Assignment No 8: The Digital Fourier Transform

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1 Introduction

The assignment deals with visualizing the Discrete Fourier Transform using Python.

2 The Assignment

2.1 Theory

Consider $f[n]$, a periodic sequence of samples with a period N. The DFT expression is

$$F[k] = \sum_{n=0}^{N-1} f[n]W^{nk}$$

$$f[n] = \sum_{n=0}^{N-1} \frac{F[k]W^{-nk}}{N}$$

where $W = exp(\frac{-2\pi j}{N})$. DFT is a sampled version of the DTFT.

The following python code is used for the implementation of the forward Fourier transform and the inverse transform.

```
numpy.fft.fft()
numpy.fft.ifft()
```

2.2 Random function

```
from pylab import *
x=rand(128)
X=fft(x)
y=ifft(X)
c_[x,y]
print (abs((x-y).max()))
```

An error of the order 10^{-15} is encountered due to the finite accuracy of the CPU.

2.3 Spectrum of $\sin(5x)$

As seen later on also, any sinusoid can be written as a sum of exponentials.

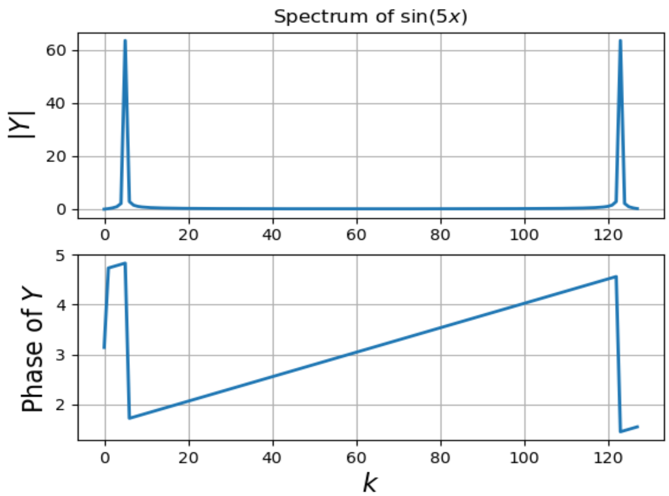
$$\sin(x) = \frac{exp(jx)-exp(-jx)}{2j}).$$

The spectrum of $\sin(5x)$ is:

$$Y(w) = \frac{\delta(w-5)-\delta(w+5)}{2j}$$

The following code is used:

```
x=linspace(0,2*pi,128)
y=sin(5*x)
Y=fft(y)
figure()
subplot(2,1,1)
plot(abs(Y),lw=2)
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\sin(5x)$")
grid(True)
subplot(2,1,2)
plot(unwrap(angle(Y)),lw=2)
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
show()
```



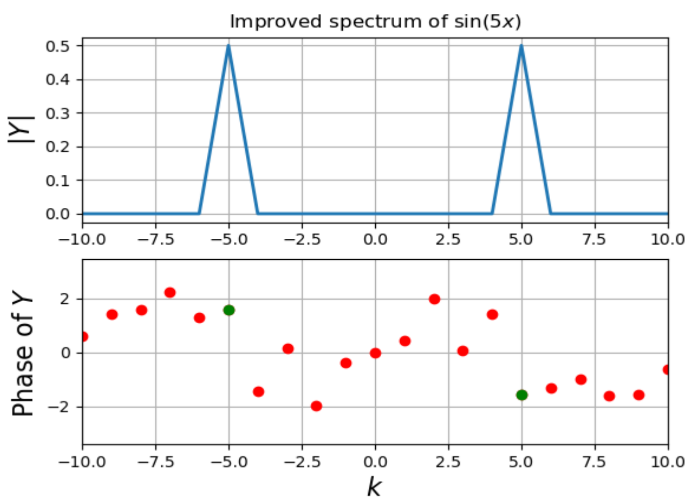
However, there is energy at nearby frequencies (the spikes aren't very sharp). Moreover, the height of the spikes must be 0.5, not 64. The frequency axis also has not been corrected for. Some corrections are clearly to be made.
With a command called fftshift, we shift the π to 2π portion to the left as it represents negative frequency. Moreover, since 0 and 2π represent the same point, it makes no sense to count both. We simply create a vector of 129 values and drop the last point.

2.4 Improved spectrum of sin(5x)

The following code improves upon the issues discussed above.

```
x=linspace(0,2*pi,129)
x=x[:-1] #we create an array of 128 points, stopping just before 2pi
y=sin(5*x)
Y=fftshift(fft(y))/128.0
w=linspace(-64,63,128)
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
title(r"Improved spectrum of $\sin(5x)$")
grid(True)
subplot(2,1,2)
```

```
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
show()
```



2.5 Spectrum of (1+0.1cost)cos10t

The given expression can be written as:

$$(1 + 0.1\cos(t))\cos(10t) = \cos(10t) + 0.05\cos(11t) + 0.05\cos(9t)$$

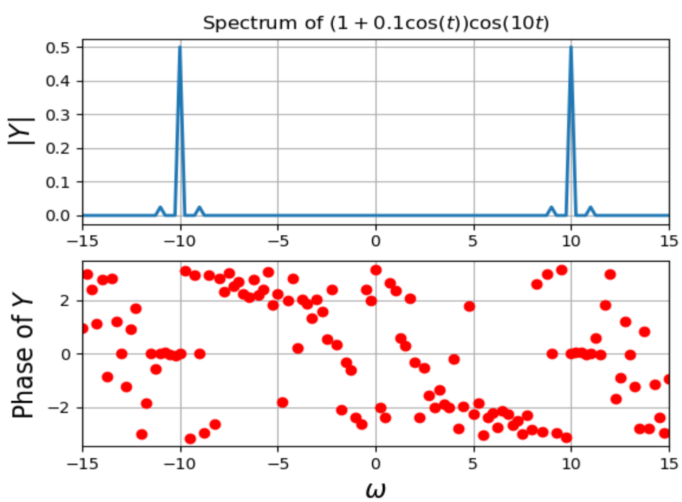
The spectrum is:

$$Y(w) = \frac{\delta(w-10)+\delta(w+10)}{2j} + \frac{\delta(w-11)+\delta(w+11)}{40j} + \frac{\delta(w-9)+\delta(w+9)}{40j}$$

The following code is used:

```
t=linspace(-4*pi,4*pi,513)
t=t[:-1]
y=(1+0.1*cos(t))*cos(10*t)
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);w=w[:-1]
figure()
```

```
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\left(1+0.1\cos\left(t\right)\right)\cos\left(10t\right)$")
grid(True)
subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
show()
```



2.6 Spectrum of $\sin^3(t)$ and $\cos^3(t)$

The above expression can be rewritten as follows.

$$\sin^3(t) = 0.75\sin(t) - 0.25\sin(3t)$$

The corresponding spectrum of the above expression is:

$$Y(w) = \frac{3(\delta(w-1)-\delta(w+1))}{8j} - \frac{1(\delta(w-3)-\delta(w+3))}{8j}$$

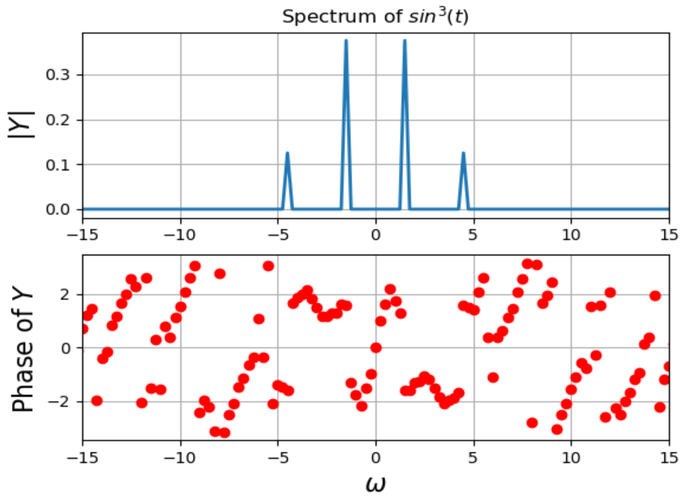
Similarly, $\cos^3(t)$ can be rewritten as follows.

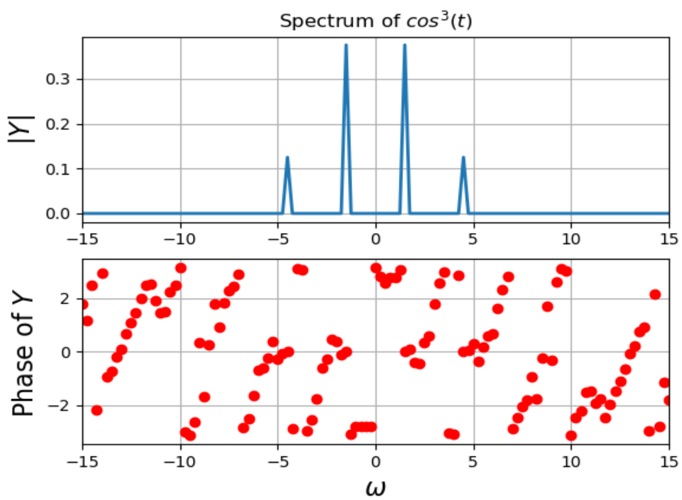
$$\cos^3(t) = 0.25\cos(3t) - 0.75\cos(t)$$

The corresponding spectrum of the above expression is:

$$Y(w) = \frac{1(\delta(w-3)+\delta(w+3))}{8j} - \frac{3(\delta(w-1)+\delta(w+1))}{8}$$

```
def spectrum1(N,k,str):
        t=linspace(-6*pi,6*pi,N+1)
        t=t[:-1]
        if k==1:
                y=pow(sin(t),3)
        if k==2:
                y=pow(cos(t),3)
        Y=fftshift(fft(y))/N
        w=linspace(-64,64,N+1)
        w=w[:-1]
        figure()
        subplot(2,1,1)
        plot(w,abs(Y),lw=2)
        ylabel(r"$|Y|$",size=16)
        title(str)
        xlim([-15,15])
        grid(True)
        subplot(2,1,2)
        plot(w,angle(Y),'ro',lw=2)
        ylabel(r"Phase of $Y$",size=16)
        xlabel(r"$\omega$",size=16)
        xlim([-15,15])
        grid(True)
        show()
spectrum1(512,1,r"Spectrum of $sin^3(t)$")
spectrum1(512,2,r"Spectrum of $cos^3(t)$")
```





2.7 Spectrum of $cos(20t + 5cos(t))$ and Gaussian function

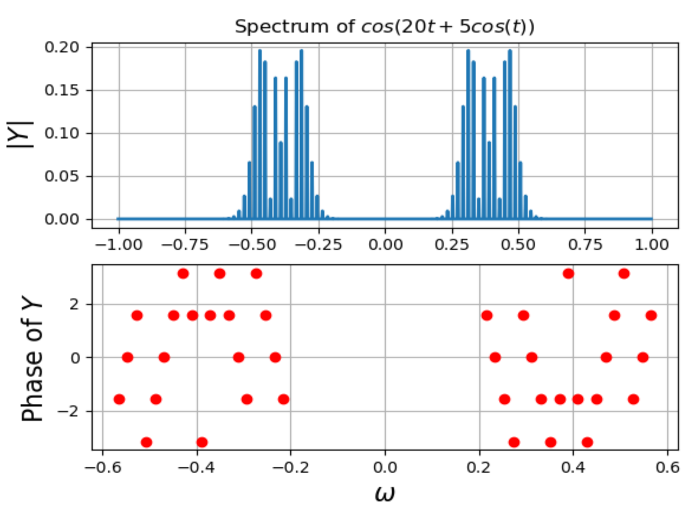
The same piece of code has been employed to compute both.

```
N=1024
def spectrum2(N,k,str):
        t=linspace(-10*pi,10*pi,N+1)
        t=t[:-1]
        if k==1:
                y=cos(20*t+5*cos(t))
```

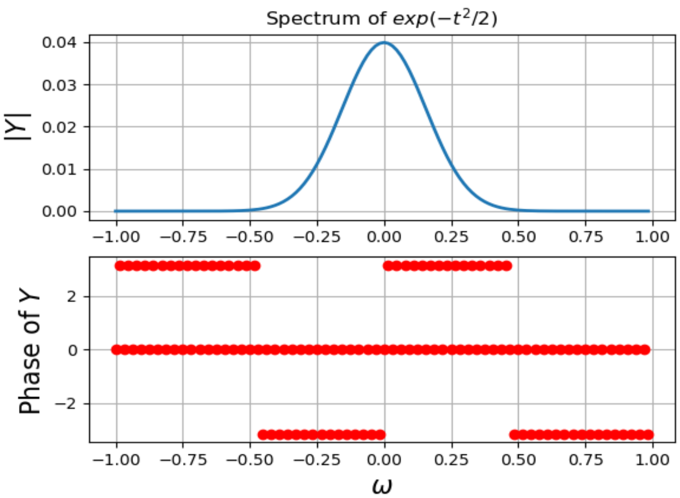
```
    if k==0:
            y=exp(-(t*t)/2)
    Y=fftshift(fft(y))/N
    w=linspace(-1,1,N+1)

    w=w[:-1]
    figure()
    subplot(2,1,1)
    plot(w,abs(Y),lw=2)
    ylabel(r"$|Y|$",size=16)
    title(str)
    grid(True)
    subplot(2,1,2)
    ii=where(abs(Y)>(k*1e-3))
    plot(w[ii],angle(Y[ii]),'ro',lw=2)
    ylabel(r"Phase of $Y$",size=16)
    xlabel(r"$\omega$",size=16)
    grid(True)
    show()

spectrum2(1024,1,r"Spectrum of $cos(20t+5cos(t))$")
spectrum2(128,0,r"Spectrum of $exp(-t^2/2)$")
```



$cos(20t+5cos(t))$ must, at its extremes, look like $cos(20t+5)$ and $cos(20t-5)$. The spectrum reflects that accordingly, with energy at a lot of frequencies.



The Fourier transform of a Gaussian is an Gaussian in ω.Therefore, the spectrum takes a similar shape. By accordingly adjusting the number of points in the time range we obtain a suitable Gaussian.

3 Conclusion

The DFT was obtained and graphically studied for various functions.