5$

For ⑥, ⑦, ⑧ : $\frac{\partial \mathcal{L}}{\partial \delta_1} > 0$; $\frac{\partial \mathcal{L}}{\partial \delta_2} > 0$; $\frac{\partial \mathcal{L}}{\partial \delta_3} > 0$

⇒ From ⑩ ; ⑪ ; ⑫ : $\delta_1 = \delta_2 = \delta_3 = 0$

System of 4 equations for 4 variables:

⑤ : $w_1 + w_2 + w_3 = 95$

① : $2w_1\sigma_{11} + 2w_2\sigma_{12} + 2w_3\sigma_{13} + \lambda = -2w_4\sigma_{14}$

② : $2w_1\sigma_{21} + 2w_2\sigma_{22} + 2w_3\sigma_{23} + \lambda = -2w_4\sigma_{24}$

③ : $2w_1\sigma_{31} + 2w_2\sigma_{23} + 2w_3\sigma_{33} + \lambda = -2w_4\sigma_{34}$

Or:
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2\sigma_{11} & 2\sigma_{12} & 2\sigma_{13} & 1 \\ 2\sigma_{21} & 2\sigma_{22} & 2\sigma_{23} & 1 \\ 2\sigma_{31} & 2\sigma_{23} & 2\sigma_{33} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 95 \\ -2w_4\sigma_{14} \\ -2w_4\sigma_{24} \\ -2w_4\sigma_{34} \end{bmatrix}$$

⇒ $w_1 = 28.371$; $w_2 = 58.13$; $w_3 = 8.498$; $\lambda = -38.1482$

From ④ : $\delta_4 = 2w_4\sigma_{44} + 2w_1\sigma_{41} + 2w_2\sigma_{42} + 2w_3\sigma_{43} + \lambda$

⇒ $\delta_4 = 9.7$

Case 13: $w_2 > 5$; $w_3 > 5$; $w_4 > 5$; $w_1 = 5$

(Similar process as Case 12)

$\delta_2 = \delta_3 = \delta_4 = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2\sigma_{22} & 2\sigma_{23} & 2\sigma_{24} & 1 \\ 2\sigma_{32} & 2\sigma_{33} & 2\sigma_{34} & 1 \\ 2\sigma_{42} & 2\sigma_{43} & 2\sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} 95 \\ -2w_1\sigma_{21} \\ -2w_1\sigma_{31} \\ -2w_1\sigma_{41} \end{bmatrix}$$

⇒ $w_2 = 73.54$; $w_3 = 14.135$; $w_4 = 7.325$; $\lambda = -44.583$

$\delta_1 = 2w_1\sigma_{11} + 2w_2\sigma_{21} + 2w_3\sigma_{31} + 2w_4\sigma_{41} + \lambda$

⇒ $\delta_1 = -20.192$

Case 14: $w_1 > 5$; $w_3 > 5$; $w_4 > 5$; $w_2 = 5$

(Similar process as Case 12)

$\delta_1 = \delta_3 = \delta_4 = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2\sigma_{11} & 2\sigma_{13} & 2\sigma_{14} & 1 \\ 2\sigma_{31} & 2\sigma_{33} & 2\sigma_{34} & 1 \\ 2\sigma_{41} & 2\sigma_{43} & 2\sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} 95 \\ -2w_2\sigma_{12} \\ -2w_2\sigma_{32} \\ -2w_2\sigma_{42} \end{bmatrix}$$

⇒ $w_1 = 56.54$; $w_3 = 31.31$; $w_4 = 7.15$, $\lambda = -60.23$

$\delta_2 = 2w_1\sigma_{21} + 2w_2\sigma_{22} + 2w_3\sigma_{23} + 2w_4\sigma_{24} + \lambda$

⇒ $\delta_2 = -36.8$

Case 15: $w_1 > 5$; $w_2 > 5$; $w_4 > 5$; $w_3 = 5$

(Similar process as Case 12)

$\delta_1 = \delta_2 = \delta_4 = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2\sigma_{11} & 2\sigma_{12} & 2\sigma_{14} & 1 \\ 2\sigma_{21} & 2\sigma_{22} & 2\sigma_{24} & 1 \\ 2\sigma_{41} & 2\sigma_{42} & 2\sigma_{44} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} 95 \\ -2w_3\sigma_{13} \\ -2w_3\sigma_{23} \\ -2w_3\sigma_{43} \end{bmatrix}$$

⇒ $w_1 = 36.411$; $w_2 = 64.947$; $w_4 = -6.357$ (Violate constraint)

Case 16: $w_1 > 5$; $w_2 > 5$; $w_3 > 5$; $w_4 > 5$

The Kuhn-Tucker problem becomes a Lagrangian problem

$$A = \begin{bmatrix} 2\tilde{\Sigma} & -\tilde{1} \\ -\tilde{1}' & 0 \end{bmatrix} \qquad z = \begin{bmatrix} \tilde{w} \\ \lambda \end{bmatrix} \qquad c = \begin{bmatrix} \tilde{0} \\ -100 \end{bmatrix}$$

⇒ $w_1 = 35.314$; $w_2 = 61.89$; $w_3 = 9.818$; $w_4 = -7.126$ (Violate constraint)